

博士論文

dS/CFT correspondence for heavy scalar field

(dS/CFT対応における重いスカラー場)

劉 凱淇

Abstract

In this thesis, we perform a comprehensive study on heavy scalar fields in de Sitter space and explore their quantum properties in relation to dS/CFT. We first study wavefunctions on de Sitter spacetime and their implications to dS/CFT correspondence. In contrast to light fields in the complementary series, heavy fields in the principal series oscillate outside the cosmological horizon. As a consequence, the quadratic term in the wavefunction does not follow a simple scaling and so it is hard to identify it with a conformal two-point function. We demonstrate that it should be interpreted as a two-point function on a cyclic RG flow which is obtained by double-trace deformations of the dual CFT. This is analogous to the situation in nonrelativistic AdS/CFT with a bulk scalar whose mass squared is below the Breitenlohner-Freedman (BF) bound. We also provide a new dS/CFT dictionary relating de Sitter two-point functions and conformal two-point functions in the would-be dual CFT.

We also study dS/CFT for global spacelike slices for heavy scalar fields by using quasinormal modes. We show that the Euclidean vacuum can be described as a thermofield double state in the dual would-be CFT description. Tracing over one copy of the would-be CFT produces a mixed thermal state describing a single static causal patch. Nonetheless, the boundary field corresponds to a mixed form of CFT operators with different conformal dimensions, making identification with a CFT difficult.

Acknowledgement

First and foremost, I would like to express my deepest gratitude to my supervisor, Prof. Yoshio Kikukawa, for his consistent support and guidance during my Ph.D. study. His expertise and insightful feedback were instrumental in shaping not just this thesis, but also my growth as a researcher and a thinker.

I am also immensely thankful to Prof. Toshifumi Noumi, for his invaluable contribution to my research projects and willingness to engage in deep discussions. The generosity with which he shared his time and expertise has played a critical role in the completion of this thesis.

Also I would especially like to thank my collaborators, Dr. Hiroshi Isono, for fruitful discussions, as the works done in collaboration with him constitute one of the most important parts of this thesis.

My appreciation extends to my fellow group members and the secretary, Toko Sasaki, in Komaba particle theory group for their help and stimulating discussions.

Last but not least, I would like to thank my family, especially my parents for their love and unwavering support, which have been the bedrock of my journey through this Ph.D. study. Their presence in my life is a blessing I am eternally grateful for.

Contents

1	Introduction and summary	8
2	Geometry of de Sitter space	12
2.1	Generality	12
2.2	Global coordinates (closed patch)	13
2.3	Poincare coordinates (Flat chart)	14
2.4	Static coordinates	15
3	Quantum field in de Sitter	17
3.1	dS unitary irreducible representations (UIRs)	17
3.2	Scalar field quantization	18
3.2.1	Poincare coordinates	20
3.2.2	Global coordinates	22
3.3	Wavefunction method and no-boundary wavefunction	25
3.4	Conformal field theory	27
3.4.1	Conformal group	27
3.4.2	CFT correlation functions	29
3.5	dS/CFT	33
3.5.1	dS_3/CFT_2	36
3.5.2	Correlation functions for light field	40
4	Wavefunctions on de Sitter space	42
4.1	Dirichlet boundary conditions	44
4.2	Mixed boundary conditions	48
4.3	Double-trace deformations	50
4.4	Wavefunctions for light fields	51
4.4.1	Dirichlet boundary conditions	51
4.4.2	Mixed boundary conditions	52
4.4.3	Relation between the two CFTs	54
4.4.4	Mixed boundary condition for general ν	54
5	dS/CFT dictionary revisited	58
5.1	General formula for correlation functions	58
5.2	Two-point functions	61
5.2.1	Heavy fields	61
5.2.2	Light fields	62
6	dS quasinormal modes and dS/CFT	63
6.1	dS quasinormal modes	64
6.2	Northern and southern modes	69

6.2.1	Quantization and vacua	72
6.3	Comments on dS/CFT correspondence	74
7	Conclusion and outlook	77
A	dS_4 isometry groups	79
B	Double-trace deformations	80
C	Hypergeometric functions	83

1 Introduction and summary

Why dS/CFT?

Since the discovery of the cosmic microwave background (CMB) half a century ago, observational cosmology has made considerable progress in developing increasingly precise understandings of the early universe. Various observations, particularly CMB's [1, 2] strongly support the theory of an early inflationary phase of the universe, characterized by rapid expansion and accounts for the origin of the large-scale structure of the universe. This phase, which closely resembles a de Sitter (dS) phase, is believed to have left quantum fluctuations imprinted in the photons observed in these studies, providing insights into the state of the universe just 10^{-36} seconds after its birth, about 14 billion years ago. Moreover, the present accelerated expansion of the universe resembling the early period of inflation has been discovered and the 2011 Nobel Prize in Physics was awarded for this discovery. This observation suggests the existence of a very small but non-zero cosmological constant, although the origin of it remains a challenge to understand in modern physics. Given the relevance of de Sitter space to at least two epochs of the universe, understanding quantum gravity in dS space is considered as crucial.

The most promising proposal for a UV-complete theory of quantum gravity is string theory, whose dynamics at low energy is captured by Einstein gravity. A significant advancement in string theory is the AdS/CFT duality conjecture, which states that certain conformal field theories (CFTs) can describe UV-complete gravitational theories in asymptotically Anti de Sitter (AdS) spacetimes [3]. This conjecture has been extensively studied since then, leading to a framework which includes a “dictionary” that enables the translation of various quantities between the two sides of the duality [4–6]. A key aspect of this framework is the ability to compute CFT correlation functions within the context of supergravity.

One crucial property of this duality conjecture is its holographic nature, where the spacetime of the CFT is of lower dimension than the dynamical spacetime in the corresponding gravitational theory. This property is an application of the holographic principle, a concept in quantum gravity that was originally developed through the study of the thermodynamic properties of black holes [7–11]. Although holographic theories were originally developed to describe the microscopic quantum properties of interactions between elementary particles, they are intricately dependent on the global structure of spacetime, thus linking the search for a UV-complete theory of quantum gravity directly to cosmology. As previously stated, the observable universe does not present itself as asymptotically AdS, which leads to the exploration of a possible holographic duality for de Sitter spacetimes [12–14].

It was found that some key features of AdS/CFT, such as the matching of symmetry groups between the dual theories, also find parallels in de Sitter space. In developing the dS/CFT framework, correspondences were established between different types of CFT

operators and various bulk fields in dS space. Additionally, it was demonstrated that the bulk correlation functions in dS space have properties analogous to those of the CFT correlation functions. However, dS/CFT is significantly less studied as compared to its AdS counterpart, due to various conceptual and technical challenges it presents. For instance, in dS/CFT, the holographically emergent dimension is temporal rather than spatial. Despite we have established this dS/CFT framework, efforts to develop a concrete model within string theory, similar to the successful constructions in AdS/CFT, have not yet been realized. Furthermore, unlike AdS/CFT, there are only a few concrete examples, such as higher spin holography [15] and 3d Einstein gravity holography [16].

Why heavy fields?

To our knowledge, most of the research on dS/CFT [14, 15, 17–59], including higher spin holography and holographic inflation, has focused on light fields because they correspond to fields with mass squared above the Breitenlohner-Freedman (BF) bound in AdS space after analytic continuation. Thus, they share the same asymptotic behavior as fields in AdS around the boundary, leading to a straightforward extension from the AdS/CFT to the dS/CFT dictionary. However, heavy fields oscillate outside the horizon and are dual to CFT operators with complex conformal dimensions at the boundary. They correspond to AdS fields with mass squared below the BF bound, where tachyonic instability appears. Moreover, a heavy field cannot be related to a single CFT operator, since both modes which are complex conjugate of each other remaining at the boundary, unlike light fields where one mode dominates. As a result, the usual dS/CFT dictionary does not apply to heavy fields even for two point functions.

However, such heavy fields are inevitable once we consider UV completion of the bulk theory, which is supposed to include the entire energy spectrum. As a matter of fact, there exist certain unique lower mass bounds for fields in de Sitter space. For example, Higuchi bound is a bound on the mass of fields with spin two or higher in de Sitter space which captures the absence of ghosts. This bound was derived under the requirement of unitarity [60]. Additionally, it is suggested that for scalar fields, quasinormal modes (whose basis is considered as a potential candidate basis for dS/CFT) for light fields result in vanishing energy flux and hence they are more well defined for heavy fields [61, 62]. Therefore, the study of heavy fields is crucial for gaining insight into the applicability and limitations of dS/CFT, and hence for advancing our understanding of quantum gravity, cosmology, and the fundamental nature of the universe.

Overview of this thesis

In this thesis we study scalar fields in de Sitter space and explore their quantum properties in relation to dS/CFT, in particular focusing on heavy fields with mass in the principle series. The sections are organized as follows. Sections 2 and 3 are review sections, and the main part starts with section 4.

In section 2, we will give a general introduction to de Sitter space on its geometry, introducing various coordinate systems.

Section 3 begins by examining the standard canonical quantization scheme for scalar fields in de Sitter space. We will explore how scalar fields behave under different coordinate systems. Then we will introduce the wavefunction method, which is an alternative approach to studying quantum fields. It provides a different perspective on how quantum states evolve and it is closely related dS/CFT correspondence. The final part of this section bridges these quantum field theoretical approaches with the dS/CFT correspondence. We will outline the statement and original proposal of dS/CFT and give a review to the dS_3/CFT_2 example presented in Ref. [12]. Then we will present the dictionary for correlation functions, which turns out to be valid only for light scalar fields.

Section 4 is one of the main parts of this thesis, where we will study wavefunctions of heavy scalars on de Sitter spacetime and their implications for dS/CFT correspondence. We analyze them by imposing various boundary conditions: Dirichlet boundary conditions and a type of mixed boundary conditions, which eliminate one asymptotic mode on the boundary and keep the other. We will demonstrate that dS wavefunctions with mixed boundary conditions are naturally identified with generating functions of the would-be dual CFTs. We will then show that wavefunctions with the Dirichlet boundary conditions correspond to the double-trace deformations of these would-be dual CFTs. We will also study the case of light fields for comparison.

In section 5, we will provide a dS/CFT dictionary applicable when the mixed boundary conditions are employed. Together with the results obtained in section 4, we find that the quadratic term in the wavefunction should be interpreted as a two-point function on a cyclic RG flow which is obtained by double-trace deformations of the dual CFT. This is analogous to the situation in nonrelativistic AdS/CFT with a bulk scalar whose mass squared is below the Breitenlohner-Freedman (BF) bound.

In section 6, we will explore a different type of dS/CFT for global de Sitter, which associates states on global spacelike slices instead of the past/future boundary, using a quasinormal mode basis. In Ref. [63], it is shown that for light scalar fields, the Euclidean vacuum is a thermofield double state in the dual CFT description, and that the global de Sitter geometry arises from quantum entanglement between two copies of the CFT. This is analogous to the situation of eternal AdS black holes in AdS/CFT and resembles the concept of ER=EPR [64]. We extend the discussion in Ref. [63] to heavy scalar fields and demonstrate that the thermofield double state description for Euclidean vacuum holds for heavy scalar fields as well. If we trace the Euclidean vacuum over southern modes, we obtain the expected northern density matrix ρ_N proportional to $e^{-2\pi H_{\text{static}}}$, where H_{static} is the static patch Hamiltonian. However, we encounter obstacles when we try to identify the entangled states with CFT operators, due to the mode mixing nature of heavy fields, as mentioned in the earlier sections. Consequently, the dictionary between the late time

two-point function in de Sitter and the tensor products of CFT operators derived in [63] does not apply to heavy fields. It requires further consideration to identify the dual would-be CFTs to the Euclidean vacuum states.

In the conclusion, section 7, we will more explicitly bring these parts together into a coherent whole, with discussion of our results.

2 Geometry of de Sitter space

De Sitter spacetime is the maximally symmetric solution to Einstein's equations with a positive cosmological constant. In this section, we present the classical properties of de Sitter space by introducing various different coordinates, which is an important preliminary to the study of quantum field theory on this background. More details and extensive discussion are well summarized in Ref. [65].

2.1 Generality

The $(d + 1)$ dimensional de Sitter geometry is realized as the hyperboloid embedded in $(d + 2)$ dimensional Minkowski space,

$$\eta_{AB}X^AX^B = H^{-2} \equiv l_{dS}^2, \quad \eta = \text{diag}(-1, 1, \dots, 1), \quad (2.1)$$

where the indices run from 0 to $d + 1$. The Hubble scale H determines the value of the cosmological constant Λ , which is given by $\Lambda = 1/d(d - 1)H^2$, and the curvature R is expressed as $R = d(d + 1)H^2$. This description makes manifest the $SO(1, d + 1)$ isometry group of dS_{d+1} , which is the same with one of the embedded Minkowski space. The Killing vectors are

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}, \quad (2.2)$$

which satisfy the algebra,

$$[L_{AB}, L_{CD}] = \eta_{BC}L_{AD} - \eta_{AC}L_{BD} + \eta_{AD}L_{BC} - \eta_{BD}L_{AC}. \quad (2.3)$$

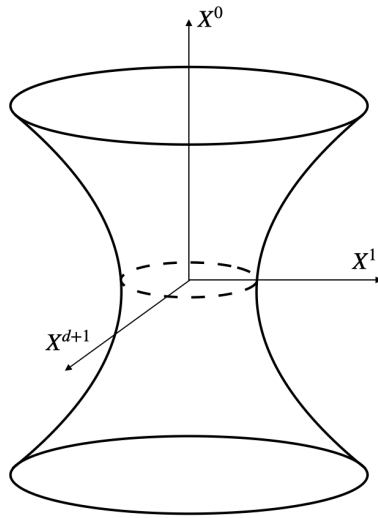


Fig 1: $(d + 1)$ dimensional de Sitter hyperboloid embedded in $(d + 2)$ dimensional Minkowski space. The dotted line denotes an extremal volume S^d .

It is well known that dS_{d+1} is related to the Euclidean sphere S^{d+1} by analytic continuation $X^0 \rightarrow iX^0$. Since S^{d+1} is compact, no IR divergences will arise through integrals over it. This property is convenient for dealing with IR divergences appear in dS correlation functions [66–68]. It is also useful for defining Euclidean vacuum and the corresponding modes, which we will present in later sections.

There are various coordinate systems available for de Sitter space. Sometimes choosing a particular coordinate system can offer distinct advantages depending on the specific issue being addressed. In the following subsections, we present an overview of different coordinate systems that can be used for de Sitter space. For simplicity, we set $H = 1$ in the following section. The factors of H can be easily reintroduced by dimensional analysis.

2.2 Global coordinates (closed patch)

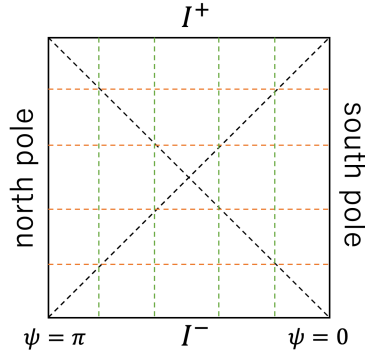


Fig 2: Penrose diagram for global coordinates. $\psi = \arccos z_1 \in (0, \pi)$ is the polar angle on the spatial S^d . The orange and green dotted lines denote the constant t and ψ slices.

First, we introduce the global coordinates that cover the entire de Sitter hyperboloid, as their name implies. The coordinate system is obtained by setting

$$\begin{aligned} X^0 &= \sinh t, \\ X^i &= z^i \cosh t, \quad i = 1, \dots, d+1, \end{aligned} \tag{2.4}$$

where the global time t runs from $-\infty$ to $+\infty$ and z_i s are constrained to the unit sphere, $\mathbf{z}^2 = 1$. The metric is given by

$$ds^2 = -dt^2 + \cosh^2 t d\Omega_d^2, \tag{2.5}$$

where $d\Omega_d^2$ is the metric of a d -dimensional sphere. In this coordinate system, the spatial sections are d dimensional closed spheres. They begin as large spheres in the past, contract to a minimal size at $t = 0$, and then begin expanding again. The Penrose diagram is depicted in Fig. 2.

The global coordinates in the context of cosmology are often referred to as the closed patch of de Sitter space. This terminology can be explained by recalling the fact that the

Friedmann-Robertson-Walker (FRW) metric describes the spacetime for any homogeneous and isotropic universe as

$$ds_{FRW}^2 = -dt^2 + a(t)^2 d\Sigma^2, \quad (2.6)$$

where Σ denotes the spatial sections and there are three possible geometries: open (hyperbolas), flat (R^d), or closed (spheres). As we can see from the metric (2.5), de Sitter space exemplifies a closed FRW universe, and the global coordinates portray closed spatial sections.

2.3 Poincare coordinates (Flat chart)

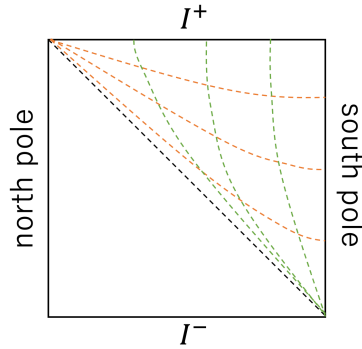


Fig 3: Penrose diagram for Poincare coordinates. The orange and green dotted lines denote the constant τ and $|\mathbf{x}|$ slices.

The Poincare coordinates are defined as

$$\begin{aligned} X^0 &= \sinh t + \frac{1}{2} x_i x^i e^t, & X^{d+1} &= \cosh t - \frac{1}{2} x_i x^i e^t, \\ X^i &= x^i e^t, & i &= 1, \dots, d, \end{aligned} \quad (2.7)$$

where $-\infty < t, x^i < \infty$. Here t is the Poincare coordinate time, even though we used the same letter for global time in the previous subsection. We find the metric as

$$ds^2 = -dt^2 + e^{2t} dx_i dx^i. \quad (2.8)$$

If we define conformal time

$$\tau \equiv -e^{-t}, \quad (2.9)$$

then the metric (2.8) becomes conformally flat

$$ds^2 = \frac{1}{\tau^2} (-d\tau^2 + dx_i dx^i). \quad (2.10)$$

Note that translations and rotations of the spatial coordinates are manifest symmetries in the Poincare coordinates. From (2.7), we see that these coordinates only cover half

of the de Sitter space, i.e. when $X^0 + X^{d+1} > 0$. Therefore, this metric is intrinsically incomplete with regards to past geodesics. Except for when they are located at the north pole, timelike trajectories will leave the flat section in a finite amount of affine time. Conversely, the future development of de Sitter space can be completely predicted when initial conditions are present across a whole flat section. Comparable coordinates can be selected to cover the other half by replacing t with $-t$ in (2.7).

The Poincare coordinates in cosmology are also known as the “flat chart”. This term is derived from (2.6), which shows that these coordinates correspond to a flat FRW universe with an exponentially growing scale factor $a(t) = e^t$. The surfaces defined by a constant value of t represent the spatial sections of de Sitter space, which are infinite volume d planes with a flat metric.

To see the relation between Poincare and global coordinates, we compare them by the embedding space coordinates (2.4) and (2.7). Eventually we arrive at a coordinate transformation,

$$e^t = \cos \theta \cosh \tau + \sinh \tau, \quad \frac{1}{|\mathbf{x}|} e^t = \sin \theta \cosh \tau, \quad (2.11)$$

where τ is the global time and $z_1 = \cos \theta$. Note that at late times, the time coordinates become the same,

$$t \sim \tau \quad \text{as} \quad t \rightarrow \infty. \quad (2.12)$$

2.4 Static coordinates

The static coordinates are constructed to have an explicit timelike Killing symmetry. If we set

$$\begin{aligned} X^0 &= \sqrt{1-r^2} \sinh T, & X^{d+1} &= \sqrt{1-r^2} \cosh T, \\ X^a &= r\omega^a, & a &= 1, \dots, d, \end{aligned} \quad (2.13)$$

then the metric becomes

$$ds^2 = -(1-r^2) dT^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega_{d-1}^2, \quad (2.14)$$

with $0 < r < 1$ and $\omega^2 = 1$. The horizons are located at $r^2 = 1$, where the metric becomes singular, and the south pole is positioned at $r = 0$. The static coordinates only cover the southern (right) diamond of the Penrose diagram, which is also called the causal patch (of an observer sitting at the south pole) or the static patch. This name is derived from the fact that the metric is static, meaning that it is independent of T . It means that $\partial/\partial T$ is a Killing vector in this coordinate system, generating the time dilatation symmetry. One motivation for seeking a timelike Killing vector is its usefulness in defining time evolution or equivalently defining the Hamiltonian. However, we observe that the norm of $\partial/\partial T$

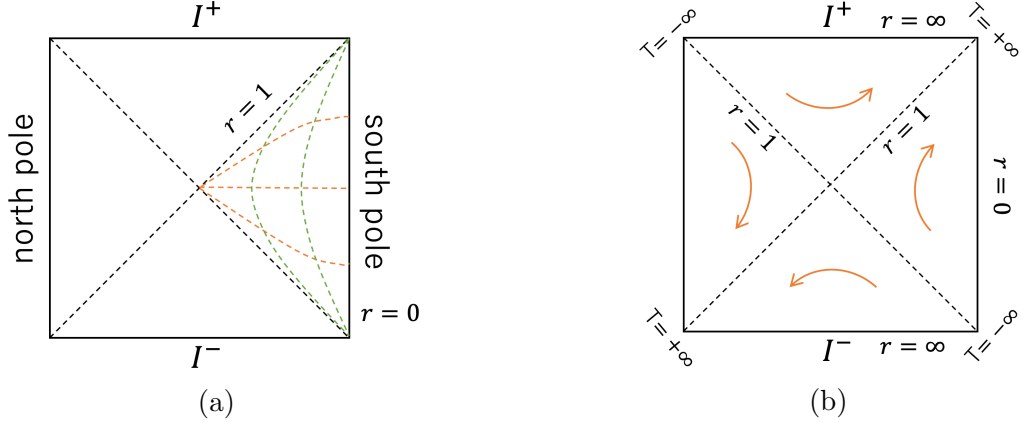


Fig 4: (a) Penrose diagram for Poincare coordinates. The orange and green dotted lines denote the constant T and r slices. (b) Penrose diagram shows the direction of the flow generated by the Killing vector $\partial/\partial T$.

vanishes at $r = 1$. Additionally, upon extending it to the top and bottom diamonds, $\partial/\partial T$ becomes spacelike. Furthermore, in the northern diamond, the vector reverses and indicates the past. Consequently, using $\partial/\partial T$ in static coordinates is suitable for defining time evolution solely within the southern diamond and not the entire de Sitter space.

Remember that de Sitter space can be characterized as the set of points X satisfying $\eta_{AB}X^AX^B = 1$ in $(d+2)$ dimensional Minkowski space. When considering any two points, X and Y one can regard them as vectors and calculate the invariant Minkowski scalar product, denoted as $Z(X, Y) = \eta_{AB}X^AY^B$. This gives an invariant measure of the separation between X and Y that is invariant under the de Sitter isometry. The relationship between the geodesic distance $D(X, Y)$ of two points is expressed by $Z(X, Y) = \cos D(X, Y)$. As a result, the specific forms of $Z(X, Y)$ in the aforementioned coordinate systems are presented as follows:

$$\begin{aligned}
 Z &= -\sinh t_x \sinh t_y + \cosh t_x \cosh t_y (\delta_{ij}x^i y^j), \quad (\text{global}) \\
 &= \sqrt{1 - r_x^2} \sqrt{1 - r_y^2} \cosh(T_x - T_y) + r_x r_y (\delta_{ab}x^a y^b), \quad (\text{static}) \\
 &= 1 + \frac{(\tau_x - \tau_y)^2 - |\vec{x} - \vec{y}|^2}{2\tau_x \tau_y}. \quad (\text{Poincare})
 \end{aligned} \tag{2.15}$$

It is important to note several features of Z . When $Z = 1$, X and Y either coincide or they are connected by a null geodesic. X and Y are connected by a timelike geodesic if $Z > 1$ and spacelike geodesic if $Z < 1$. Moreover, when $Z < -1$, it indicates that no geodesic can connect the points in real de Sitter space, implying that there is no causal relationship between them.

3 Quantum field in de Sitter

Free scalar quantum field theory on a fixed de Sitter background is already well understood and supported by numerous references [69–75]. In this section, we first give a brief review of the general classification of fields in de Sitter space in terms of unitary representation. We then demonstrate the canonical quantization of scalar fields in Poincare and global coordinates. Subsequently, we introduce the wavefunction method as an alternative approach to studying QFT. This method will also lead us to the proposal of the dS/CFT correspondence. At the end of this section, we present the original proposal for dS/CFT and the dS/CFT dictionary for light field correlation functions, after reviewing some basic properties of conformal field theory.

3.1 dS unitary irreducible representations (UIRs)

Scalar fields on de Sitter space are associated with representations of the de Sitter isometry group $SO(1, d + 1)$. For instance, considering the one-particle states of a free scalar field ϕ , we can describe them using the standard canonical Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}(m^2 - \xi R)\phi, \quad (3.1)$$

where m represents the physical mass of the scalar field, R denotes the Ricci scalar, and ξ is a numerical coupling constant. There are two special values of ξ that are of particular interest. The first is when $\xi = 0$, which corresponds to minimally coupled theory. The second is when $\xi = \frac{1}{4}\left(\frac{d-1}{d}\right)$ and, when $m = 0$, the action is invariant under conformal transformations. This case is known as the conformally coupled case. Since it is not physically evident how to distinguish between the mass term and the gravitationally coupled term, the terms “minimally coupled” and “conformally coupled” become blurred. For convenience, we use the effective (squared) mass $\tilde{m}^2 \equiv m^2 + \xi R$ and simply denote \tilde{m} as m . The states of a free scalar field form a unitary irreducible representation, which could be classified into three series: principle series, complementary series and discrete series. They are labeled by the conformal dimension Δ of the scalar field, which is related to the eigenvalue of the quadratic Casimir as

$$\mathcal{C}_2 = -\frac{1}{2}\sum_{MN}L_{MN}L^{MN} = \Delta(\Delta - d) + J(J + d - 2), \quad (3.2)$$

where L_{MN} are the Killing vectors (2.3) and J is the spin of the state. For a scalar field, the second term disappears and the eigenvalue of the quadratic Casimir is the squared mass with a minus sign. So we have the following equation:

$$m^2 = (d - \Delta)\Delta. \quad (3.3)$$

Note that the right hand side is invariant under $\Delta \rightarrow -(\Delta - d)$. Solving (3.3), we obtain two solutions for Δ ,

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 - m^2}. \quad (3.4)$$

The states of a free scalar field form a unitary irreducible representation (UIR), which could be classified into three series based on its mass as follows:

1. Principle series: known as ‘heavy field’ series, describes massive fields that are heavy compared with the dS length l_{dS} ,

$$\left(\frac{d}{2}\right)^2 \leq m^2 \quad \Rightarrow \quad \Delta_{\pm} = \frac{d}{2} \pm i\mu, \quad \{\mu \in \mathbb{R}, \mu \geq 0\} \quad (3.5)$$

2. Complementary series: known as ‘light field’ series, describes massive fields that are light compared with the dS length l_{dS} ,¹

$$0 < m^2 < \left(\frac{d}{2}\right)^2 \quad \Rightarrow \quad \Delta_{\pm} = \frac{d}{2} \pm \nu, \quad \nu \in \left(0, \frac{d}{2}\right) \quad (3.7)$$

3. Discrete series: these series only exist for some dimension. They are associated with $m^2 = 0$ and certain tachyonic masses,

$$m^2 = -n(n + d) \text{ for } n \in \mathbb{N}_0 \quad \Rightarrow \quad \Delta_+ = d + n, \quad \Delta_- = -n. \quad (3.8)$$

Here \mathbb{N}_0 denotes the non-negative integer. For our purpose in this thesis, we limit our analysis below to fields with positive squared mass $m^2 > 0$.

3.2 Scalar field quantization

Here, we first consider a quantized scalar field on dS_{d+1} and the general properties of the Green function, which contains all the information of a free field theory, without specifying any coordinates. In the subsequent subsections, we will show the explicit forms of the mode functions in particular coordinates.

We begin with the Wightman function $G_w(X, Y) = \langle 0 | \hat{\phi}(X) \hat{\phi}(Y) | 0 \rangle$, which is a two point function without any time ordering and $|0\rangle$ is an invariant vacuum state under de Sitter isometry group $SO(1, d + 1)$. Other types of two point function, such as retarded,

¹For a light field with spin s ($s \geq 1$), the ranges of complementary are instead,

$$(s - 1)(s + d - 3) < m^2 < \left(s + \frac{d}{2} - 2\right)^2, \quad (3.6)$$

with conformal dimension defined by $(\Delta + s - 2)(d + s - 2 - \Delta) = m^2$. The lower bound $\sqrt{(s - 1)(s + d - 3)}$ is known as the Higuchi bound [60].

advanced or Feynman, can all be constructed from the Wightman function. It satisfies the free field equation

$$(\nabla^2 - m^2) G_w(X, Y) = 0, \quad (3.9)$$

where ∇^2 is the Laplacian on dS_{d+1} . Note that $G_w(X, Y)$ is de Sitter invariant by definition and it can therefore only depend on the de Sitter invariant measure $Z(X, Y)$. Due to this fact, the free field equation (3.9) reduces to a differential equation in one variable Z ,

$$z(1-z)\partial_z^2 G_w + (d+1)\left(\frac{1}{2} - z\right)\partial_z G_w - m^2 G_w = 0, \quad (3.10)$$

where we defined $z = \frac{1+Z}{2}$ and made a change of variables. Notice that (3.10) is a hypergeometric equation and the solution is given by hypergeometric functions,

$$G_w(Z) = \frac{\Gamma(\Delta_+) \Gamma(\Delta_-)}{(4\pi)^{\frac{d+1}{2}} \Gamma(\frac{d+1}{2})} {}_2F_1\left[\Delta_+, \Delta_-; \frac{d+1}{2}; \frac{1+Z}{2}\right], \quad (3.11)$$

where we fixed the normalization constant by matching the overall factor of G_w at short distance singularity ($X = Y$, or equivalently $Z = 1$) with the overall factor of the Wightman function in $(d+1)$ dimensional Minkowski spacetime, because the points are insensitive to the de Sitter curvature in the short distance limit.² Note that we have to normalize it in this way only because we have not introduced any canonical normalization for the scalar fields, since we are working with embedding coordinates.

There is an additional solution to the equation (3.10), which is also independent and proportional to ${}_2F_1\left[\Delta_+, \Delta_-; \frac{d+1}{2}; \frac{1-Z}{2}\right]$, due to the $Z \rightarrow -Z$ symmetry of the free field equation.³ Consequently, the linear combinations of the above two solutions provide the general solutions of the field equation (3.9). It can be labeled by a complex parameter, which forms a one-parameter family of de Sitter invariant Green functions and the corresponding vacua.

Next let us look at the vacuum state. A vacuum state for a free scalar field which satisfies the Klein-Gordon equation with the mode expansion

$$\hat{\phi}(X) = \sum_k \left[\phi_k(X) \hat{a}_k + \phi_k^*(X) \hat{a}_k^\dagger \right] \quad (3.12)$$

can be defined by the conditions

$$\hat{a}_k |0\rangle = 0 \quad \text{for all } k. \quad (3.13)$$

²The $i\epsilon$ prescription for going around the singularity in the complex plane is also the same as in Minkowski space: by replacing $X^0 - Y^0$ with $X^0 - Y^0 - i\epsilon$.

³The Wightman function now has a singularity at $P = -1$, at which X is null-separated from Y 's antipodal point. However, because antipodal points in de Sitter space are always separated by a horizon, this seemingly unphysical singularity is undetectable by experiments.

where \hat{a}_k and \hat{a}_k^\dagger as usual obey

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \quad (3.14)$$

The modes are normalized in terms of the invariant Klein-Gordon inner product,

$$\langle \phi_i, \phi_j \rangle = i \int_{\Sigma} d\Sigma^\mu \left(\phi_i^* \overleftrightarrow{\partial}_\mu \phi_j \right) = \delta_{ij}, \quad d\Sigma^\mu = n^\mu \sqrt{-g_\Sigma} d^d \Sigma, \quad (3.15)$$

where Σ is a spacelike Cauchy hypersurface, g_Σ is the induced metric on Σ and n^μ is a forward pointing and timelike unit vector. It is easily checked that the Klein-Gordon inner product satisfies symmetries

$$\langle \phi_i, \phi_j \rangle^* = \langle \phi_j, \phi_i \rangle = -\langle \phi_i^*, \phi_j^* \rangle. \quad (3.16)$$

Using Klein-Gordon inner product, we can write \hat{a}_k and \hat{a}_k^\dagger as

$$\hat{a}_k = \langle \phi_k, \phi \rangle, \quad \hat{a}_k^\dagger = -\langle \phi_k^*, \phi \rangle. \quad (3.17)$$

Note that from a mathematical standpoint, there exist many possible ways to expand ϕ through various sets of mode functions, and different choices of positive frequency modes could lead to physically different vacuum states. In other words, dS invariance does not fix the vacuum; instead, it results in a one complex-parameter family of dS invariant vacua as well as Green functions. We will further explore this in the following sections while working with specific coordinates.

3.2.1 Poincare coordinates

In Poincare coordinates the action is written as

$$S = -\frac{1}{2} \int \frac{d\tau d^d x}{(-\tau)^{d+1}} \left[-\tau^2 (\partial_\tau \phi)^2 + \tau^2 (\vec{\nabla} \phi)^2 + m^2 \phi^2 \right]. \quad (3.18)$$

Recall that in Poincare coordinates the metric is conformally flat, in other words, if $\tau = \text{constant}$, the mode functions should have the same forms as those in Minkowski space. Therefore, we can decompose the field into spatial Fourier modes and separate the τ -dependent part and the spatial part, which can be expressed as $\phi_{\mathbf{k}}(x) = \psi_{\mathbf{k}}(\tau) e^{i\mathbf{k} \cdot \mathbf{x}}$. As a result, the equations of motion for $\psi_{\mathbf{k}}(\tau)$ can be derived as

$$(-\tau)^{d+1} \partial_\tau \left((-\tau)^{1-d} \partial_\tau \psi_{\mathbf{k}} \right) + (\tau^2 k^2 + m^2) \psi_{\mathbf{k}} = 0, \quad (3.19)$$

where $k \equiv |\mathbf{k}|$. This is a Bessel type equation, which has two linearly independent solutions in the form of Bessel functions. They are explicitly given by

$$\psi_{1k} = \frac{\sqrt{\pi}}{2} (-\tau)^{\frac{d}{2}} H_\nu^{(1)}(-k\tau), \quad \psi_{2k} = \frac{\sqrt{\pi}}{2} (-\tau)^{\frac{d}{2}} H_\nu^{(2)}(-k\tau), \quad (3.20)$$

where $\nu = \sqrt{\left(\frac{d}{2}\right)^2 - m^2}$ and we do not restrict the range of m here, namely ν can be either real or purely imaginary. Mathematically, the general solutions of the mode function are given by the linear combination of the above two solutions. However, there is only one linear combination which matches the behavior of the mode function in early time limit with the one on flat space. This comes from the fact that when $\tau \rightarrow -\infty$, $-\tau \gg k^{-1}$ is satisfied, which means the physical wavelength of the modes was way shorter than the de Sitter length and their propagation is insensitive to the curvature of spacetime. Indeed, taking the $\tau \rightarrow -\infty$ limit of (3.20), we see

$$\psi_{1k} \propto \frac{(-\tau)^{\frac{d}{2}}}{\sqrt{-k\tau}} e^{-ik\tau}, \quad \psi_{2k} \propto \frac{(-\tau)^{\frac{d}{2}}}{\sqrt{-k\tau}} e^{ik\tau} \quad (3.21)$$

which oscillate just like the positive frequency and negative frequency mode on flat space, respectively. Therefore we define $\phi_{\mathbf{k}}(x) = \psi_{1k}(\tau) e^{i\mathbf{k} \cdot \mathbf{x}}$ to be the positive frequency mode. We can see the Klein Gordon inner products of this mode is indeed positive, i.e. $\langle \phi_{\mathbf{k}}, \phi_{\mathbf{k}'} \rangle = \delta^d(\mathbf{k} - \mathbf{k}')$.

Now we can proceed with the canonical quantization after obtaining the positive and negative frequency modes. We work in momentum space and repeat the general quantization procedure we show in the previous section, we have

$$\hat{\phi}(x) = \int \frac{d^d k}{(2\pi)^{\frac{d}{2}}} \hat{\phi}_{\mathbf{k}}(\tau) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \hat{\phi}_{\mathbf{k}}(\tau) = \hat{a}_{\mathbf{k}} \psi_{\mathbf{k}}(\tau) + \hat{a}_{-\mathbf{k}}^\dagger \psi_{\mathbf{k}}^*(\tau), \quad (3.22)$$

where we define the d dimensional vector $x = (\tau, \mathbf{x})$ and choose $\psi_{\mathbf{k}} = \psi_{1k}$. Also note that ψ_{2k} is the complex conjugate of ψ_{1k} , hence $\psi_{\mathbf{k}}^* = \psi_{2k}$. The creation and annihilation operators are defined in the same way as in (3.17), which manifestly satisfy the canonical commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^d(\mathbf{k} - \mathbf{k}'). \quad (3.23)$$

The vacuum is defined by a state which is annihilated by all the $\hat{a}_{\mathbf{k}}$:

$$\hat{a}_{\mathbf{k}} |E\rangle = 0. \quad (3.24)$$

This is called the Euclidean vacuum or the Bunch-Davies vacuum, after the authors of the paper for which the two point function was first calculated in Poincare coordinates [76]. Although the Euclidean vacuum is indeed the lowest eigenstate of the Hamiltonian in Poincare coordinates which generates translations in τ , its energy is not conserved and there is no positive conserved energy in de Sitter space. As a related topic, it is known that the Euclidean vacuum on de Sitter space cannot be described as “the state without any particles” due to the Unruh effect [77]. As a result, identifying a ‘vacuum’ in Poincare coordinates is not a straightforward process. Nevertheless, if we calculate the two-point

function,

$$G_E(x, y) \equiv \langle E | \hat{\phi}(x) \hat{\phi}(y) | E \rangle = \int \frac{d^d k}{(2\pi)^d} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \psi_k(\tau) \psi_k^*(\tau') \\ \propto {}_2F_1 \left[\Delta_+, \Delta_-; \frac{d+1}{2}; \frac{1+Z}{2} \right], \quad (3.25)$$

we find it matches with the result of (3.11). So we can say the Euclidean vacuum is still distinguishable from other vacua in terms of its two point function, which is the only solution to the free field equation (3.9) that does not have the antipodal singularity.

3.2.2 Global coordinates

In global coordinates, we follow closely the conventions of [78] and work in position space. First we find the solutions to the equation of motion is separable,

$$\phi = y_L(t) Y_{Lj}(\Omega), \quad (3.26)$$

where $y_L(t)$ only depends on the global time t and Y_{Lj} is a nonstandard basis of spherical harmonics on S^d , which is defined in terms of the usual spherical harmonics S_{Lj} ,

$$Y_{Lj} = \sqrt{\frac{i}{2}} S_{Lj} + (-1)^L \sqrt{-\frac{i}{2}} S_{Lj}^*. \quad (3.27)$$

Here $L \in \mathbb{N}_0$ and j is a collective index (j_1, \dots, j_{d-2}) . Note that Y_{Lj} 's satisfy the identities

$$Y_{Lj}(\Omega_A) = Y_{Lj}^*(\Omega) = (-1)^L Y_{Lj}(\Omega). \quad (3.28)$$

and are orthonormal and complete

$$\int d\Omega Y_{Lj}(\Omega) Y_{L'j'}^*(\Omega) = \delta_{LL'} \delta_{jj'}, \quad \sum_{Lj} Y_{Lj}(\Omega) Y_{Lj}^*(\Omega') = \delta^{d-1}(\Omega, \Omega'). \quad (3.29)$$

Here Ω_A denotes the point on S^{d-1} antipodal to Ω . Then the equation of motion for the time dependent part is given by

$$\partial_t^2 y_L + d \tanh t \partial_t y_L + \left[m^2 + \frac{L(L+d-1)}{\cosh^2 t} \right] y_L = 0. \quad (3.30)$$

It requires a few more steps of rearrangement, but (3.30) turns out to be a hypergeometric equation and thus we obtain two linearly independent solutions,

$$y_L^{\text{in}\pm} = \frac{2^{L+\frac{d-1}{2}}}{\sqrt{\mp\mu}} \cosh^L t e^{(L+\frac{d}{2}\pm i\mu)t} {}_2F_1 \left[L + \frac{d}{2}, L + \frac{d}{2} \pm i\mu; 1 \pm i\mu; -e^{2t} \right], \quad (3.31)$$

where $\mu = \sqrt{m^2 - \frac{d^2}{4}}$ could be either a real or pure imaginary number. The overall coefficient is decided by demanding the orthonormality with respect to the Klein-Gordon inner product

$$\langle \phi_n, \phi_m \rangle = i \int_{\Sigma} d\Sigma^\mu \left(\phi_n^* \overleftrightarrow{\partial}_\mu \phi_m \right) = \delta_{nm}. \quad (3.32)$$

Note that when μ is real, modes with $y_L^{\text{in}-}$ and $y_L^{\text{in}+}$ can be considered as the positive and negative frequency modes of the incoming modes, respectively, since at the past boundary ($t \rightarrow -\infty$) they behave as

$$y_L^{\text{in} \pm} \rightarrow \frac{2^{\frac{d-1}{2}}}{\sqrt{\mu}} e^{(\frac{d}{2} \pm i\mu)t}. \quad (3.33)$$

Since the main focus of this thesis is on heavy scalar fields, we will restrict our discussion for the remainder of this section to $\mu \in \mathbb{R}$. The two solutions in (3.31) then become complex conjugate of each other. Thus we denote them by $y_L^{\text{in}} = y_L^{\text{in}-}$ and $y_L^{\text{in}*} = y_L^{\text{in}+}$. As for light fields, solutions in (3.31) are referred to as the Dirichlet and Neumann modes [79], which we will give a few comments at the end of this subsection.

Moreover, the equation of motion (3.30) is invariant under time reversal, there exists an alternative pair of linearly independent solutions

$$y_L^{\text{out}}(t) = y_L^{\text{in}*}(-\tau), \quad y_L^{\text{out}*}(t) = y_L^{\text{in}}(-t). \quad (3.34)$$

They are explicitly given by

$$y_L^{\text{out}} = \frac{2^{L+\frac{d-1}{2}}}{\sqrt{\mu}} \cosh^L t e^{(-L-\frac{d}{2}-i\mu)t} {}_2F_1 \left[L + \frac{d}{2}, L + \frac{d}{2} + i\mu; 1 + i\mu; -e^{-2t} \right], \quad (3.35)$$

which can be considered as the positive frequency modes of the outgoing modes, as they behave as such on the future boundary ($\tau \rightarrow \infty$)

$$y_L^{\text{out}} \rightarrow \frac{2^{d/2-1}}{\sqrt{\mu}} e^{-(\frac{d}{2}+i\mu)\tau}. \quad (3.36)$$

Using $y_L^{\text{in},\text{out}}$, we define the positive frequency modes with the angular part as

$$\phi_{Lj}^{\text{in}}(x) = e^{i\theta_L} y_L^{\text{in}}(t) Y_{Lj}(\Omega), \quad \phi_{Lj}^{\text{out}}(x) = e^{-i\theta_L} y_L^{\text{out}}(t) Y_{Lj}(\Omega), \quad (3.37)$$

where we introduced a phase factor $e^{i\theta_L}$ defined by

$$e^{-2i\theta_L} = (-)^{L-\frac{d}{2}} \frac{\Gamma(-i\mu)\Gamma(L+\frac{d}{2}+i\mu)}{\Gamma(i\mu)\Gamma(L+\frac{d}{2}-i\mu)}. \quad (3.38)$$

It looks rather complicated, but we will find it very convenient for relating the two sets of modes ϕ_{Lj}^{in} and ϕ_{Lj}^{out} , and also for relating them to the Euclidean modes, which we will show shortly. Additionally, the phase makes the modes satisfy $\phi_{Lj}^{\text{in}}(x_A) = \phi_{Lj}^{\text{in}*}(x)$. ϕ_{Lj}^{in} and ϕ_{Lj}^{out} are related through the Bogolyubov transformation,

$$\phi_{Lj}^{\text{in}}(x) = A e^{-2i\theta_L} \phi_{Lj}^{\text{out}}(x) + iB \phi_{Lj}^{\text{out}*}(x), \quad (3.39)$$

which has the form of an Mottola-Allen (MA) transformation [73, 74] and the coefficients are given by

$$A = \begin{cases} 1, & (d+1) \text{ odd} \\ \coth \pi\mu, & (d+1) \text{ even} \end{cases}, \quad B = \begin{cases} 0, & (d+1) \text{ odd} \\ (-)^{\frac{d+1}{2}} \text{csch } \pi\mu, & (d+1) \text{ even} \end{cases}. \quad (3.40)$$

This can be derived by using the hypergeometric transformation identities (C.2).

Using (3.37), we can expand the scalar field as

$$\hat{\phi}(x) = \sum_{Lj} \left[\phi_{Lj}^{\text{in}}(x) \hat{a}_{Lj} + \phi_{Lj}^{\text{in}*}(x) \hat{a}_{Lj}^\dagger \right] = \sum_{Lj} \left[\phi_{Lj}^{\text{out}}(x) \hat{b}_{Lj} + \phi_{Lj}^{\text{out}*}(x) \hat{b}_{Lj}^\dagger \right], \quad (3.41)$$

and define the vacuum states $|\text{in}\rangle$ and $|\text{out}\rangle$ as

$$\hat{a}_{Lj}|\text{in}\rangle = 0, \quad \hat{b}_{Lj}|\text{out}\rangle = 0, \quad (3.42)$$

which can be interpreted as the state with no incoming particles on the past boundary and no outgoing particles on the future boundary, respectively.

Note that for light fields, we can simply replace μ with $i\nu$ in (3.31) or (3.35), where $\nu \in \mathbb{R}$. However, the two solutions are not complex conjugate of each other anymore. Instead of oscillating, they decay in different speeds toward the past or future boundary. Therefore, they are often referred to as Dirichlet and Neumann modes. As a special case, when $d = 3$ and $\nu = \frac{1}{2}$ (which corresponds to conformally coupled case), it allows for Dirichlet and Neumann vacua with no particle production.

Euclidean modes

We have mentioned at the beginning of Sec. 2 that de Sitter space can be analytically continued to a Euclidean sphere. Specifically, this can be done in global coordinates by taking $\tau \rightarrow i\tau$, namely by letting τ run along the imaginary axis. We define the positive frequency Euclidean modes as those that are regular on the southern hemisphere when the Lorentzian de Sitter geometry is analytically continued to Euclidean signature. The explicit form of the solution is found as

$$\phi_{Lj}^E(x) = \frac{1}{f_L \sqrt{e^{2\pi\mu} - 1}} y_L^E(t) Y_{Lj}(\Omega) \quad (3.43)$$

where

$$y_L^E = \frac{2^{L+\frac{d-1}{2}} i^{-L+\frac{d}{2}}}{\sqrt{\mu}} \cosh^L t e^{(L+\frac{d}{2}+i\mu)t} {}_2F_1 \left[L + \frac{d}{2}, L + \frac{d}{2} + i\mu; 2L + d; 1 + e^{2t} \right], \quad (3.44)$$

$$f_L = \frac{\Gamma(2L + d)}{\Gamma(L + \frac{d}{2})} \left| \frac{\Gamma(i\mu)}{\Gamma(L + \frac{d}{2} - i\mu)} \right|. \quad (3.45)$$

They are normalized in terms of Klein Gordon inner product and satisfy $\phi_{Lj}^E(x_A) = \phi_{Lj}^{E*}(x)$. We find that it can also be expressed in terms of ϕ_{Lj}^{in} or ϕ_{Lj}^{out} modes as

$$\phi_{Lj}^E = \frac{1}{\sqrt{1 - e^{-2\pi\mu}}} (\phi_{Lj}^{\text{in}} + e^{i\pi\Delta_+} \phi_{Lj}^{\text{in}*}) = \frac{1}{\sqrt{1 - e^{-2\pi\mu}}} (\phi_{Lj}^{\text{out}} - e^{i\pi\Delta_+} \phi_{Lj}^{\text{out}*}). \quad (3.46)$$

Note that these are in the form of MA transformation as well, the same as (3.39). Also (3.46) can be inverted to give

$$\phi_{Lj}^{\text{in}} = \frac{1}{\sqrt{1 - e^{-2\pi\mu}}} (\phi_{Lj}^E - e^{i\pi\Delta_+} \phi_{Lj}^{E*}), \quad (3.47)$$

$$\phi_{Lj}^{\text{out}} = \frac{1}{\sqrt{1 - e^{-2\pi\mu}}} (\phi_{Lj}^E + e^{i\pi\Delta_+} \phi_{Lj}^{E*}). \quad (3.48)$$

We can also express the coefficients in terms of Δ_- by using $\Delta_- = d - \Delta_+$.

There is another solution to the equation of motion which is linearly independent of (3.44), proportional to

$$(1 + e^{2t})^{1-2L-d} {}_2F_1 \left[1 - L - \frac{d}{2}, 1 + i\mu - L - \frac{d}{2}; 2 - 2L - d; 1 + e^{2t} \right].$$

It is singular at the south pole $t = -\frac{i\pi}{2}$, so we discard this solution and choose (3.43) as the positive frequency Euclidean modes.

Up to this point, we have restricted our discussion to the case of heavy fields. Yet the Euclidean modes for light fields are derived in the same way by requiring analyticity on the southern hemisphere. Thus they have the same form as (3.43), where μ is replaced by $-i\nu$.

After canonical quantization, we can define the Euclidean vacuum which is annihilated by the positive frequency Euclidean modes. The Euclidean Green function is then given by the sum of the modes

$$G_E(x, x') = \sum_{L,j} \phi_{Lj}^E(x) \phi_{Lj}^{E*}(x'). \quad (3.49)$$

The explicit result agrees with (3.11). So far, we have examined the free field theory in both Poincare and global coordinates and defined the Euclidean vacuum states, which seem to be based on different definitions. Nevertheless, they turn out to be identical given by the fact that they yield the same Euclidean Green function.

3.3 Wavefunction method and no-boundary wavefunction

We have thus far discussed the canonical quantization for scalars field on de Sitter space. This approach is related to other formalisms of quantum field theory, such as the wave function formalism and Feynman's path integral approach, which sometimes offer advantages in describing quantum theory for the universe.

We start with quantum mechanics first. Consider a bound particle in quantum mechanics with position $x(t)$ and Hamiltonian H . We can define an orthonormal basis of eigenstates of the position operator \hat{x} as

$$\hat{x}|x\rangle = x|x\rangle. \quad (3.50)$$

Then we can expand any state by using this basis,

$$|\Psi(t)\rangle = \int \Psi(t, x) |x\rangle dx, \quad (3.51)$$

where the $\Psi(t, x)$ is called the wavefunction and it can be expressed using the orthogonality of our position basis,

$$\Psi(t, x) = \langle x | \Psi(t) \rangle = \int \Psi(t, y) \langle x | y \rangle dy. \quad (3.52)$$

In order to express $\Psi(t, x)$ in terms of path integrals, we consider the propagation of the particle from an initial position $x = 0$ to position x in time t . The probability is given by

$$\langle x | e^{-i\hat{H}t} | 0 \rangle = \int_{x(0)=0}^{x(t)=x} [\mathcal{D}x(t')][\mathcal{D}p(t')] e^{i \int p dx - H(x, p) dt'} = \int_{x(0)=0}^{x(t)=x} [\mathcal{D}x(t')] e^{iS[x(t)]}, \quad (3.53)$$

where p is the conjugate momentum of x . In the second equation, we performed the p integral, for which the result can be written in terms of the action $S[x(t)]$. We can then insert a complete set of energy eigenstates $|n\rangle$ to the left-hand side and express it with the wavefunctions $\Psi_n(x)$,

$$\langle x | e^{-i\hat{H}t} | 0 \rangle = \sum_n e^{-iE_n t} \langle x | n \rangle \langle n | 0 \rangle = \sum_n e^{-iE_n t} \Psi_n(x) \bar{\Psi}_n(0). \quad (3.54)$$

Next if we Wick rotate into Euclidean time by taking $t \rightarrow -i\tau$ and then take the limit $\tau \rightarrow \infty$, the ground state dominates over all others and becomes the only state that contributes. Hence we can express the ground state wave function in path integral language,

$$\Psi_0(x) = \frac{1}{\bar{\Psi}_0(0)} \lim_{\tau \rightarrow \infty} \sum_n e^{-E_n \tau} \langle x | n \rangle \langle n | 0 \rangle = \frac{1}{\bar{\Psi}_0(0)} \int_{x(0)=x}^{x(-\tau)=0} [\mathcal{D}x] e^{-S_E[x(\tau)]}, \quad (3.55)$$

where S_E is the Euclidean action obtained from $S_E = iS|_{t \rightarrow -i\tau}$.

This wavefunction approach can be extended to quantum field theory straightforwardly. We start by defining the wavefunction $\Psi[\phi; \eta]$ as

$$|\Psi(\eta)\rangle = \int [\mathcal{D}\phi] \Psi[\phi; \eta] |\phi; \eta\rangle, \quad (3.56)$$

which can be computed using the orthogonal basis,

$$\Psi[\bar{\phi}; \eta_0] = \langle \bar{\phi}; \eta_0 | \Psi(\eta) \rangle = \int d\phi \Psi[\phi; \eta] \langle \bar{\phi}; \eta_0 | \phi; \eta \rangle. \quad (3.57)$$

Following the same procedure, we obtain the wavefunction in terms of the action,

$$\begin{aligned} \Psi[\bar{\phi}; \eta_0] &= \mathcal{N} \lim_{\eta \rightarrow -\infty} \int_{\phi(\eta_0)=\bar{\phi}} [\mathcal{D}\phi] \int [\mathcal{D}\pi] \exp \left[i \int_{\eta}^{\eta_0} d\eta \{ \phi'(\eta) \pi(\eta) - H(\phi(\eta), \pi(\eta)) \} \right] \\ &= \mathcal{N} \lim_{\eta \rightarrow -\infty} \int_{\phi(\eta_0)=\bar{\phi}} [\mathcal{D}\phi] e^{iS[\phi]} \\ &\approx \mathcal{N} e^{iS[\phi_{\text{cl}}(\bar{\phi})]}, \end{aligned} \quad (3.58)$$

here we Wick rotated it back from the Euclidean action. π is the conjugate momentum operator and we used the semiclassical approximation in the last line, given by the fact that the path integral is dominated by the contribution from the extremum of the classical action $S[\phi_{\text{cl}}]$.

Particularly, when gravity is included, this procedure defines the ground state in quantum gravity, which is called the Hartle-Hawking state Ψ_{HH} [80,81]. In the instance of Einstein gravity in $d = 4$ dimensional spacetime with matter fields, we have the wavefunction,

$$\Psi_{HH}[h_{ij}, \phi_0] \propto \int_{\mathcal{C}} [\mathcal{D}g_{\mu\nu}][\mathcal{D}\phi] e^{-S_E[g, \phi]}, \quad (3.59)$$

where h_{ij} is a three-metric and ϕ_0 is the field configuration on the final spatial surface where the wavefunction is defined, and S_E is the Euclidean Einstein-Hilbert plus matter action,

$$S_E = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{g} (R - 2\Lambda + \mathcal{L}_{\text{matter}}(\phi)) + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} K. \quad (3.60)$$

The second term this action is the Gibbons-Hawking-York (GHY) boundary term, which is necessary to satisfy the Einstein equations of motion under various boundary conditions. It can be verified that this wavefunction indeed satisfies the Wheeler-de Witt equation [82], which is in the form of a time-independent Schrodinger equation,

$$\hat{H}\Psi_{HH}[h_{ij}, \phi_0] = 0, \quad (3.61)$$

where H is the Hamiltonian of the ADM formalism for general relativity with functional derivatives with respect to the three-metric h_{ij} . The equation indicates that the wavefunction does not depend on the coordinate time.

So far, we have not specified the geometries or the boundary conditions we should consider during the integration process in (3.59). However, it is proposed in Ref. [80,81] that instead of an open spacetime, one can consider a spacetime with closed spatial slices and the four-geometry that connects to h_{ij} on one end, can seamlessly cap off without forming an additional boundary. In this case, the geometries possess no boundary except for the surface on which Ψ_{HH} is defined. As a result of the geometries, Ψ_{HH} is often referred to as the no-boundary wavefunction.

3.4 Conformal field theory

Symmetries simplify complex phenomena and are linked to conservation laws through Noether's theorem. Field theories, especially, demonstrate the power of symmetries. The standard model of particle physics is an example where symmetries underlie the quantum field theory framework. In particular, conformal symmetry is largely related to (anti-)de Sitter space and holographic framework. In this section, we define the conformal group and examine the field theories with it as their symmetry group.

3.4.1 Conformal group

A conformal transformation is defined as a transformation that changes the metric of a spacetime by a scale factor

$$g_{ij}(x) \rightarrow \tilde{g}_{ij}(\tilde{x}) = \Omega^2(x)g_{ij}(x), \quad (3.62)$$

while keeping the angle between any two intersecting curves

$$\cos \theta = \frac{g_{\mu\nu} x^\mu y^\nu}{\sqrt{g_{\mu\nu} x^\mu x^\nu} \sqrt{g_{\mu\nu} y^\mu y^\nu}} \quad (3.63)$$

the same as before the transformation. Conformal transformations form the conformal group, and they can be classified into the following four types:

$$\begin{aligned} \text{translation: } \tilde{x}^i &= x^i + a^i, \\ \text{rotations: } \tilde{x}^i &= \Lambda_j^i x^j, \quad \Lambda \in SO(d), \\ \text{dilataions: } \tilde{x}^i &= \lambda x^i, \\ \text{special conformal transformations (SCT): } \tilde{x}^i &= \frac{x^i - b^i x^2}{1 - 2b \cdot x + b^2 x^2}, \end{aligned} \quad (3.64)$$

where a and b are constant vectors. Special conformal transformations could also be written in the form of

$$\frac{\tilde{x}^i}{\tilde{x}^2} = \frac{x^i}{x^2} - b^i, \quad (3.65)$$

which could be understood as a combination of three transformations: inversion, followed by translation, and then another inversion. The generators of the above transformations are given by

$$\begin{aligned} P_i &= \partial_i, \\ M_{ij} &= x_j \partial_i - x_i \partial_j, \\ D &= x^i \partial_i, \\ K_i &= 2x_i x^j \partial_j + x^2 \partial_i, \end{aligned} \quad (3.66)$$

correspondingly. If we relabel the generators as L_{AB} with indices $A, B = (0, i, d+1)$,

$$L_{ij} = M_{ij}, \quad L_{0,d+1} = D, \quad L_{0i} = \frac{P_i - K_i}{2}, \quad L_{d+1,i} = \frac{P_i + K_i}{2}, \quad (3.67)$$

we find the generators satisfy the commutation relations of $SO(1, d+1)$,

$$[L_{AB}, L_{CD}] = \eta_{BC} L_{AD} - \eta_{AC} L_{BD} + \eta_{AD} L_{BC} - \eta_{BD} L_{AC}, \quad (3.68)$$

where $\eta_{AB} = \text{diag}(-1, 1, \dots, 1)$.

It is known that conformal transformations of dimension $d \geq 3$ have only a finite number of the above four types, i.e., $\left(d+1 + {}_d C_2 + d = \frac{(d+2)(d+1)}{2}\right)$ transformations. However, in $d=2$, conformal transformations can all be represented by holomorphic transformations, and there are an infinite number of generators.

3.4.2 CFT correlation functions

Consider a Euclidean quantum field theory with conformal symmetry, which we usually refer to as a conformal field theory (CFT). The form of the correlation function in CFT is strictly restricted by conformal invariance. Here, we will examine specifically what kind of restrictions are imposed on the correlation functions and the detailed forms of the correlation functions. First, we work in position space. The correlation functions can be given in the form of a path integral

$$\langle O(x_1) \cdots O(x_N) \rangle = \frac{1}{Z} \int [dO] O(x_1) \cdots O(x_N) \exp(-S[O]). \quad (3.69)$$

In a conformal transformation, a field transforms in the following way is called a primary field

$$O(x) \rightarrow \tilde{O}(\tilde{x}) = \Omega(x)^{-\Delta} O(x), \quad (3.70)$$

where Δ is called the conformal dimension (weight). Thus, when $O_i(x)$ is a primary field with conformal dimension Δ_i , its correlation function is given by

$$\begin{aligned} \langle O_1(\tilde{x}_1) \cdots O_N(\tilde{x}_N) \rangle &= \frac{1}{Z} \int [dO] O_1(\tilde{x}_1) \cdots O_N(\tilde{x}_N) \exp(-S[O]) \\ &= \frac{1}{Z} \int [d\tilde{O}] \tilde{O}_1(\tilde{x}_1) \cdots \tilde{O}_N(\tilde{x}_N) \exp(-S[\tilde{O}]) \\ &= \frac{1}{Z} \int [dO] \Omega(x_1)^{-\Delta_1} O_1(\tilde{x}_1) \cdots \Omega(\tilde{x}_N)^{-\Delta_N} O_N(\tilde{x}_N) \exp(-S[O]) \\ &= \Omega(x_1)^{-\Delta_1} \cdots \Omega(x_N)^{-\Delta_N} \langle O_1(x_1) \cdots O_N(x_N) \rangle. \end{aligned} \quad (3.71)$$

Here we assumed that the action S and measure $[dO]$ are conformal invariant.

In particular, in the case of two point and three point correlation functions, the form of the functions is uniquely determined up to overall scale factors. To see this, we start with the two point function. For dilatation, Eq. (3.71) becomes

$$\langle O_1(\lambda x_1) O_2(\lambda x_2) \rangle = \lambda^{-\Delta_1} \lambda^{-\Delta_2} \langle O_1(x_1) O_2(x_2) \rangle. \quad (3.72)$$

Furthermore, the rotational and translational symmetries also determine the two point correlation function such that it only depends on $|x_1 - x_2|$. Therefore, the only possible form a two point function can take is given by

$$\langle O_1(x_1) O_2(x_2) \rangle = \frac{c_{12}}{x_{12}^{\Delta_1 + \Delta_2}} \delta_{\Delta_1, \Delta_2}, \quad (3.73)$$

where $x_{nm} \equiv |x_n - x_m|$ and c_{12} is a constant. Note that the two point function has non zero value only when $\Delta_1 = \Delta_2$.

For three point functions, they are determined in the similar manner with two point functions by rotational, translational and dilatational symmetry as

$$\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle = \frac{c_{123}}{x_{12}^a x_{23}^b x_{31}^c}, \quad (3.74)$$

where $a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \equiv \Delta_t$. Lastly, SCT requires that

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 = \Delta_t - 2\Delta_3, \\ b &= \Delta_2 + \Delta_3 - \Delta_1 = \Delta_t - 2\Delta_1, \\ c &= \Delta_3 + \Delta_1 - \Delta_2 = \Delta_t - 2\Delta_2. \end{aligned} \quad (3.75)$$

Therefore, to briefly summarize our results, the CFT two point and three point functions are restricted in the following forms

$$\langle O_1(x_1)O_2(x_2) \rangle = \frac{c_{12}}{x_{12}^{2\Delta_1}} \delta_{\Delta_1, \Delta_2}, \quad (3.76)$$

$$\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle = \frac{c_{123}}{x_{12}^{\Delta_t - 2\Delta_3} x_{23}^{\Delta_t - 2\Delta_1} x_{31}^{\Delta_t - 2\Delta_2}}. \quad (3.77)$$

As for four point functions, they cannot be completely determined up to overall factors. Instead they take the following form required by the symmetries,

$$\langle O_1 O_2 O_3 O_4 \rangle = f(u, v) \prod_{n < m}^4 x_{nm}^{\Delta_t/3 - \Delta_n - \Delta_m}, \quad (3.78)$$

where $\Delta_t \equiv \Sigma \Delta_n$ and we introduced the conformal invariant cross ratios as

$$u \equiv \left(\frac{x_{12}x_{34}}{x_{13}x_{24}} \right)^2, \quad v \equiv \left(\frac{x_{12}x_{34}}{x_{23}x_{14}} \right)^2. \quad (3.79)$$

Thus any function of these cross ratios as arguments is conformal invariant, such as $f(u, v)$ in (3.78).

Mathematically, the restriction from conformal invariance on the correlation function can be expressed in terms of the Ward-Takahashi (WT) identity. To see this, consider infinitesimal translation $x^i \rightarrow \tilde{x}^i = x^i + \epsilon^i(x)$. Define the infinitesimal transformation of the operators as $\delta O(x) \equiv \tilde{O}(x) - O(x)$, then the dilatation becomes

$$\begin{aligned} \delta_D O(x) &= \tilde{O}(x) - O(x) \\ &= \tilde{O}(\tilde{x} - \epsilon) - O(x) \\ &= (1 + \alpha)^{-\Delta} O(x) - \alpha x^j \partial_j O(x) - O(x). \\ &= -\alpha [\Delta + x^j \partial_j] O(x) \\ &\equiv \alpha D O(x). \end{aligned} \quad (3.80)$$

Similarly, the SCT takes the form as

$$\begin{aligned}
 \delta_K O(x) &= \tilde{O}(x) - O(x) \\
 &= \tilde{O}(\tilde{x} - \epsilon) - O(x) \\
 &= (1 - 2b \cdot x)^{-\Delta} O(x) - (x^2 b^j - 2(b \cdot x)x^j) \partial_j O(x) - O(x) \\
 &= b^i [2\Delta x_i + 2x_i x^j \partial_j - x^2 \partial_i] O(x) \\
 &\equiv b^i K_i O(x) .
 \end{aligned} \tag{3.81}$$

Here D and K_i are the generators for each transformation, which is consistent with (3.66) (we omit the demonstration for translation and rotation here, since one should be familiar enough with them in flat space QFT). Thus, from conformal invariance, we know that the correlation function should satisfy the WT identities,

$$\begin{aligned}
 \text{D : } 0 &= \sum_{n=1}^N \left(-\Delta_n - x_n^j \frac{\partial}{\partial x_n^j} \right) \langle O_1 \cdots O_N \rangle , \\
 \text{SCT : } 0 &= \sum_{n=1}^N \left(\Delta_n x_n^i + x_n^i x_n^j \frac{\partial}{\partial x_n^j} - \frac{x_n^2}{2} \frac{\partial}{\partial x_{n,i}} \right) \langle O_1 \cdots O_N \rangle .
 \end{aligned} \tag{3.82}$$

We should be able to obtain (3.77), (3.76) and (3.78) by solving the WT identities (including the ones for translation and rotation).

While most discussions of conformal symmetry are in position space, it is more common to work in momentum space when it comes to de Sitter space and cosmology.

The most straightforward way to obtain the CFT correlation functions in momentum space is to Fourier transform them from position space. Let us start with the two point function (3.76). Its Fourier modes are computed as

$$\begin{aligned}
 \langle O_1(\mathbf{p}_1) O_2(\mathbf{p}_2) \rangle &= (2\pi)^d \delta_{\Delta_1, \Delta_2} \delta^d(\mathbf{p}_1 + \mathbf{p}_2) \langle O_1(\mathbf{p}_1) O_2(\mathbf{p}_2) \rangle' \\
 &= (2\pi)^d \delta_{\Delta_1, \Delta_2} \delta^d(\mathbf{p}_1 + \mathbf{p}_2) \frac{c_{12} \pi^{d/2} 2^{d-2\Delta_1} \Gamma\left(\frac{d-2\Delta_1}{2}\right)}{\Gamma(\Delta_1)} p_1^{2\Delta_1-d} ,
 \end{aligned} \tag{3.83}$$

where we used the integration formula

$$\int \frac{d^d \mathbf{x}}{x^{2\Delta}} e^{-i\mathbf{p} \cdot \mathbf{x}} = \frac{\pi^{d/2} 2^{d-2\Delta} \Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)} p^{2\Delta-d} . \tag{3.84}$$

When $2\Delta = d + 2n$ ($n \in \mathbb{N}$), however, the coefficient on the right hand side is non-analytic and needs proper renormalization.

Similarly, three point function (3.77) is given by [83]

$$\begin{aligned}
 & \langle O_1(\mathbf{p}_1) O_2(\mathbf{p}_2) O_3(\mathbf{p}_3) \rangle \\
 &= (2\pi)^d \delta^d(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \langle O_1(\mathbf{p}_1) O_2(\mathbf{p}_2) O_3(\mathbf{p}_3) \rangle' \\
 &= c_{123} \pi^{\frac{3d}{2}} 2^{3d-\Delta_t} \prod_{j=1}^3 \frac{\Gamma(\delta_j)}{\Gamma(\frac{d}{2} - \delta_j)} \cdot \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{|\mathbf{k}|^{2\delta_3} |\mathbf{p}_1 - \mathbf{k}|^{2\delta_2} |\mathbf{p}_2 + \mathbf{k}|^{2\delta_1}} \\
 &= \frac{c_{123} \pi^{d 2^{4+\frac{3d}{2}-\Delta_t}}}{\Gamma(\frac{\Delta_t-d}{2}) \Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2}) \Gamma(\frac{\Delta_2+\Delta_3-\Delta_1}{2}) \Gamma(\frac{\Delta_3+\Delta_1-\Delta_2}{2})} \\
 & \quad \times p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \int_0^\infty dx x^{\frac{d}{2}-1} K_{\Delta_1-\frac{d}{2}}(p_1 x) K_{\Delta_2-\frac{d}{2}}(p_2 x) K_{\Delta_3-\frac{d}{2}}(p_3 x) , \quad (3.85)
 \end{aligned}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind.

On the other hand, as we have discussed in position space, CFT correlation functions can also be partly derived through symmetry restriction, i.e. WT identities. To obtain the generators in momentum space, we consider the the Fourier modes $O_\Delta(\mathbf{k})$ of the primary fields $O_\Delta(\mathbf{x})$ and then act the generators on them. Subsequently, we see operators in (3.80) and (3.81) can be written as

$$D \equiv -(\Delta - d) + k^j \partial_{k_j} \quad (3.86)$$

$$K^i \equiv 2(\Delta - d) \partial_{k_i} - 2k^j \partial_{k_j} \partial_{k_i} + k^i \partial_{k_j} \partial_{k_j} \quad (3.87)$$

in momentum space. Hence using (3.86) and (3.87), we obtain the WT identities

$$\begin{aligned}
 \text{D : } 0 &= \sum_{n=1}^N \left(k_n^j \frac{\partial}{\partial k_n^j} - (\Delta_n - d) \right) \langle O_1(\mathbf{k}_1) \cdots O_n(\mathbf{k}_n) \rangle , \quad (3.88) \\
 \text{SCT : } 0 &= \sum_{n=1}^N \left(k_n^i \frac{\partial^2}{\partial k_n^j \partial k_n^j} - 2k_n^j \frac{\partial^2}{\partial k_n^j \partial k_n^i} + 2(\Delta_n - d) \frac{\partial}{\partial k_n^i} \right) \langle O_1(\mathbf{k}_1) \cdots O_n(\mathbf{k}_n) \rangle . \quad (3.89)
 \end{aligned}$$

Also translational and rotational symmetries require momentum conservation, which results in the delta function.

For two point functions, the dilatation WT identity (3.88) determines that the momentum dependence must always take the form $k^{2\Delta_1-d}$. Here notice that since the delta function has dimension $-d$, we need to subtract that in the scaling dimension of k . Furthermore, the SCT invariance implies that only when $\Delta_1 = \Delta_2$, the two point functions give non-zero values. The result agrees with the one obtained through Fourier transformation (3.83) up to overall coefficients.

For three point functions, dilatation invariance (3.86) determines the scaling dimension of the momentum and it can be written as

$$\langle O_1(\mathbf{k}_1) O_2(\mathbf{k}_2) O_3(\mathbf{k}_3) \rangle' = k_3^{\Delta_t-2d} G(p, q), \quad p \equiv \frac{k_1 + k_2}{k_3}, \quad q \equiv \frac{k_1 - k_2}{k_3}, \quad (3.90)$$

where G is a dimensionless function depending only on p and q . Next we look at the WT identity for SCT. Taking the contraction of the generator (3.87) with arbitrary constant vector b^i , we obtain

$$b_i K^i \equiv 2(\Delta - d)b_i \partial_{k_i} - 2k^j \partial_{k_j} b_i \partial_{k_i} + b_i k^i \partial_{k_j} \partial_{k_j} . \quad (3.91)$$

Acting (3.91) on functions that depend only on the norm of the momentum, this operator becomes

$$b_i K^i = (b_i k^i) \left[(2\Delta - d - 1) \frac{1}{k} \partial_k - \partial_k^2 \right] . \quad (3.92)$$

However, generally it is not a simple task to solve the WT identity (3.89) analytically. Here as a particular case, we consider when $d = 3$ and $\Delta_1 = \Delta_2 = 2$. Using (3.92), the WT identity (3.89) becomes

$$(b_i \mathcal{K}_1^i + b_i \mathcal{K}_2^i + b_i \mathcal{K}_3^i) \left(k_3^{\Delta_3 - 2d} \hat{G} \right) = 0 . \quad (3.93)$$

Then by choosing b^i to be a vector orthogonal to k_3^i , and using the momentum conservation, we obtain

$$(\partial_{k_1}^2 - \partial_{k_2}^2) G(q, p) = 0 \quad \rightarrow \quad \partial_q \partial_p G(q, p) = 0 . \quad (3.94)$$

This equation suggests that G is independent of either p or q . Since (3.94) is a linear equation, we first find a q independent solution. The p independent solution is immediately obtained by analytically connecting it with $k_2 \rightarrow -k_2$. Then, substituting $G(p)$ into (3.93), we have

$$(p^2 - 1) \partial_p^2 G(p) + 2p \partial_p G(p) - [\Delta_3(\Delta_3 - 3) + 2] G(p) = 0 . \quad (3.95)$$

This is a hypergeometric equation, who has two linearly independent solutions and they are given by hypergeometric functions.

3.5 dS/CFT

The original proposal of the dS/CFT correspondence [12] suggests a duality between theories of quantum gravity on $(d + 1)$ dimensional de Sitter space and conformal field theories on the d dimensional asymptotic boundary of this space. This idea emerged as a counterpart to the well-established AdS/CFT correspondence, inspired by the similarities between AdS and dS space. The dS/CFT correspondence is still largely hypothetical and remains a topic of active research. It faces several theoretical challenges and is less well-understood than its AdS/CFT counterpart, partly due to the rather complex holographic nature of de Sitter space and the difficulty in defining a global time coordinate or a stable vacuum state in such a spacetime. Despite these challenges, it remains a significant area

of theoretical investigation for insights into the nature of our universe. In this subsection, we start with the general statement of dS/CFT correspondence and its relation with AdS/CFT. We then give a review of the example from [12], following closely the original convention. At the end, we discuss the dS/CFT dictionary for light scalar fields.

As a starting point for consistency check, let us look at the isometries of both theories. It appears that the de Sitter isometry $SO(1, d+1)$ is isomorphic to the Euclidean conformal group, as we can see from the commutation relations of their Killing vectors (2.3) and (3.68). More specifically, the corresponding generators (3.66) of de Sitter isometries in the Poincare coordinates are given by

$$\begin{aligned}\hat{P}_i &= \partial_i, \\ \hat{M}_{ij} &= x_j \partial_i - x_i \partial_j, \\ \hat{D} &= x^i \partial_{x^i} + \tau \partial_\tau, \\ \hat{K}_i &= x_i (x^j \partial_{x^j} + \tau \partial_\tau) + 2(\tau^2 - x^2) \partial_{x^i},\end{aligned}\tag{3.96}$$

which have the same form with (3.66). More details can be found in Appendix A.

Another statement of dS/CFT is the claim that correlation functions of fields in de Sitter space with certain boundary conditions can be calculated as CFT correlation functions. For example, the asymptotic form of the scalar two point function (3.11) in Poincare coordinate on the future boundary $\tau, \tau' \rightarrow 0$ is given by

$$G_E(\tau, \mathbf{x}; \tau', \mathbf{y}) = c_{\Delta_-} \left(\frac{\tau \tau'}{|\mathbf{x} - \mathbf{y}|^2} \right)^{\Delta_-} + c_{\Delta_+} \left(\frac{\tau \tau'}{|\mathbf{x} - \mathbf{y}|^2} \right)^{\Delta_+},\tag{3.97}$$

where c_{Δ_\pm} are constants independent of the coordinates. We can see that the late time two point function indeed consists of two CFT two-point functions of a scalar primary with conformal weight Δ_\pm . For light scalar fields with real conformal weights, the first term in (3.97) dominates over the second term. Therefore we can ignore the second term and the two point function could be seen as CFT scalar two-point function with conformal weight Δ_- . As for heavy scalar fields, the conformal weights are complex numbers and both terms will remain in the late time two point function. However, we will see in the next section that it is possible to isolate each term by choosing different boundary conditions for the scalar fields. This kind of correspondence between correlation functions also applies to fields with spin. For example, the bulk metric h_{ij} is related to the CFT stress tensor T_{ij} .

Moreover, the similarities between de Sitter and anti-de Sitter space also suggest the holographic correspondence. For instance, de Sitter space has conformal boundaries like AdS, which can be seen from the metric (2.5) or (2.10) by taking the future or past infinity limit. The prospect that a holographic structure similar to AdS/CFT correspondence might exist for de Sitter space appears to be a reasonable expectation, given that AdS and dS space are connected to each other through analytic continuation. We can see this

from the metric of AdS_{d+1} in the Poincare coordinates,

$$ds^2 = \frac{l_{AdS}^2}{z^2} \left(-dt^2 + dz^2 + \sum_{i=1}^{d-1} dx^i dx^i \right). \quad (3.98)$$

If we Wick rotation the time direction $t \rightarrow ix^d$, we obtain the Euclidean AdS (EAdS) metric

$$ds^2 = \frac{l_{AdS}^2}{z^2} \left(dz^2 + \sum_{i=1}^d dx^i dx^i \right). \quad (3.99)$$

Comparing it with the Poincare coordinate metric on de Sitter space (2.10) (after restoring the de Sitter length l_{dS}),

$$ds^2 = \frac{l_{dS}^2}{\tau^2} \left(-d\tau^2 + \sum_{i=1}^d dx^i dx^i \right), \quad (3.100)$$

we see that (3.99) and (3.100) are related by analytic continuation

$$z = -e^{\pm \frac{i\pi}{2}} \tau, \quad l_{AdS} = e^{\pm \frac{i\pi}{2}} l_{dS}. \quad (3.101)$$

The similar analytic continuation procedure works globally as well.

From the above insights, it seems reasonable to consider a dS/CFT correspondence as an extension and analytic continuation of AdS/CFT [25, 33, 84–87]. Then as an analogy of the original proposal with AdS/CFT, we have the relation between the partition functions of CFTs and those of QFTs on de Sitter space as $Z_{\text{CFT}}[h_{ij}, \phi_0] = Z_{\text{de Sitter}}[h_{ij}, \phi_0]$. Here $Z_{\text{de Sitter}}[h_{ij}, \phi_0]$ could be seen as an analytic continuation of the AdS partition functions. Next, to investigate the detail of dS partition function, recall that the conformal boundary we find in Section 2.4 is an S^d sphere in global coordinates. The partition function is defined by path integral ends at this boundary. Notice that the partition function shares exactly the same definition as the Hartle-Hawking state (3.59), therefore the dS/CFT could be written as

$$Z_{\text{CFT}}[h_{ij}, \phi_0] = \Psi_{HH}[h_{ij}, \phi_0]. \quad (3.102)$$

Although dS/CFT is formulated in a similar manner as AdS/CFT, there are a few critical differences between them which cause problematic issues for dS/CFT. For example, the most obvious difference between the two holographic frameworks is the role of time. In AdS/CFT, there is a timelike asymptotic Killing vector which runs parallel to the conformal boundary, and is associated with time translations in the CFT. This means the unitary time evolution in the bulk theory corresponds to the unitary time evolution in the boundary theory as well. However, as we have seen in the previous section, there is no globally timelike Killing vector and hence no positive conserved charges in de Sitter space. The boundary theory is Euclidean CFT and its time evolution is not dual to time evolution in the bulk. Therefore, the corresponding property in the CFT of the unitarity of time evolution of the dS wavefunction is not yet clear.

3.5.1 dS_3/CFT_2

Here we review a detailed dS_3/CFT_2 example presented in the original dS/CFT proposal [12]. We will first work on the analysis of the asymptotic symmetry group at the de Sitter space boundary. Then we will show that the bulk de Sitter correlators with points on the boundary are related to CFT correlators on the sphere, which is the main support for the dS/CFT proposal in the paper [12]. Furthermore, for two point functions with points on different boundaries, the points on the future boundary are mapped to the antipodal points relative to those on the past boundary. This results in the existence of only one dual CFT instead of two, despite the fact that de Sitter space has two asymptotic boundaries.

The plane and I^- correlators in Poincare coordinates

Consider a $d = 3$ dimensional de Sitter space dS_3 . First we work with the Poincare coordinates, where the metric is given by

$$\begin{aligned} ds^2 &= -dt^2 + e^{2t}(dx^2 + dy^2) \\ &= -dt^2 + e^{2t}dzd\bar{z}. \end{aligned} \quad (3.103)$$

Here we introduced the complex coordinates (z, \bar{z}) , where $z = x + iy$ and $\bar{z} = x - iy$. This metric describes the Poincare patch which includes the past boundary I^- . Here we denote this patch by \mathcal{O}^- and the past boundary is described as a flat R^2 plane. The asymptotically de Sitter geometry is obtained by taking $t \rightarrow -\infty$ in (3.103),

$$g_{z\bar{z}} = \frac{e^{-2t}}{2} + \mathcal{O}(1), \quad g_{tt} = -1 + \mathcal{O}(e^{2t}), \quad g_{zz} = \mathcal{O}(1), \quad g_{zt} = \mathcal{O}(e^{3t}). \quad (3.104)$$

The asymptotic symmetries of dS_3 are diffeomorphisms which preserve (3.104). To find its Killing vector, consider the vector fields

$$\zeta = U\partial_z + \frac{1}{2}e^{2t}U''\partial_{\bar{z}} + \frac{1}{2}U'\partial_t, \quad (3.105)$$

where $U = U(z)$ is holomorphic and the prime denotes differentiation with respect to z . Although it is necessary to add its complex conjugate to obtain a real vector field, we suppress this addition in (3.105) and all subsequent formulas for simplicity of notation. In general, the metric transforms under a diffeomorphism as the Lie derivative

$$\delta_\zeta g_{mn} = -\mathcal{L}_\zeta g_{mn}. \quad (3.106)$$

Using (3.105), we can express (3.106) in terms of U as

$$\delta_U g_{zz} = -\frac{1}{2}U''', \quad \delta_U g_{z\bar{z}} = \delta_U g_{zt} = \delta_U g_{tt} = 0. \quad (3.107)$$

It is readily confirmed that the change (3.107) in the metric satisfies (3.104) and hence (3.105) generates an asymptotic symmetry of de Sitter space on I^- .

As a special case, when U takes the form of $U = \alpha + \beta z + \gamma z^2$, where α , β and γ are complex constants, U''' vanishes. Therefore the metric is invariant. These transformations generate the $SL(2, C)$ global isometries of $2 + 1$ de Sitter. Consequently, we can see that the asymptotic symmetry group of dS_3 is the conformal group of the complex plane, and the isometry group is $SL(2, C)$ subgroup of the asymptotic symmetry group.

As we have seen that the conformal group of Euclidean R^2 has an action on the past boundary, this suggests that the rescaled gravity correlators, restricted to I^- , should match those of a Euclidean 2d conformal field theory. To verify this expectation, we consider a scalar field of mass m with wave equation given by

$$m^2 \phi = \nabla^2 \phi = -\partial_t^2 \phi + 2\partial_t \phi + 4e^{2t} \partial_z \partial_{\bar{z}} \phi. \quad (3.108)$$

Note that the last term is negligible. Near the past boundary $t \rightarrow -\infty$, the solutions behave as

$$\phi \sim e^{\Delta_{\pm} t}, \quad (3.109)$$

where $\Delta_{\pm} = 1 \pm \sqrt{1 - m^2}$. When $0 < m^2 < 1$, we can impose a boundary condition on the boundary

$$\lim_{t \rightarrow -\infty} \phi(z, \bar{z}, t) = e^{\Delta_- t} \phi_-(z, \bar{z}), \quad (3.110)$$

with subleading terms suppressed by at least one power of e^{2t} . The dS_3/CFT_2 correspondence proposes that ϕ_- is dual to an operator O_ϕ with dimension Δ_+ in the boundary CFT, in direct analogy to the AdS/CFT correspondence. The two point correlator of O_ϕ is given by the quadratic coefficient of ϕ_- in the expression [88]

$$\lim_{t \rightarrow -\infty} \int_{I^-} d^2 z d^2 v \left[e^{-2(t+t')} \phi(t, z, \bar{z}) \overleftrightarrow{\partial}_t G(t, z, \bar{z}; t', v, \bar{v}) \overleftrightarrow{\partial}_{t'} \phi(t', v, \bar{v}) \right]_{t=t'}, \quad (3.111)$$

where G is the Hadamard two point function, which can be expressed in terms of the Wightman function G_w (3.11) as $G(X, Y) = G_w(X, Y) + G_w(Y, X)$. We will simply refer to G as the Green function in the remainder of this thesis. Near I^- , G behaves as follows

$$\lim_{t, t' \rightarrow -\infty} G(t, z, \bar{z}; t', v, \bar{v}) = \frac{c_+ e^{\Delta_+(t+t')}}{|z - v|^{2\Delta_+}} + \frac{c_- e^{\Delta_-(t+t')}}{|z - v|^{2\Delta_-}}, \quad (3.112)$$

where c_{\pm} are constants. Inserting the boundary condition (3.110) and (3.112), we find that (3.111) is proportional to

$$\int_{I^-} d^2 z d^2 v \phi_-(z, \bar{z}) |z - v|^{-2\Delta_+} \phi_-(v, \bar{v}). \quad (3.113)$$

We therefore conclude that the dual operator O_ϕ should obey

$$\langle O_\phi(z, \bar{z}) O_\phi(v, \bar{v}) \rangle = \frac{C}{|z - v|^{2\Delta_+}}, \quad (3.114)$$

where C is a constant. We see this indeed matches with a CFT correlator of an operator with dimension Δ_+ .

However, it should be noted that the boundary condition (3.110) is not a general case. Usually there are also solutions with subleading behavior $\phi_+(z, \bar{z})e^{\Delta_+ t}$ at I^- . Including these terms would lead to an additional term in (3.113) proportional to

$$\int_{I^-} d^2 z d^2 v \phi_+(z, \bar{z}) |z - v|^{-2\Delta_-} \phi_+(v, \bar{v}). \quad (3.115)$$

which should be associated with an CFT operator of dimension Δ_- . We will see shortly that this additional boundary condition can be imposed on the future boundary I^+ , which is a region not covered by the Poincare patch \mathcal{O}^- . Alternatively, this second set of independent fields could be eliminated by implementing an appropriate boundary condition at the future horizon as $t \rightarrow \infty$, which would result in a different Green function.

As for heavy field with mass $m^2 > 1$, the conformal weights Δ_{\pm} become complex conjugates, indicating that the dual CFT is not unitary, and the treatment of the additional term (3.115) might be rather complicated. However, the lack of any apparent reason for the dual CFT to necessarily be unitary suggests that this issue may not be a major concern at present.

The sphere and I^{\pm} correlators in global coordinates

In this subsection, we consider dS_3 in global coordinates. The metric is given by

$$\begin{aligned} ds^2 &= -d\tau^2 + \cosh^2 \tau (d\theta^2 + \sin^2 \theta d\phi) \\ &= -d\tau^2 + 4 \cosh^2 \tau \frac{dw d\bar{w}}{(1 + w\bar{w})^2}, \end{aligned} \quad (3.116)$$

where $w = \tan \frac{\theta}{2} e^{i\phi}$ is a complex coordinate on the round sphere. Global coordinates cover the entire de Sitter space, which has two boundaries: the future and past S^2 boundaries. Hence two types of boundary two point functions can be considered here: those with points on one or both boundaries. Firstly, let us examine the two-point correlator with both points on I^- . The spherical analog of (3.111) is

$$\begin{aligned} &\lim_{\tau \rightarrow -\infty} \int_{I^-} d^2 w d^2 v \sqrt{h(w)h(v)} \\ &\quad \times \left[e^{-2(\tau+\tau')} \phi(\tau, w, \bar{w}) \overleftrightarrow{\partial}_{\tau} G(\tau, w, \bar{w}; \tau', v, \bar{v}) \overleftrightarrow{\partial}_{\tau'} \phi(\tau', v, \bar{v}) \right]_{\tau=\tau'}, \end{aligned} \quad (3.117)$$

where $h(w) = 2(1 + w\bar{w})^{-2}$ is the measure on the sphere. The asymptotic behavior of ϕ near I^- can be decomposed as

$$\lim_{\tau \rightarrow -\infty} \phi(\tau, w, \bar{w}) = \phi_+^{\text{in}}(w, \bar{w}) e^{\Delta_+ \tau} + \phi_-^{\text{in}}(w, \bar{w}) e^{\Delta_- \tau}. \quad (3.118)$$

The superscripts “in” (“out”) denote quantities on I^- (I^+). The Green function G then behaves as

$$\begin{aligned} \lim_{\tau, \tau' \rightarrow -\infty} G(\tau, w, \bar{w}; \tau', v, \bar{v}) = & c_+ e^{\Delta_+(\tau+\tau')} \frac{(1+w\bar{w})^{\Delta_+} (1+v\bar{v})^{\Delta_+}}{|w-v|^{2\Delta_+}} \\ & + c_- e^{\Delta_-(\tau+\tau')} \frac{(1+w\bar{w})^{\Delta_-} (1+v\bar{v})^{\Delta_-}}{|w-v|^{2\Delta_-}}. \end{aligned} \quad (3.119)$$

Substituting it into (3.117), we obtain

$$\begin{aligned} \int_{I^-} d^2v d^2w \sqrt{h(v)h(w)} [c_+ \phi_-^{\text{in}}(w, \bar{w}) G_{\Delta_+}(w, \bar{w}; v, \bar{v}) \phi_-^{\text{in}}(v, \bar{v}) \\ + c_- \phi_+^{\text{in}}(w, \bar{w}) G_{\Delta_-}(w, \bar{w}; v, \bar{v}) \phi_+^{\text{in}}(v, \bar{v})] . \end{aligned} \quad (3.120)$$

where $G_{\Delta_{\pm}}$ is the two point function for a conformal field of dimension Δ_{\pm} on the sphere. It is explicitly given by

$$G_{\Delta_{\pm}} = \left[\frac{(1+w\bar{w})(1+v\bar{v})}{|w-v|^2} \right]^{\Delta_{\pm}}. \quad (3.121)$$

As a result, it becomes evident that the two-point scalar correlators can be identified with correlators of conformal fields of dimension Δ_{\pm} just as in Poincare coordinates. The difference is that these correlators are now situated on a sphere instead of a plane. The above procedure can be repeated to obtain a similar expression for the future boundary I^+ .

There is also the case of correlators for one point on I^- and one on I^+ , which exhibits intriguing features. Here we must compute

$$\begin{aligned} \lim_{\tau \rightarrow -\infty} \int_{I^-} d^2w \int_{I^+} d^2v \sqrt{h(w)h(v)} \\ \times \left[e^{2(\tau-\tau')} \phi(\tau, w, \bar{w}) \overleftrightarrow{\partial}_{\tau} G(\tau, w, \bar{w}; \tau', v, \bar{v}) \overleftrightarrow{\partial}_{\tau'} \phi(\tau', v, \bar{v}) \right]_{\tau=-\tau'}. \end{aligned} \quad (3.122)$$

The relevant limit of the Green function is given by

$$\begin{aligned} \lim_{\tau \rightarrow -\infty, \tau' \rightarrow +\infty} G(\tau, w, \bar{w}; \tau', v, \bar{v}) = c_+ \cos(\pi\Delta_+) e^{\Delta_+(\tau-\tau')} G_{\Delta_+} \left(w, \bar{w}; -\frac{1}{\bar{v}}, -\frac{1}{v} \right) \\ + (+ \leftrightarrow -). \end{aligned} \quad (3.123)$$

Again substituting (3.123), (3.122) becomes

$$\begin{aligned} \int_{S^2} d^2w d^2v \sqrt{h(w)h(v)} [c_+ \cos(\pi\Delta_+) \phi_-^{\text{in}}(w, \bar{w}) G_{\Delta_+}(w, \bar{w}; v, \bar{v}) \tilde{\phi}_-^{\text{out}}(v, \bar{v}) \\ + c_- \cos(\pi\Delta_-) \phi_+^{\text{in}}(w, \bar{w}) G_{\Delta_-}(w, \bar{w}; v, \bar{v}) \tilde{\phi}_+^{\text{out}}(v, \bar{v})] . \end{aligned} \quad (3.124)$$

Here we have defined the inverted boundary field as $\tilde{\phi}_+^{\text{out}}(v, \bar{v}) = \phi_+^{\text{out}}(-\frac{1}{\bar{v}}, -\frac{1}{v})$. One significant feature we observe is that ϕ^{in} and $\tilde{\phi}^{\text{out}}$ exhibit a non-trivial two-point function,

even though they live on widely separated boundaries. (3.124) indicates that for bulk gravity correlators, inserting additional gravity operators on I^+ is equivalent to inserting dual CFT operators at the antipodal points on the sphere of I^- . This is due to the unique causal structure of de Sitter space, where a light ray from a point on I^- reaches its antipodal point at I^+ , and a singularity of a correlator between a point on I^- and one on I^+ can occur only if the two points are null separated. This causal connection affects the symmetry group as well; instead of two separate conformal groups for the two boundaries, the Green functions are aware of this causal connection and thus transform under only a single subgroup, which leads to a single CFT on one sphere. This can also be seen from the fact that only two of the four boundary fields ϕ_{\pm}^{in} , ϕ_{\pm}^{out} are independent. They are related to each other through Bogolubov transformations.

3.5.2 Correlation functions for light field

Regardless of the unsolved issues of dS/CFT, the holographic formula itself is a powerful method for calculations. For example, in the previous subsection 3.5.1, we have seen it provides dS/CFT dictionary between correlation functions of fields on the boundary of dS and CFT correlation functions. To further discuss this, we work in Poincare coordinates and focus on theory with one scalar field. Nevertheless, the following discussion and results could be extended to multiple scalar field theory straightforwardly. Recall the wavefunction method we introduced in section 3.3, we can express wavefunction in terms of the action, using semiclassical approximation

$$\Psi[\bar{\phi}; \eta_0] \approx \mathcal{N} e^{iS[\phi_{\text{cl}}(\bar{\phi})]}. \quad (3.125)$$

Here we are interested in momentum space correlation functions, in which the wavefunction of a bulk scalar field ϕ can be expanded in terms of its boundary value of $\bar{\phi} = \phi(\tau_*)$,

$$\begin{aligned} \Psi[\bar{\phi}] = \exp & \left[\frac{1}{2} \int \frac{d^d \mathbf{k}_1 d^d \mathbf{k}_2}{(2\pi)^d} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} \rangle' \bar{\phi}_{\mathbf{k}_1} \bar{\phi}_{\mathbf{k}_2} \right. \\ & \left. + \frac{1}{3!} \int \frac{d^d \mathbf{k}_1 d^d \mathbf{k}_2 d^d \mathbf{k}_3}{(2\pi)^{\frac{3}{2}d}} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} O_{\mathbf{k}_3} \rangle' \bar{\phi}_{\mathbf{k}_1} \bar{\phi}_{\mathbf{k}_2} \bar{\phi}_{\mathbf{k}_3} + \mathcal{O}(\bar{\phi}^4) \right], \end{aligned} \quad (3.126)$$

where the $\langle O_{\mathbf{k}_1} \dots O_{\mathbf{k}_n} \rangle'$ s are delta function dropped coefficients defined by

$$\langle O_{\mathbf{k}_1} \dots O_{\mathbf{k}_n} \rangle \equiv \delta^d \left(\sum_{i=1}^n \mathbf{k}_i \right) \langle O_{\mathbf{k}_1} \dots O_{\mathbf{k}_n} \rangle', \quad (3.127)$$

and $\langle O_{\mathbf{k}_1} \dots O_{\mathbf{k}_n} \rangle$ can be considered as correlation functions of certain operators O proportional to the momentum conservation delta functions, which come from homogeneity and isotropy of τ constant slices. The equal-time correlators of an operator $F[\phi_{\mathbf{k}}(\tau)]$ at $\tau = \tau_*$ are given by [14]

$$\langle F[\phi_{\mathbf{k}}(\tau_*)] \rangle = \int [d\bar{\phi}] |\Psi[\bar{\phi}]|^2 F[\bar{\phi}_{\mathbf{k}}], \quad (3.128)$$

where $F[\phi_{\mathbf{k}}(\tau)]$ is made from products of the field ϕ at the equal time τ . For example, we have $F[\phi_{\mathbf{k}}(\tau)] = \phi_{\mathbf{k}}(\tau)\phi_{\mathbf{k}'}(\tau)$ for the computation of two-point functions $\langle\phi_{\mathbf{k}}(\tau_*)\phi_{\mathbf{k}'}(\tau_*)\rangle$. Then from (3.126), we have

$$\langle\phi_{\mathbf{k}}(\tau_*)\phi_{-\mathbf{k}}(\tau_*)\rangle' = -\frac{1}{2\text{Re}\langle O_{\mathbf{k}}O_{-\mathbf{k}}\rangle'}, \quad (3.129)$$

$$\langle\prod_{i=1}^3\phi_{\mathbf{k}_i}(\tau_*)\rangle' = 2\prod_{i=1}^3\frac{1}{2\text{Re}\langle O_{\mathbf{k}_i}O_{-\mathbf{k}_i}\rangle'}\text{Re}\langle O_{\mathbf{k}_1}O_{\mathbf{k}_2}O_{\mathbf{k}_3}\rangle', \quad (3.130)$$

$$\begin{aligned} \langle\prod_{i=1}^4\phi_{\mathbf{k}_i}(\tau_*)\rangle' &= 2\prod_{i=1}^4\frac{1}{2\text{Re}\langle O_{\mathbf{k}_i}O_{-\mathbf{k}_i}\rangle'}\left[\text{Re}\langle O_{\mathbf{k}_1}O_{\mathbf{k}_2}O_{\mathbf{k}_3}O_{\mathbf{k}_4}\rangle' \right. \\ &\quad \left. - \frac{\text{Re}\langle O_{\mathbf{k}_1}O_{\mathbf{k}_2}O_{-\mathbf{k}_{12}}\rangle'\text{Re}\langle O_{\mathbf{k}_{12}}O_{\mathbf{k}_3}O_{\mathbf{k}_4}\rangle'}{\text{Re}\langle O_{\mathbf{k}_{12}}O_{\mathbf{k}_{12}}\rangle'} + 2\text{ permutations}\right], \end{aligned} \quad (3.131)$$

where $\mathbf{k}_{ij} \equiv \mathbf{k}_i + \mathbf{k}_j$. The famous dS/CFT dictionary suggests that $O_{\mathbf{k}}$ is the dual CFT operator of the bulk field ϕ . Therefore we have above simple dictionary for late time dS correlators in terms of CFT correlators. For example, from (3.11), the two-point functions of light scalars with the mass $0 \leq m < \frac{d}{2}H$ in the superhorizon regime $-k\tau_* \ll 1$ could be calculated as

$$\langle\phi_{\mathbf{k}}(\tau_*)\phi_{-\mathbf{k}}(\tau_*)\rangle' \propto (-\tau_*)^d(-k\tau_*)^{-2\nu} \quad \text{with} \quad \nu = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}, \quad (3.132)$$

Indeed this can naturally be identified with the inverse of two-point functions of the dual CFT operator with the scaling dimension $\Delta = \frac{d}{2} + \nu$. However, this dictionary (3.129) does not work naively for heavy scalars in the principal series. In contrast to light fields, heavy fields oscillate outside the cosmological horizon and so do their two-point functions, which could be obtained the same way through (3.11),

$$\langle\phi_{\mathbf{k}}(\tau_*)\phi_{-\mathbf{k}}(\tau_*)\rangle' \propto (-\tau_*)^d \left|1 + e^{-\pi\mu}e^{i\alpha(\mu)}(-k\tau_*)^{2i\mu}\right|^2 \quad \text{with} \quad \mu = \sqrt{\frac{m^2}{H^2} - \frac{d^2}{4}}, \quad (3.133)$$

where $\alpha(\mu)$ is a mass-dependent phase factor. At least naively, it is hard to identify them with inverse of (a real part of) conformal two-point functions. Instead of a simple inverse as (3.129), we will demonstrate in the following sections that they should be interpreted as two-point functions on a cyclic RG flow generated by double-trace deformations, similarly to the AdS tachyon with the mass squared below the Breitenlohner-Freedman (BF) bound [89].

4 Wavefunctions on de Sitter space

We have seen in the previous section that two point functions of heavy fields (3.133) oscillates outside the cosmological horizon and cannot be identified as the inverse of CFT correlation functions. It is therefore important to find an appropriate dS/CFT interpretation for it. This and next sections are the reprints of [90], where we revisited holographic interpretation of dS wavefunctions and late time two point functions, mainly focusing on heavy scalars in the principal series. In this section, we first show that for heavy fields, wavefunctions evaluated with mixed boundary conditions are naturally identified with a generating function of the would-be dual CFT. Then, we demonstrate that wavefunctions evaluated with the Dirichlet boundary conditions are identified with generating functions of QFTs on a cyclic RG flow obtained by double-trace deformations. For comparison, we also discuss mixed boundary conditions for light fields at the end of the section, following Ref. [36].

AdS/CFT	$m^2/H_{\text{AdS}}^2 > -\frac{d^2}{4} + 1$	$-\frac{d^2}{4} < m^2/H_{\text{AdS}}^2 < -\frac{d^2}{4} + 1$	$m^2/H_{\text{AdS}}^2 < -\frac{d^2}{4}$
Δ_+	$\Delta_+ > \frac{d}{2} + 1$	$\frac{d}{2} < \Delta_+ < \frac{d}{2} + 1$	$\Delta_+ = \frac{d}{2} + i\mu_{\text{AdS}}$
Δ_-	$\Delta_- < \frac{d}{2} - 1$	$\frac{d}{2} - 1 < \Delta_- < \frac{d}{2}$	$\Delta_- = \frac{d}{2} - i\mu_{\text{AdS}}$

Table 1: AdS scalars have two modes with the asymptotic behavior $\phi \sim z^{\Delta_{\pm}}$ near the AdS boundary, where $\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{m^2}{H_{\text{AdS}}^2} + \frac{d^2}{4}}$. Canonical quantization in AdS shows that the mode $\phi \sim z^{\Delta}$ has a finite energy only when $\Delta > \frac{d}{2} - 1$, which coincides with the unitarity bound in the dual CFT. When the mass squared is below the BF bound $m^2 < -\frac{d^2}{4}$, there appears tachyonic instability. Also, in this regime, the dual CFT operator has a complex conformal dimension $\Delta_{\pm} = \frac{d}{2} \pm i\mu_{\text{AdS}}$ with $\mu_{\text{AdS}} = \sqrt{-\frac{m^2}{H_{\text{AdS}}^2} - \frac{d^2}{4}}$, which is prohibited by unitarity in the dual CFT.

Before proceeding to the details of the discussion, it is convenient to summarize our results in the connection with the story in AdS/CFT (see also Table 1). For this purpose, let us recall boundary conditions for quantization of scalar fields on AdS [91–93]. First, when the mass squared m^2 is in the range $m^2 > (-\frac{d^2}{4} + 1)H_{\text{AdS}}^2$, the unique admissible boundary condition is the Dirichlet boundary condition that kills the mode with the asymptotic behavior,

$$\phi \sim z^{\Delta_-}, \quad \Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{m^2}{H_{\text{AdS}}^2} + \frac{d^2}{4}}, \quad (4.1)$$

where H_{AdS} is inverse of the AdS radius l_{AdS} and z is the scale direction of the Poincare coordinate ($z = 0$ is the AdS boundary). For $-\frac{d^2}{4}H_{\text{AdS}}^2 < m^2 < (-\frac{d^2}{4} + 1)H_{\text{AdS}}^2$, on the other hand, there are two possible boundary conditions essentially because the two modes

dS/CFT	$m^2/H^2 < 0$	$\frac{d^2}{4} > m^2/H^2 > 0$	$m^2/H^2 > \frac{d^2}{4}$
Δ_+	$\Delta_+ > d$	$\frac{d}{2} < \Delta_+ < d$	$\Delta_+ = \frac{d}{2} + i\mu$
Δ_-	$\Delta_- < 0$	$0 < \Delta_- < \frac{d}{2}$	$\Delta_- = \frac{d}{2} - i\mu$

Table 2: Scalar fields on dS have two modes with the asymptotic late time behavior $\phi \sim (-\tau)^{\Delta_{\pm}}$, where $\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$ for $m^2 < \frac{d^2}{4}H^2$ and $\Delta_{\pm} = \frac{d}{2} \pm i\mu$ for $m^2 > \frac{d^2}{4}H^2$. Unitary representations on de Sitter space are classified into the principal series $m^2 > \frac{d^2}{4}H^2$ and the complementary series $0 < m^2 < \frac{d^2}{4}H^2$. In contrast to the AdS/CFT case, the time direction in the bulk is not along the CFT direction, so that unitarity in the bulk does not imply that in the would-be dual CFT. In particular, for the principal series, the dual CFT operator has a complex conformal dimension, similarly to AdS scalars with the mass squared below the BF bound.

$\phi \sim z^{\Delta_{\pm}}$ carry a finite energy. In particular, one may employ a mixed boundary condition that kills the mode $\phi \sim z^{\Delta_+}$. It is known that the CFT dual to the mixed boundary condition is related to the CFT dual to the Dirichlet boundary condition through an RG flow induced by double-trace deformations [94–98].⁴ See more details in Appendix B. Note that, as is known as the BF bound, tachyonic instability appears in the mass range $m^2 < -\frac{d^2}{4}H_{\text{AdS}}^2$. Correspondingly, the dual CFT operator has a complex conformal dimension, which violates unitarity.

Now let us get back to the dS story. First, let us recall that in the computation of dS correlators, we do not impose any boundary condition at the future infinity, but rather we impose boundary conditions deep inside the horizon alone. This reflects the closed time path nature of the in-in formalism (the Schwinger-Keldysh formalism). On the other hand, the dS/CFT dictionary equates dS wavefunctions at late time with generating functions of the would-be dual CFTs. While we normally employ the Dirichlet boundary condition $\phi_{\mathbf{k}}(\tau_*) = \bar{\phi}_{\mathbf{k}}$ for the computation of wavefunctions, there is no a priori reason for this: a different choice of boundary conditions simply means a different representation of wavefunctions, so that in contrast to the AdS case, there is no physical criterion to select boundary conditions. Indeed, in the context of higher spin dS/CFT [15, 35], one may consider two different boundary conditions dual to the free $Sp(N)$ model and the critical $Sp(N)$ model, which are interpolated via an RG flow induced by double-trace deformations [35]. See also Ref. [36] for double-trace deformations in dS/CFT with bulk scalars in the complementary series.

In this section, we study a similar problem for heavy scalars in the principal series. First, we point out that for heavy fields, a mixed boundary condition gives a natural

⁴The RG induced by double-trace deformations is also discussed in [99–101] in the context of Wilsonian holographic renormalization group.

identification of dS wavefunctions with generating functions of the would-be dual CFTs. In particular, QFTs dual to wavefunctions with the Dirichlet boundary condition are their double-trace deformations, similarly to the AdS story mentioned earlier. Another property characteristic to heavy fields on dS is that the RG flow generated by double-trace deformations is cyclic, simply because the dual CFT operator has a complex conformal dimension (see Table 2). This situation is analogous to AdS tachyons which violate the BF bound: Recall that dS and AdS are related via the analytic continuation $H^2 \rightarrow -H_{\text{AdS}}^2$ together with double Wick rotations. Therefore, AdS scalars corresponding to dS scalars in the principal series $m^2/H^2 > \frac{d^2}{4}$ have the mass squared $m^2/H_{\text{AdS}}^2 < -\frac{d^2}{4}$, which violates the BF bound. Such fields are prohibited in the standard AdS/CFT, but Ref. [89] argued that they are acceptable in nonrelativistic AdS/CFT and studied their relation to RG limit cycles. In dS/CFT, essentially the same situation appears even without considering nonrelativistic setups, simply by taking heavy fields in the bulk which are inevitable when we consider UV completion of the bulk theory.

Furthermore, based on these observations on dS wavefunctions, we provide a dictionary between dS two-point functions and CFT two-point functions of the form,

$$\langle \phi_{\mathbf{k}}(\tau_*) \phi_{-\mathbf{k}}(\tau_*) \rangle' = -(-\tau_*)^{2d} \frac{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-}}{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} + \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-}}, \quad (4.2)$$

where $\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm}$ are two-point functions in QFTs dual to wavefunctions with mixed boundary conditions. The subscripts \pm are associated to the time ordered and anti-time ordered integration contours in the in-in formalism. For light scalars in the complementary series, $\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm}$ coincide with conformal two-point functions by themselves. On the other hand, for heavy scalars in the principal series, they coincide with conformal two-point functions up to analytic terms which can be subtracted by local counterterms in a similar manner to holographic renormalization in the AdS/CFT context [102, 103]. This provides a new dS/CFT dictionary applicable when mixed boundary conditions are employed.

4.1 Dirichlet boundary conditions

We begin by the Dirichlet boundary problem on de Sitter space, which is normally used to calculate dS correlators directly from the wavefunction. Consider a free scalar ϕ on $(d+1)$ -dimensional de Sitter space in Poincare coordinates. The action is given by

$$\begin{aligned} S[\phi] &= -\frac{1}{2} \int d^{d+1}x \sqrt{-g} [(\partial_\mu \phi)^2 + m^2 \phi^2] \\ &= \frac{1}{2} \int \frac{d\tau}{(-\tau)^{d+1}} \int d^d \mathbf{x} [(-\tau)^2 ((\partial_\tau \phi)^2 - (\partial_i \phi)^2) - m^2 \phi^2]. \end{aligned} \quad (4.3)$$

It is easier working in momentum space, so we Fourier transform the action and it becomes

$$S[\phi] = \frac{1}{2} \int \frac{d\tau}{(-\tau)^{d+1}} \int \frac{d^d \mathbf{k}}{(2\pi)^d} [(\theta_\tau \phi_{-\mathbf{k}})(\theta_\tau \phi_{\mathbf{k}}) - (m^2 + k^2 \tau^2) \phi_{-\mathbf{k}} \phi_{\mathbf{k}}]. \quad (4.4)$$

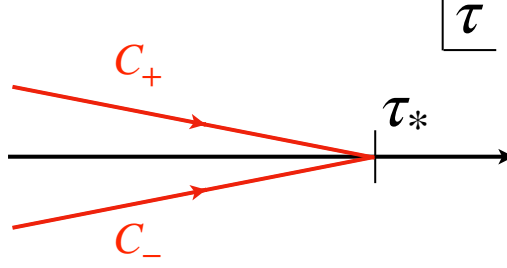


Fig 5: Time integration contours \mathcal{C}_\pm : The path integral measure in the in-in formalism is given by $\exp(i \int_{\mathcal{C}_+ - \mathcal{C}_-} d\tau \int d^d \mathbf{x} \sqrt{-g} \mathcal{L}) = \exp(iS_+ - iS_-)$, where we defined $S_\pm = \int_{\mathcal{C}_\pm} d\tau \int d^d \mathbf{x} \sqrt{-g} \mathcal{L}$. In particular, $\mathcal{C}_+ - \mathcal{C}_-$ forms a closed time path. We call \mathcal{C}_+ and \mathcal{C}_- the time ordered path and the anti-time ordered path, respectively.

Here for convenience, we introduced the Euler operator $\theta_x = x \frac{\partial}{\partial x}$, which counts the power in x .

The usual way for computing the bulk de Sitter correlation function is through the in-in formalism (the Schwinger-Keldysh formalism), where the time integral is defined along a closed path, which starts from the past infinity, then goes all the way to the future boundary, and then goes back to the past infinity. As depicted in Fig. 5, the two time contour in the integration can be divided into two parts: the time ordered contour \mathcal{C}_+ and the anti-time ordered contour \mathcal{C}_- , which extend over $(-(1-i\epsilon)\infty, \tau_*]$ and $(-(1+i\epsilon)\infty, \tau_*]$, respectively. Here τ_* is a late time near the future boundary and we introduced the $i\epsilon$ prescription to ensure the convergence of the path integral. We denote the action defined with each integration contour by

$$S^\pm[\phi] = \frac{1}{2} \int_{\mathcal{C}_\pm} \frac{d\tau}{(-\tau)^{d+1}} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left[(\theta_\tau \phi_{-\mathbf{k}})(\theta_\tau \phi_{\mathbf{k}}) - (m^2 + k^2 \tau^2) \phi_{-\mathbf{k}} \phi_{\mathbf{k}} \right]. \quad (4.5)$$

Using (4.5), we define the Dirichlet de Sitter wavefunction Ψ_D^\pm as

$$\Psi_D^\pm[\bar{\phi}] = \int_{\bar{\phi}, \text{BD}_\pm} [d\phi] e^{\pm i S^\pm[\phi]}, \quad (4.6)$$

where the path integral for $\Psi_D^\pm[\bar{\phi}]$ is performed with the Dirichlet boundary conditions at $\tau = \tau_*$,

$$\phi_{\mathbf{k}}(\tau_*) = \bar{\phi}_{\mathbf{k}}, \quad (4.7)$$

and the Bunch-Davies vacuum⁵ conditions at the past infinity,

$$\lim_{\tau \rightarrow -(1 \mp i\epsilon)\infty} \phi_{\mathbf{k}}(\tau) = 0. \quad (4.8)$$

⁵In Sec. 4 and Sec. 5, we refer to it as the Bunch-Davies vacuum instead of the Euclidean vacuum, since it is the more conventional term when working in Poincare coordinates.

Here $\bar{\phi}$ is a boundary scalar field independent of τ . We have seen in (3.125) that wavefunctions could be approximated as the exponential of the on-shell action. Hence throughout this thesis, we use the semiclassical approximation, under which the wavefunction is evaluated as

$$\Psi_D^\pm[\bar{\phi}] = e^{\pm i S^\pm[\phi^\pm]}, \quad (4.9)$$

where ϕ^\pm is a solution of the equation of motion,

$$[\theta_\tau(\theta_\tau - d) + (m^2 + k^2\tau^2)] \phi_\mathbf{k}^\pm = 0, \quad (4.10)$$

which satisfies the boundary conditions (4.7) and (4.8). Note that after integrating by parts, (4.5) could be rearranged as

$$\begin{aligned} S^\pm[\phi] = & -\frac{1}{2} \int_{\mathcal{C}_\pm} \frac{d\tau}{(-\tau)^{d+1}} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \phi_{-\mathbf{k}} [\theta_\tau(\theta_\tau - d) + (m^2 + k^2\tau^2)] \phi_\mathbf{k} \\ & - \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} [(-\tau)^{-d} \phi_{-\mathbf{k}} \theta_\tau \phi_\mathbf{k}]_{\tau=-(1\mp i\epsilon)\infty}^{\tau_*}. \end{aligned} \quad (4.11)$$

Using (4.10), the first term vanishes and only the second term remains. In other words, the on-shell Lagrangian vanishes up to total derivatives and so the on-shell action is localized at the boundaries just as usual, which is given by

$$S^\pm[\phi^\pm] = -\frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} [(-\tau)^{-d} \phi_{-\mathbf{k}}^\pm \theta_\tau \phi_\mathbf{k}^\pm]_{\tau=-(1\mp i\epsilon)\infty}^{\tau_*}. \quad (4.12)$$

To find the exact solution of the equation of motion (4.10), we introduce bulk-to-boundary propagators for each time path using the mode functions (3.20) we derived in Section 3.2.1,

$$\mathcal{B}_D^+(k; \tau) = \frac{\psi_k^*(\tau)}{\psi_k^*(\tau_*)}, \quad \mathcal{B}_D^-(k; \tau) = \frac{\psi_k(\tau)}{\psi_k(\tau_*)}. \quad (4.13)$$

These propagators manifestly satisfy the equation of motion (4.10) because of the numerators, while the denominators are merely normalization factors,

$$[\theta_\tau(\theta_\tau - d) + (m^2 + k^2\tau^2)] \mathcal{B}_D^\pm(k; \tau) = 0, \quad (4.14)$$

and the following boundary conditions,

$$\lim_{\tau \rightarrow \tau_*} \mathcal{B}_D^\pm(k; \tau) = 1, \quad \lim_{\tau \rightarrow -(1\mp i\epsilon)\infty} \mathcal{B}_D^\pm(k; \tau) = 0. \quad (4.15)$$

Then the solution ϕ^\pm is given by

$$\phi_\mathbf{k}^\pm(\tau) = \mathcal{B}_D^\pm(k; \tau) \bar{\phi}_\mathbf{k}. \quad (4.16)$$

It is easy to check (4.16) indeed satisfies (4.10) and the boundary conditions (4.7), (4.8). For heavy fields in the principal series $m > \frac{d}{2}$ (see Sec. 4.4 for light fields), we have explicitly

$$\mathcal{B}_D^+(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_{-i\mu}^{(2)}(-k\tau)}{(-\tau_*)^{\frac{d}{2}} H_{-i\mu}^{(2)}(-k\tau_*)}, \quad \mathcal{B}_D^-(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_{i\mu}^{(1)}(-k\tau)}{(-\tau_*)^{\frac{d}{2}} H_{i\mu}^{(1)}(-k\tau_*)}, \quad (4.17)$$

where $\mu = \sqrt{m^2 - \frac{d^2}{4}}$. In the superhorizon regime $-k\tau_* \leq -k\tau \ll 1$, the bulk-to-boundary propagator behaves as

$$\mathcal{B}_D^\pm(k; \tau) \simeq (\tau/\tau_*)^{\frac{d}{2} \mp i\mu} \frac{1 + e^{-\pi\mu} e^{\pm i\alpha(\mu)} (-k\tau)^{\pm 2i\mu}}{1 + e^{-\pi\mu} e^{\pm i\alpha(\mu)} (-k\tau_*)^{\pm 2i\mu}} \quad \text{with} \quad e^{i\alpha(\mu)} = \frac{\Gamma(-i\mu)}{2^{2i\mu} \Gamma(i\mu)}, \quad (4.18)$$

where we introduced the mass-dependent phase factor $\alpha(\mu)$. Note that \mathcal{B}_D^- is the complex conjugate of \mathcal{B}_D^+ by its definition. Also, for \mathcal{B}_D^+ , the first and second terms in the numerator describe the positive and negative frequency modes at late time, respectively, and vice versa for \mathcal{B}_D^- . The factor $e^{-\pi\mu} e^{\pm i\alpha(\mu)}$ is nothing but the one appearing in the thermal Bogoliubov coefficients.

Now we have every tool to evaluate the wavefunction. Substituting the solution (4.16) into Eq. (4.9) gives

$$\begin{aligned} \Psi_D^\pm[\bar{\phi}] &= \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} [\theta_\tau \mathcal{B}_D^\pm(k; \tau)]_{\tau=\tau_*} \bar{\phi}_{\mathbf{k}} \bar{\phi}_{-\mathbf{k}} \right] \\ &\simeq \exp \left[\mp \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \left(\frac{d}{2} i + \mu \cdot \frac{1 - e^{\mp \pi\mu} e^{i\alpha(\mu)} (-k\tau_*)^{2i\mu}}{1 + e^{\mp \pi\mu} e^{i\alpha(\mu)} (-k\tau_*)^{2i\mu}} \right) \bar{\phi}_{\mathbf{k}} \bar{\phi}_{-\mathbf{k}} \right]. \end{aligned} \quad (4.19)$$

The holographic dictionary then states that

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{D\pm} = \mp (-\tau_*)^{-d} \left[\frac{d}{2} i + \mu \cdot \frac{1 - e^{\mp \pi\mu} e^{i\alpha(\mu)} (-k\tau_*)^{2i\mu}}{1 + e^{\mp \pi\mu} e^{i\alpha(\mu)} (-k\tau_*)^{2i\mu}} \right], \quad (4.20)$$

where $\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{D\pm}$ are the two-point functions of the dual QFT operators O . Also the subscripts D and \pm specify the Dirichlet boundary condition and the time path. Essentially because of the oscillatory behavior of the heavy field outside the horizon, it is hard to identify Eq. (4.20) with a conformal two-point function which respects the dilatation Ward-Takahashi identity, unlike what was suggested in section 3.5. Note that there is a freedom to add a local boundary action (or equivalently, total derivative terms in Lagrangian), which corresponds to renormalization in the dual QFT [102, 103]. At the level of our interest, this is equivalent with adding a pure imaginary constant to Eq. (4.20), but obviously it does not help with the identification of CFT correlators. Also, this freedom does not change the wavefunction squared $|\Psi|^2$ and so the result of canonical quantization is of course reproduced for whatever choice of holographic renormalization scheme.

In the rest of the section, we shall demonstrate that late time two point functions for heavy scalar fields should be identified with a two-point functions on a cyclic RG flow induced by a double-trace deformation of the dual CFT.

4.2 Mixed boundary conditions

In this section, we show that wavefunctions with mixed boundary conditions are naturally identified with CFT generating functions. As we have discussed, with the Dirichlet boundary condition, the two modes $\phi \sim (-\tau)^{\frac{d}{2} \pm i\mu}$ mix with each other at late time in the correlation functions, which was the main obstruction to identifying dS wavefunctions with CFT generating functions. However mathematically, there is no particular reason for imposing the Dirichlet boundary condition, only that imposing other boundary conditions would result in different correlation functions other than the one directly related to the cosmological observables. Hence, there is freedom for choosing the future boundary conditions for wavefunctions theoretically. Here instead of the Dirichlet boundary condition, we employ the following mixed boundary conditions,

$$\left(\theta_\tau - \frac{d}{2} - i\mu\right) \phi_{\mathbf{k}}^\pm(\tau) \Big|_{\tau=\tau_*} = \chi_{\mathbf{k}}^\pm, \quad (4.21)$$

where $\chi_{\mathbf{k}}^\pm$ are source terms introduced for each time path. Note that χ^\pm are independent of the boundary scalar fields $\bar{\phi}$ in the Dirichlet boundary problem. The conditions (4.21) kill the mode $\phi \sim (-\tau_*)^{\frac{d}{2} + i\mu}$ and pick up the mode $\phi \sim (-\tau_*)^{\frac{d}{2} - i\mu}$, which makes it possible to identify the resulting wavefunctions with CFT generating functions, as we will explain shortly. One could also pick up the other mode by imposing a different mixed boundary conditions,

$$\left(\theta_\tau - \frac{d}{2} + i\mu\right) \phi_{\mathbf{k}}^\pm(\tau) \Big|_{\tau=\tau_*} = \chi_{\mathbf{k}}^\pm, \quad (4.22)$$

which can be realized by simply replacing $\mu \rightarrow -\mu$ in the following analysis. Also, one could even impose different boundary conditions for each time path, such as

$$\left(\theta_\tau - \frac{d}{2} - i\mu\right) \phi_{\mathbf{k}}^+(\tau) \Big|_{\tau=\tau_*} = \chi_{\mathbf{k}}^+, \quad \left(\theta_\tau - \frac{d}{2} + i\mu\right) \phi_{\mathbf{k}}^-(\tau) \Big|_{\tau=\tau_*} = \chi_{\mathbf{k}}^-, \quad (4.23)$$

which would result in different dictionaries for correlation functions. But here we employ the conditions (4.21) for technical simplicity.

To make the variation problem of the action consistent with the mixed boundary conditions (4.21), let us modify the action (4.5) by adding appropriate boundary terms,

$$\begin{aligned} S^\pm[\phi] &= S_{\text{bulk}}^\pm[\phi] + S_{\text{bd}}^\pm[\phi], \\ S_{\text{bulk}}^\pm[\phi] &= \frac{1}{2} \int_{\mathcal{C}_\pm} \frac{d\tau}{(-\tau)^{d+1}} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left[(\theta_\tau \phi_{-\mathbf{k}})(\theta_\tau \phi_{\mathbf{k}}) - (m^2 + k^2 \tau^2) \phi_{-\mathbf{k}} \phi_{\mathbf{k}} \right], \\ S_{\text{bd}}^\pm[\phi] &= \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \left[\left(\frac{d}{2} + i\mu \right) \phi_{-\mathbf{k}} \phi_{\mathbf{k}} + 2 \phi_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm \right]_{\tau=\tau_*}. \end{aligned} \quad (4.24)$$

Similarly, repeating the procedures with Dirichlet boundary conditions, we introduce

the bulk-to-boundary propagators using the mode functions as follows:

$$\mathcal{B}_M^+(k; \tau) = \frac{\psi_k^*(\tau)}{(\theta_{\tau_*} - \frac{d}{2} - i\mu)\psi_k^*(\tau_*)} = \frac{(-\tau)^{\frac{d}{2}} H_{-i\mu}^{(2)}(-k\tau)}{(\theta_{\tau_*} - \frac{d}{2} - i\mu) \left[(-\tau_*)^{\frac{d}{2}} H_{-i\mu}^{(2)}(-k\tau_*) \right]}, \quad (4.25)$$

$$\mathcal{B}_M^-(k; \tau) = \frac{\psi_k(\tau)}{(\theta_{\tau_*} - \frac{d}{2} - i\mu)\psi_k(\tau_*)} = \frac{(-\tau)^{\frac{d}{2}} H_{i\mu}^{(1)}(-k\tau)}{(\theta_{\tau_*} - \frac{d}{2} - i\mu) \left[(-\tau_*)^{\frac{d}{2}} H_{i\mu}^{(1)}(-k\tau_*) \right]}. \quad (4.26)$$

The classical solutions satisfying the mixed boundary conditions (4.21) and the Bunch-Davies conditions are then given by

$$\phi_{\mathbf{k}}^{\pm}(\tau) = \mathcal{B}_M^{\pm}(k; \tau) \chi_{\mathbf{k}}^{\pm} \quad (4.27)$$

In the superhorizon regime $-k\tau_* \leq -k\tau \ll 1$, the bulk-to-boundary propagators behave as

$$\mathcal{B}_M^+(k; \tau_*) \simeq -(\tau/\tau_*)^{\frac{d}{2}-i\mu} \frac{1 + e^{-\pi\mu} e^{i\alpha(\mu)} (-k\tau)^{2i\mu}}{2i\mu}, \quad (4.28)$$

$$\begin{aligned} \mathcal{B}_M^-(k; \tau_*) &\simeq -(\tau/\tau_*)^{\frac{d}{2}+i\mu} \frac{1 + e^{-\pi\mu} e^{-i\alpha(\mu)} (-k\tau)^{-2i\mu}}{2i\mu \cdot e^{-\pi\mu} e^{-i\alpha(\mu)} (-k\tau_*)^{-2i\mu}} \\ &= -(\tau/\tau_*)^{\frac{d}{2}-i\mu} \frac{1 + e^{\pi\mu} e^{i\alpha(\mu)} (-k\tau)^{2i\mu}}{2i\mu}. \end{aligned} \quad (4.29)$$

One can see in Eq. (4.28) and the first line of Eq. (4.29) that the positive frequency modes in the denominator of Eq. (4.18) are projected out by the boundary conditions. Also, as the second expression of Eq. (4.29) shows, the two bulk-to-boundary propagators match with each other except for the Boltzmann factor $e^{\mp\pi\mu}$ in the numerator.

Now the on-shell action with the solutions (4.27) reads

$$\begin{aligned} S^{\pm}[\phi^{\pm}] &= \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \phi_{-\mathbf{k}}^{\pm} \chi_{\mathbf{k}}^{\pm} \Big|_{\tau=\tau_*} \\ &= \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \mathcal{B}_M^{\pm}(k; \tau_*) \chi_{\mathbf{k}}^{\pm} \chi_{-\mathbf{k}}^{\pm} \\ &= \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \left[\frac{e^{\mp\pi\mu} e^{i\alpha(\mu)} (-k\tau_*)^{2i\mu} + 1}{-2i\mu} \right] \chi_{\mathbf{k}}^{\pm} \chi_{-\mathbf{k}}^{\pm}. \end{aligned} \quad (4.30)$$

Consequently, the wavefunction $\Psi_M^{\pm}[\chi^{\pm}]$ with the mixed boundary conditions (4.21) reads

$$\Psi_M^{\pm}[\chi^{\pm}] = e^{\pm i S^{\pm}[\phi^{\pm}]} = \exp \left[\mp \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \frac{e^{\mp\pi\mu} e^{i\alpha(\mu)} (-k\tau_*)^{2i\mu} + 1}{2\mu} \chi_{\mathbf{k}}^{\pm} \chi_{-\mathbf{k}}^{\pm} \right]. \quad (4.31)$$

Hence the dual QFT two-point functions are given by

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm} = \mp (-\tau_*)^{-d} \frac{e^{\mp\pi\mu} e^{i\alpha(\mu)} (-k\tau_*)^{2i\mu} + 1}{2\mu}. \quad (4.32)$$

Note that the second term in the numerator is analytic, so that we may eliminate it by adding an extra boundary term $\sim \chi_{-\mathbf{k}}^\pm \chi_{\mathbf{k}}^\pm$ to the action. More explicitly, we define the renormalized action with an additional boundary term as

$$\begin{aligned}\tilde{S}^\pm[\phi] &= S_{\text{bulk}}^\pm[\phi] + \tilde{S}_{\text{bd}}^\pm[\phi], \\ \tilde{S}_{\text{bd}}^\pm[\phi] &= \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \left[\left(\frac{d}{2} + i\mu \right) \phi_{-\mathbf{k}} \phi_{\mathbf{k}} + 2\phi_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm + (2i\mu)^{-1} \chi_{-\mathbf{k}}^\pm \chi_{\mathbf{k}}^\pm \right]_{\tau=\tau_*},\end{aligned}\quad (4.33)$$

where $S_{\text{bulk}}^\pm[\phi]$ are given in the second line of (4.24). Note that the newly added counterterms do not affect the boundary conditions. As a result, the renormalized wavefunctions $\tilde{\Psi}_M^\pm[\chi^\pm]$ are given by

$$\tilde{\Psi}_M^\pm[\chi^\pm] = e^{\pm i\tilde{S}^\pm[\phi^\pm]} = \exp \left[\mp \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{e^{\mp \pi \mu} e^{i\alpha(\mu)}}{2\mu} (-\tau_*)^{-d} (-k\tau_*)^{2i\mu} \chi_{\mathbf{k}}^\pm \chi_{-\mathbf{k}}^\pm \right]. \quad (4.34)$$

As we advertised earlier, the coefficients in the wavefunctions are naturally identified with conformal two-point functions of the dual operator O with the conformal dimension $\Delta = \frac{d}{2} + i\mu$:

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}\pm} = \mp \frac{e^{\mp \pi \mu} e^{i\alpha(\mu)}}{2\mu} (-\tau_*)^{-d} (-k\tau_*)^{2i\mu}. \quad (4.35)$$

Note that the dual CFTs are non-unitary since the conformal dimension Δ is complex.

4.3 Double-trace deformations

We have argued that the renormalized wavefunctions $\tilde{\Psi}_M^\pm[\chi^\pm]$ with the mixed boundary conditions may be identified with CFT generating functions. However, since the wavefunctions $\Psi_D^\pm[\bar{\phi}]$ with the Dirichlet boundary conditions are the ones which directly related to cosmological observables and the most commonly discussed, we want to derive the proper interpretation of them in terms of CFT language. In this subsection, we demonstrate that the wavefunctions $\Psi_D^\pm[\bar{\phi}]$ are identified with the generating functions of QFTs on a cyclic RG flow induced by double-trace deformations.

To see this, let us first notice that $\Psi_D^\pm[\bar{\phi}]$ and $\tilde{\Psi}_M^\pm[\chi^\pm]$ are related to each other by

$$\Psi_D^\pm[\bar{\phi}] = \int [d\chi^\pm] \tilde{\Psi}_M^\pm[\chi^\pm] \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\left(\frac{d}{2} + i\mu \right) \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}} + 2\bar{\phi}_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm + (2i\mu)^{-1} \chi_{-\mathbf{k}}^\pm \chi_{\mathbf{k}}^\pm}{(-\tau_*)^d} \right]. \quad (4.36)$$

According to the result of the last subsection, we are free to identify $\tilde{\Psi}_M^\pm[\chi^\pm]$ with CFT generating functions $Z_{\text{CFT}}^\pm[\chi^\pm]$ as

$$\tilde{\Psi}_M^\pm[\chi^\pm] = Z_{\text{CFT}}^\pm[\chi^\pm] = \int [d\Phi^\pm] \exp \left[-S_{\text{CFT}}^\pm + \int \frac{d^d \mathbf{k}}{(2\pi)^d} \chi_{-\mathbf{k}}^\pm O_{\mathbf{k}} \right], \quad (4.37)$$

where we schematically introduced the path integral for the would-be dual CFTs. In particular, S_{CFT}^\pm are CFT actions and $[d\Phi^\pm]$ is the path integral measure. The standard holographic dictionary says that χ^\pm source the dual CFT operator O . Substituting this into Eq. (4.36) gives a CFT interpretation of the wavefunctions $\Psi_D^\pm[\bar{\phi}]$:

$$\Psi_D^\pm[\bar{\phi}] = \int [d\Phi^\pm] \exp \left[-S_{\text{QFT}}^\pm \mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left(\frac{\frac{d}{2} - i\mu}{(-\tau_*)^d} \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}} \pm 4\mu \bar{\phi}_{-\mathbf{k}} O_{\mathbf{k}} \right) \right], \quad (4.38)$$

where we defined

$$S_{\text{QFT}}^\pm = S_{\text{CFT}}^\pm \mp \mu (-\tau_*)^d \int \frac{d^d \mathbf{k}}{(2\pi)^d} O_{-\mathbf{k}} O_{\mathbf{k}}. \quad (4.39)$$

Note that from the last term, now $\bar{\phi}$ plays the role of the source after integrating χ^\pm . And in holographic renormalization, one may add arbitrary local counterterms of the source $\bar{\phi}$ to the boundary action. So the $\bar{\phi}^2$ term in Eq. (4.38) can be understood as such a counterterm and hence there is not much to further discuss. Thus in general, the correlation functions of the dual QFT can be written as

$$\langle \dots \rangle_{D^\pm} = \left\langle \dots \exp \left(\pm \mu (-\tau_*)^d \int \frac{d^d \mathbf{k}}{(2\pi)^d} O_{-\mathbf{k}} O_{\mathbf{k}} \right) \right\rangle_{\text{CFT}^\pm}. \quad (4.40)$$

This shows that the QFTs dual to Ψ_D^\pm are obtained by double-trace deformations of the CFTs dual to $\tilde{\Psi}_M^\pm$. Also the corresponding RG flow is cyclic because the double-trace operator O^2 has a conformal dimension $d + 2i\mu$, which is also implied by the pure imaginary scaling of the two-point function (4.35). This is analogous to the AdS case with a bulk scalar below the BF bound [91].

4.4 Wavefunctions for light fields

For comparison, we discuss mixed boundary conditions for light scalars in the complementary series $0 \leq m < \frac{d}{2}$, following Ref. [36]. In contrast to the heavy field case, the wavefunctions evaluated with mixed boundary conditions can be identified with CFT generating functions and the scalar field sources the shadow operator. For technical simplicity, we focus on the mass range $\sqrt{\frac{d^2}{4} - 1} < m < \frac{d}{2}$ in this section. More general cases are discussed in Sec. 4.4.4.

4.4.1 Dirichlet boundary conditions

First, let us briefly summarize the Dirichlet boundary conditions for light fields, for which the bulk-to-boundary propagators are introduced in the same manner as heavy field cases and they read explicitly

$$\mathcal{B}_D^+(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_\nu^{(2)}(-k\tau)}{(-\tau_*)^{\frac{d}{2}} H_\nu^{(2)}(-k\tau_*)}, \quad \mathcal{B}_D^-(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_\nu^{(1)}(-k\tau)}{(-\tau_*)^{\frac{d}{2}} H_\nu^{(1)}(-k\tau_*)}, \quad (4.41)$$

where $\nu = \sqrt{\frac{d^2}{4} - m^2}$. In the superhorizon regime $-k\tau_* \leq -k\tau \ll 1$, the propagators behave as

$$\mathcal{B}_D^\pm(k; \tau) \simeq (\tau/\tau_*)^{\frac{d}{2}-\nu}. \quad (4.42)$$

Note that there is only one term remaining because the mode $\sim (-\tau)^{\frac{d}{2}-\nu}$ dominates over the other in the superhorizon limit, which is analogous to the AdS case. Consequently, the bulk-to-boundary operators have a simple asymptotic behavior as (4.42) outside the horizon. Then substituting (4.41), the wavefunctions are given by

$$\begin{aligned} \Psi_D^\pm[\bar{\phi}] &= \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} [\theta_\tau \mathcal{B}_D^\pm(k; \tau)]_{\tau=\tau_*} \bar{\phi}_{\mathbf{k}} \bar{\phi}_{-\mathbf{k}} \right] \\ &= \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \right. \\ &\quad \times \left(\left(\frac{d}{2} - \nu \right) + \mathcal{O}(k^2 \tau_*^2) + e^{\pm i\pi\nu} \cdot 2\nu \cdot \frac{\Gamma(-\nu)}{2^{2\nu} \Gamma(\nu)} (-k\tau_*)^{2\nu} + \dots \right) \bar{\phi}_{\mathbf{k}} \bar{\phi}_{-\mathbf{k}} \left. \right], \end{aligned} \quad (4.43)$$

where the first two terms are analytic in $(-k\tau_*)^2$ and pure imaginary. The dots stand for non-analytic terms which are subleading in the limit $k\tau_* \ll 1$. When we compute the wavefunction squared $|\Psi_D|^2 = \Psi_D^+ \Psi_D^-$, these pure imaginary terms disappear due to the projection onto the imaginary part of the action. In addition, we can also eliminate these analytic terms by adding proper boundary terms to the action. Either way they would not contribute to the dual QFT (CFT) correlators.

More explicitly, if we focus on the mass regime $\sqrt{\frac{d^2}{4} - 1} < m < \frac{d}{2}$, which corresponds to $0 < \nu < 1$, holographic renormalization can be performed as

$$\Psi_D^\pm[\bar{\phi}] = \exp \left[\frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left(\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{D\pm}} \bar{\phi}_{\mathbf{k}} \bar{\phi}_{-\mathbf{k}} \mp i \frac{(\frac{d}{2} - \nu) \bar{\phi}_{\mathbf{k}} \bar{\phi}_{-\mathbf{k}}}{(-\tau_*)^d} \right) \right], \quad (4.44)$$

where the second term is the counterterm. Also we introduced

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{D\pm}} = \mp i \frac{2\nu e^{\pm i\pi\nu} \Gamma(-\nu)}{2^{2\nu} \Gamma(\nu)} (-\tau_*)^{-d+2\nu} k^{2\nu}, \quad (4.45)$$

which can be identified straightforwardly with the conformal two-point function of a primary field with conformal dimension $\Delta = \frac{d}{2} + \nu$, just as the original dS/CFT proposal suggests. Note that for $\nu > 1$, there appear more counterterms, but the final result (4.45) still remains valid.

4.4.2 Mixed boundary conditions

Similarly, we study mixed boundary conditions for light fields and impose

$$\left(\theta_\tau - \frac{d}{2} + \nu \right) \phi_{\mathbf{k}}^\pm(\tau) \Big|_{\tau=\tau_*} = \chi_{\mathbf{k}}^\pm, \quad (4.46)$$

which kill the leading mode $\phi \sim (-\tau_*)^{\frac{d}{2}-\nu}$ and consequently the subleading mode $\phi \sim (-\tau_*)^{\frac{d}{2}+\nu}$ now becomes the leading mode. Together with the Bunch-Davies conditions, the classical solutions are given by

$$\phi_{\mathbf{k}}^{\pm}(\tau) = \mathcal{B}_M^{\pm}(k; \tau) \chi_{\mathbf{k}}^{\pm} \quad (4.47)$$

with the bulk-to-boundary propagators,

$$\mathcal{B}_M^+(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_{\nu}^{(2)}(-k\tau)}{(\theta_{\tau_*} - \frac{d}{2} + \nu) \left[(-\tau_*)^{\frac{d}{2}} H_{\nu}^{(2)}(-k\tau_*) \right]}, \quad (4.48)$$

$$\mathcal{B}_M^-(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_{\nu}^{(1)}(-k\tau)}{(\theta_{\tau_*} - \frac{d}{2} + \nu) \left[(-\tau_*)^{\frac{d}{2}} H_{\nu}^{(1)}(-k\tau_*) \right]}. \quad (4.49)$$

In the superhorizon regime $-k\tau_* \leq -k\tau \ll 1$, the bulk-to-boundary propagators behave as

$$\mathcal{B}_M^{\pm}(k; \tau_*) \simeq \frac{2^{2\nu} \Gamma(\nu)}{2\nu e^{\pm i\pi\nu} \Gamma(-\nu)} (-k\tau_*)^{-2\nu}, \quad (4.50)$$

where again we assumed $0 < \nu < 1$. To make the variation problem consistent with the modified mixed boundary conditions (4.46), we add a boundary term to the action (4.5) as follows:

$$S_{\text{bd}}^{\pm}[\phi] = \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \left[\left(\frac{d}{2} - \nu \right) \phi_{-\mathbf{k}} \phi_{\mathbf{k}} + 2\phi_{-\mathbf{k}} \chi_{\mathbf{k}}^{\pm} \right]_{\tau=\tau_*}. \quad (4.51)$$

Thus, the on-shell action reads

$$S^{\pm}[\phi^{\pm}] = \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau_*)^{-d} \left[\frac{2^{2\nu} \Gamma(\nu)}{2\nu e^{\pm i\pi\nu} \Gamma(-\nu)} (-k\tau_*)^{-2\nu} + \dots \right] \chi_{\mathbf{k}}^{\pm} \chi_{-\mathbf{k}}^{\pm}, \quad (4.52)$$

where the dots stand for subleading terms in the limit $k\tau_* \ll 1$. Then the corresponding wavefunctions $\Psi_M^{\pm}[\chi^{\pm}]$ are given by

$$\Psi_M^{\pm}[\chi^{\pm}] = \exp \left[\pm \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{2^{2\nu} \Gamma(\nu)}{2\nu e^{\pm i\pi\nu} \Gamma(-\nu)} (-\tau_*)^{-d} (-k\tau_*)^{-2\nu} \chi_{\mathbf{k}}^{\pm} \chi_{-\mathbf{k}}^{\pm} \right]. \quad (4.53)$$

As one can see immediately, the coefficient in the wavefunction can be identified with conformal two-point function as

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{M^{\pm}}} = \pm i \frac{2^{2\nu} \Gamma(\nu)}{2\nu e^{\pm i\pi\nu} \Gamma(-\nu)} (-\tau_*)^{-d-2\nu} k^{-2\nu}, \quad (4.54)$$

where the operator O has a scaling dimension $\Delta = \frac{d}{2} - \nu$. This corresponds to the shadow operator of the CFT operator which appeared in Eq. (4.45) in the context of the Dirichlet boundary problem [35, 36, 104, 94–98, 105]. Notice that in contrast to the heavy field case, no counterterm $\sim \chi^2$ in the boundary action is needed to have conformal two-point functions.

4.4.3 Relation between the two CFTs

Finally, we discuss how the two CFTs dual to the Dirichlet boundary condition and the mixed boundary condition are related with each other for light scalar fields. For this purpose, let us first note the relation,

$$\Psi_D^\pm[\bar{\phi}] = \int [d\chi^\pm] \Psi_M^\pm[\chi^\pm] \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(\frac{d}{2} - \nu) \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}} + 2 \bar{\phi}_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm}{(-\tau_*)^d} \right], \quad (4.55)$$

which is analogous to Eq. (4.36) in the heavy field case. We also identify $\Psi_D^\pm[\bar{\phi}]$ and $\Psi_M^\pm[\bar{\phi}]$ with CFT generating functions $Z_{\text{CFT}_D}^\pm[\bar{\phi}]$ and $Z_{\text{CFT}_M}^\pm[\bar{\phi}]$ as

$$\Psi_D^\pm[\bar{\phi}] = Z_{\text{CFT}_D}^\pm[\bar{\phi}] \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(\frac{d}{2} - \nu) \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}}}{(-\tau_*)^d} \right], \quad (4.56)$$

$$\Psi_M^\pm[\chi^\pm] = Z_{\text{CFT}_M}^\pm[\chi^\pm], \quad (4.57)$$

where the second factor on the right hand side of the first equation reflects holographic renormalization (see discussion around Eq. (4.44)). The subscripts D and M indicate that these CFTs are dual to the Dirichlet and the mixed boundary conditions, respectively. Then, the relation (4.55) implies that the two CFT generating functions are related with each other through the Legendre transformation,

$$Z_{\text{CFT}_D}^\pm[\bar{\phi}] = \int [d\chi^\pm] Z_{\text{CFT}_M}^\pm[\chi^\pm] \exp \left[\mp i \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\bar{\phi}_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm}{(-\tau_*)^d} \right]. \quad (4.58)$$

Notice here that the $\bar{\phi}^2$ term in Eq. (4.55) and Eq. (4.56) cancel out each other and that is why we manage to arrive at a simple relation (4.58). As we mentioned earlier, for $\nu > 1$, there appear more counterterms in the Dirichlet boundary problem. Also, the mixed boundary condition (4.46) is modified as we discuss in Sec. 4.4.4. Accordingly, Eqs. (4.55) and (4.56) acquire more counterterms. However, using the results in Sec. 4.4.4, one can easily show that these counterterms cancel each other out and the relation (4.58) holds even for $\nu > 1$.

Moreover, (4.58) directly shows that the two-point functions in two CFTs are inverse of each other,

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_D \pm} = (-\tau_*)^{-2d} \left(\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_M \pm} \right)^{-1}, \quad (4.59)$$

as we observed earlier. One can also show explicitly that the two CFTs are connected by an RG flow generated by double-trace deformations [36], generalizing the AdS results in Ref. [94–98] via an appropriate analytic continuation.

4.4.4 Mixed boundary condition for general ν

In the preceding subsections, we provided the relation between the wavefunctions with the mixed boundary condition and the dual conformal two-point functions by analyzing

the asymptotic (superhorizon) behavior of the bulk to boundary propagators for $0 < \nu < 1$. Here we extend the discussion to general values of ν . When $\nu > 1$, the superhorizon limit of the bulk to boundary propagators (4.48) and (4.49) become rather complicated. To see this, we use the series expansion of the Hankel functions:

$$H_\nu^{(2)}(-k\tau) = i \frac{2^\nu \Gamma(\nu)}{\pi} \sum_{n=0}^{\infty} [a_n(-k\tau)^{2n-\nu} + b_n(-k\tau)^{2n+\nu}] , \quad (4.60)$$

$$H_\nu^{(1)}(-k\tau) = -i \frac{2^\nu \Gamma(\nu)}{\pi} \sum_{n=0}^{\infty} [a_n(-k\tau)^{2n-\nu} + \tilde{b}_n(-k\tau)^{2n+\nu}] , \quad (4.61)$$

where the coefficients a_n , b_n and \tilde{b}_n are given by

$$a_n = \frac{(-1)^n}{2^{2n} n! (1-\nu)_n} , \quad (4.62)$$

$$b_n = \frac{(-1)^n}{2^{2n} n! (1+\nu)_n} \frac{e^{i\nu\pi} \Gamma(-\nu)}{2^{2\nu} \Gamma(\nu)} , \quad \tilde{b}_n = \frac{(-1)^n}{2^{2n} n! (1+\nu)_n} \frac{e^{-i\nu\pi} \Gamma(-\nu)}{2^{2\nu} \Gamma(\nu)} \quad (4.63)$$

with the shifted factorial $(x)_n = x(x+1)\cdots(x+n-1)$. In terms of them, the propagators (4.48) and (4.49) can be written as

$$\mathcal{B}_M^+(k; \tau) = (\tau/\tau_*)^{\frac{d}{2}} \frac{\sum_{n=0}^{\infty} [a_n(-k\tau)^{2n-\nu} + b_n(-k\tau)^{2n+\nu}]}{\sum_{n=1}^{\infty} 2na_n(-k\tau_*)^{2n-\nu} + \sum_{n=0}^{\infty} (2n+2\nu)b_n(-k\tau_*)^{2n+\nu}} , \quad (4.64)$$

$$\mathcal{B}_M^-(k; \tau) = (\tau/\tau_*)^{\frac{d}{2}} \frac{\sum_{n=0}^{\infty} [a_n(-k\tau)^{2n-\nu} + \tilde{b}_n(-k\tau)^{2n+\nu}]}{\sum_{n=1}^{\infty} 2na_n(-k\tau_*)^{2n-\nu} + \sum_{n=0}^{\infty} (2n+2\nu)\tilde{b}_n(-k\tau_*)^{2n+\nu}} . \quad (4.65)$$

We can now see that when $\nu > 1$, in each denominator, the terms $2na_n(-k\tau_*)^{2n-\nu}$ with $1 \leq n \leq \lfloor \nu \rfloor$ dominate over $2\nu b_0(-k\tau_*)^\nu$, where $\lfloor \nu \rfloor$ is the floor function, namely the integer k satisfying $k \leq \lfloor \nu \rfloor < k+1$. As a result, in the superhorizon regime $-k\tau_* \leq -k\tau \ll 1$, the leading terms of (4.64) and (4.65) become proportional to $(-k\tau_*)^{-2}$, in sharp contrast with the $\nu < 1$ case. Note that this problem does not arise in the AdS/CFT case as long as we focus on CFT operators above the unitarity bound $\Delta > \frac{d}{2} - 1$, i.e., $\nu < 1$.

In order to regain the desired leading behavior $\sim (k\tau_*)^{-2\nu}$, we need to eliminate these unwanted terms, but this forces us to change the boundary conditions. Concretely, we modify the boundary condition (4.46) in the following manner:

$$\left(\theta_\tau - \frac{d}{2} + \nu - \sum_{n=1}^{\lfloor \nu \rfloor} c_n^\pm (-k\tau)^{2n} \right) \phi_{\mathbf{k}}^\pm(\tau) \Big|_{\tau=\tau_*} = \chi_{\mathbf{k}}^\pm , \quad (4.66)$$

where c_n^\pm are real constants. We shall show shortly that c_n^\pm can always be chosen to eliminate the unwanted terms in the denominators of (4.48) and (4.49), so that the terms proportional to $(-k\tau_*)^\nu$ become leading again. Under the modification, the bulk-to-

boundary propagators become

$$\mathcal{B}_M^+(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_\nu^{(2)}(-k\tau)}{(\theta_{\tau_*} - \frac{d}{2} + \nu - \sum_{n=1}^{[\nu]} c_n^+ (-k\tau_*)^{2n}) \left[(-\tau_*)^{\frac{d}{2}} H_\nu^{(2)}(-k\tau_*) \right]}, \quad (4.67)$$

$$\mathcal{B}_M^-(k; \tau) = \frac{(-\tau)^{\frac{d}{2}} H_\nu^{(1)}(-k\tau)}{(\theta_{\tau_*} - \frac{d}{2} + \nu - \sum_{n=1}^{[\nu]} c_n^- (-k\tau_*)^{2n}) \left[(-\tau_*)^{\frac{d}{2}} H_\nu^{(1)}(-k\tau_*) \right]}. \quad (4.68)$$

The coefficients c_n^\pm can be determined by employing their recurrence relations. Let us first look into c_n^+ . The denominator of (4.67) is expanded as

$$i \frac{2^\nu \Gamma(\nu)}{\pi} (-\tau_*)^{\frac{d}{2}} \left[\sum_{n=1}^{\infty} 2na_n (-k\tau_*)^{2n-\nu} - \sum_{n=0}^{[\nu]} \sum_{m=1}^{\infty} a_n c_m^+ (-k\tau_*)^{2n+2m-\nu} + \sum_{n=0}^{\infty} (2n+2\nu)b_n (-k\tau_*)^{2n+\nu} - \sum_{n=0}^{[\nu]} \sum_{m=1}^{\infty} b_n c_m^+ (-k\tau_*)^{2n+2m+\nu} \right]. \quad (4.69)$$

To cancel all terms dominating over $2\nu b_0 (-k\tau_*)^\nu$, we impose the following recurrence relation: for $1 \leq n \leq [\nu]$,

$$2na_n - a_0 c_n^+ - \sum_{i=1}^{n-1} a_{n-i} c_i^+ = 0, \quad (4.70)$$

where notice that it is independent of b_n . Since $a_0 = 1$, this immediately gives $c_1^+ = 2a_1$, with which we can solve the recurrence relation. The solution c_n^+ is then given by

$$\begin{aligned} c_n^+ &= 2na_n - \sum_{i=1}^{n-1} a_{n-i} c_i^+ \\ &= 2na_n - \sum_{i_1=1}^{n-1} a_{n-i_1} \left(2i_1 a_{i_1} - \sum_{i_2=1}^{i_1-1} a_{i_1-i_2} c_{i_2}^+ \right) \\ &\vdots \\ &= \sum_{m=1}^n \sum_{\substack{i_1+i_2+\dots+i_m=n \\ i_1, i_2, \dots, i_m \geq 1}} (-1)^{m+1} 2i_1 A_{i_1 i_2 \dots i_m}, \end{aligned} \quad (4.71)$$

where we introduced

$$A_{i_1 i_2 \dots i_m} \equiv a_{i_1} a_{i_2} \dots a_{i_m}. \quad (4.72)$$

The coefficients c_n^- can be fixed exactly in the same manner: Since the unwanted terms in (4.65) are exactly the same as in (4.64), the recurrence relation for c_n^- takes exactly the same form as (4.70). Therefore, we find $c_n^- = c_n^+$, hence we denote both of them by c_n for simplicity.

Now, even for $\nu > 1$, in the superhorizon regime, the bulk-to-boundary propagators behave again as

$$\mathcal{B}_M^\pm(k; \tau_*) \simeq \frac{2^{2\nu} \Gamma(\nu)}{2\nu e^{\pm i\pi\nu} \Gamma(-\nu)} (-k\tau_*)^{-2\nu}. \quad (4.73)$$

To make the variation problem consistent with the modified boundary conditions (4.66), the boundary action must be chosen to be

$$S_{\text{bd}}^\pm[\phi] = \frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} (-\tau)^{-d} \left[\left(\frac{d}{2} - \nu + \sum_{n=1}^{[\nu]} c_n (-k\tau)^{2n} \right) \phi_{-\mathbf{k}} \phi_{\mathbf{k}} + 2\phi_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm \right]_{\tau=\tau_*}. \quad (4.74)$$

Namely, the modification of the boundary condition forces the extra local counterterms. As a result, we reproduce the wavefunction (4.53), so that the results we derived in Sec 4.4.2 apply to general value of ν . On the other hand, the modification of the boundary actions (4.74) affects the relation between the wavefunctions Ψ_D^\pm and Ψ_M^\pm ,

$$\begin{aligned} \Psi_D^\pm[\bar{\phi}] &= \int [d\chi^\pm] \Psi_M^\pm[\chi^\pm] \\ &\times \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\left(\frac{d}{2} - \nu + \sum_{n=1}^{[\nu]} c_n (-k\tau_*)^{2n} \right) \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}} + 2\bar{\phi}_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm}{(-\tau_*)^d} \right]. \end{aligned} \quad (4.75)$$

The summation over n in the exponent, however, cancels out in the wavefunction squared because they are pure imaginary, thereby they do not contribute to the wavefunction squared.

5 dS/CFT dictionary revisited

We have seen that wavefunctions of heavy fields with Dirichlet boundary conditions can be interpreted as the generating functions of the dual QFTs, which are double-trace deformations of the CFTs dual to wavefunctions with the mixed boundary conditions. In this section, we use the relation to express equal-time correlation functions on de Sitter space (which are normally computed with wavefunctions with Dirichlet boundary conditions) in terms of the dual conformal correlators appearing in wavefunctions with mixed boundary condition. In particular, this provides a proper dS/CFT dictionary for heavy fields, which has a rather different form from the original proposal (3.129).

5.1 General formula for correlation functions

First, we derive a formula relating equal-time correlation functions on de Sitter space to the wavefunctions with mixed boundary conditions. Recall that as we mentioned briefly in (3.128), the equal-time correlators of an operator $F[\phi_{\mathbf{k}}(\tau)]$ at $\tau = \tau_*$ are given by

$$\langle F[\phi_{\mathbf{k}}(\tau_*)] \rangle_{\text{dS}} = \int [d\bar{\phi}] |\Psi_D[\bar{\phi}]|^2 F[\bar{\phi}_{\mathbf{k}}] \quad \text{with} \quad |\Psi_D[\bar{\phi}]|^2 = \Psi_D^+[\bar{\phi}] \Psi_D^-[\bar{\phi}]. \quad (5.1)$$

Next, to find a more explicit form for (5.1), we rewrite Ψ_D^\pm in terms of the wavefunctions with mixed boundary conditions. For the heavy field case, we may consider either Ψ_M^\pm or the renormalized version $\tilde{\Psi}_M^\pm$ dual to the CFT generating functions. Here we consider the former for convenience. Similarly to Eq. (4.36), Ψ_D^\pm and Ψ_M^\pm are related to each other as

$$\Psi_D^\pm[\bar{\phi}] = \int [d\chi^\pm] \Psi_M^\pm[\chi^\pm] \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(\frac{d}{2} + i\mu) \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}} + 2\bar{\phi}_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm}{(-\tau_*)^d} \right]. \quad (5.2)$$

The wavefunction squared then reads

$$|\Psi_D[\bar{\phi}]|^2 = \int [d\chi^+ d\chi^-] \Psi_M^+[\chi^+] \Psi_M^-[\chi^-] \exp \left[-i \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(\chi_{-\mathbf{k}}^+ - \chi_{-\mathbf{k}}^-) \bar{\phi}_{\mathbf{k}}}{(-\tau_*)^d} \right], \quad (5.3)$$

where we notice that the $\bar{\phi}^2$ terms cancel out when the $+$ and $-$ contributions are multiplied. Also, essentially because of this type of cancellation, the same relation (5.3) holds in the light field case too, as one may explicitly check using Eq. (4.55) and its $\nu > 1$ generalization (4.75). An important point is that both χ^+ and χ^- are coupled to $\bar{\phi}$, and so Ψ_M^+ and Ψ_M^- nontrivially mix with each other upon integration over $\bar{\phi}$. This reflects the closed time path nature of the in-in formalism. For this reason, it is convenient to introduce two new variables instead of χ^\pm ,

$$\chi_{R,\mathbf{k}} = \frac{\chi_{\mathbf{k}}^+ + \chi_{\mathbf{k}}^-}{2}, \quad \chi_{A,\mathbf{k}} = \chi_{\mathbf{k}}^+ - \chi_{\mathbf{k}}^-. \quad (5.4)$$

In this $\chi_{R,A}$ language, (5.3) becomes

$$|\Psi_D[\bar{\phi}]|^2 = \int [d\chi_A] \exp \left[-i \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\chi_{A,-\mathbf{k}} \bar{\phi}_{\mathbf{k}}}{(-\tau_*)^d} \right] \int [d\chi_R] \Psi_M^+ [\chi_R + \tfrac{1}{2}\chi_A] \Psi_M^- [\chi_R - \tfrac{1}{2}\chi_A]. \quad (5.5)$$

Then substituting Eq. (5.5) into Eq. (5.1) and performing the integration over $\bar{\phi}$, we obtain

$$\begin{aligned} & \langle F[\phi_{\mathbf{k}}(\tau_*)] \rangle_{\text{dS}} \\ &= \int [d\chi_A d\bar{\phi}] \exp \left[-i \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\chi_{A,-\mathbf{k}} \bar{\phi}_{\mathbf{k}}}{(-\tau_*)^d} \right] F[\bar{\phi}_{\mathbf{k}}] \int [d\chi_R] \Psi_M^+ [\chi_R + \tfrac{1}{2}\chi_A] \Psi_M^- [\chi_R - \tfrac{1}{2}\chi_A] \\ &= \int [d\chi_A] \delta(\chi_A) F[-i(-\tau_*)^d \partial_{\chi_{A,-\mathbf{k}}}] \int [d\chi_R] \Psi_M^+ [\chi_R + \tfrac{1}{2}\chi_A] \Psi_M^- [\chi_R - \tfrac{1}{2}\chi_A], \end{aligned} \quad (5.6)$$

where we performed integration by parts over χ_A at the second equality. In other words, we have derived the following master formula:

$$\begin{aligned} & \langle F[\phi_{\mathbf{k}}(\tau_*)] \rangle_{\text{dS}} \\ &= F[-i(-\tau_*)^d \partial_{\chi_{A,-\mathbf{k}}}] \int [d\chi_R] \Psi_M^+ [\chi_R + \tfrac{1}{2}\chi_A] \Psi_M^- [\chi_R - \tfrac{1}{2}\chi_A] \Big|_{\chi_A=0}, \end{aligned} \quad (5.7)$$

where we notice again that the integration over χ_R nontrivially mixes Ψ_M^+ and Ψ_M^- reflecting the closed time path nature.

Comments on integral path with different mixed boundary conditions

In section 4.2, we have mentioned that it is possible to impose different boundary conditions on each integral path such as in (4.23). This results in a different dictionary for the correlation function and provides a CFT interpretation for the boundary fields. Let us look at this in more detail before proceeding with the explicit calculation of correlation functions.

Boundary conditions (4.23) indicates that we eliminate the mode $\phi \sim (-\tau_*)^{\frac{d}{2} \pm i\mu}$ and keep $\phi \sim (-\tau_*)^{\frac{d}{2} \mp i\mu}$ for the \pm contour. It means that the corresponding boundary fields χ^\pm source the dual CFT operator with the conformal dimension $\frac{d}{2} \pm i\mu$, respectively. For this choice of boundary conditions, our previous relation between $\Psi_D^\pm[\bar{\phi}]$ and $\tilde{\Psi}_M^\pm[\chi^\pm]$ (4.36) is modified as

$$\begin{aligned} \Psi_D^\pm[\bar{\phi}] &= \int [d\chi^\pm] \tilde{\Psi}_M^\pm[\chi^\pm] \\ &\times \exp \left[\mp \frac{i}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(\frac{d}{2} \pm i\mu) \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}} + 2\bar{\phi}_{-\mathbf{k}} \chi_{\mathbf{k}}^\pm + (\pm 2i\mu)^{-1} \chi_{-\mathbf{k}}^\pm \chi_{\mathbf{k}}^\pm}{(-\tau_*)^d} \right]. \end{aligned} \quad (5.8)$$

Note that here we have \pm in front of each term which contains $i\mu$. Under the new boundary conditions, $\tilde{\Psi}_M^\pm[\chi^\pm]$ are now complex conjugate of each other. Taking the square of $\Psi_D^\pm[\bar{\phi}]$,

(5.3) is modified with several additional terms as

$$|\Psi_D[\bar{\phi}]|^2 = \int [d\chi^+ d\chi^-] Z_{\text{CFT}^+}^+ [\chi^+] Z_{\text{CFT}^-}^- [\chi^-] \exp \left[-i \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(\chi_{-\mathbf{k}}^+ - \chi_{-\mathbf{k}}^-) \bar{\phi}_{\mathbf{k}}}{(-\tau_*)^d} \right] \\ \times \exp \left[\int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\mu \bar{\phi}_{-\mathbf{k}} \bar{\phi}_{\mathbf{k}} - (4\mu)^{-1} \chi_{-\mathbf{k}}^+ \chi_{\mathbf{k}}^+ - (4\mu)^{-1} \chi_{-\mathbf{k}}^- \chi_{\mathbf{k}}^-}{(-\tau_*)^d} \right], \quad (5.9)$$

where we used the relation $\tilde{\Psi}_M^\pm[\chi^\pm] = Z_{\text{CFT}^\pm}^\pm[\chi^\pm]$. It could also be rewritten as

$$|\Psi_D[\bar{\phi}]|^2 = \int [d\chi^+ d\chi^-] Z_{\text{CFT}^+}^+ [\chi^+] Z_{\text{CFT}^-}^- [\chi^-] \exp \left[-\frac{1}{2\mu} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\chi_{-\mathbf{k}}^+ \chi_{\mathbf{k}}^-}{(-\tau_*)^d} \right] \\ \times \exp \left[\int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\mu (\bar{\phi}_{-\mathbf{k}} + (2i\mu)^{-1} (\chi_{-\mathbf{k}}^+ - \chi_{-\mathbf{k}}^-)) (\bar{\phi}_{\mathbf{k}} + (2i\mu)^{-1} (\chi_{\mathbf{k}}^+ - \chi_{\mathbf{k}}^-))}{(-\tau_*)^d} \right]. \quad (5.10)$$

When computing correlators (5.1) using (5.10), we can write $\bar{\psi}$ in terms of χ^\pm and their derivatives from the second line of (5.10) as

$$\bar{\phi}_{\mathbf{k}} = -(2i)^{-1} (-\tau_*)^d \left(\frac{\partial}{\partial \chi_{-\mathbf{k}}^+} - \frac{\partial}{\partial \chi_{-\mathbf{k}}^-} \right) - (2i\mu)^{-1} (\chi_{\mathbf{k}}^+ - \chi_{\mathbf{k}}^-). \quad (5.11)$$

We want to act this operator on the CFT generating functions. By performing partial integrals, we have

$$\bar{\phi}_{\mathbf{k}} = (2i)^{-1} (-\tau_*)^d \left(\frac{\partial}{\partial \chi_{-\mathbf{k}}^+} - \frac{\partial}{\partial \chi_{-\mathbf{k}}^-} \right) - (4i\mu)^{-1} (\chi_{\mathbf{k}}^+ - \chi_{\mathbf{k}}^-) \quad (5.12)$$

acting on $|Z_{\text{CFT}^\pm}^\pm[\chi^\pm]|^2$. Therefore, (5.7) becomes

$$\langle F[\phi_{\mathbf{k}}(\tau_*)] \rangle_{\text{dS}} = \int [d\chi^+ d\chi^-] \exp \left[-\frac{1}{2\mu} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\chi_{-\mathbf{k}}^+ \chi_{\mathbf{k}}^-}{(-\tau_*)^d} \right] \\ \times F \left[-i(-\tau_*)^d (\partial_{\chi_{+,-\mathbf{k}}} - \partial_{\chi_{-,-\mathbf{k}}}) \right] Z_{\text{CFT}^+}^+ [\chi^+] Z_{\text{CFT}^-}^- [\chi^-] \quad (5.13)$$

which can be expressed as

$$\langle F[\phi_{\mathbf{k}}(\tau_*)] \rangle_{\text{dS}} = \left\langle F [O_{\mathbf{k}}^+ + O_{\mathbf{k}}^-] \exp \left[2\mu \int \frac{d^d \mathbf{k}}{(2\pi)^d} O_{-\mathbf{k}}^+ O_{\mathbf{k}}^- \right] \right\rangle, \quad (5.14)$$

where we defined the source term as $\exp \pm i \int \frac{d^d \mathbf{k}}{(2\pi)^d} \chi_{-\mathbf{k}}^\pm O_{\mathbf{k}}^\pm$. This formula suggests that heavy fields on the future boundary correspond to a mixed form of CFT operators with different conformal dimensions,

$$\bar{\phi}_{\mathbf{k}} = O_{\mathbf{k}}^+ + O_{\mathbf{k}}^- . \quad (5.15)$$

While it is a simple dictionary for light fields, i.e. $\bar{\phi}_{\mathbf{k}} = O_{\mathbf{k}}^-$, (5.15) can be seen as a natural generalization of the dictionary.

5.2 Two-point functions

We now apply the general formula (5.7) to two-point functions. First suppose that the wavefunctions with mixed boundary conditions are given by

$$\Psi_M^\pm[\chi^\pm] = \exp \left[\frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm} \chi_{\mathbf{k}}^\pm \chi_{-\mathbf{k}}^\pm \right]. \quad (5.16)$$

Then using $\chi_{R,A}$ (5.4), we find

$$\begin{aligned} & \ln \left[\Psi_M^+ \left[\chi_R + \frac{1}{2} \chi_A \right] \Psi_M^- \left[\chi_R - \frac{1}{2} \chi_A \right] \right] \\ &= \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left[\frac{1}{2} \left(\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} + \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-} \right) \left(\chi_{R,\mathbf{k}} \chi_{R,-\mathbf{k}} + \frac{1}{4} \chi_{A,\mathbf{k}} \chi_{A,-\mathbf{k}} \right) \right. \\ & \quad \left. + \frac{1}{4} \left(\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} - \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-} \right) \left(\chi_{R,\mathbf{k}} \chi_{A,-\mathbf{k}} + \chi_{A,\mathbf{k}} \chi_{R,-\mathbf{k}} \right) \right], \quad (5.17) \end{aligned}$$

which implies

$$\begin{aligned} & \int [d\chi_R] \Psi_M^+ \left[\chi_R + \frac{1}{2} \chi_A \right] \Psi_M^- \left[\chi_R - \frac{1}{2} \chi_A \right] \\ &= \exp \left[\frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-}}{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} + \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-}} \chi_{A,\mathbf{k}} \chi_{A,-\mathbf{k}} \right]. \quad (5.18) \end{aligned}$$

Therefore, the de Sitter two-point function reads

$$\langle \phi_{\mathbf{k}}(\tau_*) \phi_{-\mathbf{k}}(\tau_*) \rangle'_{\text{dS}} = -(-\tau_*)^{2d} \frac{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-}}{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M+} + \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M-}}. \quad (5.19)$$

This is the holographic dictionary when mixed boundary conditions are employed and how two point functions for scalar fields with general mass are related to CFT two point functions. Below we show that this dictionary reproduces the correct two-point functions as it should be.

5.2.1 Heavy fields

Let us begin by the heavy field case:

$$\Psi_M^\pm[\chi^\pm] = \exp \left[\frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm} \chi_{\mathbf{k}}^\pm \chi_{-\mathbf{k}}^\pm \right], \quad (5.20)$$

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm} = \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}\pm} \mp \frac{(-\tau_*)^{-d}}{2\mu}, \quad (5.21)$$

where the conformal two-point functions are given by

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}\pm} = \mp \frac{e^{\mp \pi \mu} e^{i\alpha(\mu)}}{2\mu} (-\tau_*)^{-d} (-k\tau_*)^{2i\mu}. \quad (5.22)$$

Here we keep the analytic terms in (5.21), which were subtracted by holographic renormalization in Sec. 4.2. Thus (5.21) cannot be identified as CFT correlators anymore, although we will see shortly that this procedure is necessary. Then, the de Sitter two-point function reads

$$\begin{aligned} & \langle \phi_{\mathbf{k}}(\tau_*) \phi_{-\mathbf{k}}(\tau_*) \rangle'_{\text{dS}} \\ &= (-\tau_*)^{2d} \frac{\left(\frac{(-\tau_*)^{-d}}{2\mu} - \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}+} \right) \left(\frac{(-\tau_*)^{-d}}{2\mu} + \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}-} \right)}{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}+} + \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}-}}. \end{aligned} \quad (5.23)$$

Notice here that the analytic terms in the numerator which come from (5.21) make the expression in terms of conformal two-point functions less simple compared to the original dictionary (5.19). However, such terms are necessary to reproduce the correct two-point functions. Indeed, using the expression (5.22), we find

$$\langle \phi_{\mathbf{k}}(\tau_*) \phi_{-\mathbf{k}}(\tau_*) \rangle'_{\text{dS}} = \frac{(-\tau_*)^d}{2\mu} \frac{|1 + e^{-\pi\mu} e^{i\alpha} (-k\tau_*)^{2i\mu}|^2}{1 - e^{-2\pi\mu}}, \quad (5.24)$$

which agrees with the result in canonical quantization (3.133). Note that it is convenient to use $\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}+} = -e^{-2\pi\mu} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}-}$ in the intermediate step for computing (5.24).

5.2.2 Light fields

Lastly, let us look at the light field case:

$$\Psi_M^\pm[\chi^\pm] = \exp \left[\frac{1}{2} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm} \chi_{\mathbf{k}}^\pm \chi_{-\mathbf{k}}^\pm \right], \quad (5.25)$$

$$\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{M\pm} = \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{M\pm}} = \pm i \frac{2^{2\nu} \Gamma(\nu)}{2\nu e^{\pm i\pi\nu} \Gamma(-\nu)} (-\tau_*)^{-d-2\nu} k^{-2\nu}, \quad (5.26)$$

where we recall that no counterterm term $\sim \chi^2$ in the boundary action was required in the light field case. Thus (5.26) could be identified as CFT correlators directly. Using the dictionary (5.19), we obtain

$$\begin{aligned} \langle \phi_{\mathbf{k}}(\tau_*) \phi_{-\mathbf{k}}(\tau_*) \rangle'_{\text{dS}} &= -(-\tau_*)^{2d} \frac{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{M+}} \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{M-}}}{\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{M+}} + \langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{M-}}} \\ &= (-\tau_*)^d \frac{2^{2\nu} \Gamma(\nu)^2}{4\pi} (-k\tau_*)^{-2\nu}, \end{aligned} \quad (5.27)$$

which again is consistent with the result of canonical quantization (3.132). Notice that the light field de Sitter two-point functions have the same scaling as the dual CFT two point functions $\langle O_{\mathbf{k}} O_{-\mathbf{k}} \rangle'_{\text{CFT}_{M\pm}}$, which appears in the wavefunctions with mixed boundary conditions.

6 dS quasinormal modes and dS/CFT

With respect to AdS/CFT, a key aspect of this kind of holographic framework is the role of quantum entanglement in the emergence of spacetime geometry, as exemplified by the connection between black hole states and their corresponding entanglement, summarized by the term “ER = EPR”. This duality becomes particularly interesting in the context of eternal AdS black holes, where the black hole geometry featuring two causal patches with two boundaries is understood as an entangled state within the framework of two CFTs. And two entangled particles in this state are called an Einstein-Podolsky-Rosen (EPR) pair. The interior of the black hole and the Einstein-Rosen (ER) bridge connecting the two boundaries can be interpreted as arising from the entanglement of these boundary CFTs. On the other hand, de Sitter space has a similar structure to the external black hole - it also has two causal static patches. A unique feature of de Sitter space is that there are geodesically complete topologically R^3 spacelike slices that end on an S^2 on the future boundary I^+ . And every S^2 on I^+ is in general the boundary of two R^3 slices - a “northern” slice and a “southern” slice, whose topological sum gives back S^3 . An example of this are the hyperbolic slices, whose quantization has been studied in previous studies [106]. This exactly resembles the relation of the Hilbert spaces of the northern and southern static patches to that of the global dS, suggesting the double copy holographic structure for de Sitter space.

Ref. [63] extends the “ER = EPR” discussion to de Sitter space, proposing that the ER bridge can emerge from the entanglement between two boundary CFTs. This is framed within the dS/CFT context, where bulk states in de Sitter space correspond to entangled states in two boundary CFTs. It is argued that the global de Sitter space can be viewed as an entangled sum of pairs of CFT states, with each state associated with a slice of de Sitter space. A major focus is on the use of a quasinormal mode basis for scalar fields, which is in line with the CFT perspective and brings up unconventional features such as imaginary frequency and unitarity issues.

In general, quasinormal modes (QNMs) refer to a particular type of mode that decays over time, radiating energy in the process. They are particularly important in the study of black holes and other astrophysical objects. For de Sitter space, QNMs are often studied in static coordinates and satisfy two boundary conditions [62, 107, 108]: they remain regular at the observer’s position and are purely outgoing towards the horizon. These conditions cause QNMs to have discrete and complex energy values, in contrast to normal modes, which have real and continuous energy values. It is suggested that QNMs could serve as a suitable basis for considering dS/CFT, given their behavior around horizons and boundaries, along with their symmetry properties.

In this section, we extend the holographic framework in [63] for heavy fields. We expect to get a global view of the QNMs and how they relate to the dS/CFT. Specifically, we start by working in global coordinates and construct QNMs in terms of the global modes

introduced in section (3.2.2). We then demonstrate that the thermofield double state description for Euclidean vacuum holds for heavy scalar fields by constructing a kind of “northern modes” and “southern modes” basis using the QNMs. Finally, we discuss the potential holographic interpretation for this configuration.

6.1 dS quasinormal modes

In this section, we follow closely with the convention in [79, 109] to construct QNMs in de Sitter space, but mainly focus on heavy fields rather than light fields. Here we restrict our discussion to dS_4 . With global coordinates $x = (t, \Omega) = (t, \psi, \theta, \phi)$, the metric is given by

$$ds^2 = -dt^2 + \cosh^2 t d\Omega_3^2 = -dt^2 + \cosh^2 t [d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)] . \quad (6.1)$$

Here we denote the north and south pole by

$$\Omega_{SP} \sim \psi = 0, \quad \Omega_{NP} \sim \psi = \pi . \quad (6.2)$$

Using (2.15), the dS invariant distance function is given by

$$Z(x, x') = \cosh t \cosh t' \cos \Theta_3(\Omega, \Omega') - \sinh t \sinh t' , \quad (6.3)$$

where

$$\cos \Theta_3(\Omega, \Omega') = \cos \psi \cos \psi' + \sin \psi \sin \psi' [\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')] . \quad (6.4)$$

To construct QNMs, we need to construct modes which decay exponentially towards the future for a physical observer in the northern static patch (even though the possibly more common choice is a southern patch observer). To do this, we use the Euclidean Green function (3.49) to construct Feynman propagator by imposing the appropriate $i\epsilon$ prescription for going around singularities. When $d = 3$, it gives

$$G_E(x; x') = \frac{\Gamma(\Delta_+) \Gamma(\Delta_-)}{16\pi^2} {}_2F_1 \left[\Delta_+, \Delta_-, 2, \frac{1 + Z(x; x') - i s(x; x') \epsilon}{2} \right] , \quad (6.5)$$

where $s(x; y) = s(X; Y) \equiv X^0 - Y^0$ comes from sending $X^0 - Y^0 \rightarrow X^0 - Y^0 - i\epsilon$ in (3.49). Notice that in $t' \rightarrow \infty$ limit, there are two leading terms corresponding to Δ_{\pm} respectively in the asymptotic form of the Euclidean Green function with $\Omega' = \Omega_{SP}$

$$\begin{aligned} \lim_{t' \rightarrow \infty} G_E(t, \Omega; t', \Omega_{SP}) &= \frac{\Gamma(\Delta_- - \Delta_+) \Gamma(\Delta_+)}{2^{4-2\Delta_+} \pi^2 \Gamma(2 - \Delta_+)} \frac{e^{-\Delta_+ t'}}{(\sinh t - i\epsilon - \cosh t \cos \psi)^{\Delta_+}} \\ &\quad + (\Delta_+ \leftrightarrow \Delta_-) . \end{aligned} \quad (6.6)$$

To eliminate one term and keep the other, we consider $G_E(x; x'_A)$,

$$\begin{aligned} \lim_{t' \rightarrow \infty} G_E(t, \Omega; -t', \Omega_{NP}) &= e^{-i\pi\Delta_+} \frac{\Gamma(\Delta_- - \Delta_+) \Gamma(\Delta_+)}{2^{4-2\Delta_+} \pi^2 \Gamma(2 - \Delta_+)} \frac{e^{-\Delta_+ t'}}{(\sinh t - i\epsilon - \cosh t \cos \psi)^{\Delta_+}} \\ &\quad + (\Delta_+ \leftrightarrow \Delta_-) . \end{aligned} \quad (6.7)$$

Now we can see by defining their difference with appropriate coefficients⁶

$$G_{\pm}(x; x') \equiv G_E(x; x') - e^{i\pi\Delta_{\mp}} G_E(x; x'_A), \quad (6.8)$$

we obtain mode functions with simple asymptotic oscillating behaviors

$$\Phi^{\pm\text{QN}}(x) \equiv \lim_{t' \rightarrow \infty} e^{\Delta_{\pm} t'} G_{\pm}(t, \Omega; t', \Omega_{SP}) = \frac{1}{4\pi^{5/2}} \frac{\Gamma(\mp i\mu) \Gamma(\Delta_{\pm}) (1 - e^{\pm 2\pi\mu})}{[\sinh t - i\epsilon - \cosh t \cos \psi]^{\Delta_{\pm}}}. \quad (6.9)$$

This could be considered as a rescaled Green function with one argument placed at the south pole and pushed to the future boundary I^+ . We can also define another pair of modes in the similar form as

$$\Phi^{\pm\text{AQN}}(x) \equiv \lim_{t' \rightarrow \infty} e^{\Delta_{\pm} t'} G_{\pm}(t, \Omega; t', \Omega_{NP}) = \frac{1}{4\pi^{5/2}} \frac{\Gamma(\mp i\mu) \Gamma(\Delta_{\pm}) (1 - e^{\pm 2\pi\mu})}{[\sinh t - i\epsilon + \cosh t \cos \psi]^{\Delta_{\pm}}}, \quad (6.10)$$

Note that they are the highest and lowest weight conformal primary solutions of the wave equation

$$(\nabla^2 - m^2) \Phi = 0, \quad (6.11)$$

in terms of dS_4 $SO(4, 1)$ isometry group representations, hence they can be associated to highest and lowest weight primary operators O inserted at the south and north pole of the future boundary, respectively. To confirm this, we show that these modes indeed have the same symmetries as an insertion of primary operators. Firstly, selecting a point on I^+ results in the breaking of de Sitter symmetry $SO(4, 1)$ into $SO(3) \times SO(1, 1)$. It is evident that $\Phi^{\pm\text{QN}}$ and $\Phi^{\pm\text{AQN}}$ remain invariant with respect to the $SO(3)$ spatial rotations. As for $SO(1, 1)$, first we note that from (2.4), the denominators of (6.9) and (6.10) in terms of embedding coordinates can be written as

$$\Phi^{\pm\text{QN}} \propto (X^0 - X^1)^{-\Delta_{\pm}}, \quad \Phi^{\pm\text{AQN}} \propto (X^0 + X^1)^{-\Delta_{\pm}}. \quad (6.12)$$

They are manifestly eigenfunctions of the $SO(1, 1)$ dilatation operator $L_0 = -\cos \psi \partial_t + \tanh t \sin \psi \partial_{\psi}$. Because in terms of static coordinates, $L_0 = \partial_T$, where

$$T = -\text{arctanh}(\sec \psi \tanh t) \quad (6.13)$$

is the northern static patch time, (6.12) becomes

$$\begin{aligned} \Phi^{\pm\text{QN}} &\propto (-\sinh T - \cosh T)^{-\Delta_{\pm}} = (-1)^{-\Delta_{\pm}} e^{-\Delta_{\pm} T}, \\ \Phi^{\pm\text{AQN}} &\propto (-\sinh T + \cosh T)^{-\Delta_{\pm}} = e^{\Delta_{\pm} T}. \end{aligned} \quad (6.14)$$

Note that the sign of the static time T is different from (2.13), because we are considering northern static patch time here. Therefore, we have

$$L_0 \Phi^{\pm\text{QN}} = -\Delta_{\pm} \Phi^{\pm\text{QN}}, \quad L_0 \Phi^{\pm\text{AQN}} = \Delta_{\pm} \Phi^{\pm\text{AQN}}. \quad (6.15)$$

⁶Combinations of the Euclidean Green function with similar properties have been previously investigated regarding the elliptic Z_2 -identification of de Sitter space [110], which could be connected to our construction.

Next, to show that they are the highest and lowest weight primary solutions, note that acting Laplacian ∇^2 on the $SO(3)$ invariant symmetric function $\Phi^{\pm\text{QN}}$ yields

$$\nabla^2 = -L_0(L_0 + 3) + \sum_{k=1}^3 M_{+k}M_{-k}, \quad (6.16)$$

where $M_{\pm k}$ are the 6 raising and lowering operators for L_0 . Substituting this to the wave equation (6.11), it becomes

$$\sum_{k=1}^3 M_{+k}M_{-k}\Phi^{\pm\text{QN}}(x) = (m^2 + \Delta_{\pm}(\Delta_{\pm} - 3))\Phi^{\pm\text{QN}}(x) = 0. \quad (6.17)$$

Same procedure applies for $\Phi^{\pm\text{AQN}}$ as well. Hence we see they satisfy the highest and lowest weight conditions,

$$M_{-k}\Phi^{\pm\text{QN}} = M_{+k}\Phi^{\pm\text{AQN}} = 0. \quad (6.18)$$

It could also be shown that these symmetries uniquely determine the primary weight solutions, which are equivalent to $\Phi^{\pm\text{QN}}$ and $\Phi^{\pm\text{AQN}}$. As a result, we confirmed that they are identified as the classical wavefunction associated to the insertion of the primary operators O at the south pole.

The descendants and ascendants of $\Phi^{\pm\text{QN}}$ and $\Phi^{\pm\text{AQN}}$ can be obtained by applying the operators $M_{\pm k}$, where $k \in \{1, 2, 3\}$,

$$\Phi_B^{\pm\text{QN}}(x) \equiv M_{+k_1} \cdots M_{+k_n} \Phi^{\pm\text{QN}}(x), \quad \Phi_B^{\pm\text{AQN}}(x) \equiv M_{-k_1} \cdots M_{-k_n} \Phi^{\pm\text{AQN}}(x) \quad (6.19)$$

where B represents a multi-index that indicates the action of powers of $M_{\pm k}$, and we denote the primary field as $\Phi_0^{\pm\text{QN}} = \Phi^{\pm\text{QN}}$.

We see from their explicit forms that $\Phi^{\pm\text{QN}}$ ($\Phi^{\pm\text{AQN}}$) are indeed singular at the past (future) horizon and decay exponentially towards the future (past) for physical observers in the northern static patch (instead of the possibly more standard choice of southern patch).⁷ For $\Phi^{\pm\text{AQN}}$, this can be read more easily from the following equivalent expression,

$$\Phi^{\pm\text{AQN}}(x) = \lim_{t' \rightarrow -\infty} e^{\Delta_{\pm} t'} G_{\pm}(t, \Omega; t', \Omega_{SP}). \quad (6.20)$$

Therefore, $\Phi^{\pm\text{QN}}$ and $\Phi^{\pm\text{AQN}}$, along with all their descendants and ascendants (6.19) and also their complex conjugates, form the quasinormal modes and anti-quasinormal modes (AQNMs) of the northern static patch, respectively.

Although QNMs and AQNMs are not smooth everywhere, they are defined in global coordinates and have analytic values everywhere except on the horizons. Hence we are able to derive the relation between (A)QNMs and the global modes we introduced in

⁷However unlike with light fields, heavy fields also oscillate everywhere throughout all the coordinates. This makes the behaviors of QNMs rather complicated and more difficult to apply a proper dS/CFT interpretation.

section (3.2.2). To see this, we use the first equation in (6.9) and the definition of the Euclidean Green function (3.49), then we have

$$\Phi^{\pm\text{QN}}(t, \Omega) = \lim_{t' \rightarrow \infty} e^{\Delta_{\pm} t'} \sum_{L,j} \phi_{Lj}^E(t, \Omega) (\phi_{Lj}^{E*}(t', \Omega_{SP}) - e^{i\pi\Delta_{\mp}} \phi_{Lj}^E(t', \Omega_{SP})) . \quad (6.21)$$

Furthermore, using (3.48), they can be expressed as in terms of the out modes,

$$\Phi^{+\text{QN}}(t, \Omega) = \lim_{t' \rightarrow \infty} e^{\Delta_+ t'} e^{-i\pi\Delta_+} \sqrt{1 - e^{-2\pi\mu}} \sum_{L,j} \phi_{Lj}^E(t, \Omega) \phi_{Lj}^{\text{out}}(t', \Omega_{SP}) , \quad (6.22)$$

$$\Phi^{-\text{QN}}(t, \Omega) = \lim_{t' \rightarrow \infty} e^{\Delta_- t'} \sqrt{1 - e^{-2\pi\mu}} \sum_{L,j} \phi_{Lj}^E(t, \Omega) \phi_{Lj}^{\text{out}*}(t', \Omega_{SP}) . \quad (6.23)$$

For AQNMs, we simply replace Ω_{SP} with Ω_{NP} in (6.22) and (6.23). Therefore, the (A)QNMs are proportional to Euclidean modes and hence they are Euclidean modes themselves too. Additionally, we can also express the QNMs solely through the out modes by using (3.46). For example,

$$\Phi^{+\text{QN}}(t, \Omega) = \lim_{t' \rightarrow \infty} e^{\Delta_+ t'} e^{-i\pi\Delta_+} \sum_{L,j} (\phi_{Lj}^{\text{out}}(t, \Omega) - e^{i\pi\Delta_+} \phi_{Lj}^{\text{out}*}(t, \Omega)) \phi_{Lj}^{\text{out}}(t', \Omega_{SP}) . \quad (6.24)$$

This expression is valuable for determining the asymptotic behaviors of the QNMs when t approaches infinity, which can later be used for computing the Klein-Gordon inner product. Substituting (3.36) and (3.29) into (6.24), we find that when $t \rightarrow \infty$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \Phi^{\pm\text{QN}}(t, \Omega) &= \frac{2^{3-\Delta_{\pm}}}{\sqrt{\pi}} \Gamma(\mp i\mu) \Gamma(\Delta_{\pm}) (1 - e^{\pm 2\pi\mu}) \Delta_{\pm}(\Omega, \Omega_{SP}) e^{-\Delta_{\pm} t} \\ &\quad \mp \frac{4 \delta^3(\Omega - \Omega_{SP})}{\mu \sqrt{h}} e^{-\Delta_{\mp} t}, \end{aligned} \quad (6.25)$$

where $\sqrt{h} = \sin^2 \psi \sin \theta$ is the measure and

$$\begin{aligned} \Delta_{\pm}(\Omega, \Omega') &= \frac{2^{3(\Delta_{\pm}-1)}}{\pi} \Gamma(2 - 2\Delta_{\pm}) \sin(\Delta_{\pm}\pi) \sum \frac{\Gamma(\Delta_{\pm} + L)}{\Gamma(\Delta_{\mp} + L)} Y_{Lj}(\Omega) Y_{Lj}^*(\Omega') \\ &= \frac{1}{2^{5-2\Delta_{\pm}} \pi^2} \frac{1}{[1 - \cos \Theta_3(\Omega, \Omega')]^{\Delta_{\pm}}} . \end{aligned} \quad (6.26)$$

The first and second terms in (6.25) come from the the first and second terms in (6.24) correspondingly.

One step before we proceed to the Klein-Gordon inner product, we choose a different set of normalizations of (6.9) for later convenience and define the primary (A)QNMs as

$$\Phi^{\pm\text{QN}} = \frac{i (\Gamma(\mp i\mu) \Gamma(\Delta_{\pm}))^{\frac{1}{2}} e^{-i\pi\Delta_{\pm}}}{[\sinh t - \cosh t \cos \psi - i\epsilon]^{\Delta_{\pm}}}, \quad \Phi^{\pm\text{AQN}} = \frac{i (\Gamma(\mp i\mu) \Gamma(\Delta_{\pm}))^{\frac{1}{2}} e^{-i\pi\Delta_{\pm}}}{[\sinh t + \cosh t \cos \psi - i\epsilon]^{\Delta_{\pm}}}. \quad (6.27)$$

It is known that Klein Gordon inner product is independent of the choice of the integrating spacelike Cauchy surface, namely independent of the global time t . Thus it is convenient

to evaluate it on the future boundary using (6.25). And we obtain all the non-zero Klein Gordon inner products as

$$\langle \Phi^{\pm \text{QN}}, \Phi^{\mp \text{AQN}} \rangle_{\text{KG}} = 4\pi^{\frac{5}{2}}, \quad (6.28)$$

$$\langle \Phi^{\pm \text{QN}}, \Phi^{\mp \text{QN}} \rangle_{\text{KG}} = \frac{2^{2+\Delta_{\mp}} \pi^{\frac{5}{2}}}{(1 - \cos \Theta_3(\Omega, \Omega_{\text{SP}}))^{\Delta_{\mp}}} \Big|_{\Omega=\Omega_{\text{SP}}}, \quad (6.29)$$

$$\langle \Phi^{\pm \text{QN}}, \Phi^{\pm \text{QN}} \rangle_{\text{KG}} = \pm \left(\frac{\Gamma(i\mu)\Gamma(-i\mu)}{\Gamma(\Delta_+)\Gamma(\Delta_-)} \right)^{\frac{1}{2}} \frac{2^4 \pi^4 (e^{\pm 2\pi\mu} - 1)}{\sinh \pi\mu} \frac{\delta^3(\Omega - \Omega_{\text{SP}})}{\sqrt{h}} \Big|_{\Omega=\Omega_{\text{SP}}}. \quad (6.30)$$

Note that the Klein Gordon inner products between (A)QNMs diverge, which is one of the well-known problematic properties of QNMs, although Ref. [109] suggests that this can be solved by introducing a different normalization scheme called 'R-norm'.⁸ However, in this thesis we argue that for non-integer Δ_{\pm} including light fields, (6.29) and (6.30) can be regularized to zero by properly treating the integration around the horizons. Therefore, there is no need to consider the R-norm except for integer Δ_{\pm} . Also, unlike light fields, which have finite inner products between $\Phi^{\pm \text{QN}}$ and $\Phi^{\pm \text{AQN}}$, heavy fields have finite inner products between $\Phi^{\pm \text{QN}}$ and $\Phi^{\mp \text{AQN}}$. This implies the mixing of modes for heavy fields, and both modes are required to construct a complete set.⁹

Last we would like to make some comments about the completeness. The descendants and ascendants of $\Phi^{\pm \text{QN}}$ and $\Phi^{\pm \text{AQN}}$ (6.19) are themselves QNMs and AQNMs as well. Thus taking all of these modes together we have eight highest-weight representations of the de Sitter isometry group $SO(4, 1)$,

$$\Phi_A^{\pm \text{QN}}, \quad \Phi_A^{\pm \text{QN}*}, \quad \Phi_A^{\pm \text{AQN}}, \quad \Phi_A^{\pm \text{AQN}*}. \quad (6.32)$$

Similar to the case of light fields, these eight towers are actually an overcomplete basis of solutions to the wave equation, and only half of them are needed to form a complete basis. However, it is rather complicated and obscure which combination of the four sets is qualified due to the modes mixing feature of heavy fields. For instance, it can be demonstrated explicitly that the sum over $\Phi_A^{+ \text{QN}}, \Phi_A^{- \text{QN}}, \Phi_A^{+ \text{QN}*}, \Phi_A^{- \text{QN}*}$ reproduces the Euclidean Green function following the same steps in Appendix C of [109]. Therefore, they form a complete basis, just like light particles. Further, [63] claims that $\Phi_A^{+ \text{QN}}, \Phi_A^{+ \text{AQN}}, \Phi_A^{+ \text{QN}*}, \Phi_A^{+ \text{AQN}*}$ also form a complete basis, corresponding to $\Phi_A^{+ \text{QN}}, \Phi_A^{+ \text{AQN}}, \Phi_A^{- \text{QN}*}, \Phi_A^{- \text{AQN}*}$ for heavy fields. However, we are not able to construct real scalar fields with this basis, this is obviously not a qualified set.

⁸R-norm is defined by

$$\langle \Phi_1, \Phi_2 \rangle_R \equiv \langle \Phi_1, R\Phi_2 \rangle_{\text{KG}}, \quad (6.31)$$

where $R: (\psi, \theta, \phi) \rightarrow (\pi - \psi, \theta, \phi)$, namely exchanging QNMs and ANMs, i.e. $R\Phi^{\pm \text{QN}}(x) = \Phi^{\pm \text{AQN}}(x)$.

⁹To construct modes have finite R-norms with themselves like light fields, we can redefine quasinormal modes as $\Phi_+ = \frac{1}{2}(\Phi^{+ \text{QN}} + \Phi^{- \text{QN}})$, $\Phi_- = \frac{1}{2}(\Phi^{+ \text{QN}} - \Phi^{- \text{QN}})$. Then their R-norms are given by $\langle \Phi_{\pm}, \Phi_{\pm} \rangle_R = \pm 4\pi^{\frac{5}{2}}$ and $\langle \Phi_{\pm}, \Phi_{\mp} \rangle_R = 0$. Although these modes will no longer be eigenfunctions of the dilatation operator L_0 .

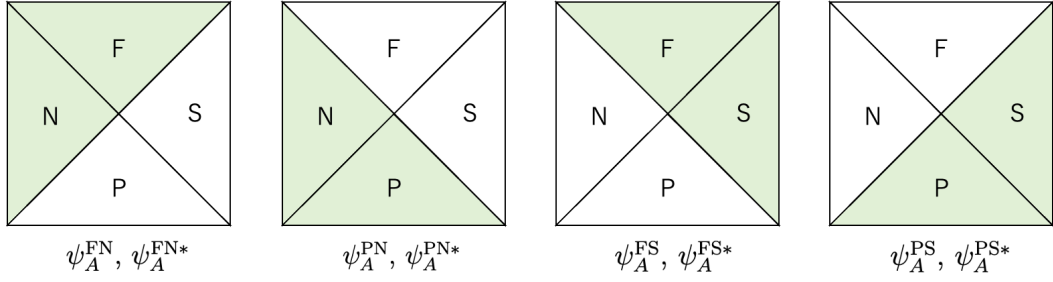


Fig 6: Regions that $\psi_B^{\text{FN}}, \psi_B^{\text{PN}}, \psi_B^{\text{FS}}, \psi_B^{\text{PS}}$ and their complex conjugates support. N and S denote the northern and southern static patches, and F and P denote the future and past Milne regions.

6.2 Northern and southern modes

To further investigate QNMs in global coordinates, we construct modes that have support on only half of the de Sitter space,

$$\begin{aligned} \psi_A^{\text{FN}} &= \Phi_A^{+\text{QN}} + \Phi_A^{-\text{QN}*}, & \psi_A^{\text{PN}} &= - \left(e^{i\pi\Delta_+} \Phi_A^{+\text{AQN}} + e^{-i\pi\Delta_+} \Phi_A^{-\text{AQN}*} \right) \\ \psi_A^{\text{FS}} &= \Phi_A^{+\text{AQN}} + \Phi_A^{-\text{AQN}*}, & \psi_A^{\text{PS}} &= - \left(e^{i\pi\Delta_+} \Phi_A^{+\text{QN}} + e^{-i\pi\Delta_+} \Phi_A^{-\text{QN}*} \right). \end{aligned} \quad (6.33)$$

along with their complex conjugates. Note that this construction is only possible for fields with non-integer Δ_{\pm} , otherwise singularities appear. However, this is not our concern here, since all heavy fields have complex conformal weights. Also the overall coefficients in (6.27) are chosen to satisfy $\Phi^{\pm\text{QN}}(x_A) = -e^{-i\pi\Delta_{\pm}} \Phi^{\mp\text{QN}*}(x)$, so that

$$\psi_A^{\text{FN}}(x_A) = \psi_A^{\text{PS}}(x), \quad \psi_A^{\text{FS}}(x_A) = \psi_A^{\text{PN}}(x). \quad (6.34)$$

Be careful that $(\Delta_+)^* = \Delta_-$ for heavy fields. As an example, the explicit form of ψ_0^{FN} is given by

$$\psi_0^{\text{FN}} = \frac{i (\Gamma(-i\mu)\Gamma(\Delta_+))^{\frac{1}{2}} e^{-i\pi\Delta_+}}{[\sinh t - \cosh t \cos \psi - i\epsilon]^{\Delta_+}} - \frac{i (\Gamma(-i\mu)\Gamma(\Delta_+))^{\frac{1}{2}} e^{i\pi\Delta_+}}{[\sinh t - \cosh t \cos \psi + i\epsilon]^{\Delta_+}}, \quad (6.35)$$

which have no support in the southern static patch and the past Milne region due to the extra $e^{\pm i\pi\Delta_+}$ the denominators pick up when crossing the horizon. Fig. 6 depicts the regions that each mode in (6.33) supports.

Note that using (6.22), (6.23) and (3.46) (while taking care of the different normalization

schemes), we can express the primary fields in (6.33) in terms of the in/out modes as

$$\begin{aligned}
\psi_0^{\text{FN}} &\propto \lim_{t' \rightarrow \infty} e^{\Delta+t'} \sum_{L,j} \phi_{Lj}^{\text{in}}(t, \Omega) \phi_{Lj}^{\text{out}}(t', \Omega_{SP}) , \\
\psi_0^{\text{PN}} &\propto \lim_{t' \rightarrow \infty} e^{\Delta+t'} \sum_{L,j} \phi_{Lj}^{\text{in}}(t, \Omega) \phi_{Lj}^{\text{out}}(t', \Omega_{NP}) , \\
\psi_0^{\text{FS}} &\propto \lim_{t' \rightarrow \infty} e^{\Delta+t'} \sum_{L,j} \phi_{Lj}^{\text{out}*}(t, \Omega) \phi_{Lj}^{\text{out}}(t', \Omega_{NP}) , \\
\psi_0^{\text{PS}} &\propto \lim_{t' \rightarrow \infty} e^{\Delta+t'} \sum_{L,j} \phi_{Lj}^{\text{out}*}(t, \Omega) \phi_{Lj}^{\text{out}}(t', \Omega_{SP}) .
\end{aligned} \tag{6.36}$$

Thus although it is not obvious from their explicit (resummation) forms, they exhibit the same asymptotic behaviors on the boundary as the in/out modes. On the other hand, their explicit forms after resummation provide more details about their analyticities.

However unlike light scalar fields, where the corresponding modes of (6.33) are manifestly defined as real functions due to Δ_{\pm} being real numbers, it is impossible to construct real functions in a similar manner because of the need to mix the $+$ and $-$ modes. Despite this, we compute the inner products of (6.33) using (6.28). For modes do not have any overlapping support regions, their inner product should vanish by construction. In spite of that, if we compute products naively by expanding them in terms of QNMs and AQNMs, we find for instance

$$\langle \psi_0^{\text{FN}}, \psi_0^{\text{PS}} \rangle = -e^{i\pi\Delta_+} \langle \Phi^{+\text{QN}}, \Phi^{+\text{QN}} \rangle - e^{-i\pi\Delta_+} \langle \Phi^{-\text{QN}*}, \Phi^{-\text{QN}*} \rangle . \tag{6.37}$$

Using (6.30), it turns out that this product diverges, i.e. is proportional to $\delta^3(\Omega - \Omega_{SP})|_{\Omega=\Omega_{SP}}$. To resolve this discrepancy, we look at the details of each function and the integral (6.37). Recall that when we computed the Klein Gordon inner product for QNMs and AQNMs, we used the asymptotic forms (6.25) and evaluated the inner product on the future limit. Using (6.33), we see the corresponding asymptotic form of ψ_0^{FN} and ψ_0^{PS} as $t \rightarrow \infty$ becomes

$$\begin{aligned}
\psi_0^{\text{FN}} &= A_1 \Delta_+(\Omega, \Omega_{SP}) e^{-\Delta+t} + B_1 \delta^3(\Omega - \Omega_{SP}) e^{-\Delta-t}, \\
\psi_0^{\text{PS}} &= B_2 \delta^3(\Omega - \Omega_{SP}) e^{-\Delta-t},
\end{aligned} \tag{6.38}$$

where A and B are coefficients independent of the coordinates. The delta function divergences appearing in (6.30) and (6.37) come from the second term of (6.25), namely when integrating around the south pole. However, this term should be treated carefully when approaching the horizons (when $t \rightarrow \infty$, it means around the south pole). In particular, we obtain the asymptotic forms (6.36) by first taking the limit $t \rightarrow \infty$ and then $\Omega \rightarrow \Omega_{SP}$. However, this procedure is hardly appropriate for ψ_0^{PS} , since the function is strictly zero everywhere on the future boundary except for the south pole. To consider the behavior around the singularity more properly, we consider finite t and take $\Omega \rightarrow \Omega_{SP}$ first. ψ_0^{PS} behaves like

$$\psi_0^{\text{PS}} = -\frac{i(\Gamma(-i\mu)\Gamma(\Delta_+))^{\frac{1}{2}}}{[-e^{-t} - i\epsilon]^{\Delta_+}} + \frac{i(\Gamma(-i\mu)\Gamma(\Delta_+))^{\frac{1}{2}}}{[-e^{-t} + i\epsilon]^{\Delta_+}} = \frac{2(\Gamma(-i\mu)\Gamma(\Delta_+))^{\frac{1}{2}} \sin \pi \Delta_+}{e^{-\Delta+t}} . \tag{6.39}$$

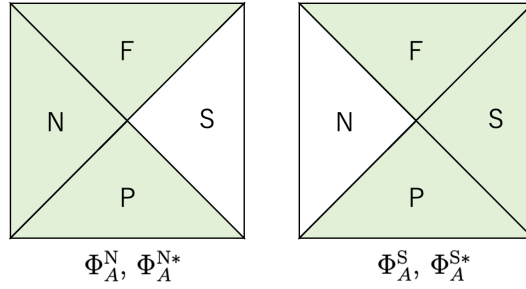


Fig 7: Regions that Φ_A^N, Φ_A^S and their complex conjugates support. Φ_A^N, Φ_A^{N*} have no support in the southern static patch and Φ_A^S, Φ_A^{S*} have no support in the northern one.

We see that the function diverges exponentially as we approach the past boundary $t \rightarrow \infty$, instead of a delta function type of divergence. Also, the function drops from infinity to zero instantly as it crosses the horizon. The same behavior also applies to ψ_0^{FN} , which has a step function drop on the other side of the horizon. Thus, if we consider inner products between ψ_0^{FN} and ψ_0^{PS} on a finite time slice, ψ_0^{FN} diverges as we approach the horizon from the north side, and then drops to zero right after crossing the horizon; while ψ_0^{PS} diverges when we approach the horizon from the south side, and then drops to zero right after crossing the horizon. The two functions do not overlap as long as we impose a proper regularization on the horizon. In consequence, (6.36) becomes zero and so do $\langle \Phi^{\pm \text{QN}}, \Phi^{\pm \text{QN}} \rangle$. (6.29) can be regularized to zero in the same way if we consider $\langle \psi_0^{\text{FN}}, \psi_0^{\text{PS}} \rangle$. Note that this regularization is only possible if we are able to construct modes like (6.33), in other words, if Δ_{\pm} are non-integer. As a result, the only inner products with non-zero values are the following

$$\langle \psi_0^{\text{FN}}, \psi_0^{\text{PN}*} \rangle = \langle \psi_0^{\text{FS}}, \psi_0^{\text{PS}*} \rangle = i2^3 \pi^{\frac{5}{2}} \cosh \pi \mu, \quad (6.40)$$

along with the ones associated with their descendants and ascendants, which can be computed using $SO(4, 1)$ group theory from (6.40).

Next we want to choose the appropriate set of modes to construct a complete basis. One of the obvious options would be (6.33) and its complex conjugates. However, this is equivalent to taking all eight towers of modes in (6.32), which we argued was an overcomplete basis in the previous section. Here, instead, we combine the future and past modes to construct the northern and southern modes as

$$\Phi_A^N = \psi_A^{\text{FN}} + e^{i\theta_N} \psi_A^{\text{PN}*}, \quad \Phi_A^S = \psi_A^{\text{FS}} - e^{i\theta_N} \psi_A^{\text{PS}*}, \quad (6.41)$$

along with their complex conjugates, where $e^{i\theta_N}$ is a pure phase factor and θ_N can take on arbitrary values. The regions these modes support are depicted in Fig. 7.

The above modes and their complex conjugates form an orthonormal set with finite Klein-Gordon norms,¹⁰

$$\langle \Phi_A^N, \Phi_B^N \rangle_{\text{KG}} = N_{AB} = -\langle \Phi_A^S, \Phi_B^S \rangle_{\text{KG}}, \quad (6.42)$$

¹⁰The completeness should be able to prove following the same procedure as in Appendix C of Ref. [109]

and the rest are all zero. Notice that we can simply absorb the phase factors by redefining ψ_B^{PN} and ψ_B^{PS} , so for simplicity we choose $\theta_N = \frac{\pi}{2}$ in the following context. While the explicit form of matrix N_{AB} will not be needed in this thesis, it can be calculated from that of the primaries using $SO(4, 1)$ group theory, where the Klein-Gordon norms of the primaries are given by

$$\langle \Phi_0^{\text{N}}, \Phi_0^{\text{N}} \rangle_{\text{KG}} = -2^4 \pi^{\frac{5}{2}} \cosh \pi \mu = -\langle \Phi_0^{\text{S}}, \Phi_0^{\text{S}} \rangle_{\text{KG}}. \quad (6.43)$$

6.2.1 Quantization and vacua

Now we can construct the bulk field operator with the northern and southern modes in the form of a bulk reconstruction formula. Although there remains the last question - how should we determine the positive frequency modes? To see this, recall (6.14), which suggests that in terms of the northern static patch time T , $\Phi_0^{\pm \text{QN}} \propto e^{-h_{\pm} T} = e^{-(\frac{3}{2} \pm i\mu)T}$ and $\Phi_0^{\pm \text{AQN}} \propto e^{h_{\pm} T} = e^{(\frac{3}{2} \pm i\mu)T}$. Thus it is natural to associate $\Phi_A^{+\text{QN}}$ and $\Phi_A^{-\text{AQN}}$ with the positive frequency modes and $\Phi_A^{-\text{QN}}$ and $\Phi_A^{+\text{AQN}}$ with the negative frequency modes. Accordingly Φ_A^{N} and Φ_A^{S} should be considered as the positive frequency modes too and their complex conjugates are the negative frequency modes. Therefore, we expand the bulk field operator as

$$\hat{\Phi}(x) = N^{AB} \left[\Phi_A^{\text{N}}(x) \hat{\Phi}_B^{\text{N}} + \Phi_A^{\text{N}*}(x) \hat{\Phi}_B^{\text{N}\dagger} - \Phi_A^{\text{S}}(x) \hat{\Phi}_B^{\text{S}} - \Phi_A^{\text{S}*}(x) \hat{\Phi}_B^{\text{S}\dagger} \right] \quad (6.44)$$

where N_{AB} is the same with the one given in (6.42), and $N^{AB} N_{BC} = \delta_C^A$. Note that if x is in the northern static patch, only the first two terms remain and the bulk field operator becomes $\hat{\Phi}(x) = N^{AB} \left[\Phi_A^{\text{N}}(x) \hat{\Phi}_B^{\text{N}} + \Phi_A^{\text{N}*}(x) \hat{\Phi}_B^{\text{N}\dagger} \right]$. The operators are defined by

$$\hat{\Phi}_A^{\text{N/S}} = \left\langle \Phi_A^{\text{N/S}}(x), \hat{\Phi} \right\rangle_{\text{KG}}, \quad \hat{\Phi}_A^{\text{N/S}\dagger} = -\left\langle \Phi_A^{\text{N/S}*}(x), \hat{\Phi} \right\rangle_{\text{KG}}, \quad (6.45)$$

which can be considered as annihilation and creation operators of certain vacuums. The nontrivial commutators are given by

$$\left[\hat{\Phi}_A^{\text{N}}, \hat{\Phi}_B^{\text{N}\dagger} \right] = N_{AB} = -\left[\hat{\Phi}_A^{\text{S}}, \hat{\Phi}_B^{\text{S}\dagger} \right]. \quad (6.46)$$

Using the annihilation operators, we can define the northern and southern vacuum states by

$$\hat{\Phi}_A^{\text{N}} |0_{\text{N}}\rangle = 0, \quad (6.47)$$

and

$$\hat{\Phi}_A^{\text{S}} |0_{\text{S}}\rangle = 0. \quad (6.48)$$

Excited quantum states are then constructed as powers or functions of $\hat{\Phi}_A^{\text{N}\dagger}$ and $\hat{\Phi}_A^{\text{S}\dagger}$ acting on $|0_{\text{N}}\rangle$ and $|0_{\text{S}}\rangle$. Note that unlike with light fields [63], we have mixed the past

and future modes while constructing the northern and southern modes, hence we could not decompose the Fock spaces into further subspaces.

Next we want to derive the relation between the $|0_N\rangle$, $|0_S\rangle$ vacuum and the Euclidean vacuum. To see this, notice that we can express QNMs and AQNMs in terms of ψ by inverting (6.33). Furthermore, we find combinations of QNMs and AQNMs that can be expressed in terms of the northern and southern modes,

$$\begin{aligned}\Phi_A^{+QN} - e^{\pi\mu}\Phi_A^{-AQN} &= \frac{1}{1 - e^{-2\pi\mu}} (\Phi_A^N - e^{-\pi\mu}\Phi_A^{S*}) , \\ \Phi_A^{-QN} + e^{-\pi\mu}\Phi_A^{+AQN} &= \frac{1}{e^{\pi\mu} + e^{-\pi\mu}} (\Phi_A^S + e^{-\pi\mu}\Phi_A^{N*}) .\end{aligned}\quad (6.49)$$

Recall that QNMs and AQNMs are themselves Euclidean modes, hence their annihilation operators satisfy $\hat{\Phi}_A^{+QN} - e^{\pi\mu}\hat{\Phi}_A^{-AQN}|0_E\rangle = 0$ and $\hat{\Phi}_A^{-QN} + e^{-\pi\mu}\hat{\Phi}_A^{+AQN}|0_E\rangle = 0$. Substituting (6.49), we have

$$\hat{\Phi}_A^N + e^{-\pi\mu}\hat{\Phi}_A^{S\dagger}|0_E\rangle = 0, \quad \hat{\Phi}_A^S - e^{-\pi\mu}\hat{\Phi}_A^{N\dagger}|0_E\rangle = 0. \quad (6.50)$$

These two equations suggest that the Euclidean vacuum can be expressed as a superposition of states in the northern and southern Fock space,

$$|0_E\rangle \propto e^{-e^{-\pi\mu}N^{AB}\hat{\Phi}_A^{S\dagger}\hat{\Phi}_B^{N\dagger}}|0_N\rangle \otimes |0_S\rangle. \quad (6.51)$$

This turns out to be similar to expressing the Minkowski vacuum as a thermofield double state in Rindler space, in a basis where Rindler energies are imaginary. This is only possible when we construct the northern and southern modes as (6.41). If we change the relative scale between $\psi^{FN/S}$ and $\psi^{PN/S}$ when we define Φ^N and Φ^S , we will see that there does not exist a formula like (6.51) satisfies both equations in (6.50). Recall that only the southern diamond is causally accessible to an observer located at the north pole. Thus, we would like to study more about the quantum state that includes this region, which is described by a density matrix ρ_N . It can be obtained by tracing the Euclidean vacuum over the southern modes. For the Euclidean vacuum, we define the density matrix ρ_E as

$$\begin{aligned}\rho_E = |0_E\rangle\langle 0_E| &\propto \sum_{m,n=0}^{\infty} \frac{(-e^{-\pi\mu})^{m+n}}{m!n!} N^{A_1C_1} \dots N^{A_mC_m} N^{B_1D_1} \dots N^{B_nD_n} \\ &\times \hat{\Phi}_{A_1}^{S\dagger} \dots \hat{\Phi}_{A_m}^{S\dagger} |0_S\rangle \langle 0_S| \hat{\Phi}_{B_1}^S \dots \hat{\Phi}_{B_n}^S \otimes \hat{\Phi}_{C_1}^{N\dagger} \dots \hat{\Phi}_{C_m}^{N\dagger} |0_N\rangle \langle 0_N| \hat{\Phi}_{D_1}^N \dots \hat{\Phi}_{D_n}^N .\end{aligned}\quad (6.52)$$

Then tracing over the southern modes, we have

$$\rho_N \equiv \text{tr}_S(\rho) \propto \sum_{n=0}^{\infty} \frac{e^{-2n\pi\mu}}{n!^2} N^{A_1B_1} \dots N^{A_nB_n} \hat{\Phi}_{A_1}^{N\dagger} \dots \hat{\Phi}_{A_n}^{N\dagger} |0_N\rangle \langle 0_N| \hat{\Phi}_{B_n}^N \dots \hat{\Phi}_{B_1}^N. \quad (6.53)$$

Denote the unnormalized right-hand as $\tilde{\rho}_N$, we can write

$$\tilde{\rho}_N \hat{\Phi}_{A_1}^{N\dagger} \dots \hat{\Phi}_{A_n}^{N\dagger} |0_N\rangle = e^{-2n\pi\mu} \hat{\Phi}_{A_1}^{N\dagger} \dots \hat{\Phi}_{A_n}^{N\dagger} |0_N\rangle. \quad (6.54)$$

Recalling (6.15) and combining with

$$\Phi_B^N = \psi_B^{\text{FN}} + i\psi_B^{\text{PN}*} = \Phi_B^{+\text{QN}} + \Phi_B^{-\text{QN}*} - i \left(e^{i\pi\Delta_-} \Phi_B^{-\text{AQN}} + e^{-i\pi\Delta_-} \Phi_B^{+\text{AQN}*} \right),$$

we find that even though Φ_B^N are not eigenfunctions of L_0 , the fields they include have eigenvalues either $\frac{3}{2} - i\mu + n$ or $-\frac{3}{2} - i\mu - n$, where $n \in \mathbb{N}_0$. Thus we can write

$$e^{-2\pi i L_0} \Phi_B^N = -e^{-2\pi\mu} \Phi_B^N. \quad (6.55)$$

Therefore, we are able to express (6.54) as

$$\tilde{\rho}_N \hat{\Phi}_{A_1}^{N\dagger} \dots \hat{\Phi}_{A_n}^{N\dagger} |0_N\rangle = (-1)^n e^{-2\pi i L_0} \hat{\Phi}_{A_1}^{N\dagger} \dots \hat{\Phi}_{A_n}^{N\dagger} |0_N\rangle. \quad (6.56)$$

It shows that (6.53) becomes a thermal density matrix $\tilde{\rho}_N = e^{-2\pi i L_0} = e^{-2\pi H_{\text{static}}}$ with temperature $\frac{1}{2\pi}$, where $H_{\text{static}} = iL_0 = i\partial_T$ is the Hamiltonian of the northern static patch. This implies that the Euclidean vacuum state can be realized as a thermofield double state. Note that the density matrix ρ_N here is not Hermitian with respect to the Hilbert space inner product, which results from using a QNM basis. However, the precise understanding or interpretation of this characteristic remains unclear.

6.3 Comments on dS/CFT correspondence

In the previous section, we defined vacuum and excited states using northern modes and southern modes, which are constructed by QNMs and AQNMs. It is suggested that QNMs are suitable candidates for dS/CFT and they are dual to particular CFT operators, benefiting from their simple decaying behaviors on the boundaries. We have explained in the beginning of this section that every S^2 at I^+ possesses two geodesically complete R^3 slices which end on it. These slices can be divided into “northern” (R_N^3) and “southern” (R_S^3) slices, with their topological sum forming an S^3 slice, i.e. $R_S^3 \cup R_N^3 = S^3$, which corresponds to the global S^3 slices where the Hilbert space is built on. This division resembles the relationship between the left and right Rindler wedges in global Minkowski space and also reflects the dynamics of northern and southern causal diamonds in global dS space. Besides, we have seen from (6.51) that the Euclidean state behaves like a thermal state double. Therefore, similar to AdS/CFT for external black, it seems natural to consider a double copy of CFTs dual to our bulk theory.

Generally, CFT states can be constructed by acting with a CFT operator O with a certain conformal weight and all of its descendants, or the power of them. We see that the bulk states we have introduced $\hat{\Phi}_{A_1}^{N\dagger} \dots \hat{\Phi}_{A_n}^{N\dagger} |0_N\rangle$ have the same structure as those CFT states, if we can find a CFT operator O dual to $\hat{\Phi}_A^{N\dagger}$. However, notice that $\hat{\Phi}_A^{N\dagger}$ is constructed by modes with different conformal weights. For example, $\hat{\Phi}_0^{N\dagger}$ contains (A)QNMs with both conformal weight Δ_+ and Δ_- . Recall that from (6.36), ψ_0^{FN} and $\psi_0^{\text{PN}*}$ are proportional to ϕ^{in} and $\phi^{\text{in}*}$, respectively, due to the mixing of incoming and

outgoing modes which come from the future and past Milne regions. This kind of mixing turns out to be necessary for constructing a bulk heavy field because of its oscillating behavior. Moreover, in terms of global coordinates, we have $\phi^{\text{in}} \propto e^{\Delta-t}$ and $\phi^{\text{in}*} \propto e^{\Delta+t}$ on the past boundary, which suggests ψ_0^{FN} and $\psi_0^{\text{PN*}}$ dual to CFT operators $O_{\Delta+}$ and $O_{\Delta-}$, respectively. This is consistent with our conclusion in section 5.1, where we suggested in (5.15) that heavy fields on the boundary correspond to a mixed form of CFT operators with different conformal dimensions

$$\bar{\phi}_{\mathbf{k}} = O_{\mathbf{k}}^+ + O_{\mathbf{k}}^-.$$

The same discussion applies to the southern modes as well. However, unlike the usual dS/CFT framework, we are considering two copies of boundary CFTs dual to the entire de Sitter space and we do not specify the CFT on the past/future boundary; instead, the CFTs should correspond to boundary theories on the static patches. Therefore, we should be more careful with our holographic interpretation. This configuration aligns with the description of eternal AdS black holes in AdS/CFT, which are understood as thermofield double states.

Nevertheless, recall that we argued QFTs dual to a theory with Dirichlet boundary conditions are obtained by double-trace deformations of CFTs. Here since we have seen the same behavior of Φ_A^{N} and Φ_A^{S} as (5.15), we assume that a similar holographic framework applies. There exist two dual QFTs which are related to CFTs in some way, for example, through double trace transformation.

Although we do not have enough concrete evidence yet, we consider the two dual QFTs as would-be CFTs, in a sense that they are related to CFTs in some ways. We denote the vacuum state associates to them as $|0\rangle_{\text{wb-CFT}}$, then their states are defined as

$$|A_1, \dots, A_n\rangle \equiv \frac{1}{\sqrt{n!}} O_{A_1} \cdots O_{A_n} |0\rangle_{\text{wb-CFT}}, \quad (6.57)$$

where O s are the operators dual to $\hat{\Phi}_A^{\text{N}\dagger}$ and $\hat{\Phi}_A^{\text{S}\dagger}$, which have a mixed form of CFT operators. Then we make the following identifications of the two would-be CFTs:

$$\text{First CFT : } \frac{1}{\sqrt{n!}} \hat{\Phi}_{A_1}^{\text{N}\dagger} \cdots \hat{\Phi}_{A_n}^{\text{N}\dagger} |0_{\text{N}}\rangle \longleftrightarrow |A_1, \dots, A_n\rangle \quad (6.58)$$

$$\text{Second CFT : } \frac{1}{\sqrt{n!}} \hat{\Phi}_{B_1}^{\text{S}\dagger} \cdots \hat{\Phi}_{B_n}^{\text{S}\dagger} |0_{\text{S}}\rangle \longleftrightarrow |B_1, \dots, B_n\rangle \quad (6.59)$$

We have seen that the Euclidean vacuum is dual to an entangled state in a thermal field double theory, thus we can write

$$|\text{TFD}\rangle \propto \sum_{n=0}^{\infty} e^{-\pi n \mu} N^{A_1 B_1} \cdots N^{A_n B_n} |A_1, \dots, A_n\rangle \otimes |B_1, \dots, B_n\rangle. \quad (6.60)$$

This suggests that the two dual would-be CFTs form a thermal field double state. Particularly, the modes Φ_A^{N} and $\Phi_A^{\text{N}*}$ form a basis of functions in the northern static patch which

does not support the southern static patch. Therefore, in the northern static patch, bulk fields can be represented as states in the Fock space of the dual would-be CFT, which are created by the products of the dual operator O_A . This implies that the Hilbert space associated with a single static patch is contained within the broader Hilbert space of the corresponding dual would-be CFT.

7 Conclusion and outlook

In this thesis we examined the properties of heavy scalar fields in the context of holographic principle, and extended a part of the dS/CFT framework from light scalar fields to scalar fields with general mass.

First, we revisited holographic interpretation of de Sitter wavefunctions for free scalars. We demonstrated that especially for heavy scalars in the principal series, mixed boundary conditions provide natural identification of wavefunctions with generating functions of the would-be dual CFTs. In particular, wavefunctions with Dirichlet boundary conditions are dual to QFTs on an RG flow generated by double-trace deformations. Since CFT operators dual to heavy fields have complex conformal dimensions, the resultant RG flow is cyclic. This is in sharp contrast to the light field case, for which double-trace deformations interpolate two CFTs associated with Dirichlet and mixed boundary conditions. Besides, we provided a new dS/CFT dictionary of two-point functions that are applicable when mixed boundary conditions are employed.

Second, we considered dS/CFT correspondence for global S^3 slices instead of the future boundary. To this end, we studied quasinormal modes for heavy scalar fields in global coordinates and used this basis to construct the northern and southern modes. We showed that the Euclidean vacuum can be described as an entangled state of the northern and southern modes. Also tracing the Euclidean vacuum over the southern modes gave the expected northern density matrix ρ_N proportional to $e^{-2\pi H_{\text{static}}}$, which resembles the features of a thermal field double state. However, ρ_N is not hermitian here. We also found the dual boundary interpretation of these constructions, which required two copies of boundary theories corresponding to the northern and southern Hilbert spaces, respectively. In addition, the boundary field corresponds to a mixed form of CFT operators with different conformal dimensions, which complicates the identification to a CFT. Nevertheless, we expect them to be dual to a double copy of CFTs through certain transformations.

There are several interesting future directions along the line of the present work. The first thing to do is to generalize our analysis of wavefunctions with mixed boundary conditions to interacting theories and also to theories with spinning fields. Especially for heavy fields, mixed boundary conditions manifest conformal symmetry of dual QFT correlators, which could be useful when studying symmetry and analytic structures of cosmological correlators in the context of the cosmological collider program (see, e.g., [111–117, 85, 86, 118–121, 87, 122–124]). It would also be interesting to study relations between dS Witten diagrams and AdS Witten diagrams based on our findings. More conceptually, our results imply that wavefunctions defined with Dirichlet boundary conditions are dual to QFTs on RG limit cycles whenever we take into account heavy fields required by UV completion of gravity in the bulk. It could be an obstruction to constructing realistic dS/CFT setups in string theory, at least as long as the standard holographic dictionary is employed. It would be interesting to reconsider why it is difficult to turn on stringy mass

in higher spin dS/CFT from this perspective, which could be useful in understanding de Sitter space in string theory.

Furthermore, it is important to find an appropriate CFT description for our northern and southern Hilbert space and to derive the general dictionary for late time correlation functions in terms of CFT correlation functions. It is expected that quantum entanglement involving two CFTs plays a role in the emergence of geometry including time in global de Sitter. Also, the eigenvalues of the static patch Hamiltonian are complex due to the use of the quasinormal mode basis. This raises technical and conceptual issues about hermiticity that require further understanding.

Our ultimate aim is to uncover the fundamental properties and structural complexities of the dS/CFT correspondence in order to achieve a UV complete quantum theory that is consistent with quantum gravity in de Sitter space. We hope that our analysis makes significant progress towards this goal.

A dS_4 isometry groups

We write down some details about generators of dS_4 isometry groups, which would be useful in section 6. There are 10 Killing vectors of dS_4 , which can be defined in terms of the embedded coordinates Lorentz generators (2.2) $L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}$ by

$$\begin{aligned} L_0 &= L_{40} \\ M_{\pm k} &= L_{4k} \mp L_{0k} \\ J_i &= -\epsilon_{ijk} L_{jk} . \end{aligned} \tag{A.1}$$

Comparing with (3.67), we find that

$$L_0 = -D, \quad M_{+i} = K_i, \quad M_{-i} = P_i, \quad J_i = -\epsilon_{ijk} M_{jk} . \tag{A.2}$$

Hence they correspond to the dilatation, translation, SCT and $SO(3)$ rotation generators, respectively. In global coordinates, these Killing vectors are explicitly given by:

$$\begin{aligned} L_0 &= -\cos \psi \partial_t + \tanh t \sin \psi \partial_\psi , \\ M_{\pm 1} &= \pm \sin \psi \sin \theta \sin \phi \partial_t + (1 \pm \tanh t \cos \psi) \sin \theta \sin \phi \partial_\psi , \\ &\quad + (\cot \psi \pm \tanh t \csc \psi) (\cos \theta \sin \phi \partial_\theta + \csc \theta \cos \phi \partial_\phi) , \\ M_{\pm 2} &= \pm \sin \psi \sin \theta \cos \phi \partial_t + (1 \pm \tanh t \cos \psi) \sin \theta \cos \phi \partial_\psi , \\ &\quad + (\cot \psi \pm \tanh t \csc \psi) (\cos \theta \cos \phi \partial_\theta - \csc \theta \sin \phi \partial_\phi) , \\ M_{\pm 3} &= \pm \sin \psi \cos \theta \partial_t + (1 \pm \tanh t \cos \psi) \cos \theta \partial_\psi - (\cot \psi \pm \tanh t \csc \psi) \sin \theta \partial_\theta , \\ J_1 &= \cos \phi \partial_\theta - \sin \phi \cot \theta \partial_\phi , \\ J_2 &= -\sin \phi \partial_\theta - \cos \phi \cot \theta \partial_\phi , \\ J_3 &= \partial_\phi . \end{aligned} \tag{A.3}$$

Their commutators satisfy

$$\begin{aligned} [J_i, J_j] &= \sum_{k=1}^3 \epsilon_{ijk} J_k , \quad [J_i, M_{\pm j}] = \sum_{k=1}^3 \epsilon_{ijk} M_{\pm k} , \\ [L_0, M_{\pm i}] &= \mp M_{\pm i} , \quad [M_{+i}, M_{-j}] = 2L_0 \delta_{ij} + 2 \sum_{k=1}^3 \epsilon_{ijk} J_k , \end{aligned} \tag{A.4}$$

which indeed form the $SO(4, 1)$ algebra. The commutators in the first line suggest that J_i indicate an $SO(3)$ subalgebra, which results in the transformation of M_{+i} and M_{-i} as vectors. The second line shows that the killing vectors $M_{\pm i}$ for any $i \in 1, 2, 3$ and L_0 create an $SO(2, 1)$ subalgebra, whose commutators that satisfy (without summing over i)

$$[M_{+i}, M_{-i}] = 2L_0, \quad [L_0, M_{\pm i}] = \mp M_{\pm i} . \tag{A.5}$$

Furthermore, the scalar Laplacian is a Casimir operator, which reads

$$\nabla^2 = -L_0 (L_0 - 3) + \sum_{i=1}^3 M_{-i} M_{+i} + J^2 = -L_0 (L_0 + 3) + \sum_{i=1}^3 M_{+i} M_{-i} + J^2 , \tag{A.6}$$

where $J^2 = -L(L+1)$ when it acts on the spherical harmonics Y_{Lj} .

B Double-trace deformations

Since the large N limit of $SU(N)$ gauge theory was recognized as a promising foundation for understanding the dynamics of four-dimensional quantum gauge theories, there has been significant interest in studying large N limits for matrix-valued fields. When working with a collection of matrix-valued fields Φ_i , constructing a theory with a large N limit involves considering normalized trace operators

$$O_\alpha = \frac{1}{N} \text{Tr} F_\alpha(\Phi_i) , \quad (\text{B.1})$$

and an action functional

$$I = N^2 W(O_\alpha) , \quad (\text{B.2})$$

where F_α are arbitrary functions of Φ_i and their derivatives, and W is an arbitrary function of O_α . Since F_α has no explicit dependence on N and is defined without any traces, O_α is called a single-trace operator. If W is a linear function of the O_α 's, it is referred to as a single-trace action, while non-linear terms in W are known as multi-trace interactions. In particular, consider a CFT_d that admits a large N expansion and possesses a single trace scalar operator O with dimension Δ . Then we can consider the double-trace deformation

$$S_f = S_{\text{CFT}} + \frac{f}{2} \int d^d x O^2 , \quad (\text{B.3})$$

where f is a constant. In order for the deformation to be relevant, we assume that the conformal dimension of O in the unperturbed theory falls in the range $(d/2 - 1, d/2)$, where the lower bound is required by unitarity. In the large N limit, this double-trace term $\frac{f}{2} \int d^d x O^2$ triggers a renormalization group flow from the unperturbed CFT in the UV to a new CFT in the IR in which the operator O has dimension $d - \Delta + \mathcal{O}(1/N)$. This argument can be understood via the Hubbard-Stratonovich transformation. First we consider the Euclidean partition function of a CFT_d in the presence of the double-trace deformation, and with a source J for O

$$Z_f[J] = \int \mathcal{D}\phi e^{-S_{\text{CFT}}[\phi] - \frac{f}{2} \int O^2 + \int J O} = \left\langle e^{-\frac{f}{2} \int O^2 + \int J O} \right\rangle_0 , \quad (\text{B.4})$$

where $\langle \dots \rangle_0$ denotes a correlator in the unperturbed CFT. The Hubbard-Stratonovich transformation amounts to introducing an auxiliary field σ and modifying the action as

$$S_f \rightarrow S_f - \frac{1}{2f} \int (\sigma + f O)^2 . \quad (\text{B.5})$$

The physics remains unchanged by the additional term - the non-dynamical σ field can be integrated out, giving back the original theory. The partition function (B.4) then becomes

$$Z_f[J] = \sqrt{\det \left(-\frac{1}{f} \mathbf{1} \right)} \int \mathcal{D}\sigma \left\langle e^{\int \left(\frac{1}{2f} \sigma^2 + \sigma O + J O \right)} \right\rangle_0 . \quad (\text{B.6})$$

The determinant is defined so that $\sqrt{\det\left(-\frac{1}{f}\mathbf{1}\right)} \int \mathcal{D}\sigma e^{\frac{1}{2f}\sigma^2} = 1$. Dropping the subleading terms in $1/N$, the assumption that higher point functions of O are suppressed allows us to write

$$\left\langle e^{f(\sigma+J)O} \right\rangle_0 \approx e^{\frac{1}{2}\langle (f(\sigma+J)O)^2 \rangle_0}. \quad (\text{B.7})$$

The σ integral that remains for computing $Z_f[J]$ is now strictly Gaussian. Next, to write a closed form expression for $Z_f[J]$, it is convenient to introduce three linear operators by the following relations

$$(\hat{G}\sigma)(x) = \int d^d\xi \sqrt{g} \langle \mathcal{O}(x) \mathcal{O}(\xi) \rangle_0 \sigma(\xi), \quad (\text{B.8})$$

$$\hat{K} = 1 + f\hat{G}, \quad (\text{B.9})$$

$$\hat{Q} = -\frac{1}{f} \left(\hat{K}^{-1} - 1 \right) = \frac{\hat{G}}{1 + f\hat{G}}. \quad (\text{B.10})$$

Using the above operators, the path integral over σ gives

$$Z_f[J] = \frac{1}{\sqrt{\det \hat{K}}} e^{\frac{1}{2}\langle J, \hat{Q}J \rangle}. \quad (\text{B.11})$$

Hence in particular, the two-point function for O in the presence of the deformation is given by

$$\langle O(x_1) O(x_2) \rangle_f = \left. \frac{\delta^2 \log Z_f[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} = Q(x_1, x_2), \quad (\text{B.12})$$

where Q is the position space representation of the operator \hat{Q} .

The three linear operators (B.8), (B.9) and (B.10) are diagonal in a momentum space basis and they are explicitly given by

$$G(k) = \int d^d x \frac{e^{ik \cdot x}}{x^{2\Delta}} = 2^{d-2\Delta} \pi^{d/2} \frac{\Gamma(\frac{d}{2} - \Delta)}{\Gamma(\Delta)} k^{2\Delta-d}, \quad (\text{B.13})$$

$$K(k) = 1 + fG(k), \quad (\text{B.14})$$

$$Q(k) = \frac{G(k)}{1 + fG(k)}. \quad (\text{B.15})$$

In the IR limit, where $fG \gg 1$, expanding $Q(k)$, we found

$$Q(k) = \frac{1}{f} - \frac{1}{f^2 G(k)} + \frac{1}{f^3 G(k)^2} - \dots \quad (\text{B.16})$$

The leading non-analytic term of this expansion is the second term $-\frac{1}{f^2 G(k)}$, whose Fourier transformation gives the two-point function for O

$$\langle O(x) O(0) \rangle_f \approx -\frac{1}{f^2 \pi^d} \frac{\Gamma(\Delta) \Gamma(d - \Delta)}{\Gamma(\frac{d}{2} - \Delta) \Gamma(\Delta - \frac{d}{2})} \frac{1}{x^{2(d-\Delta)}} \quad \text{for } x \gg f^{-\frac{1}{d-2\Delta}}. \quad (\text{B.17})$$

The observed power law behavior of the two point function in the infrared suggests the presence of an infrared fixed point. Additionally, from (B.17) we found that the dimension of the operator O has changed from Δ in the UV to $d - \Delta$ in the IR, and the RG flow triggered by the double-trace deformation terminates at this infrared fixed point. In fact, a simple manipulation of the generating functional (B.6) shows that the UV and IR large N CFTs are related by the Legendre transform [97].

C Hypergeometric functions

Here we present some properties and formulas for hypergeometric functions. More details may be found in Ref. [125].

To relate hypergeometric functions of z with different values of parameters, we have

$${}_2F_1[a, b; c; z] = (1 - z)^{c-a-b} {}_2F_1[c - a, c - b; c; z]. \quad (\text{C.1})$$

Also the following formula relates hypergeometric functions of different variables as

$$\begin{aligned} {}_2F_1[a, b; c; z] &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} (-z)^{-a} {}_2F_1[a, a+1-c; a+1-b; z^{-1}] \\ &\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} {}_2F_1[b, b+1-c; b+1-a; z^{-1}] \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1[a, b; 1+a+b-c; 1-z] \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1[c-a, c-b; c-a-b+1; 1-z]. \end{aligned} \quad (\text{C.2})$$

Hypergeometric function ${}_2F_1$ has a branch cut with respect to z along $(1, +\infty)$. Additionally, around $z = 1$, it behaves as

$$\lim_{z \rightarrow 1} {}_2F_1[a, b; c; z] = \begin{cases} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \text{Re}(a+b-c) < 0 \\ -\frac{\log(1-z)+R(a,b)}{B(a,b)} + O(1) & \text{Re}(a+b-c) = 0, \\ (1-z)^{c-a-b} {}_2F_1[c-a, c-b; c; z] & \text{Re}(a+b-c) > 0 \end{cases} \quad (\text{C.3})$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $R(a, b) = \psi(a) + \psi(b) + 2\gamma_E$.

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