

Doctoral Dissertation

博士論文

Fusion category symmetries in 1+1 dimensions
and their fermionization

(1+1 次元におけるフュージョン圏対称性とそのフェルミオン化)

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Abstract

We study bosonic and fermionic gapped phases with finite generalized symmetries in 1+1 dimensions. We first construct all 1+1d bosonic topological field theories (TFTs) with general finite symmetries using the state sum construction. When the symmetries are non-anomalous, we write down concrete lattice Hamiltonians that realize gapped phases whose universal properties are captured by the above TFTs. We also perform a similar construction for fermionic TFTs and fermionic lattice models. The bosonic and fermionic TFTs that we construct are related to each other by bosonization and fermionization. Based on this relation, we derive a formula that determines the symmetry of a fermionic system from the symmetry of its bosonic counterpart. This fermionization formula is applicable to any finite generalized symmetries regardless of whether they are anomalous or non-anomalous. As concrete examples, we compute the fermionization of finite group symmetries, symmetries of finite group gauge theories, and Kramers-Wannier-like self-dualities. Combined with the classification of 1+1d bosonic gapped phases with finite generalized symmetries, our fermionization formula allows us to classify 1+1d fermionic gapped phases with arbitrary finite generalized symmetries.

List of publications

Chapter 4 of this dissertation is based on

- Kansei Inamura, *On lattice models of gapped phases with fusion category symmetries*, Journal of High Energy Physics **03** (2022) 036 [arXiv:2110.12882].

Chapter 5 and Chapter 6 of this dissertation are based on

- Kansei Inamura, *Fermionization of fusion category symmetries in $1+1$ dimensions*, Journal of High Energy Physics **10** (2023) 101 [arXiv:2206.13159].

The list of my publications other than the above is as follows:

- Kansei Inamura and Xiao-Gang Wen, *$2+1D$ symmetry-topological-order from local symmetric operators in $1+1D$* , arXiv:2310.05790.
- Kansei Inamura and Kantaro Ohmori, *Fusion Surface Models: $2+1d$ Lattice Models from Fusion 2-Categories*, arXiv:2305.05774.
- Kansei Inamura, *Topological field theories and symmetry protected topological phases with fusion category symmetries*, Journal of High Energy Physics **05** (2021) 204 [arXiv:2103.15588].
- Kansei Inamura, Ryohei Kobayashi, and Shinsei Ryu, *Non-local Order Parameters and Quantum Entanglement for Fermionic Topological Field Theories*, Journal of High Energy Physics **01** (2020) 121 [arXiv:1911.00653].

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Chapter 1

Introduction

Quantum many-body systems exhibit a vast variety of phases. For example, the transverse field Ising model in 1+1 spacetime dimensions has two distinct gapped phases, i.e., the ordered phase and disordered phase, which are separated by a gapless critical point in its phase diagram. There are also more exotic phases such as topologically ordered phases [1], with a prototypical example being the one realized in Kitaev's toric code model [2]. Classifying all possible phases of quantum many-body systems is a big challenge in theoretical condensed matter physics.

Symmetry plays a pivotal role in the classification of quantum phases of matter. In the example of the transverse field Ising model, two gapped phases are distinguished by different patterns of spontaneous symmetry breaking. Characterizing various phases based on symmetries is the central idea of the famous Landau paradigm [3]. Although the original Landau paradigm only works for symmetry broken phases, recent studies suggest that the Landau paradigm is also applicable to even more exotic phases once we generalize the notion of symmetry [4–8]. Therefore, studying symmetries of many-body systems is a crucial step toward the classification of quantum phases of matter.

In recent years, the notion of symmetry has been greatly generalized based on the correspondence between symmetry operators and topological defects. The idea behind the generalization is to interpret symmetry operators as topological defects that act on local operators surrounded by them, see Figure 1.1. The defect implementing the symmetry action has to be topological because the symmetry charge of the local operator surrounded by the defect does not change under continuous deformations of the defect. This suggests that the symmetry of a physical system can be characterized in terms of topological defects. Indeed, from a modern perspective, the symmetry of a physical system is *defined* by the algebraic structure of topological defects. The symmetries defined in terms of topological defects are called generalized symmetries [9].

Generalized symmetries include ordinary symmetries as special examples. Specifically, the ordinary symmetry described by a group is associated with codimension one invertible topological defects labeled by group elements.¹ Here, codimension one means that the defects have dimension $d - 1$ in a d -dimensional spacetime. The group structure of the symmetry is reflected in the fusion rules of these defects. Namely, topological defects labeled by group elements g and h fuse into the one labeled by gh . We can also incorporate 't Hooft anomalies [10] into subtle algebraic data of topological defects [9]. Therefore, we can think of generalized symmetries as generalizations of ordinary symmetries and their 't Hooft anomalies.

Remarkably, generalized symmetries are not necessarily described by groups because topological defects of a physical system do not form a group in general. In particular, a topological

¹These defects are the Poincaré duals of background gauge fields.

$$X \text{ (circle with dot } \mathcal{O} \text{ inside)} = \bullet X \cdot \mathcal{O}$$

Figure 1.1: A topological defect X surrounding a local operator \mathcal{O} is thought of as a symmetry operator acting on \mathcal{O} . Due to the state-operator correspondence, the symmetry operator acting on a state can be understood as a topological defect put on a time slice.

defect can be non-invertible, that is, it may not have its inverse. Symmetries generated by such non-invertible defects are called non-invertible symmetries [11–13]. These symmetries have originally been studied in the context of two-dimensional quantum field theories before the term “non-invertible symmetry” was coined, see, e.g., [14–21]. A simple example of this sort of symmetry appears at the critical point of the transverse field Ising model in 1+1 dimensions. Indeed, the Kramers-Wannier self-duality [22] of the critical Ising model gives rise to a non-invertible topological line defect \mathcal{N} , which obeys the fusion rule $\mathcal{N} \times \mathcal{N} = 1 + \eta$ where 1 is the trivial defect and η is the \mathbb{Z}_2 spin-flip defect [23, 24]. More generally, non-invertible symmetries turn out to be ubiquitous particularly in 1+1d conformal field theories [12, 14, 16, 25–28].

Furthermore, topological defects of a generalized symmetry can have codimensions higher than one. Topological defects of higher codimensions act on extended operators as opposed to ordinary symmetry operators that act on local operators. Generalized symmetries generated by such higher codimensional defects are called higher-form symmetries [9]. In the context of condensed matter physics, these symmetries typically appear in the low energy limits of topologically ordered phases. Specifically, the deep infrared of a 2+1d topologically ordered phase has a higher-form symmetry generated by the worldlines of anyons, which have codimension two in three-dimensional spacetime and act on line operators by linking. The topological ground state degeneracy of a topologically ordered phase is attributed to the spontaneous symmetry breaking of this higher-form symmetry [4, 5]. More generally, within the generalized Landau paradigm, topologically ordered phases in any dimension are characterized by the spontaneous breaking of emergent higher-form symmetries.

In the most general situation, symmetries of physical systems can be mixtures of non-invertible symmetries and higher-form symmetries. The algebraic framework to describe such symmetries is believed to be the higher category theory. For instance, finite symmetries in 1+1d bosonic systems are generally described by fusion categories [11, 29], which are natural generalizations of finite groups. Similarly, finite symmetries in 2+1d bosonic systems are generally described by fusion 2-categories [30], which generalize finite groups and finite 2-groups [31–35]. More generally, finite symmetries in $(d + 1)$ d bosonic systems are described by what should be called fusion d -categories, which are mathematical structures encoding the fusion rules and crossing relations of finitely many topological defects of various codimensions.² Symmetries of fermionic systems can have even richer structures [39–47], which cannot be captured by ordinary fusion (higher) categories [48]. Recent developments in the study of generalized symmetries have revealed higher categorical structures of symmetries in various physical systems both in the continuum and on the lattice, see, e.g., [49–52] for pioneering works and [6, 53–56] for reviews of recent developments.

There are also other types of generalized symmetries. For example, one can further gen-

²To the best of the author’s knowledge at the time of writing this dissertation, there is no consensus on a rigorous definition of fusion d -categories for $d \geq 3$, although recent years have witnessed remarkable developments in the theory of higher category inspired by physics [36–38].

eralize the notion of symmetry by relaxing the condition that symmetry operators are fully topological. When the symmetry operators are non-topological in some spatial directions, the corresponding symmetry is called subsystem symmetry. It turns out that subsystem symmetries are closely related to fracton phases of matter [57, 58] via gauging [59, 60]. One can also consider symmetries whose symmetry operators are well-defined only on local patches. These symmetry operators become globally well-defined once we specify transition functions of symmetry operators on overlaps of the patches. The symmetries associated with such symmetry operators are called bundle symmetries [61]. Examples of bundle symmetries include multipole symmetries such as dipole symmetries and quadrupole symmetries, which are also related to fracton phases [62, 63]. We will not pursue these generalizations in this dissertation.

In this dissertation, based on the author's papers [45, 64], we discuss a systematic way to construct both bosonic and fermionic topological field theories (TFTs) with finite generalized symmetries in 1+1 dimensions. These topological field theories describe the low energy limits of gapped phases including symmetry broken phases, symmetry protected topological (SPT) phases [65, 66], and mixtures thereof.³ In the bosonic case, the classification of 1+1d TFTs with finite generalized symmetries is known in the literature [13, 43], and our construction exhausts all topological field theories that appear in the classification. Another approach to the construction of these TFTs was discussed in an independent work of Huang, Lin, and Seifnashri [68]. On the other hand, in the fermionic case, the classification of 1+1d TFTs with finite generalized symmetries had not been figured out in the literature. Nevertheless, the bosonization and fermionization duality implies that our construction gives us all 1+1d fermionic TFTs with these symmetries. As such, our construction can be regarded as a constructive approach to the classification of 1+1d fermionic topological phases with finite generalized symmetries.

Based on the construction of topological field theories, we derive an explicit formula for the fermionization of finite generalized symmetries. This formula enables us to compute the symmetry of the fermionic theory from the symmetry of its bosonization. Despite being derived in the context of topological field theories, we expect that the formula is applicable to any physical systems that may not be topological. Indeed, our formula reproduces a well-known relation between the Kramers-Wannier self-duality of the Ising conformal field theory and the anomalous chiral fermion parity symmetry of the massless Majorana fermion [69–72].

We also construct concrete lattice models for all gapped phases with non-anomalous finite generalized symmetries. Here, symmetries are said to be non-anomalous if and only if they admit gapped phases with unique ground states on a circle [13]. In particular, our lattice models can realize all SPT phases with finite generalized symmetries. The lattice models that we construct are complementary to the anyon chain models [73, 74], which are other 1+1d lattice models with finite generalized symmetries.⁴ The relation between these models will be discussed briefly in the main text.

Summary of main results

The main results of this dissertation consist of (1) the construction of all 1+1d bosonic and fermionic topological field theories with finite generalized symmetries, (2) the derivation of the fermionization formula of finite generalized symmetries in 1+1 dimensions, and (3) the construction of 1+1d bosonic and fermionic lattice models for all gapped phases with non-anomalous finite generalized symmetries. Let us briefly summarize each of these results below.

³In 1+1 dimensions, there are no intrinsic topological orders [67].

⁴See also [75–79] for the use of anyon chain models for the study of exotic phases with finite generalized symmetries and dualities on the lattice.

Topological field theories with finite generalized symmetries. One can systematically construct 1+1d bosonic topological field theories from semisimple algebras by the state sum construction [80,81]. In Chapter 4, we show that the bosonic TFT \mathfrak{B}_K constructed from a semisimple algebra K has a finite generalized symmetry when K is equipped with an additional structure. Furthermore, we find that any 1+1d bosonic TFTs with arbitrary finite generalized symmetries can be obtained by choosing the input algebra K appropriately. For example, when the input algebra K is equipped with a G -grading where G is a finite group, the output TFT \mathfrak{B}_K has a finite generalized symmetry described by the category $\text{Rep}(G)$ of representations of G . We note that $\text{Rep}(G)$ is no longer group-like when G is non-abelian, see Section 2.1 for more details of this category. Bosonic TFTs constructed from G -graded algebras capture the universal properties of gapped phases of finite group gauge theories. Our result generalizes the construction of bosonic TFTs with finite group symmetries discussed in [82,83]. As we will see in Chapter 6, the construction of bosonic TFTs with finite generalized symmetries can further be extended to fermionic TFTs. Namely, we can construct all 1+1d fermionic topological field theories with finite generalized symmetries in a similar way.

Fermionization formula of finite generalized symmetries. A 1+1d bosonic system can be fermionized if and only if it has a (non-anomalous) \mathbb{Z}_2 symmetry [69,70,84,85]. The resulting fermionic system \mathfrak{F} depends on the choice of a \mathbb{Z}_2 symmetry that we use for the fermionization. In Chapter 5, we derive the formula to determine the symmetry \mathcal{C}_f of the fermionic system \mathfrak{F} from the symmetry \mathcal{C}_b of the original bosonic system \mathfrak{B} . The symmetry \mathcal{C}_f depends on the choice of a \mathbb{Z}_2 subgroup of \mathcal{C}_b because the fermionic system \mathfrak{F} does. For example, we find that the fermionization of a finite group symmetry G strongly depends on whether the chosen \mathbb{Z}_2 subgroup of G is central or not. Specifically, when the \mathbb{Z}_2 subgroup is central, the fermionized symmetry is also G , whose \mathbb{Z}_2 subgroup is replaced by the fermion parity symmetry \mathbb{Z}_2^F . On the other hand, if we fermionize G by using its non-central \mathbb{Z}_2 subgroup, the fermionized symmetry is no longer described by a group, see Section 5.3 for more details. Since the fermionization is a bijective map between bosonic and fermionic systems, the classification of fermionic gapped phases with symmetry \mathcal{C}_f reduces to the classification of bosonic gapped phases with the corresponding symmetry \mathcal{C}_b . Thus, combined with the classification of bosonic gapped phases shown in [13], our fermionization formula allows us to classify 1+1d fermionic gapped phases with finite generalized symmetries.

Lattice models with finite generalized symmetries. Along with the construction of topological field theories, we also construct concrete 1+1d lattice models that realize general gapped phases with non-anomalous finite generalized symmetries. The on-site state space of the model is a semisimple algebra K and the Hamiltonian is given by the sum of local commuting projectors consisting of structure maps of K ,⁵ see Section 4.4. The low energy limit of this model is described by the topological field theory constructed from the same algebra K . We emphasize that our lattice models can realize any gapped phases with non-anomalous symmetries, which include all symmetry protected topological phases and symmetry broken phases. For example, if we choose the algebra K to be a twisted group algebra $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]^\omega$ with a non-trivial twist $\omega \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1))$,⁶ our lattice model realizes a symmetry protected topological phase

⁵The Hamiltonian of our model can be written explicitly as a matrix once we choose a basis of each on-site state space. Given that we know the explicit form of the Hamiltonian, it should be possible to design an experiment to realize our model in principle.

⁶The non-trivial twist ω is unique because $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \text{U}(1)) \cong \mathbb{Z}_2$.

with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, which is the same phase as that of the celebrated Affleck-Kennedy-Lieb-Tasaki model [86]. Remarkably, the symmetries of our lattice models are not limited to those described by groups. As in the case of topological field theories, the construction of the lattice models can be generalized to fermionic systems.

Structure of the dissertation

The rest of this dissertation is organized as follows. In Chapter 2, we review fusion categories and related algebraic structures that are necessary to describe finite generalized symmetries of 1+1d bosonic systems. We also introduce the string diagram notation, which will be heavily used in the later chapters. In Chapter 3, we review superfusion categories, which describe finite generalized symmetries of 1+1d fermionic systems. Based on these mathematical backgrounds, we describe our main results in Chapters 4, 5, and 6. In Chapter 4, we discuss bosonic topological phases with fusion category symmetries in 1+1d. After recalling the classification of these topological phases, we construct all 1+1d topological field theories with fusion category symmetries by using the state sum construction and the pullback of fusion category symmetries. Concrete lattice models for topological phases with non-anomalous symmetries are provided at the end of the same chapter. We also mention their relation to the anyon chain models. In Chapter 5, we propose a fermionization formula of general fusion category symmetries and apply it to several examples such as finite group symmetries and Kramers-Wannier-like self-dualities. In Chapter 6, we construct 1+1d fermionic topological field theories and corresponding lattice models with superfusion category symmetries. The discussions in Chapter 6 are almost parallel to those in Chapter 4. Finally, in Chapter 7, we conclude with a summary of the results and future prospects. Detailed computations of superfusion categories that appear in Chapter 5 are relegated to Appendix A.

Before proceeding, we emphasize that all the symmetries that we discuss in this dissertation are finite internal global symmetries. In particular, we do not discuss continuous symmetries and spacetime symmetries. We also note that gapped phases and topological phases are used interchangeably in this dissertation.

Relation to publications

Chapter 4 is based on the author's published paper [64]. Chapters 5 and 6 are based on another piece of the author's published work [45].

Chapter 2

Fusion categories

Symmetries of physical systems are described by the algebraic structures of topological defects of various dimensions [9]. In particular, in 1+1 dimensional bosonic systems, symmetries are associated with topological line defects and their topological junctions. These topological lines and junctions form a mathematical structure known as a fusion category when the number of topological defects is finite [11]. This means that finite symmetries in 1+1d bosonic systems are generally described by fusion categories, and therefore, such symmetries are called fusion category symmetries [13]. These symmetries not only generalize the conventional symmetries described by finite groups but also generalize the notion of 't Hooft anomalies. Indeed, finite group symmetries with and without 't Hooft anomalies can be regarded as fusion category symmetries whose topological lines are all invertible. In general, a fusion category symmetry is said to be non-anomalous if there exists a gapped phase with a unique ground state that preserves the symmetry [13]. On the other hand, a fusion category symmetry is said to be anomalous if there does not exist such a gapped phase. Mathematically, non-anomalous and anomalous fusion category symmetries are described by the representation categories of Hopf algebras and weak Hopf algebras respectively.

In this chapter, we review the basics of fusion categories and related mathematical backgrounds such as Hopf algebras and weak Hopf algebras over the field \mathbb{C} of complex numbers. A thorough review on this subject can be found in [29]. Throughout this chapter, algebras are supposed to be finite dimensional unless otherwise stated.

2.1 Definitions

2.1.1 Fusion categories and tensor functors

We first recall the definitions of fusion categories and tensor functors. To this end, we begin with the definitions of categories and functors. We refer the reader to [29] for more details.

Categories and functors. A category \mathcal{C} consists of objects and morphisms between them. The collection of objects of \mathcal{C} is denoted by $\text{Obj}(\mathcal{C})$, which is simply written as \mathcal{C} when no confusion can arise. For each pair of objects $X, Y \in \text{Obj}(\mathcal{C})$, the collection of morphisms from X to Y is denoted by $\text{Hom}_{\mathcal{C}}(X, Y)$. The subscript \mathcal{C} will be omitted when it is clear from the context. A morphism $f \in \text{Hom}(X, Y)$ is also written as $f : X \rightarrow Y$. The collection of objects and morphisms is called a category if it satisfies the following conditions:

- For any two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, there exists a morphism $g \circ f :$

$X \rightarrow Z$ called the composition of f and g . The composition of morphisms has to be associative, i.e., it satisfies $h \circ (g \circ f) = (h \circ g) \circ f$.

- For every object $X \in \mathcal{C}$, there exists a morphism id_X that satisfies $f \circ \text{id}_X = f$ and $\text{id}_X \circ g = g$ for any $f : X \rightarrow Y$ and $g : Y \rightarrow X$. The morphism id_X is called the identity morphism.

A morphism $f \in \text{Hom}(X, Y)$ is called an isomorphism if it has the inverse $f^{-1} \in \text{Hom}(Y, X)$, i.e., a morphism that satisfies $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$. Objects X and Y are said to be isomorphic to each other if there exists an isomorphism between them. When X is isomorphic to Y , we write $X \cong Y$.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between categories \mathcal{C} and \mathcal{D} assigns an object $F(X) \in \mathcal{D}$ to each object $X \in \mathcal{C}$ and a morphism $F(f) \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ to each morphism $f \in \text{Hom}_{\mathcal{C}}(X, Y)$. This assignment F is called a functor if it preserves the composition and the identity morphisms:

$$F(g \circ f) = F(g) \circ F(f), \quad F(\text{id}_X) = \text{id}_{F(X)}. \quad (2.1.1)$$

A functor from \mathcal{C} to itself is called an endofunctor of \mathcal{C} . The identity functor $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ is defined by $1_{\mathcal{C}}(X) = X$ and $1_{\mathcal{C}}(f) = f$ for any $X \in \mathcal{C}$ and $f \in \text{Hom}_{\mathcal{C}}(X, Y)$.

A natural transformation $\eta : F \rightarrow G$ between two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ assigns a morphism $\eta_X : F(X) \rightarrow G(X)$ to each object $X \in \mathcal{C}$ in such a way that the following equation holds for any morphism $f : X \rightarrow Y$:

$$\eta_Y \circ F(f) = G(f) \circ \eta_X. \quad (2.1.2)$$

A natural isomorphism is a natural transformation $\eta : F \rightarrow G$ such that $\eta_X : F(X) \rightarrow G(X)$ is an isomorphism for every object $X \in \mathcal{C}$.

The collection of endofunctors of \mathcal{C} and natural transformations between them form a category, which is denoted by $\text{End}(\mathcal{C})$. Specifically, objects of $\text{End}(\mathcal{C})$ are endofunctors of \mathcal{C} and morphisms in $\text{Hom}_{\text{End}(\mathcal{C})}(F, G)$ are natural transformations from F to G . The composition of morphisms is given by the composition of natural transformations, which is defined by $(\xi \circ \eta)_X := \xi_X \circ \eta_X$ for any $\eta \in \text{Hom}_{\text{End}(\mathcal{C})}(F, G)$ and $\xi \in \text{Hom}_{\text{End}(\mathcal{C})}(G, H)$. The identity morphism of $F \in \text{End}(\mathcal{C})$ is the identity natural transformation $1_F : F \rightarrow F$ defined by $(1_F)_X := \text{id}_{F(X)}$.

Monoidal categories and monoidal functors. A category \mathcal{C} is called a monoidal category if it is equipped with a tensor product structure. The tensor product of objects X and Y is denoted by $X \otimes Y$, and the tensor product of morphisms $f \in \text{Hom}(X, Y)$ and $g \in \text{Hom}(X', Y')$ is denoted by $f \otimes g \in \text{Hom}(X \otimes X', Y \otimes Y')$. The tensor product has to be compatible with the composition of morphisms, namely,

$$(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k) \quad (2.1.3)$$

for any morphisms f, g, h , and k such that the left-hand side and right-hand side of the above equation make sense. In other words, the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a bifunctor. The tensor product has to be associative up to natural isomorphism called an associator $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$, which satisfies the following pentagon equation:

$$\alpha_{X,Y,Z \otimes W} \circ \alpha_{X \otimes Y, Z, W} = (\text{id}_X \otimes \alpha_{Y,Z,W}) \circ \alpha_{X,Y \otimes Z, W} \circ (\alpha_{X,Y,Z} \otimes \text{id}_W). \quad (2.1.4)$$

The above equation can also be written in terms of a commutative diagram as follows:¹

$$\begin{array}{ccc}
 & (X \otimes Y) \otimes (Z \otimes W) & \\
 \alpha_{X \otimes Y, Z, W} \nearrow & & \searrow \alpha_{X, Y, Z \otimes W} \\
 ((X \otimes Y) \otimes Z) \otimes W & & X \otimes (Y \otimes (Z \otimes W)) \quad (2.1.5) \\
 \alpha_{X, Y, Z} \otimes \text{id}_W \downarrow & & \uparrow \text{id}_X \otimes \alpha_{Y, Z, W} \\
 (X \otimes (Y \otimes Z)) \otimes W & \xrightarrow{\alpha_{X, Y \otimes Z, W}} & X \otimes ((Y \otimes Z) \otimes W)
 \end{array}$$

The unit object of a monoidal category \mathcal{C} is denoted by $1_{\mathcal{C}}$, which is a distinguished object equipped with natural isomorphisms $l_X : 1_{\mathcal{C}} \otimes X \rightarrow X$ and $r_X : X \otimes 1_{\mathcal{C}} \rightarrow X$ that satisfy the following unit axiom:

$$(\text{id}_X \otimes l_Y) \circ \alpha_{X, 1_{\mathcal{C}}, Y} = r_X \otimes \text{id}_Y. \quad (2.1.6)$$

The natural isomorphisms l and r are called the left unit isomorphism and right unit isomorphism respectively. The subscript \mathcal{C} of the unit object will be omitted when there is no confusion.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories \mathcal{C} and \mathcal{D} is called a monoidal functor if it preserves the monoidal structure. More specifically, a monoidal functor F is equipped with a natural isomorphism $J_{X,Y} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ and an isomorphism $\phi : 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}})$ that satisfy the following consistency conditions:

$$F(\alpha_{X,Y,Z}) \circ J_{X \otimes Y, Z} \circ (J_{X,Y} \otimes \text{id}_{F(Z)}) = J_{X, Y \otimes Z} \circ (\text{id}_{F(X)} \otimes J_{Y,Z}) \circ \alpha_{F(X), F(Y), F(Z)}, \quad (2.1.7)$$

$$\begin{aligned}
 l_{F(X)}^{\mathcal{D}} &= F(l_X^{\mathcal{C}}) \circ J_{1_{\mathcal{C}}, X} \circ (\phi \otimes \text{id}_{F(X)}), \\
 r_{F(X)}^{\mathcal{D}} &= F(r_X^{\mathcal{C}}) \circ J_{X, 1_{\mathcal{C}}} \circ (\text{id}_{F(X)} \otimes \phi).
 \end{aligned} \quad (2.1.8)$$

Here, $l^{\mathcal{C}}$ and $r^{\mathcal{C}}$ are the left and right unit isomorphisms of \mathcal{C} , while $l^{\mathcal{D}}$ and $r^{\mathcal{D}}$ are the left and right unit isomorphisms of \mathcal{D} . Equation (2.1.7) is called the monoidal structure axiom.

A monoidal natural transformation is a natural transformation $\eta : F \rightarrow G$ that is compatible with the structure isomorphisms $J_{X,Y}^F : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ and $J_{X,Y}^G : G(X) \otimes G(Y) \rightarrow G(X \otimes Y)$ in the following sense:

$$\eta_{X \otimes Y} \circ J_{X,Y}^F = J_{X,Y}^G \circ (\eta_X \otimes \eta_Y). \quad (2.1.9)$$

A monoidal natural transformation is called a monoidal natural isomorphism if it is also a natural isomorphism.

The category $\text{End}(\mathcal{C})$ of endofunctors of \mathcal{C} is a monoidal category. The tensor product of object is given by the composition of functors, i.e., $F \otimes G := G \circ F$ for any functors $F, G \in \text{End}(\mathcal{C})$. Similarly, the tensor product of morphisms is given by the horizontal composition of natural transformations, which is defined by $(\eta \otimes \xi)_X := \xi_{G(X)} \circ F'(\eta_X)$ for any natural transformations $\eta \in \text{Hom}_{\text{End}(\mathcal{C})}(F, G)$ and $\xi \in \text{Hom}_{\text{End}(\mathcal{C})}(F', G')$. The associator $\alpha_{F,G,H} : (F \otimes G) \otimes H \rightarrow F \otimes (G \otimes H)$ is the identity natural transformation on $H \circ G \circ F$.

¹Consistency conditions on various structure isomorphisms that appear in the rest of this section can also be expressed in the form of commutative diagrams.

Fusion categories and tensor functors. A multifusion category is a monoidal category \mathcal{C} equipped with a direct sum structure in addition to the tensor product structure such that the following conditions are satisfied:

- (Finiteness.) There are only finitely many isomorphism classes of simple objects. Here, a simple object is an object whose endomorphism space is one-dimensional. Equivalently, a simple object is an object that cannot be decomposed into a direct sum of two non-zero objects.
- (Semisimplicity.) Every object $X \in \mathcal{C}$ is isomorphic to a finite direct sum of simple objects. In particular, the tensor product of two simple objects X and Y can be decomposed as $X \otimes Y \cong \bigoplus_Z N_{XY}^Z Z$, where the direct sum on the right-hand side is taken over all (isomorphism classes of) simple objects and N_{XY}^Z is a non-negative integer called a fusion coefficient. A simple object $Z \in \mathcal{C}$ is called a fusion channel of $X \otimes Y$ if $N_{XY}^Z \geq 1$.
- (\mathbb{C} -linearity.) Morphisms between any two objects form a finite dimensional \mathbb{C} -vector space, and the composition of morphisms is \mathbb{C} -linear. The tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is also bilinear in morphisms.
- (Rigidity.) Every object X has its left dual X^* and right dual *X together with a left evaluation morphism $\text{ev}_X^L : X^* \otimes X \rightarrow \mathbf{1}$, left coevaluation morphism $\text{coev}_X^R : \mathbf{1} \rightarrow X \otimes X^*$, right evaluation morphism $\text{ev}_X^R : X \otimes {}^*X \rightarrow \mathbf{1}$, and right coevaluation morphism $\text{coev}_X^L : \mathbf{1} \rightarrow {}^*X \otimes X$. These morphisms satisfy the following consistency conditions:

$$\begin{aligned}
\text{id}_X &= r_X \circ (\text{id}_X \otimes \text{ev}_X^L) \circ \alpha_{X, X^*, X} \circ (\text{coev}_X^L \otimes \text{id}_X) \circ l_X^{-1}, \\
\text{id}_{X^*} &= l_{X^*} \circ (\text{ev}_X^L \otimes \text{id}_{X^*}) \circ \alpha_{X^*, X, X^*}^{-1} \circ (\text{id}_{X^*} \otimes \text{coev}_X^L) \circ r_{X^*}^{-1}, \\
\text{id}_X &= l_X \circ (\text{ev}_X^R \otimes \text{id}_X) \circ \alpha_{X, {}^*X, X}^{-1} \circ (\text{id}_X \otimes \text{coev}_X^R) \circ r_X^{-1}, \\
\text{id}_{{}^*X} &= r_{{}^*X} \circ (\text{id}_{{}^*X} \otimes \text{ev}_X^R) \circ \alpha_{{}^*X, X, {}^*X} \circ (\text{coev}_X^R \otimes \text{id}_{{}^*X}) \circ l_{{}^*X}.
\end{aligned} \tag{2.1.10}$$

The left dual of the right dual of X is X , i.e., $({}^*X)^* = X$. Similarly, the right dual of the left dual of X is also X , i.e., $*(X^*) = X$.

Precisely, a multifusion category is a finite semisimple \mathbb{C} -linear abelian rigid monoidal category. A fusion category is a multifusion category whose unit object is simple. A \mathbb{C} -linear monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between (multi)fusion categories \mathcal{C} and \mathcal{D} is called a tensor functor. In particular, a tensor functor from a fusion category \mathcal{C} to the category Vec of finite dimensional vector spaces is called a fiber functor. We will give a more detailed definition of Vec at the end of this subsection.

A fusion category is said to be pivotal if it is equipped with a monoidal natural isomorphism $a_X : X \rightarrow X^{**}$ called a pivotal structure, which enables us to identify the left dual X^* with the right dual *X . Each object of a pivotal fusion category is associated with a complex number called the quantum dimension, which is defined by

$$\dim(X) := \text{ev}_{X^*}^L \circ (a_X \otimes \text{id}_{X^*}) \circ \text{coev}_X^L \in \text{End}(\mathbf{1}) = \mathbb{C}. \tag{2.1.11}$$

A pivotal fusion category is said to be spherical if the quantum dimension of X is equal to that of X^* for all objects X . The pivotal structure on a spherical fusion category is called a spherical structure.

A fusion category \mathcal{C} is said to be unitary if it is equipped with an adjoint $\dagger : \mathcal{C} \rightarrow \mathcal{C}$ that satisfies the conditions listed below (see, e.g., section 2.1 of [87] for the definition of a unitary fusion category).

- The adjoint of an object $X \in \mathcal{C}$ is X itself.
- The adjoint of a morphism $f \in \text{Hom}(X, Y)$ is a morphism $f^\dagger \in \text{Hom}(Y, X)$.
- The adjoint is involutive, anti-linear in morphisms, and compatible with the composition of morphisms, i.e.,

$$f^{\dagger\dagger} = f, \quad (\lambda f)^\dagger = \lambda^* f^\dagger, \quad (g \circ f)^\dagger = g^\dagger \circ f^\dagger, \quad \text{id}_X^\dagger = \text{id}_X, \quad (2.1.12)$$

where λ^* is the complex conjugate of $\lambda \in \mathbb{C}$.

- The adjoint is compatible with the tensor product of morphisms:

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger. \quad (2.1.13)$$

- The Hom space $\text{Hom}(X, Y)$ is a Hilbert space equipped with a norm that satisfies

$$\|g \circ f\| \leq \|g\| \|f\|, \quad \|f\|^2 = \|f^\dagger \circ f\|. \quad (2.1.14)$$

- The associator and unit isomorphisms are unitary:

$$\alpha_{X,Y,Z}^\dagger = \alpha_{X,Y,Z}^{-1}, \quad l_X^\dagger = l_X^{-1}, \quad r_X^\dagger = r_X^{-1}. \quad (2.1.15)$$

For any object X of a unitary fusion category \mathcal{C} , we can identify the left dual X^* and the right dual *X by a canonical spherical structure $a_{*X} : {}^*X \rightarrow X^*$ defined by

$$a_{*X} = r_{X^*} \circ (\text{id}_{X^*} \otimes \text{ev}_{*X}^L) \circ \alpha_{X^*,X,*X} \circ ((\text{ev}_X^L)^\dagger \otimes \text{id}_{*X}) \circ l_{*X}^{-1}. \quad (2.1.16)$$

With this identification of X^* and *X , we can think of the right evaluation and coevaluation morphisms as the adjoint of the left coevaluation and evaluation morphisms, i.e., we have $\text{ev}_X^R = (\text{coev}_X^L)^\dagger$ and $\text{coev}_X^R = (\text{ev}_X^L)^\dagger$. The quantum dimensions with respect to the above spherical structure turn out to be positive real numbers greater than one.

Physically, fusion categories describe finite generalized symmetries of 1+1d bosonic systems. In this context, objects and morphisms of a fusion category label topological lines and topological junctions respectively. Taking the dual of an object amounts to reversing the orientation of a topological line. The tensor product of objects corresponds to the fusion of topological lines, while the direct sum of objects corresponds to the superposition of topological lines. This physical interpretation will become clearer in Section 2.5 where we introduce the string diagram representations of objects and morphisms.

Braided and symmetric fusion categories A braided fusion category is a fusion category equipped with a natural isomorphism $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ that satisfies the following consistency conditions known as the hexagon equations:

$$\begin{aligned} \alpha_{Y,Z,X} \circ c_{X,Y \otimes Z} \circ \alpha_{X,Y,Z} &= (\text{id}_Y \otimes c_{X,Z}) \circ \alpha_{Y,X,Z} \circ (c_{X,Y} \otimes \text{id}_Z), \\ \alpha_{Z,X,Y}^{-1} \circ c_{X \otimes Y,Z} \circ \alpha_{X,Y,Z}^{-1} &= (c_{X,Z} \otimes \text{id}_Y) \circ \alpha_{X,Z,Y}^{-1} \circ (\text{id}_X \otimes c_{Y,Z}). \end{aligned} \quad (2.1.17)$$

A braided fusion category is called a symmetric fusion category if its braiding satisfies $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ for all objects X and Y .

Example: Vec . The simplest example of a fusion category is the category Vec of finite dimensional \mathbb{C} -vector spaces. As the name suggests, objects of Vec are finite dimensional \mathbb{C} -vector spaces, and morphisms are \mathbb{C} -linear maps between vector spaces. The tensor product and direct sum structures on Vec are given by the usual tensor product and direct sum of vector spaces. The structure isomorphisms such as the associator and unit isomorphisms are all trivial. The unit object of Vec is a one-dimensional vector space \mathbb{C} , which is the unique simple object of Vec up to isomorphism. The dual of an object V is the dual vector space V^* . The evaluation and coevaluation morphisms are given by the standard pairing and embedding, namely, for any $\phi \in V^*$, $v \in V$, and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} \text{ev}_V^L(\phi \otimes v) &= \phi(v), & \text{coev}_V^L(\lambda) &= \lambda \sum_i v_i \otimes v^i, \\ \text{ev}_V^R(v \otimes \phi) &= \phi(v), & \text{coev}_V^R(\lambda) &= \lambda \sum_i v^i \otimes v_i, \end{aligned} \quad (2.1.18)$$

where $\{v_i \in V \mid i = 1, 2, \dots, \dim(V)\}$ and $\{v^i \in V^* \mid i = 1, 2, \dots, \dim(V)\}$ are the dual bases that satisfy $v^i(v_j) = \delta_{ij}$. The left and right duals are implicitly identified in the above equations. We note that Vec is a symmetric fusion category equipped with the trivial braiding defined by $c_{\text{triv}}(v \otimes w) := w \otimes v$ for all $v \in V$ and $w \in W$.

Example: Vec_G . Another example of a fusion category is the category Vec_G of finite dimensional G -graded vector spaces. Objects and morphisms of this category are finite dimensional G -graded vector spaces and linear maps that preserve the G -grading. The tensor product of two objects $V = \bigoplus_g V_g$ and $W = \bigoplus_g W_g$ is given by $V \otimes W = \bigoplus_g (V \otimes W)_g$, where $(V \otimes W)_g = \bigoplus_h (V_{gh^{-1}} \otimes W_h)$. Similarly, the direct sum of V and W is defined by $V \oplus W = \bigoplus_g (V \oplus W)_g$, where $(V \oplus W)_g = V_g \oplus W_g$. A simple object V of Vec_G is a one-dimensional G -graded vector space, i.e., $V_g \cong \delta_{g,h} \mathbb{C}$ for some $h \in G$. Two simple objects are isomorphic to each other if and only if they have the same G -grading. In particular, the number of isomorphism classes of simple objects is equal to the order of G . The dual of a simple object graded by $h \in G$ is the one graded by h^{-1} . The structure morphisms such as the associator and unit morphisms are the obvious ones. We note that Vec_G reduces to Vec when G is the trivial group. Physically, the category Vec_G describes a non-anomalous finite group symmetry G .²

Example: $\text{Rep}(G)$. Finite dimensional representations of a finite group G form a fusion category denoted by $\text{Rep}(G)$. Objects and morphisms of this fusion category are finite dimensional representations and intertwiners respectively. The tensor product and direct sum of objects are the usual tensor product and direct sum of representations. Physically, $\text{Rep}(G)$ describes the symmetry generated by Wilson lines of finite G -gauge theories. We emphasize that $\text{Rep}(G)$ is not group-like when the gauge group G is non-abelian.

2.1.2 Module categories

A left module category over a fusion category \mathcal{C} , or a left \mathcal{C} -module category, is a category on which \mathcal{C} acts from the left. More specifically, a left \mathcal{C} -module category is a finite semisimple \mathbb{C} -linear category \mathcal{M} equipped with a bilinear bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ that is associative and unital up to natural isomorphisms $m_{X,Y,M} : (X \otimes Y) \overline{\otimes} M \rightarrow X \overline{\otimes} (Y \overline{\otimes} M)$ and $\ell_M :$

²An anomalous finite group symmetry is described by Vec_G^ω [11], which is the category of G -graded vector spaces with the associator twisted by a 3-cocycle $\omega \in H^3(G, \text{U}(1))$.

$1 \otimes M \rightarrow M$, where $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$. The associativity isomorphism $m_{X,Y,M}$ and unit isomorphism ℓ_M are required to satisfy the following consistency conditions:

$$\begin{aligned} m_{X,Y,Z \otimes M} \circ m_{X \otimes Y, Z, M} &= (\text{id}_X \otimes m_{Y,Z,M}) \circ m_{X,Y \otimes Z, M} \circ (\alpha_{X,Y,Z} \otimes \text{id}_M), \\ (\text{id}_X \otimes \ell_M) \circ m_{X,1,M} &= r_X \otimes \text{id}_M. \end{aligned} \quad (2.1.19)$$

A right module category over a fusion category is also defined similarly. In what follows, a module category means a left module category unless otherwise stated.

A \mathcal{C} -module category can equivalently be defined as a finite semisimple \mathbb{C} -linear category \mathcal{M} that is equipped with a tensor functor $F : \mathcal{C} \rightarrow \text{End}(\mathcal{M})$, where $\text{End}(\mathcal{M})$ is the category of endofunctors of \mathcal{M} . Indeed, given a tensor functor $F : \mathcal{C} \rightarrow \text{End}(\mathcal{M})$ with the structure isomorphisms $J_{X,Y} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ and $\phi : 1_{\mathcal{M}} \rightarrow F(1_{\mathcal{C}})$, we can endow \mathcal{M} with a \mathcal{C} -module category structure as follows: for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$, we define

$$X \otimes M := F(X)(M), \quad m_{X,Y,M} := (J_{X,Y}^{-1})_M, \quad \ell_M := \phi_M^{-1}. \quad (2.1.20)$$

Conversely, when \mathcal{M} is a \mathcal{C} -module category, we can define a tensor functor $F : \mathcal{C} \rightarrow \text{End}(\mathcal{M})$ by the same equation as above. Thus, a \mathcal{C} -module category structure on \mathcal{M} is represented by a tensor functor from \mathcal{C} to $\text{End}(\mathcal{M})$. This is analogous to the fact that the action of an algebra A on a module M is represented by an algebra homomorphism from A to the endomorphism algebra $\text{End}(M)$.

When \mathcal{M}_1 and \mathcal{M}_2 are module categories over \mathcal{C} , the direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ is also a \mathcal{C} -module category. A module category \mathcal{M} is said to be indecomposable if it cannot be decomposed into a direct sum of two non-zero module categories. Any semisimple module category is a finite direct sum of indecomposable module categories.

A \mathcal{D} -module category \mathcal{M} can be regarded as a \mathcal{C} -module category if there is a tensor functor $G : \mathcal{C} \rightarrow \mathcal{D}$. The \mathcal{C} -module action on \mathcal{M} is given by the composition of $G : \mathcal{C} \rightarrow \mathcal{D}$ and the left \mathcal{D} -module action $F : \mathcal{D} \rightarrow \text{End}(\mathcal{M})$. The composition $G \circ F : \mathcal{C} \rightarrow \text{End}(\mathcal{M})$ is a tensor functor because both F and G are tensor functors, and therefore $G \circ F$ represents a left \mathcal{C} -action on \mathcal{M} . The corresponding action bifunctor $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ is given by $X \otimes_{\mathcal{C}} M = F(X) \otimes_{\mathcal{D}} M$, where $X \in \mathcal{C}$, $M \in \mathcal{M}$, and $\otimes_{\mathcal{D}} : \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$ is the bifunctor representing the action of \mathcal{D} on \mathcal{M} . The associativity isomorphism $m^{\mathcal{C}}$ and unit isomorphism $\ell^{\mathcal{C}}$ of a \mathcal{C} -module category \mathcal{M} can be written in terms of those of a \mathcal{D} -module category \mathcal{M} and the structure isomorphisms of a tensor functor $G : \mathcal{C} \rightarrow \mathcal{D}$. Concretely, we have

$$m_{X,Y,M}^{\mathcal{C}} = m_{F(X),F(Y),M}^{\mathcal{D}} \circ (J_{X,Y}^{-1} \otimes_{\mathcal{D}} \text{id}_M), \quad \ell_M^{\mathcal{C}} = \ell_M^{\mathcal{D}} \circ (\phi^{-1} \otimes_{\mathcal{D}} \text{id}_M), \quad (2.1.21)$$

where $m^{\mathcal{D}}$ and $\ell^{\mathcal{D}}$ are the associativity and unit isomorphisms of the \mathcal{D} -module category \mathcal{M} , and J and ϕ are structure isomorphisms of the tensor functor F .

2.2 Semisimple algebras and modules

In this section, we review the basic properties of semisimple algebras over \mathbb{C} and related notions such as modules and bimodules over an algebra. Semisimple algebras play a crucial role in the construction of bosonic topological field theories as we will see in Chapter 4. Unless otherwise stated, algebras discussed in this section are supposed to be finite dimensional.

2.2.1 Frobenius algebra structures on semisimple algebras

An associative unital algebra K over \mathbb{C} is a \mathbb{C} -vector space equipped with linear maps $m_K : K \otimes K \rightarrow K$ and $\eta_K : \mathbb{C} \rightarrow K$ that satisfy the following equalities:

$$m_K \circ (m_K \otimes \text{id}_K) = m_K \circ (\text{id}_K \otimes m_K), \quad m_K \circ (\eta_K \otimes \text{id}_K) = m_K \circ (\text{id}_K \otimes \eta_K) = \text{id}_K. \quad (2.2.1)$$

The linear maps m_K and η_K are called multiplication and unit of K respectively. In what follows, an associative unital algebra will be simply called an algebra.

A simple algebra over \mathbb{C} is an algebra isomorphic to a full matrix algebra $\text{End}(\mathbb{C}^n)$ for some $n \in \mathbb{Z}_{>0}$, which is an algebra of all $n \times n$ matrices. The multiplication on a matrix algebra $\text{End}(\mathbb{C}^n)$ is given by the usual matrix multiplication, and the unit of $\text{End}(\mathbb{C}^n)$ is a linear map that sends $\lambda \in \mathbb{C}$ to $\lambda I_n \in \text{End}(\mathbb{C}^n)$, where I_n is the $n \times n$ identity matrix. An algebra isomorphic to a direct sum of matrix algebras is called a semisimple algebra. Namely, a semisimple algebra K is an algebra of the form

$$K \cong \bigoplus_i \text{End}(\mathbb{C}^{n_i}), \quad n_i \in \mathbb{Z}_{>0}. \quad (2.2.2)$$

An algebra K is called a Frobenius algebra if it is equipped with linear maps $\Delta_K : K \rightarrow K \otimes K$ and $\epsilon_K : K \rightarrow \mathbb{C}$ that satisfy

$$(\Delta_K \otimes \text{id}_K) \circ \Delta_K = (\text{id}_K \otimes \Delta_K) \circ \Delta_K, \quad (\epsilon_K \otimes \text{id}_K) \circ \Delta_K = (\text{id}_K \otimes \epsilon_K) \circ \Delta_K, \quad (2.2.3)$$

$$(m_K \otimes \text{id}_K) \circ (\text{id}_K \otimes \Delta_K) = (\text{id}_K \otimes m_K) \circ (\Delta_K \otimes \text{id}_K) = \Delta_K \circ m_K. \quad (2.2.4)$$

The linear maps Δ_K and ϵ_K are called comultiplication and counit respectively. Equation (2.2.3) describes the coassociativity and counitality, while equation (2.2.4) is known as the Frobenius relation. A \mathbb{C} -vector space equipped with a comultiplication and a counit that satisfy eq. (2.2.3) is called a coassociative counital coalgebra, which we simply call a coalgebra. A Frobenius algebra is an algebra that is simultaneously a coalgebra whose multiplication and comultiplication satisfy the Frobenius relation (2.2.4).

A Frobenius algebra K is said to be Δ -separable if the comultiplication Δ_K and the multiplication m_K satisfy

$$m_K \circ \Delta_K = \text{id}_K. \quad (2.2.5)$$

A Δ -separable Frobenius algebra is also called a special Frobenius algebra in the literature [15]. A Frobenius algebra K is said to be symmetric if $\epsilon_K \circ m_K : K \otimes K \rightarrow \mathbb{C}$ and $\Delta_K \circ \eta_K : \mathbb{C} \rightarrow K \otimes K$ are symmetric under the exchange of two copies of K :

$$\epsilon_K \circ m_K = \epsilon_K \circ m_K \circ c_{\text{triv}}, \quad \Delta_K \circ \eta_K = c_{\text{triv}} \circ \Delta_K \circ \eta_K, \quad (2.2.6)$$

where $c_{\text{triv}} : K \otimes K \rightarrow K \otimes K$ is the trivial braiding of K .

Every semisimple algebra (K, m_K, η_K) over \mathbb{C} turns out to be a Δ -separable symmetric Frobenius algebra [15, 88]. The comultiplication and counit of K are defined by

$$\Delta_K := (\text{id}_K \otimes m_K) \circ (\text{id}_K \otimes \Phi^{-1} \otimes \text{id}_K) \circ (\text{coev}_K^L \otimes \text{id}_K), \quad (2.2.7)$$

$$\epsilon_K := \text{ev}_K^R \circ (m_K \otimes \text{id}_{K^*}) \circ (\text{id}_K \otimes \text{coev}_K^L), \quad (2.2.8)$$

where ev_K^R and coev_K^L are the evaluation and coevaluation morphisms defined in eq. (2.1.18), and $\Phi : K \rightarrow K^*$ is an algebra isomorphism defined by

$$\Phi = ((\epsilon_K \circ m_K) \otimes \text{id}_{K^*}) \circ (\text{id}_K \otimes \text{coev}_K^L). \quad (2.2.9)$$

The Δ -separable symmetric Frobenius algebra structure on a semisimple algebra is unique [15].

2.2.2 Modules and bimodules

A left module over an algebra K is a vector space M equipped with a left K -action $\rho^L : K \otimes M \rightarrow M$ that is compatible with the algebra structure on K . More concretely, the left K -action ρ^L on M and the structure maps (m_K, η_K) of the algebra K satisfy

$$\rho^L \circ (\text{id}_K \otimes \rho^L) = \rho^L \circ (m_K \otimes \text{id}_M), \quad \rho^L \circ (\eta_K \otimes \text{id}_M) = \text{id}_M. \quad (2.2.10)$$

Equivalently, a left K -module M is a vector space equipped with an algebra homomorphism from K to $\text{End}(M)$. A linear map $f : M \rightarrow N$ between left K -modules M and N is called a left K -module map if it commutes with the left K -action, i.e., $\rho_N^L \circ (\text{id}_K \otimes f) = f \circ \rho_M^L$, where ρ_M^L and ρ_N^L are the left K -actions on M and N respectively.

A left module M over a semisimple algebra K is said to be simple if there is no non-zero module map from M to itself other than (the scalar multiple of) the identity map. A simple module over a simple algebra $\text{End}(\mathbb{C}^n)$ is isomorphic to \mathbb{C}^n , which is unique up to isomorphism. The action of $\text{End}(\mathbb{C}^n)$ on \mathbb{C}^n is defined by $\rho(a \otimes v) = a(v) \in \mathbb{C}^n$ for $a \in \text{End}(\mathbb{C}^n)$ and $v \in \mathbb{C}^n$. More generally, a semisimple algebra K of the form (2.2.2) has a simple module \mathbb{C}^{n_j} for each direct sum component $\text{End}(\mathbb{C}^{n_j})$: the action of $\text{End}(\mathbb{C}^{n_j})$ on \mathbb{C}^{n_j} is the same as above, while the actions of the other direct sum components are zero. In particular, the number of isomorphism classes of simple modules over K is equal to the number of direct sum components of K . A left K -module is called a semisimple module if it is isomorphic to a finite direct sum of simple modules.

A right module over an algebra K is a vector space M on which K acts from the right. The right K -action on M is denoted by $\rho^R : M \otimes K \rightarrow K$, which is a linear map that is compatible with the algebra structure on K , i.e.,

$$\rho^R \circ (\rho^R \otimes \text{id}_K) = \rho^R \circ (\text{id}_M \otimes m_K), \quad \rho^R \circ (\text{id}_M \otimes \eta_K) = \text{id}_M. \quad (2.2.11)$$

A right K -module map $f : M \rightarrow N$ between right K -modules M and N is a linear map that satisfies $\rho_N^R \circ (f \otimes \text{id}_K) = f \circ \rho_M^R$, where ρ_M^R and ρ_N^R denote the right K -actions on M and N . When M is a left K -module, its dual M^* is a right K -module. The right K -action $\rho_{M^*}^R$ on M^* is given by

$$\rho_{M^*}^R = (\text{ev}_M^L \otimes \text{id}_{M^*}) \circ (\text{id}_{M^*} \otimes \rho_M^L \otimes \text{id}_{M^*}) \circ (\text{id}_{M^*} \otimes \text{id}_K \otimes \text{coev}_M^L), \quad (2.2.12)$$

where ρ_M^L denotes the left K -action on M and the evaluation and coevaluation maps are given by eq. (2.1.18).

A left K -module that is simultaneously a right K -module is called a (K, K) -bimodule if the left K -action ρ^L and the right K -action ρ^R are compatible with each other in the following sense:

$$\rho^L \circ (\text{id}_K \otimes \rho^R) = \rho^R \circ (\rho^L \otimes \text{id}_K). \quad (2.2.13)$$

We can also define a (K_1, K_2) -bimodule similarly for two different algebras K_1 and K_2 . A bimodule map $f : M \rightarrow N$ between (K_1, K_2) -bimodules M and N is a linear map that is a left K_1 module map and a right K_2 -module map at the same time.

The dual notion of a module is known as a comodule. Specifically, a left comodule over a coalgebra K is defined as a vector space M equipped with a left K -coaction $\lambda^L : M \rightarrow K \otimes M$ that satisfies

$$(\text{id}_K \otimes \lambda^L) \circ \lambda^L = (\Delta_K \otimes \text{id}_M) \circ \lambda^L, \quad (\epsilon_K \otimes \text{id}_M) \circ \lambda^L = \text{id}_M, \quad (2.2.14)$$

where Δ_K and ϵ_K denote the comultiplication and counit of the coalgebra K . Right comodules and bicomodules are also defined similarly. We note that a left module M over a Frobenius algebra K is also a left K -comodule with the left K -coaction λ^L given by

$$\lambda^L = (\text{id}_K \otimes \rho^L) \circ ((\Delta_K \circ \eta_K) \otimes \text{id}_M). \quad (2.2.15)$$

The same holds for the right modules and bicomodules. The superscripts L and R will be omitted when they are clear from the context.

Category of bimodules. Semisimple bimodules over a finite dimensional semisimple algebra K form a multifusion category ${}_K\mathcal{M}_K$ called the category of (K, K) -bimodules. We review this category in some detail for later convenience. The objects and morphisms of ${}_K\mathcal{M}_K$ are (K, K) -bimodules and (K, K) -bimodule maps respectively. The monoidal structure on ${}_K\mathcal{M}_K$ is given by the tensor product over K , which is usually denoted by \otimes_K . Specifically, the tensor product $X_1 \otimes_K X_2$ of (K, K) -bimodules X_1 and X_2 is the image of a linear map $p_{X_1, X_2} : X_1 \otimes X_2 \rightarrow X_1 \otimes_K X_2$ defined by

$$p_{X_1, X_2} = (\rho_1^R \otimes \rho_2^L) \circ (\text{id}_{X_1} \otimes (\Delta_K \circ \eta_K) \otimes \text{id}_{X_2}), \quad (2.2.16)$$

which is a projector, i.e., $p_{X_1, X_2}^2 = p_{X_1, X_2}$, due to eqs. (2.2.4) and (2.2.5). We note that the unit object for the tensor product over K is K itself viewed as a regular (K, K) -bimodule. The splitting maps of the projector (2.2.16) are denoted by $\pi_{X_1, X_2} : X_1 \otimes X_2 \rightarrow X_1 \otimes_K X_2$ and $\iota_{X_1, X_2} : X_1 \otimes_K X_2 \rightarrow X_1 \otimes X_2$, which obey $\iota_{X_1, X_2} \circ \pi_{X_1, X_2} = p_{X_1, X_2}$ and $\pi_{X_1, X_2} \circ \iota_{X_1, X_2} = \text{id}_{X_1 \otimes_K X_2}$. The associator $\alpha_{X_1, X_2, X_3} : (X_1 \otimes_K X_2) \otimes_K X_3 \rightarrow X_1 \otimes_K (X_2 \otimes_K X_3)$ is given by a composition of these splitting maps as follows:

$$\alpha_{X_1, X_2, X_3} = \pi_{X_1, X_2 \otimes_K X_3} \circ (\text{id}_{X_1} \otimes \pi_{X_2, X_3}) \circ (\iota_{X_1, X_2} \otimes \text{id}_{X_3}) \circ \iota_{X_1 \otimes_K X_2, X_3}. \quad (2.2.17)$$

The tensor product of morphisms $f \in \text{Hom}_{KK}(X_1, X'_1)$ and $g \in \text{Hom}_{KK}(X_2, X'_2)$ is defined in terms of the splitting maps as $f \otimes_K g := \pi_{X'_1, X'_2} \circ (f \otimes g) \circ \iota_{X_1, X_2}$, where $\text{Hom}_{KK}(X, X')$ denotes the space of (K, K) -bimodule maps from X to X' .

We notice that the category ${}_K\mathcal{M}$ of left K -modules is a ${}_K\mathcal{M}_K$ -module category. The category ${}_K\mathcal{M}_K$ acts on ${}_K\mathcal{M}$ by the tensor product over K , that is, $X \otimes_K M := X \otimes_K M$ for any $X \in {}_K\mathcal{M}_K$ and $M \in {}_K\mathcal{M}$. The module associativity isomorphism $m_{X_1, X_2, M} : (X_1 \otimes_K X_2) \otimes_K M \rightarrow X_1 \otimes_K (X_2 \otimes_K M)$ for $X_1, X_2 \in {}_K\mathcal{M}_K$ and $M \in {}_K\mathcal{M}$ is given by the composition of the splitting maps as follows:

$$m_{X_1, X_2, M} = \pi_{X_1, X_2 \otimes_K M} \circ (\text{id}_{X_1} \otimes \pi_{X_2, M}) \circ (\iota_{X_1, X_2} \otimes \text{id}_M) \circ \iota_{X_1 \otimes_K X_2, M}. \quad (2.2.18)$$

2.3 Hopf algebras

Hopf algebras play a pivotal role in the study of fusion categories because any fusion category that admits a fiber functor is equivalent to the category of representations of a Hopf algebra. Indeed, given a fusion category \mathcal{C} equipped with a fiber functor, we can reconstruct a Hopf algebra H whose representation category is equivalent to \mathcal{C} , see, e.g., chapter 5 of [29]. Physically, the representation categories of Hopf algebras describe non-anomalous finite generalized symmetries in 1+1d bosonic systems [13]. In this section, we recall the definition of Hopf algebras and their representation categories. Unless otherwise stated, we will only consider finite dimensional semisimple Hopf algebras.

Definition. Let H be an associative unital algebra that is also a coassociative counital coalgebra and is equipped with a linear map $S : H \rightarrow H$ called an antipode. The multiplication and the unit of H are denoted by $m : H \otimes H \rightarrow H$ and $\eta : \mathbb{C} \rightarrow H$. Similarly, the comultiplication and the counit of H are denoted by $\Delta : H \rightarrow H \otimes H$ and $\epsilon : H \rightarrow \mathbb{C}$. The algebra H is called a Hopf algebra if the structure maps $(m, \eta, \Delta, \epsilon, S)$ satisfy the following conditions [89]:

- The comultiplication Δ is a unit-preserving algebra homomorphism, i.e.

$$\Delta \circ m = (m \otimes m) \circ (\text{id} \otimes c_{\text{triv}} \otimes \text{id}) \circ (\Delta \otimes \Delta), \quad \Delta \circ \eta = \eta \otimes \eta, \quad (2.3.1)$$

where $c_{\text{triv}} : H \otimes H \rightarrow H \otimes H$ is the trivial braiding.

- The counit ϵ is a unit-preserving algebra homomorphism, i.e.

$$\epsilon \circ m = \epsilon \otimes \epsilon, \quad \epsilon \circ \eta(1) = 1. \quad (2.3.2)$$

- The antipode S satisfies the following antipode axiom:

$$m \circ (\text{id} \otimes S) \circ \Delta = m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon. \quad (2.3.3)$$

We refer the reader to Section 2.5.2 for the string diagram representations of the above equations.

The antipode S turns out to be an algebra and coalgebra anti-homomorphism. Furthermore, the antipode of a semisimple Hopf algebra squares to the identity, i.e. $S^2 = \text{id}$. Every semisimple (weak) Hopf algebra over \mathbb{C} is finite dimensional [90].

We note that the dual vector space H^* of a finite dimensional Hopf algebra H is also a Hopf algebra. The structure maps of the dual Hopf algebra H^* are given by the duals of the structure maps of the original Hopf algebra H . More specifically, if we denote the structure maps of H^* by $(m_{H^*}, \eta_{H^*}, \Delta_{H^*}, \epsilon_{H^*}, S_{H^*})$, we have

$$m_{H^*} = \Delta^*, \quad \eta_{H^*} = \epsilon^*, \quad \Delta_{H^*} = m^*, \quad \epsilon_{H^*} = \eta^*, \quad S_{H^*} = S^*. \quad (2.3.4)$$

Here, the dual $f^* : Y^* \rightarrow X^*$ of a linear map $f : X \rightarrow Y$ is defined by $f^* := (\text{ev}_Y^L \otimes \text{id}_{X^*}) \circ (\text{id}_{Y^*} \otimes f \otimes \text{id}_{X^*}) \circ (\text{id}_{Y^*} \otimes \text{coev}_X^L)$.

Representation categories. The category of representations of a semisimple Hopf algebra H is a fusion category, which is denoted by $\text{Rep}(H)$. The tensor product of representations V and W is defined by the usual tensor product $V \otimes W$ equipped with the H -action $\rho_{V \otimes W} : H \otimes (V \otimes W) \rightarrow V \otimes W$ given as follows:

$$\rho_{V \otimes W} = (\rho_V \otimes \rho_W) \circ (\text{id}_H \otimes c_{\text{triv}} \otimes \text{id}_W) \circ (\Delta \otimes \text{id}_{V \otimes W}), \quad (2.3.5)$$

where ρ_V and ρ_W denote the H -actions on V and W respectively. The action of $h \in H$ on $v \otimes w \in V \otimes W$ is sometimes written as $\Delta(h) \cdot v \otimes w := \rho_{V \otimes W}(h)(v \otimes w)$. The unit object is a one-dimensional representation \mathbb{C} on which H acts by the formula $h \cdot v = \epsilon(h)v$ for any $h \in H$ and $v \in \mathbb{C}$. The dual of a representation $V \in \text{Rep}(H)$ is the dual vector space V^* . The action of H on the dual representation V^* is defined by

$$\rho_{V^*} = (\text{ev}_V^L \otimes \text{id}_{V^*}) \circ (\text{id}_{V^*} \otimes \rho_V \otimes \text{id}_{V^*}) \circ (c_{\text{triv}} \otimes \text{id}_V \otimes \text{id}_{V^*}) \circ (S \otimes \text{id}_{V^*} \otimes \text{coev}_V^L). \quad (2.3.6)$$

The other structure isomorphisms of $\text{Rep}(H)$ are trivial. For later convenience, we mention that a left H -module V is also equipped with a left $(H^*)^{\text{cop}}$ -comodule structure $\lambda_V : V \rightarrow (H^*)^{\text{cop}} \otimes V$ defined by

$$\lambda_V := (S^* \otimes \rho_V) \circ (\text{coev}_H^R \otimes \text{id}_V). \quad (2.3.7)$$

Here, $(H^*)^{\text{cop}}$ denotes the coopposite coalgebra of H^* ,³ which is also a Hopf algebra.

Any indecomposable module category over $\text{Rep}(H)$ is equivalent to the category ${}_K\mathcal{M}$ of left K -modules, where K is an indecomposable left H -comodule algebra [91]. Here, a left H -comodule algebra K is an algebra equipped with a left H -comodule action $\lambda_K^L : K \rightarrow H \otimes K$ that is compatible with the multiplication $m_K : K \otimes K \rightarrow K$ and the unit $\eta_K : \mathbb{C} \rightarrow K$. More specifically, an algebra (K, m_K, η_K) equipped with an H -comodule action λ_K^L is called an H -comodule algebra if the linear maps λ_K^L , m_K , and η_K satisfy

$$\lambda_K^L \circ m_K = (m \otimes m_K) \circ (\text{id}_H \otimes c_{\text{triv}} \otimes \text{id}_K) \circ (\lambda_K^L \otimes \lambda_K^L), \quad \lambda_K^L \circ \eta_K = \eta \otimes \eta_K, \quad (2.3.8)$$

where m and η are the multiplication and the unit of a Hopf algebra H . The action of a left H -module $V \in \text{Rep}(H)$ on a left K -module $M \in {}_K\mathcal{M}$ is denoted by $V \overline{\otimes} M \in {}_K\mathcal{M}$. The underlying vector space of $V \overline{\otimes} M$ is the usual tensor product $V \otimes M$ and the left K -action on $V \overline{\otimes} M$ is defined in a way analogous to eq. (2.3.5):

$$\rho_{V \overline{\otimes} M}^L = (\rho_V \otimes \rho_M^L) \circ (\text{id}_H \otimes c_{\text{triv}} \otimes \text{id}_M) \circ (\lambda_K^L \otimes \text{id}_{V \overline{\otimes} M}), \quad (2.3.9)$$

where $\rho_M^L : K \otimes M \rightarrow M$ and $\rho_{V \overline{\otimes} M}^L : K \otimes (V \overline{\otimes} M) \rightarrow V \overline{\otimes} M$ denote the left K -actions on M and $V \overline{\otimes} M$ respectively. The $\text{Rep}(H)$ -module category structure on ${}_K\mathcal{M}$ is encoded in a tensor functor $F_K : \text{Rep}(H) \rightarrow \text{End}({}_K\mathcal{M}) \cong {}_K\mathcal{M}_K$. This tensor functor maps a left H -module $V \in \text{Rep}(H)$ to a (K, K) -bimodule $V \overline{\otimes} K \in {}_K\mathcal{M}_K$, where the right K -module action on $V \overline{\otimes} K$ is given by the multiplication of K from the right. The equivalence $\text{End}({}_K\mathcal{M}) \cong {}_K\mathcal{M}_K$ follows from the Eilenberg-Watts theorem [92, 93]. We note that $F_K(V) \otimes_K M \cong V \overline{\otimes} M$.

Example: the group algebra. The first example of a Hopf algebra is the group algebra $\mathbb{C}[G]$ for a finite group G . We denote a basis of $\mathbb{C}[G]$ by $\{g \mid g \in G\}$. The structure maps of this Hopf algebra are given by

$$m(g \otimes h) = gh, \quad \eta(\lambda) = \lambda e, \quad \Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}, \quad (2.3.10)$$

where $\lambda \in \mathbb{C}$ is an arbitrary complex number and $e \in G$ is the unit element. The representation category of this Hopf algebra is $\text{Rep}(G)$.

Example: the dual group algebra. The coopposite dual group algebra $(\mathbb{C}[G]^*)^{\text{cop}}$ is another example of a Hopf algebra. Equation (2.3.10) implies that the structure maps of $(\mathbb{C}[G]^*)^{\text{cop}}$ are given by

$$m(\widehat{g} \otimes \widehat{h}) = \delta_{g,h} \widehat{g}, \quad \eta(\lambda) = \sum_{g \in G} \widehat{g}, \quad \Delta(\widehat{g}) = \sum_{h \in G} \widehat{gh^{-1}} \otimes \widehat{h}, \quad \epsilon(\widehat{g}) = \delta_{g,e}, \quad S(\widehat{g}) = \widehat{g^{-1}}, \quad (2.3.11)$$

where $\{\widehat{g} \mid g \in G\}$ denotes the dual basis of $(\mathbb{C}[G]^*)^{\text{cop}}$ satisfying $\widehat{g}(h) = \delta_{g,h}$. Representations of this Hopf algebra form the category of G -graded vector spaces, i.e., we have an equivalence between $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ and Vec_G . See also Section 5.3.1 for more details on this category.

³In general, the coopposite coalgebra A^{cop} is defined by replacing the comultiplication Δ of the original coalgebra A by $c_{\text{triv}} \circ \Delta$.

2.4 Weak Hopf algebras

Although Hopf algebras provide us with a large class of fusion categories, not every fusion category can be realized by the representation category of a Hopf algebra. In general, a fusion category is equivalent to the category of representations of a weak Hopf algebra [94, 95], which is a generalization of a Hopf algebra [96]. Physically, fusion categories that do not come from Hopf algebras describe anomalous finite symmetries of 1+1d bosonic systems [13]. In this section, we review the definition and representation categories of weak Hopf algebras.

Definition. Weak Hopf algebras are defined by relaxing the defining properties of ordinary Hopf algebras. Specifically, a weak Hopf algebra H is an associative unital algebra (H, m, η) equipped with a structure of a coassociative counital coalgebra (H, Δ, ϵ) and an antipode $S : H \rightarrow H$ that satisfy the following properties [96]:

- The comultiplication Δ is multiplicative, i.e.

$$\Delta \circ m = (m \otimes m) \circ (\text{id} \otimes c_{\text{triv}} \otimes \text{id}) \circ (\Delta \otimes \Delta). \quad (2.4.1)$$

- The counit ϵ satisfies

$$\epsilon \circ m \circ (m \otimes \text{id}) = (\epsilon \otimes \epsilon) \circ (m \otimes m) \circ (\text{id} \otimes \Delta \otimes \text{id}) = (\epsilon \otimes \epsilon) \circ (m \otimes m) \circ (\text{id} \otimes (c_{\text{triv}} \circ \Delta) \otimes \text{id}). \quad (2.4.2)$$

- The unit η satisfies

$$(\Delta \otimes \text{id}) \circ \Delta \circ \eta = (\text{id} \otimes m \otimes \text{id}) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta) = (\text{id} \otimes (m \circ c_{\text{triv}}) \otimes \text{id}) \circ (\Delta \otimes \Delta) \circ (\eta \otimes \eta). \quad (2.4.3)$$

- The antipode S satisfies

$$\begin{aligned} m \circ (\text{id} \otimes S) \circ \Delta &= ((\epsilon \circ m) \otimes \text{id}) \circ (\text{id} \otimes c_{\text{triv}}) \circ ((\Delta \circ \eta) \otimes \text{id}), \\ m \circ (S \otimes \text{id}) \circ \Delta &= (\text{id} \otimes (\epsilon \circ m)) \circ (c_{\text{triv}} \otimes \text{id}) \circ (\text{id} \otimes (\Delta \circ \eta)), \\ S &= m \circ (m \otimes \text{id}) \circ (S \otimes \text{id} \otimes S) \circ (\Delta \otimes \text{id}) \circ \Delta. \end{aligned} \quad (2.4.4)$$

The string diagram representations of the above equations will be provided in Section 2.5.3.

The antipode S of a weak Hopf algebra is an algebra and coalgebra homomorphism. The linear maps defined by the first and the second equations of eq. (2.4.4) are called the target counital map ϵ_t and the source counital map ϵ_s respectively. These counital maps are idempotents, i.e. we have $\epsilon_t \circ \epsilon_t = \epsilon_t$ and $\epsilon_s \circ \epsilon_s = \epsilon_s$. The images of these idempotents are called the target and source counital subalgebras, which are denoted by H_t and H_s respectively. When H is a Hopf algebra, the source and target counital subalgebras are trivial: $H_s = H_t \cong \mathbb{C}$. The dual H^* of a finite dimensional weak Hopf algebra H is also a weak Hopf algebra with the structure maps defined as in eq. (2.3.4).

Representation categories. Representations of a semisimple weak Hopf algebra H form a multifusion category $\text{Rep}(H)$ [97]. Conversely, any multifusion category is equivalent to the representation category of an appropriate weak Hopf algebra H [94, 95]. The category $\text{Rep}(H)$ becomes a fusion category if and only if $Z(H) \cap H_t = \mathbb{C}$, where $Z(H)$ is the center of H [90]. The tensor product $V \boxtimes W$ of representations $V, W \in \text{Rep}(H)$ is a subspace of $V \otimes W$ on which

the unit element $\eta(1) \in H$ acts as the identity. More specifically, the underlying vector space of $V \boxtimes W$ is the image of the action of $\eta(1)$ on $V \otimes W$, that is, $V \boxtimes W = \Delta(\eta(1)) \cdot V \otimes W$, where the action of H on $V \otimes W$ is defined by eq. (2.3.5).⁴ The action of H on $V \boxtimes W$ is given by the restriction of the H -action (2.3.5) on $V \otimes W$. The unit object of $\text{Rep}(H)$ is the target counital subalgebra H_t , where H acts on H_t by the multiplication followed by the projection to H_t , i.e., $h \cdot v = \epsilon_t(hv)$ for all $h \in H$ and $v \in H_t$. The dual of a representation $V \in \text{Rep}(H)$ is the dual vector space V^* on which H acts by eq. (2.3.6). The details of evaluation and coevaluation morphisms and the other structure isomorphisms of $\text{Rep}(H)$ can be found in [97].

Any indecomposable module category over a unitary (multi)fusion category $\text{Rep}(H)$ can be written as the category ${}_K\mathcal{M}$ of left K -modules, where K is an indecomposable left H -comodule algebra [98]. Here, a left H -comodule algebra K for a weak Hopf algebra H equipped with structure maps $(m, \eta, \Delta, \epsilon, S)$ is an algebra equipped with a left H -comodule action $\lambda_K^L : K \rightarrow H \otimes K$ that is compatible with the algebra structure (K, m_K, η_K) in the following sense:

$$\lambda_K^L \circ m_K = (m \otimes m_K) \circ (\text{id}_H \otimes c_{\text{triv}} \otimes \text{id}_K) \circ (\lambda_K^L \otimes \lambda_K^L), \quad \lambda_K^L \circ \eta_K = (\epsilon_s \otimes \text{id}_K) \circ \lambda_K^L \circ \eta_K. \quad (2.4.5)$$

We note that this equation reduces to eq. (2.3.8) when H is a Hopf algebra. The action of $V \in \text{Rep}(H)$ on $M \in {}_K\mathcal{M}$ is denoted by $V \overline{\otimes} M \in {}_K\mathcal{M}$, which is the subspace of $V \otimes M$ on which the unit element $\eta_K(1) \in K$ acts as the identity. Namely, we define $V \overline{\otimes} M := \eta_K(1) \cdot V \otimes M$, where the action of K on $V \otimes M$ is given by eq. (2.3.9). The left K -action on $V \overline{\otimes} M$ is the restriction of the left K -action on $V \otimes M$. The tensor functor $F_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$ representing the $\text{Rep}(H)$ -module category structure on ${}_K\mathcal{M}$ maps a left H -module V to a (K, K) -bimodule $V \overline{\otimes} K$. In particular, we have an isomorphism of (K, K) -bimodules $F_K(V) \otimes_K M \cong V \overline{\otimes} M$.

2.5 String diagrams

For later convenience, we introduce the string diagram representations of morphisms in fusion categories.⁵ Based on the string diagram representation, we will give the definitions of Hopf algebras and weak Hopf algebras in general symmetric fusion categories.

2.5.1 String diagrams in fusion categories

A string diagram is a convenient tool to pictorially represent a morphism of a fusion category. In string diagrams, a morphism is typically expressed by a small dot or box sitting at the junction of strands labeled by the source and target objects. For example, a morphism $f : X \rightarrow Y$ is represented by the following string diagram:

$$\begin{array}{c} Y \\ \uparrow \\ f \bullet \\ \uparrow \\ X \end{array} \quad \text{or} \quad \begin{array}{c} Y \\ \uparrow \\ \boxed{f} \\ \uparrow \\ X \end{array}. \quad (2.5.1)$$

Here, the strand at the bottom is labeled by the source object, while the strand at the top is labeled by the target object, and we read the string diagram from the bottom to the top. We

⁴The vector space $V \boxtimes W$ can also be viewed as the tensor product of V and W over H_t .

⁵See [99] for an early use of string diagrams in monoidal categories.

often omit the arrows on the strands. The identity morphism is represented by a single strand without a black dot on it. Similarly, the unit object is represented by an invisible strand in string diagrams. The composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is represented by the concatenation of string diagrams for f and g , i.e.,

$$g \circ f \quad \begin{array}{c} Z \\ | \\ \bullet \\ | \\ X \end{array} = \begin{array}{c} Z \\ | \\ \bullet \quad g \\ | \\ \bullet \quad f \\ | \\ X \end{array}, \quad (2.5.2)$$

where the middle strand on the right-hand side is labeled by Y . The tensor product of morphisms $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ is represented by the juxtaposition of string diagrams:

$$f \otimes g \quad \begin{array}{c} Y \otimes Y' \\ | \\ \bullet \\ | \\ X \otimes X' \end{array} = \begin{array}{cc} Y & Y' \\ | & | \\ \bullet & \bullet \\ | & | \\ X & X' \end{array} g. \quad (2.5.3)$$

The relative height of f and g on the right-hand side does not matter because we have the equality $(f \otimes \text{id}_{Y'}) \circ (\text{id}_X \otimes g) = (\text{id}_Y \otimes g) \circ (f \otimes \text{id}_{X'})$ due to the compatibility (2.1.3) of the tensor product and composition of morphisms. Morphisms $f : Z \rightarrow X \otimes Y$ and $g : X \otimes Y \rightarrow Z$ are often represented by trivalent junctions as follows:

$$\begin{array}{c} X \quad Y \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ Z \end{array} \quad f, \quad \begin{array}{c} Z \\ | \\ \bullet \\ \diagup \quad \diagdown \\ X \quad Y \end{array} \quad g. \quad (2.5.4)$$

Physically, strands and their junctions can be interpreted as topological line defects and topological point-like defects respectively.

The structure morphisms of a fusion category can also be written in terms of string diagrams. For example, the data of the associator $\alpha_{XYZ} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ are encoded in the following diagrammatic equation:

$$\begin{array}{c} X \quad Y \quad Z \\ | \quad | \quad | \\ \boxed{\alpha_{XYZ}} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \mu \quad \nu \\ \diagdown \quad \diagup \\ U \quad V \\ | \\ W \end{array} = \sum_V \sum_{\rho, \sigma} (F_W^{XYZ})_{(U; \mu, \nu), (V; \rho, \sigma)} \begin{array}{c} X \quad Y \quad Z \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \rho \quad \sigma \\ \diagdown \quad \diagup \\ V \\ | \\ W \end{array}. \quad (2.5.5)$$

Here, the objects X, Y, Z, W, U , and V are supposed to be simple. The sum on the right-hand side is taken over all fusion channels V of $Y \otimes Z$ and basis vectors $\rho \in \text{Hom}(W, X \otimes V)$ and $\sigma \in \text{Hom}(V, Y \otimes Z)$. The complex numbers $(F_W^{XYZ})_{(U; \mu, \nu), (V; \rho, \sigma)}$ are called F -symbols [100], which satisfy the consistency conditions that originate from the pentagon equation (2.1.4). The associator on the left-hand side of eq. (2.5.5) is usually not written explicitly. This is justified by the Mac Lane strictness theorem, which states that every monoidal category is equivalent to the one whose associators are trivial [101].⁶

If the fusion category is braided, the braiding $c_{XY} : X \otimes Y \rightarrow Y \otimes X$ and the inverse braiding $c_{YX}^{-1} : X \otimes Y \rightarrow Y \otimes X$ are represented by

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ X \quad Y \end{array} \quad c_{XY}, \quad \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ X \quad Y \end{array} \quad c_{YX}^{-1}. \quad (2.5.6)$$

⁶This does not imply that the F -symbols are trivial.

In a symmetric fusion category, we do not need to specify which strand is above the other strand because c_{XY} and c_{YX}^{-1} are equal to each other. In this case, the braiding isomorphism c_{XY} is represented simply by a 4-valent junction as follows:

$$c_{XY} \quad \text{.} \quad (2.5.7)$$

2.5.2 Hopf algebra objects

In Section 2.3, we defined a Hopf algebra as a vector space H equipped with structure maps $(m, \eta, \Delta, \epsilon, S)$ that satisfy several conditions. More generally, we can define a Hopf algebra object in a general symmetric fusion category as an object equipped with structure morphisms that satisfy appropriate conditions. From this point of view, Hopf algebras defined in Section 2.3 can be understood as Hopf algebra objects in the category Vec of vector spaces.

Let \mathcal{C} be a symmetric fusion category. A Hopf algebra object $(H, m, \eta, \Delta, \epsilon, S)$ is an object $H \in \mathcal{C}$ equipped with morphisms $m : H \otimes H \rightarrow H$, $\eta : \mathbf{1} \rightarrow H$, $\Delta : H \rightarrow H \otimes H$, $\epsilon : H \rightarrow \mathbf{1}$, and $S : H \rightarrow H$ that satisfy the following conditions:

- (H, m, η) is an associative unital algebra object in \mathcal{C} , i.e., m and η satisfy

$$\begin{array}{c} m \\ \text{---} \\ \text{---} \\ m \end{array} = \begin{array}{c} m \\ \text{---} \\ \text{---} \\ m \end{array}, \quad \begin{array}{c} \text{---} \\ \eta \circlearrowleft \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \eta \end{array} = \left| \right|. \quad (2.5.8)$$

The associator $\alpha_{HHH} : (H \otimes H) \otimes H \rightarrow H \otimes (H \otimes H)$ is omitted in the above equation. We will also omit the associator in what follows.

- (H, Δ, ϵ) is a coassociative counital coalgebra object in \mathcal{C} , i.e., Δ and ϵ satisfy

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array}, \quad \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array} = \left| \right|. \quad (2.5.9)$$

- The comultiplication morphism Δ is a unit-preserving algebra homomorphism:

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array}, \quad \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array} \quad (2.5.10)$$

The 4-valent junction on the right-hand side of the first equation represents the symmetric braiding $c_{HH} : H \otimes H \rightarrow H \otimes H$.

- The counit morphism ϵ is a unit-preserving algebra homomorphism:

$$\begin{array}{c} \epsilon \\ \circ \\ \text{---} m \end{array} = \begin{array}{c} \epsilon \\ \circ \end{array} \begin{array}{c} \circ \\ \epsilon \end{array}, \quad \begin{array}{c} \epsilon \\ \circ \\ \text{---} \eta \\ \circ \end{array} = \text{id}_1. \quad (2.5.11)$$

- The antipode morphism S satisfies the following antipode axiom:

$$\text{Diagram 1} = S \cdot \text{Diagram 2} = \begin{matrix} \circ \\ \circ \end{matrix} \begin{matrix} \eta \\ \epsilon \end{matrix}. \quad (2.5.12)$$

The above definition will be used in Chapter 3 where we define Hopf superalgebras as Hopf algebra objects in the category of super vector spaces.

2.5.3 Weak Hopf algebra objects

Weak Hopf algebras defined in Section 2.4 can also be generalized to weak Hopf algebra objects in general symmetric fusion categories. Concretely, a weak Hopf algebra object $(H, m, \eta, \Delta, \epsilon, S)$ in a symmetric fusion category \mathcal{C} is an object $H \in \mathcal{C}$ equipped with structure morphisms $m : H \otimes H \rightarrow H$, $\eta : 1 \rightarrow H$, $\Delta : H \rightarrow H \otimes H$, $\epsilon : H \rightarrow 1$, and $S : H \rightarrow H$ that satisfy the following conditions:

- (H, m, η) is an associative unital algebra object that satisfies eq. (2.5.8).
- (H, Δ, ϵ) is a coassociative counital coalgebra object that satisfies eq. (2.5.9).
- The comultiplication morphism Δ is multiplicative, i.e.,

$$\begin{array}{c} \text{Diagram: A cup with } \Delta \text{ on the left and } m \text{ on the right.} \end{array} = \begin{array}{c} \text{Diagram: A cup with } m \text{ on the left and } \Delta \text{ on the right, with a crossing.} \end{array} \quad (2.5.13)$$

- The counit morphism ϵ satisfies

$$\begin{array}{c} \text{Diagram: A cup with } m \text{ on the left and } \epsilon \text{ on the right.} \end{array} = \begin{array}{c} \text{Diagram: A cup with } \epsilon \text{ on the left and } m \text{ on the right.} \end{array} = \begin{array}{c} \text{Diagram: A cup with } \epsilon \text{ on the left and } \Delta \text{ on the right.} \end{array} \quad (2.5.14)$$

- The unit morphism η satisfies

$$\begin{array}{c} \text{Diagram: A cup with } \Delta \text{ on the left and } \eta \text{ on the right.} \end{array} = \begin{array}{c} \text{Diagram: A cup with } \eta \text{ on the left and } \Delta \text{ on the right.} \end{array} = \begin{array}{c} \text{Diagram: A cup with } \eta \text{ on the left and } m \text{ on the right.} \end{array} \quad (2.5.15)$$

- The antipode morphism S satisfies

$$\begin{array}{c} \text{Diagram: A circle with } S \text{ on the left.} \end{array} = \begin{array}{c} \text{Diagram: A circle with } S \text{ on the right.} \end{array}, \quad \begin{array}{c} \text{Diagram: A circle with } S \text{ on the left.} \end{array} = \begin{array}{c} \text{Diagram: A circle with } S \text{ on the right.} \end{array}, \quad \begin{array}{c} \text{Diagram: A circle with } S \text{ on the left.} \end{array} = \begin{array}{c} \text{Diagram: A circle with } S \text{ on the right.} \end{array} \quad (2.5.16)$$

The 4-valent junctions in the above equations represent the symmetric braiding $c_{HH} : H \otimes H \rightarrow H \otimes H$. Weak Hopf algebras defined in Section 2.4 are weak Hopf algebra objects in the category Vec of vector spaces. Weak Hopf algebra objects in more general braided tensor categories are discussed in [102, 103].

Chapter 3

Superfusion categories

Although fusion categories describe general finite symmetries in 1+1 dimensional bosonic systems, they are not the most general mathematical structures to describe finite symmetries in 1+1d fermionic systems. This is because topological lines in fermionic systems can have fermionic topological junctions, which modify the algebraic structure of topological defects. Accordingly, the classification of symmetries and anomalies of fermionic systems differs from that of bosonic systems. In general, finite symmetries in 1+1d fermionic systems are described by superfusion categories. We call these symmetries superfusion category symmetries, which are closely related to Hopf superalgebras and weak Hopf superalgebras.

In this chapter, we review the basic definitions of superfusion categories, Hopf superalgebras, and weak Hopf superalgebras over \mathbb{C} . Superalgebras discussed in this chapter are supposed to be finite dimensional and semisimple unless otherwise stated.

3.1 Definitions

We first review superfusion categories and related notions such as supertensor functors and supermodule categories following [40, 48, 104], see also [105] for earlier discussions. Let us begin with the definitions of supercategories and superfunctors. A supercategory is a category such that the set $\text{Hom}(X, Y)$ of morphisms between any objects X and Y is a \mathbb{Z}_2 -graded vector space and the composition of morphisms preserves the \mathbb{Z}_2 -grading. The \mathbb{Z}_2 -grading of a homogeneous morphism f is denoted by $|f|$, which is 0 or 1 depending on whether f is even or odd. A superfunctor between supercategories is a functor that preserves the \mathbb{Z}_2 -grading of morphisms.

A supercategory \mathcal{C} is called a monoidal supercategory if it is equipped with a tensor product structure $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, the unit object 1 , and even natural isomorphisms called associators and unit morphisms that satisfy the usual pentagon equation (2.1.4) and unit axiom (2.1.6). The tensor product of morphisms preserves the \mathbb{Z}_2 -grading, which means $|f \otimes g| = |f| + |g| \bmod 2$ for any homogeneous morphisms f and g . The compatibility of the tensor product and the composition of morphisms is encoded in the following relation called the super interchange law:

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k). \quad (3.1.1)$$

This equation implies that \mathbb{Z}_2 -odd morphisms obey the anti-commutation relation because $(f \otimes$

$\text{id}) \circ (\text{id} \otimes g) = (-1)^{|f||g|} (\text{id} \otimes g) \circ (f \otimes \text{id})$, which is represented by the following string diagrams:

$$\begin{array}{c} f \\ \bullet \\ | \\ X \end{array} \begin{array}{c} | \\ \bullet \\ g \\ Y \end{array} = (-1)^{|f||g|} \begin{array}{c} | \\ \bullet \\ f \\ X \end{array} \begin{array}{c} g \\ \bullet \\ | \\ Y \end{array}. \quad (3.1.2)$$

We note that a monoidal supercategory is not a monoidal category in the usual sense due to the extra minus sign on the right-hand side of the above equation.

A monoidal supercategory is said to be rigid if every object has both left dual and right dual. Here, a left dual X^* of an object X is an object equipped with even morphisms $\text{ev}_X^L : X^* \otimes X \rightarrow \mathbf{1}$ and $\text{coev}_X^L : \mathbf{1} \rightarrow X \otimes X^*$ that satisfy the following axioms:

$$(\text{id}_X \otimes \text{ev}_X^L) \circ (\text{coev}_X^L \otimes \text{id}_X) = \text{id}_X, \quad (\text{ev}_X^L \otimes \text{id}_{X^*}) \circ (\text{id}_{X^*} \otimes \text{coev}_X^L) = \text{id}_{X^*}. \quad (3.1.3)$$

We omitted to write the associators and unit morphisms in the above equation. This does not introduce any ambiguity due to the Mac Lane coherence theorem [101]. The morphisms ev_X^L and coev_X^L are called left evaluation and left coevaluation morphisms respectively as in the case of ordinary monoidal categories. A right dual *X is also defined similarly.

A superfusion category is a finite semisimple \mathbb{C} -linear rigid monoidal supercategory such that the unit object is simple. In particular, any object X of a superfusion category is isomorphic to a finite direct sum of simple objects, and the Hom space $\text{Hom}(X, Y)$ between any two objects X and Y is a finite dimensional super vector space. The Hom space $\text{Hom}(X, X)$ for a simple object X is isomorphic to either $\mathbb{C}^{1|0}$ or $\mathbb{C}^{1|1}$.¹ A simple object of the former type is called an m-type object, whereas a simple object of the latter type is called a q-type object [40]. We note that the unit object is always an m-type object [104].

We assume that superfusion categories are pivotal. Namely, they are equipped with an even natural isomorphism $a_X : X \rightarrow X^{**}$ called a pivotal structure. We use this natural isomorphism to identify the right dual *X with the left dual X^* . Accordingly, we write the right evaluation and coevaluation morphisms in terms of X^* rather than *X . Specifically, the right evaluation and coevaluation morphisms are defined as even morphisms $\text{ev}_X^R : X \otimes X^* \rightarrow \mathbf{1}$ and $\text{coev}_X^R : \mathbf{1} \rightarrow X^* \otimes X$ that satisfy the axiom analogous to eq. (3.1.3).

Since a superfusion category is rigid, we can define the left and right duals of a morphism. For a pivotal superfusion category, the left dual f^* and the right dual *f of a morphism $f \in \text{Hom}(X, Y)$ are morphisms from Y^* to X^* defined by

$$\begin{aligned} f^* &:= (\text{ev}_Y^L \otimes \text{id}_{X^*}) \circ (\text{id}_{Y^*} \otimes f \otimes \text{id}_{X^*}) \circ (\text{id}_{Y^*} \otimes \text{coev}_X^L), \\ {}^*f &:= (\text{id}_{X^*} \otimes \text{ev}_Y^R) \circ (\text{id}_{X^*} \otimes f \otimes \text{id}_{Y^*}) \circ (\text{coev}_X^R \otimes \text{id}_{Y^*}). \end{aligned} \quad (3.1.4)$$

Following [40], we require that the left dual and the right dual of a morphism f are related by

$${}^*f = (-1)^{|f|} f^*. \quad (3.1.5)$$

Physically, the above equation corresponds to the fact that a fermion acquires a minus sign when it is rotated by 2π .

Superfusion categories that describe the symmetries of 1+1d fermionic systems have a further structure known as Π -superfusion categories. Here, a Π -superfusion category is a superfusion category equipped with a distinguished object π and an odd isomorphism $\zeta : \pi \rightarrow \mathbf{1}$.²

¹ A vector space $\mathbb{C}^{p|q}$ denotes a super vector space of superdimension (p, q) .

² Physically, an odd isomorphism ζ corresponds to a probe local fermion and π corresponds to its worldline.

In particular, every object X of a Π -superfusion category has an oddly isomorphic object $\Pi X := \pi \otimes X$. The odd isomorphism from ΠX to X is denoted by ζ_X . The evaluation and coevaluation morphisms of ΠX can be expressed in terms of those of X as

$$\begin{aligned} \text{ev}_{\Pi X}^L &= -\text{ev}_X^L \circ ((\zeta_X^*)^{-1} \otimes \zeta_X), & \text{coev}_{\Pi X}^L &= (\zeta_X^{-1} \otimes \zeta_X^*) \circ \text{coev}_X^L, \\ \text{ev}_{\Pi X}^R &= \text{ev}_X^R \circ (\zeta_X \otimes (*\zeta_X)^{-1}), & \text{coev}_{\Pi X}^R &= -(*\zeta_X \otimes \zeta_X^{-1}) \circ \text{coev}_X^R, \end{aligned} \quad (3.1.6)$$

which follow from equations (3.1.2), (3.1.3), and (3.1.4). We note that in a pivotal Π -superfusion category, the quantum dimension of ΠX agrees with that of X due to equations (3.1.5) and (3.1.6):

$$\dim(\Pi X) = \text{ev}_{\Pi X}^R \circ \text{coev}_{\Pi X}^L = \text{ev}_X^R \circ \text{coev}_X^L = \dim(X). \quad (3.1.7)$$

In what follows, a superfusion category means a Π -superfusion category unless otherwise stated.

A supertensor functor $(\mathcal{F}, \mathcal{J}, \varphi)$ between superfusion categories \mathcal{C} and \mathcal{D} is a \mathbb{C} -linear superfunctor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ equipped with an even natural isomorphism $\mathcal{J}_{X,Y} : \mathcal{F}(X) \otimes \mathcal{F}(Y) \rightarrow \mathcal{F}(X \otimes Y)$ and an even isomorphism $\varphi : \mathbf{1}_{\mathcal{D}} \rightarrow \mathcal{F}(\mathbf{1}_{\mathcal{C}})$ that satisfy the same consistency conditions (2.1.7) and (2.1.8) as in the case of an ordinary tensor functor. We often write a supertensor functor $(\mathcal{F}, \mathcal{J}, \varphi)$ simply as \mathcal{F} .

A supermodule category \mathcal{M} over a superfusion category \mathcal{C} is a supercategory equipped with a \mathcal{C} -action denoted by $\bar{\otimes} : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$. The structure of a \mathcal{C} -supermodule category on \mathcal{M} is represented by a supertensor functor $\mathcal{F} : \mathcal{C} \rightarrow \text{End}(\mathcal{M})$, where $\text{End}(\mathcal{M})$ is the supercategory of superfunctors from \mathcal{M} to itself.

Example. The simplest example of a superfusion category is the supercategory sVec of super vector spaces. Objects and morphisms of sVec are finite dimensional super vector spaces and (not necessarily even) linear maps between them. The \mathbb{Z}_2 -even part and the \mathbb{Z}_2 -odd part of a super vector space V are denoted by V_0 and V_1 respectively. The direct sum and the tensor product of super vector spaces V and W are given by

$$(V \oplus W)_0 = V_0 \oplus W_0, \quad (V \oplus W)_1 = V_1 \oplus W_1, \quad (3.1.8)$$

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1), \quad (V \otimes W)_1 = (V_1 \otimes W_0) \oplus (V_0 \otimes W_1). \quad (3.1.9)$$

The tensor product of morphisms involves a non-trivial sign coming from the braiding of \mathbb{Z}_2 -odd elements. For homogeneous morphisms $f \in \text{Hom}(V, V')$ and $g \in \text{Hom}(W, W')$, the tensor product $f \otimes g \in \text{Hom}(V \otimes W, V' \otimes W')$ is defined by

$$f \otimes g(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w), \quad \forall v \in V, \forall w \in W, \quad (3.1.10)$$

which obeys the super interchange law (3.1.1). The dual object of a super vector space V is the dual vector space $V^* := \text{Hom}(V, \mathbb{C})$ equipped with the obvious \mathbb{Z}_2 -grading. The evaluation and coevaluation morphisms are defined in the usual way, cf. eq. (2.1.18). We can express the \mathbb{Z}_2 -grading automorphism $p_V : V \rightarrow V$, which is defined by $p_V(v) = (-1)^{|v|}v$, in terms of the evaluation and coevaluation morphisms as follows:

$$p_V = (\text{ev}_V^L \otimes \text{id}_V) \circ (\text{id}_{V^*} \otimes c_{\text{super}}) \circ (\text{coev}_V^R \otimes \text{id}_V) = (\text{id}_V \otimes \text{ev}_V^R) \circ (c_{\text{super}} \otimes \text{id}_{V^*}) \circ (\text{id}_V \otimes \text{coev}_V^L). \quad (3.1.11)$$

Here, $c_{\text{super}} : V \otimes W \rightarrow W \otimes V$ is the symmetric braiding of super vector spaces defined by $c_{\text{super}}(v \otimes w) = (-1)^{|v||w|} w \otimes v$. The subscript V of the \mathbb{Z}_2 -grading automorphism p_V will be omitted if it is clear from the context. Combined with the axiom (3.1.3) and its counterpart

for the right dual, equation (3.1.11) implies that the left (co)evaluation morphism and the right (co)evaluation morphism are related to each other in the following way:

$$\begin{aligned} \text{ev}_V^L &= \text{ev}_V^R \circ c_{\text{super}} \circ (\text{id} \otimes p) = \text{ev}_V^R \circ c_{\text{super}} \circ (p \otimes \text{id}), \\ \text{coev}_V^L &= (p \otimes \text{id}) \circ c_{\text{super}} \circ \text{coev}_V^R = (\text{id} \otimes p) \circ c_{\text{super}} \circ \text{coev}_V^R. \end{aligned} \quad (3.1.12)$$

We can use this equation to show that sVec is a pivotal superfusion category satisfying the condition (3.1.5). Furthermore, sVec is a Π -superfusion category whose distinguished object π is $\mathbb{C}^{0|1}$ and the odd isomorphism $\zeta : \mathbb{C}^{0|1} \rightarrow \mathbb{C}^{1|0}$ is the identity map of the underlying vector space.

We note that the subcategory of sVec consisting of super vector spaces and even linear maps is an ordinary fusion category because the super interchange law (3.1.1) for even morphisms reduces to the ordinary compatibility condition (2.1.3). This subcategory is denoted by $\underline{\text{sVec}}$ and is called the underlying category of sVec . The underlying category $\underline{\text{sVec}}$ is, in particular, a symmetric fusion category equipped with the symmetric braiding c_{super} .³

3.2 Semisimple superalgebras and supermodules

A superalgebra is an algebra equipped with a \mathbb{Z}_2 -grading such that the multiplication and unit are even.⁴ A simple superalgebra is isomorphic to either $\text{End}(\mathbb{C}^{p|q})$ or $\text{End}(\mathbb{C}^{p|0}) \otimes \text{Cl}(1)$, where $\text{End}(\mathbb{C}^{p|q})$ is the endomorphism superalgebra of $\mathbb{C}^{p|q}$ and $\text{Cl}(1)$ is the complex Clifford algebra with a single odd generator [106]. We note that $\text{End}(\mathbb{C}^{p|0}) \otimes \text{Cl}(1)$ is not simple as an algebra because we have an isomorphism $\text{End}(\mathbb{C}^{p|0}) \otimes \text{Cl}(1) \cong \text{End}(\mathbb{C}^{p|0}) \oplus \text{End}(\mathbb{C}^{p|0})$. However, this isomorphism does not preserve the \mathbb{Z}_2 -grading, and therefore, this is not an isomorphism of superalgebras. A semisimple superalgebra K can be decomposed into a direct sum of simple superalgebras, namely, we have an isomorphism of superalgebras [107]

$$K \cong \left(\bigoplus_i \text{End}(\mathbb{C}^{p_i|q_i}) \right) \oplus \left(\bigoplus_j \text{End}(\mathbb{C}^{p_j|0}) \otimes \text{Cl}(1) \right). \quad (3.2.1)$$

In particular, every semisimple superalgebra is a semisimple algebra.

A left supermodule M over a superalgebra K is a \mathbb{Z}_2 -graded vector space on which K acts by an even linear map $\rho_M^L : K \otimes M \rightarrow M$. A linear map between K -supermodules is called a K -supermodule morphism if it commutes with the action of K . More specifically, a K -supermodule morphism $f : M \rightarrow N$ is a linear map that satisfies $f \circ \rho_M^L = \rho_N^L \circ (\text{id}_K \otimes f)$, where the tensor product on the right-hand side is defined by eq. (3.1.10). We emphasize that supermodule morphisms can be odd. Simple supermodules over simple superalgebras $\text{End}(\mathbb{C}^{p|q})$ and $\text{End}(\mathbb{C}^{p|0}) \otimes \text{Cl}(1)$ are $\mathbb{C}^{p|q}$ and $\mathbb{C}^{p|p}$ respectively, which are unique up to isomorphism of supermodules. If a simple superalgebra K_0 is a direct summand of a semisimple superalgebra K , we can think of a K_0 -supermodule M_0 as a K -supermodule by demanding that the direct summands other than K_0 act trivially on M_0 . Any simple K -supermodule is of this form. Therefore, simple supermodules over a semisimple superalgebra K are in one-to-one correspondence with the direct summands on the right-hand side of eq. (3.2.1) [107].

We can also define right K -supermodules and (K, K) -superbimodules in a similar fashion. Concretely, a right K -supermodule M is a \mathbb{Z}_2 -graded vector space on which K acts from the

³Note that the symmetric fusion category $\underline{\text{sVec}}$ is often denoted by sVec in the literature. In this dissertation, sVec always refers to the superfusion category consisting of finite dimensional super vector spaces and (not necessarily even) linear maps between them.

⁴The unit is automatically even when the multiplication is even.

right via an even linear map $\rho_M^R : M \otimes K \rightarrow M$. Similarly, a (K, K) -superbimodule is a left K -supermodule that is simultaneously a right K -supermodule whose right K -action ρ^R commutes with the left K -action ρ^L , i.e., $\rho^L \circ (\text{id}_K \otimes \rho^R) = \rho^R \circ (\rho^L \otimes \text{id}_K)$. A linear map between (K, K) -superbimodules is called a (K, K) -superbimodule morphism if it commutes with both the left K -action and right K -action. Supercoalgebras and supercomodules over them are also defined similarly.

Since a semisimple superalgebra K is a semisimple algebra, it is a Δ -separable symmetric Frobenius algebra as we saw in Section 2.2.1. Namely, K is equipped with a comultiplication (2.2.7) and a counit (2.2.8) that satisfy the Δ -separability (2.2.5), symmetricity (2.2.6), and the Frobenius relation (2.2.4). The above Frobenius algebra structure is defined regardless of the \mathbb{Z}_2 -grading on K . We note that the symmetricity (2.2.6) can also be written as

$$\begin{aligned} \epsilon_K \circ m_K &= \epsilon_K \circ m_K \circ c_{\text{super}} \circ (\text{id}_K \otimes p_K) = \epsilon_K \circ m_K \circ c_{\text{super}} \circ (p_K \otimes \text{id}_K), \\ \Delta_K \circ \eta_K &= (p_K \otimes \text{id}_K) \circ c_{\text{super}} \circ \Delta_K \circ \eta_K = (\text{id}_K \otimes p_K) \circ c_{\text{super}} \circ \Delta_K \circ \eta_K, \end{aligned} \quad (3.2.2)$$

where $m_K : K \otimes K \rightarrow K$ is the multiplication, $\eta_K : \mathbb{C} \rightarrow K$ is the unit, and $p_K : K \rightarrow K$ is the \mathbb{Z}_2 -grading automorphism of K .

Supercategory of superbimodules The supercategory ${}_K\mathcal{SM}_K$ of semisimple superbimodules over a semisimple superalgebra K is a monoidal supercategory [48], which we expect to be a multisuperfusion category, i.e., a finite semisimple \mathbb{C} -linear abelian rigid monoidal supercategory whose unit object may not be simple. This supercategory reduces to sVec when K is a trivial superalgebra $\mathbb{C}^{1|0}$. The monoidal structure on ${}_K\mathcal{SM}_K$ is given by the tensor product over K . The unit object of ${}_K\mathcal{SM}_K$ is the regular (K, K) -superbimodule K . The tensor product $f \otimes_K g$ of superbimodule morphisms $f : X_1 \rightarrow X'_1$ and $g : X_2 \rightarrow X'_2$ is given by $f \otimes_K g = \pi_{X'_1, X'_2} \circ (f \otimes g) \circ \iota_{X_1, X_2}$, where $\pi_{X'_1, X'_2}$ and ι_{X_1, X_2} are splitting maps of the projector (2.2.16), and $f \otimes g$ is the tensor product of \mathbb{Z}_2 -graded linear maps defined by eq. (3.1.10). The structure morphisms of ${}_K\mathcal{SM}_K$ are analogous to those of the category ${}_K\mathcal{M}_K$ of (K, K) -bimodules, which is an ordinary multifusion category that we reviewed in Section 2.2.2. Moreover, the monoidal supercategory ${}_K\mathcal{SM}_K$ is a Π -supercategory. The distinguished object $\Pi \in {}_K\mathcal{SM}_K$ is obtained by flipping the \mathbb{Z}_2 -grading on the regular superbimodule K and twisting the left K -action on it by the \mathbb{Z}_2 -grading automorphism $p_K : K \rightarrow K$. More specifically, the \mathbb{Z}_2 -even subspace Π_0 and \mathbb{Z}_2 -odd subspace Π_1 are given by $\Pi_0 := K_1$ and $\Pi_1 := K_0$, and the left and right K -actions on Π are $m_K \circ (p_K \otimes \text{id})$ and m_K respectively. The odd isomorphism between Π and K is the identity map of the underlying vector space.

3.3 Hopf superalgebras and weak Hopf superalgebras

Superfusion categories are closely related to Hopf superalgebras and weak Hopf superalgebras in the same way as fusion categories are related to Hopf algebras and weak Hopf algebras. In this section, we recall the definitions of Hopf superalgebras and weak Hopf superalgebras and briefly describe their representation categories and supermodule categories over them. The discussions in this section will be brief because we already discussed similar algebraic structures in detail in Sections 2.3 and 2.4.

Hopf superalgebras. Hopf superalgebras are Hopf algebra objects in the symmetric fusion category sVec whose objects are super vector spaces and whose morphisms are even linear maps. More specifically, a Hopf superalgebra \mathcal{H} is an associative unital superalgebra (\mathcal{H}, m, η)

equipped with a structure of a coassociative counital supercoalgebra $(\mathcal{H}, \Delta, \epsilon)$ and an even linear map $S : \mathcal{H} \rightarrow \mathcal{H}$ called an antipode such that the structure maps $(m, \eta, \Delta, \epsilon, S)$ satisfy the conditions (2.5.10), (2.5.11), and (2.5.12) where the string diagrams are those in $\underline{\text{sVec}}$ [108]. As in the case of Hopf algebras, the antipode of a Hopf superalgebra is a superalgebra and supercoalgebra anti-homomorphism. Furthermore, when a Hopf superalgebra \mathcal{H} is semisimple, the antipode S squares to the \mathbb{Z}_2 -grading automorphism of \mathcal{H} [109]. In particular, the antipode S satisfies $S^4 = \text{id}$. A semisimple Hopf superalgebra reduces to an ordinary semisimple Hopf algebra without a \mathbb{Z}_2 -grading if and only if $S^2 = \text{id}$. The dual of a Hopf superalgebra \mathcal{H} is also a Hopf superalgebra equipped with the dual structure maps defined by eq. (2.3.4).

Representation supercategories of Hopf superalgebras. The supercategory of super representations of a semisimple Hopf superalgebra \mathcal{H} is a superfusion category, which we denote by $\text{sRep}(\mathcal{H})$. Objects and morphisms of $\text{sRep}(\mathcal{H})$ are left \mathcal{H} -supermodules (i.e., super representations of \mathcal{H}) and \mathcal{H} -supermodule morphisms respectively. The monoidal structure on $\text{sRep}(\mathcal{H})$ is given by the tensor product of super representations. The underlying super vector space of the tensor product representation is the usual tensor product of super vector spaces. The \mathcal{H} -action on the tensor product representation is given by eq. (2.3.5) where the trivial braiding c_{triv} is replaced by the symmetric braiding c_{super} . The unit object is a one-dimensional super representation $\mathbb{C}^{1|0}$ on which \mathcal{H} acts by $h \cdot v = \epsilon(h)v$ for all $h \in \mathcal{H}$ and $v \in \mathbb{C}^{1|0}$. The dual object of a super representation $V \in \text{sRep}(\mathcal{H})$ is the dual super vector space V^* . The action of \mathcal{H} on V^* is defined by

$$\rho_{V^*}(h \otimes \phi)(v) = (-1)^{|h||\phi|} \phi(\rho_V(S(h) \otimes v)) \quad (3.3.1)$$

for homogeneous elements $h \in \mathcal{H}$, $\phi \in V^*$, and $v \in V$. This action is linearly extended to inhomogeneous elements. The evaluation and coevaluation morphisms and the other structure isomorphisms of $\text{sRep}(\mathcal{H})$ are induced by those of sVec .

The supercategory ${}_K\mathcal{SM}$ of left K -supermodules is an $\text{sRep}(\mathcal{H})$ -supermodule category when K is a left \mathcal{H} -supercomodule algebra. Here, a left \mathcal{H} -supercomodule algebra K is a superalgebra equipped with a left \mathcal{H} -supercomodule structure that satisfies the compatibility condition (2.3.8) where the trivial braiding c_{triv} is replaced by the symmetric braiding c_{super} . The supertensor functor $\mathcal{F}_K : \text{sRep}(\mathcal{H}) \rightarrow \text{End}({}_K\mathcal{SM}) \cong {}_K\mathcal{SM}_K$ corresponding to the $\text{sRep}(\mathcal{H})$ -supermodule structure on ${}_K\mathcal{SM}$ is analogous to the tensor functor $F_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$ discussed in Section 2.3.

Weak Hopf superalgebras. Weak Hopf superalgebras are weak Hopf algebra objects in the symmetric fusion category $\underline{\text{sVec}}$. Namely, a weak Hopf superalgebra \mathcal{H} is an associative unital superalgebra (\mathcal{H}, m, η) equipped with a structure of a coassociative counital supercoalgebra $(\mathcal{H}, \Delta, \epsilon)$ and an antipode $S : \mathcal{H} \rightarrow \mathcal{H}$ that satisfy the properties (2.5.13), (2.5.14), (2.5.15), and (2.5.16). As in the case of weak Hopf algebras, the target counital map ϵ_t and the source counital map ϵ_s of a weak Hopf superalgebra are defined by $\epsilon_t := m \circ (\text{id} \otimes S) \circ \Delta$ and $\epsilon_s := m \circ (S \otimes \text{id}) \circ \Delta$. The images of these maps are called the target and source counital subalgebras and are denoted by \mathcal{H}_t and \mathcal{H}_s respectively.

Representation supercategories of weak Hopf superalgebras. The supercategory $\text{sRep}(\mathcal{H})$ of super representations of a weak Hopf superalgebra \mathcal{H} is defined in a similar way to the representation category $\text{Rep}(H)$ of a weak Hopf algebra H . Supermodule categories over $\text{sRep}(\mathcal{H})$ and the corresponding supertensor functors are also similar to module categories

over $\text{Rep}(H)$ and the corresponding tensor functors. In particular, the supercategory ${}_K\mathcal{SM}$ of left K -supermodules is an $\text{sRep}(\mathcal{H})$ -supermodule category if K is a left \mathcal{H} -supercomodule algebra, which is a superalgebra equipped with a left \mathcal{H} -supercomodule structure satisfying the compatibility condition (2.4.5) with c_{triv} replaced by c_{super} . Accordingly, we have a supertensor functor $\mathcal{F}_K : \text{sRep}(\mathcal{H}) \rightarrow {}_K\mathcal{SM}_K$ when K is a left \mathcal{H} -supercomodule algebra.

Chapter 4

Bosonic topological phases with fusion category symmetries

Bosonic topological phases in 1+1 dimensions, such as symmetry protected topological phases, symmetry broken phases, and mixtures thereof, are described by two-dimensional bosonic topological field theories (TFTs) in the deep infrared.¹ Therefore, the classification of 1+1d bosonic topological phases reduces to the classification of two-dimensional bosonic TFTs. In this chapter, after we argue that bosonic TFTs with fusion category symmetry \mathcal{C} are classified by \mathcal{C} -module categories, we explicitly construct these TFTs by using the state sum construction and pullback of fusion category symmetries. We also construct the corresponding lattice models with fusion category symmetries on a one-dimensional chain when the fusion category symmetries are non-anomalous. The string diagrams appearing in this chapter should be understood as those in Vec . In particular, the braiding of strands always represents the trivial braiding c_{triv} . This chapter is based on the author's original paper [64] except for Section 4.1.

4.1 Review: classification of bosonic TFTs with fusion category symmetries

In this section, we review the classification of 2d bosonic TFTs with fusion category symmetries following [13, 43]. There are at least two approaches to this classification, namely, a three-dimensional approach [13] and a two-dimensional approach [43]. We will discuss both of these approaches in the following subsections.

4.1.1 Three-dimensional approach

The first approach is based on the correspondence between fusion category symmetries in 1+1 dimensions and three-dimensional topological field theories [13]. Let us first recall this correspondence. As illustrated in Figure 4.1, a 1+1d bosonic system with fusion category symmetry \mathcal{C} is obtained by putting a 3d bosonic TFT $\text{TV}(\mathcal{C})$ on a slab $[0, 1] \times \Sigma$, where $\text{TV}(\mathcal{C})$ is the Turaev-Viro-Barrett-Westbury TFT constructed from \mathcal{C} [110, 111], $[0, 1]$ is a finite interval, and Σ is a two-dimensional closed oriented manifold. The boundary condition on the left boundary $\{0\} \times \Sigma$ is a specific topological boundary condition known as the Dirichlet boundary condition, which determines the symmetry of the 1+1d system that is obtained by squashing the 3d

¹In this dissertation, we use topological phases and gapped phases interchangeably.

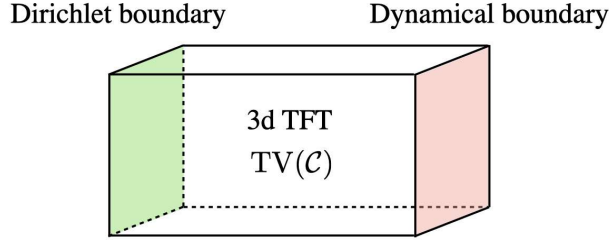


Figure 4.1: A 3d topological field theory $\text{TV}(\mathcal{C})$ on a slab $[0, 1] \times \Sigma$ gives rise to a 1+1 dimensional system with a fusion category symmetry \mathcal{C} . The symmetry of the 1+1d system is generated by topological defects on the Dirichlet boundary on the left, while the dynamics of the system is encoded in the dynamical boundary condition on the right.

bulk. On the other hand, the boundary condition on the right boundary $\{1\} \times \Sigma$ is chosen arbitrarily and this boundary condition determines the dynamics of the 1+1d system. The defining feature of the Dirichlet boundary condition is that topological lines on this boundary form the input fusion category \mathcal{C} . This implies that the symmetry of the 1+1d system obtained by the above construction is described by \mathcal{C} because topological defects in 1+1d originate from those on the topological boundary of the 3d TFT [13, 75, 112–114]. The topological field theory in the 3d bulk is called a symmetry TFT [115] because it dictates the symmetry in one lower dimension. The symmetry TFT is also called a categorical symmetry [116, 117] or a symmetry topological order [118–120] in the condensed matter literature. See also, e.g., [7, 121–124] for various applications of 3d symmetry TFTs to 2d problems.²

The 1+1d system obtained above becomes topological if and only if the boundary condition on the right boundary of the 3d symmetry TFT is also topological. Therefore, 1+1d bosonic topological phases with fusion category symmetry \mathcal{C} are in one-to-one correspondence with topological boundary conditions of the 3d topological field theory $\text{TV}(\mathcal{C})$. Furthermore, topological boundary conditions of $\text{TV}(\mathcal{C})$ are in bijective correspondence with module categories over \mathcal{C} [133–135]. Thus, there is a one-to-one correspondence between 1+1d topological phases with symmetry \mathcal{C} and \mathcal{C} -module categories [13]. The 1+1d topological phase corresponding to a \mathcal{C} -module category \mathcal{M} will be denoted by $\mathcal{T}_{\mathcal{M}}$. The topological phase $\mathcal{T}_{\mathcal{M}}$ is indecomposable if and only if the corresponding \mathcal{C} -module category \mathcal{M} is indecomposable.

The topological phase $\mathcal{T}_{\mathcal{M}}$ has as many ground states on a circle as the number of simple objects of \mathcal{M} [13]. This can be seen from the Hamiltonian formalism of the above symmetry TFT construction. From the Hamiltonian point of view, the symmetry TFT $\text{TV}(\mathcal{C})$ is realized in the low energy limit of the Levin-Wen model $\text{LW}(\mathcal{C})$ constructed from the same fusion category \mathcal{C} [136]. Therefore, a 1+1d system with symmetry \mathcal{C} on a circle is obtained by putting the Levin-Wen model $\text{LW}(\mathcal{C})$ on an annulus as shown in Figure 4.2a. If we impose the Dirichlet boundary condition on the inner boundary, this 1+1d system has a fusion category symmetry \mathcal{C} whatever boundary condition we chose on the outer boundary. In particular, we can obtain a topological phase $\mathcal{T}_{\mathcal{M}}$ if we choose a topological boundary labeled by \mathcal{M} as the outer boundary.³ Since the ground states of the Levin-Wen model are string-net condensates [134, 136], the number of ground states is equal to the number of inequivalent configurations of string-net

²A similar relation between finite symmetries and topological field theories holds in any dimensions [125, 126] and is sometimes referred to as boundary-bulk relation [127–129] or topological holography [130–132]. The study of this relation dates back at least to the early 2000s [15].

³The topological boundary labeled by \mathcal{M} is defined by taking the dynamical variables on the boundary to be simple objects of \mathcal{M} and modifying the Hamiltonian near the boundary accordingly [134].

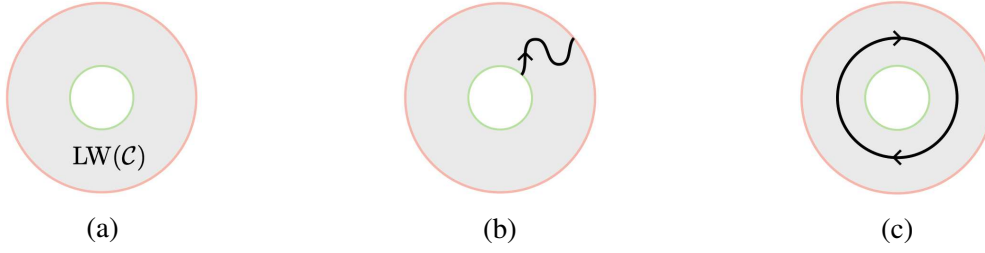


Figure 4.2: (a) The Levin-Wen model $LW(\mathcal{C})$ on an annulus gives rise to a 1+1d system with symmetry \mathcal{C} on a circle. Here, we impose the Dirichlet boundary condition on the inner boundary and a dynamical boundary condition on the outer boundary. (b) A prohibited configuration of a string. (c) An allowed configuration of a string.

on an annulus. To count the number of inequivalent configurations of string-net, we first notice that the configuration of a string depicted in Figure 4.2b is not allowed in the string-net condensate because the Dirichlet boundary does not condense any string. On the other hand, a string can wrap around a non-trivial cycle on an annulus as shown in Figure 4.2c, which is the only non-trivial configuration of string-net on an annulus. A different type of string wrapping around the non-trivial cycle gives rise to a different state. However, the states obtained in this way are not necessarily linearly independent of each other because the string in the bulk can be absorbed into the outer boundary and thus only changes the boundary labels leaving nothing behind in the bulk. As a result, the number of inequivalent configurations of string-net is equal to the number of different labels on the outer boundary, i.e., the number of simple objects of \mathcal{M} . This shows that the ground state degeneracy of the topological phase $\mathcal{T}_{\mathcal{M}}$ agrees with the number of simple objects of \mathcal{M} .

SPT phases with fusion category symmetries. A topological phase $\mathcal{T}_{\mathcal{M}}$ with symmetry \mathcal{C} is called a symmetry protected topological phase if it has a unique ground state on a circle.⁴ The above result on the classification of TFTs implies that 1+1d bosonic SPT phases with fusion category symmetry \mathcal{C} are classified by \mathcal{C} -module categories that have a single simple object. Equivalently, 1+1d bosonic SPT phases with symmetry \mathcal{C} are classified by fiber functors of \mathcal{C} [13], i.e., tensor functors from \mathcal{C} to Vec . This is because a \mathcal{C} -module category \mathcal{M} is associated with a tensor functor $F_{\mathcal{M}} : \mathcal{C} \rightarrow \text{End}(\mathcal{M})$, which becomes a fiber functor when \mathcal{M} has a single simple object because $\text{End}(\mathcal{M})$ is equivalent to Vec in this case. The fiber functor $F_{\mathcal{M}}$ associated with \mathcal{M} is isomorphic to a fiber functor $F_{\mathcal{N}}$ associated with another \mathcal{C} -module category \mathcal{N} if and only if \mathcal{M} and \mathcal{N} are equivalent to each other as a \mathcal{C} -module category [29]. Thus, the correspondence between SPT phases and fiber functors is one-to-one up to isomorphism. We note that a fusion category symmetry \mathcal{C} admits SPT phases only when it is non-anomalous, i.e., when \mathcal{C} is equivalent to $\text{Rep}(H)$ for a semisimple Hopf algebra H .

Spontaneous breaking of fusion category symmetries. Every fusion category \mathcal{C} is a module category over itself. This module category is called a regular \mathcal{C} -module category. The gapped phase $\mathcal{T}_{\mathcal{C}}$ corresponding to the regular \mathcal{C} -module category spontaneously breaks the symmetry \mathcal{C} down to the trivial symmetry Vec . This symmetry broken phase always exists.

⁴A trivial phase is also called a symmetry protected topological phase based on this definition.



Figure 4.3: (a) The linear map $\theta_X : \text{End}_{\mathcal{B}}(X) \rightarrow \mathbb{C}$ is the transition amplitude on a half disk that is a bordism from an interval I_{XX} to an empty space, where I_{XX} denotes an interval with boundary condition X on both ends. (b) The non-degenerate pairing $\theta_{XY} : \text{Hom}_{\mathcal{B}}(X, Y) \otimes \text{Hom}_{\mathcal{B}}(Y, X) \rightarrow \mathbb{C}$ is the transition amplitude on a strip that is a bordism from the disjoint union of intervals $I_{XY} \sqcup I_{YX}$ to an empty space.

4.1.2 Two-dimensional approach

The classification of 1+1d topological phases with fusion category symmetries can also be derived from a purely two-dimensional perspective [43]. In order to derive the classification, we first recall that a semisimple two-dimensional bosonic topological field theory without symmetry is completely characterized by the category of boundary conditions [137], which we denote by \mathcal{B} . Here, objects of \mathcal{B} are topological boundary conditions of the TFT, and the Hom space $\text{Hom}_{\mathcal{B}}(X, Y)$ is the state space on an interval with boundary conditions X and Y imposed on the two ends. The Hom space $\text{Hom}_{\mathcal{B}}(X, Y)$ can also be thought of as the space of topological boundary-changing operators between X and Y due to the state-operator correspondence. A remarkable property of the category \mathcal{B} is that it is equipped with a linear map called a trace $\theta_X : \text{End}_{\mathcal{B}}(X) \rightarrow \mathbb{C}$ that gives rise to a symmetric non-degenerate pairing $\theta_{XY} : \text{Hom}_{\mathcal{B}}(X, Y) \otimes \text{Hom}_{\mathcal{B}}(Y, X) \rightarrow \mathbb{C}$ defined by $\theta_{XY}(f \otimes g) := \theta_X(g \circ f)$ for $f \in \text{Hom}_{\mathcal{B}}(X, Y)$ and $g \in \text{Hom}_{\mathcal{B}}(Y, X)$. That the pairing θ_{XY} is symmetric means that it satisfies $\theta_{XY}(g \circ f) = \theta_{YX}(f \circ g)$. See Figure 4.3 for the physical interpretation of this pairing. The category \mathcal{B} equipped with such a linear map $\theta_{X \in \mathcal{B}}$ is called a Frobenius category or a Calabi-Yau category in [137]. As shown in [137], the complete data of a semisimple 2d TFT can be reconstructed from the category of its topological boundary conditions.⁵ Furthermore, for any semisimple Frobenius category \mathcal{B} , there exists a semisimple 2d TFT whose category of boundary conditions is \mathcal{B} . Therefore, semisimple 2d bosonic TFTs without symmetry are classified by semisimple Frobenius categories.

When a 2d TFT has a fusion category symmetry \mathcal{C} , the category of its topological boundary conditions \mathcal{B} has a structure of a \mathcal{C} -module category because the symmetry \mathcal{C} naturally acts on topological boundary conditions by the fusion of topological lines with boundary. In particular, the category of boundary conditions should be indecomposable as a \mathcal{C} -module category if the TFT is indecomposable as a TFT with symmetry \mathcal{C} . The trace on the category \mathcal{B} has to be compatible with the \mathcal{C} -module structure, that is, the trace has to satisfy $\theta_{X \otimes M} = \theta_M \circ \text{tr}_{X, M}$ for $X \in \mathcal{C}$ and $M \in \mathcal{B}$, where $\text{tr}_{X, M}$ is to take the trace of any morphism $f \in \text{End}_{\mathcal{B}}(X \otimes M)$ only on X , i.e., $\text{tr}_{X, M}(f) = (\text{ev}_X^L \otimes \text{id}_M) \circ (\text{id}_X \otimes f) \circ (\text{coev}_X^R \otimes \text{id}_M)$.⁶ Such a trace on a \mathcal{C} -module category is called a module trace [139]. It is known that any module category over a unitary fusion category \mathcal{C} admits a module trace, which is unique up to scaling if \mathcal{M} is indecomposable. Therefore, a semisimple indecomposable 2d TFT with unitary fusion category symmetry \mathcal{C}

⁵For example, the algebra A of local operators is obtained as the algebra of natural transformations from the identity functor of \mathcal{B} to itself. Equivalently, A is isomorphic to the center of the endomorphism algebra $\text{End}_{\mathcal{B}}(X)$ for a sufficiently “large” boundary condition X . It is well-known that A is a commutative Frobenius algebra [138].

⁶Here, we implicitly identified the left dual and right dual of X assuming that \mathcal{C} is pivotal.

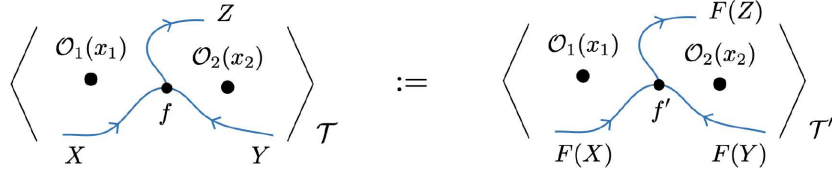


Figure 4.4: The pullback of a fusion category symmetry. The correlation functions of the new QFT \mathcal{T} with symmetry \mathcal{C} are defined by those of the original QFT \mathcal{T}' with symmetry \mathcal{C}' . The topological defect networks on the left-hand side and right-hand side are related by the tensor functor $F : \mathcal{C} \rightarrow \mathcal{C}'$. Specifically, the topological junction $f' \in \text{Hom}_{\mathcal{C}'}(F(X) \otimes F(Y), F(Z))$ on the right-hand side is given by $f' = F(f) \circ J_{XY}$, where $J_{XY} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ is the natural isomorphism associated with the tensor functor F .

should be completely characterized by an indecomposable \mathcal{C} -module category. Conversely, any indecomposable \mathcal{C} -module category should serve as a category of boundary conditions of a semisimple indecomposable 2d TFT with symmetry \mathcal{C} . Thus, we end up with the same classification as in the previous subsection: semisimple 2d bosonic TFTs with fusion category symmetry \mathcal{C} are classified by \mathcal{C} -module categories.

A more direct correspondence between the data of a \mathcal{C} -module category and those of 2d TFTs with symmetry \mathcal{C} was elucidated in [68]. Specifically, the explicit formula for general correlation functions of 2d TFTs with fusion category symmetry \mathcal{C} are given in terms of the data of \mathcal{C} -module categories. However, the correlation functions given in [68] have not been proven to satisfy some of the consistency conditions of topological field theories. Later in this chapter, we provide another way to construct 2d bosonic TFTs with fusion category symmetry by using the state sum construction and pullback of fusion category symmetries. Although we will not give an explicit formula for general correlation functions, our approach enables us to compute general correlation functions concretely.

4.2 Pullback of fusion category symmetries

Pullback of a symmetry of a quantum field theory (QFT) is a procedure to change the symmetry without affecting the underlying QFT. More specifically, given a QFT \mathcal{T}' with a fusion category symmetry \mathcal{C}' , we can define a new QFT \mathcal{T} with another symmetry \mathcal{C} by pulling back the symmetry \mathcal{C}' by a tensor functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ as follows: the correlation functions of the new QFT \mathcal{T} in the presence of a defect network D of the symmetry \mathcal{C} are defined by the correlation functions of the original QFT \mathcal{T}' in the presence of a defect network that is obtained by applying the functor F to the defect network D . That is, the correlation function of the new QFT is schematically written as

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle_{\mathcal{T}}^D := \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle_{\mathcal{T}'}^{F(D)}, \quad (4.2.1)$$

where the left-hand side is the correlation function of the new QFT \mathcal{T} , while the right-hand side is the correlation function of the original QFT \mathcal{T}' , see also Figure 4.4 for a pictorial representation of the above equation. We note that the pullback of a fusion category symmetry does not change the correlation functions of local operators in the absence of symmetry defects. This implies that the pullback only changes the symmetry of a QFT.

In this section, we explicitly show that given a 2d TFT \mathcal{T}' with symmetry \mathcal{C}' and a tensor functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, we can construct a 2d TFT \mathcal{T} with symmetry \mathcal{C} by pulling back the

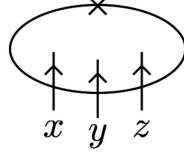


Figure 4.5: The Hilbert space on the above spatial circle is given by $Z((x \otimes y) \otimes z)$, where the base point on the circle is represented by the cross mark in the above figure. We can also assign a Hilbert space to a circle with an arbitrary number of topological defects in a similar way.

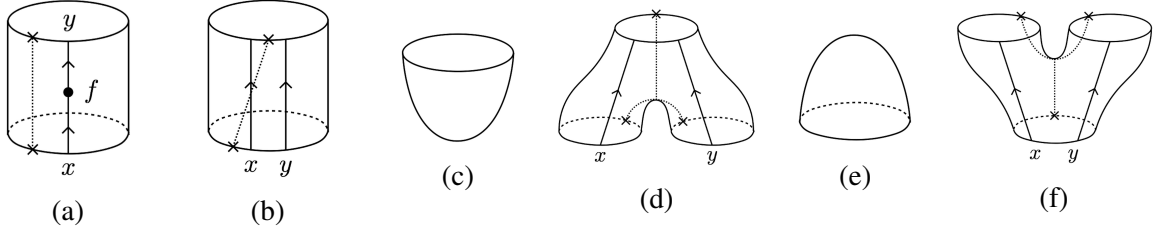


Figure 4.6: The building blocks of linear maps. (a) The cylinder amplitude $Z(f)$. (b) The change of the base point $X_{x,y}$. (c) The unit η . (d) The multiplication $M_{x,y}$. (e) The counit ϵ . (f) The comultiplication $\Delta_{x,y}$. Each diagram represents a linear map from the Hilbert space assigned to the bottom circles to the Hilbert space assigned to the top circles.

symmetry of the TFT \mathcal{T}' by the tensor functor F . Although the pullback of a fusion category symmetry may look innocuous, it turns out that this procedure is essential for the construction of topological field theories with general fusion category symmetry. Indeed, as we will see in Section 4.3, the pullback of a fusion category symmetry in combination with the state sum construction of 2d bosonic TFTs enables us to construct all 2d bosonic TFTs with general fusion category symmetry. Although we only consider bosonic TFTs in this section, we believe that the pullback should also be valid for fermionic TFTs as long as fusion categories and tensor functors are replaced by superfusion categories and supertensor functors.

Before proceeding, we notice that the symmetry \mathcal{C} of the new TFT \mathcal{T} may not act faithfully even if the action of the original symmetry \mathcal{C}' is faithful. This is not an issue but a general feature of topological field theories. For instance, a fusion category symmetry \mathcal{C} acts unfaithfully on local operators of a TFT unless the symmetry \mathcal{C} is spontaneously broken. Such an unfaithful action occurs in the low energy limit of a gapped phase of a UV theory (e.g., a lattice model) where the symmetry acts faithfully on physical degrees of freedom.

4.2.1 Topological field theories with fusion category symmetries

We first review the axiomatic formulation of 2d unitary TFTs with fusion category symmetry \mathcal{C} following [11].⁷ A 2d TFT assigns a Hilbert space $Z(x)$ to a spatial circle with the insertion of a topological defect $x \in \mathcal{C}$ running along the time direction.⁸ When the spatial circle has multiple topological defects x, y, z, \dots , the Hilbert space is given by $Z(((x \otimes y) \otimes z) \otimes \dots)$, where the order of the tensor product is determined by the position of the base point on the circle, see Figure 4.5. A 2d TFT also assigns a linear map to a two-dimensional surface decorated by a network of topological defects. The linear map assigned to an arbitrary surface is composed of the following building blocks, see also Figure 4.6:

⁷A unitary 2d bosonic TFT is automatically semisimple [68, 82, 137].

⁸In this subsection, objects of a fusion category are denoted by small letters to avoid a conflict of notation.

1. the cylinder amplitude $Z(f) : Z(x) \rightarrow Z(y)$ for $f \in \text{Hom}_{\mathcal{C}}(x, y)$
2. the change of the base point $X_{x,y} : Z(x \otimes y) \rightarrow Z(y \otimes x)$
3. the unit $\eta : \mathbb{C} \rightarrow Z(\mathbf{1}_{\mathcal{C}})$
4. the multiplication $M_{x,y} : Z(x) \otimes Z(y) \rightarrow Z(x \otimes y)$
5. the counit $\epsilon : Z(\mathbf{1}_{\mathcal{C}}) \rightarrow \mathbb{C}$
6. the comultiplication $\Delta_{x,y} : Z(x \otimes y) \rightarrow Z(x) \otimes Z(y)$

For unitary TFTs, the counit ϵ and the comultiplication $\Delta_{x,y}$ are the adjoints of the unit η and the multiplication $M_{x,y}$ respectively, i.e. $\epsilon = \eta^\dagger$ and $\Delta_{x,y} = M_{x,y}^\dagger$. In particular, the counit ϵ and the comultiplication $\Delta_{x,y}$ are no longer independent data of a TFT.

For the well-definedness of the cylinder amplitude, we require that $Z(f)$ is \mathbb{C} -linear in morphisms and preserves the composition of morphisms:

$$Z(\lambda f + \lambda' f') = \lambda Z(f) + \lambda' Z(f'), \quad \forall \lambda, \lambda' \in \mathbb{C}, \forall f, f' \in \text{Hom}_{\mathcal{C}}(x, y), \quad (4.2.2)$$

$$Z(g \circ f) = Z(g) \circ Z(f), \quad \forall f \in \text{Hom}_{\mathcal{C}}(x, y), \forall g \in \text{Hom}_{\mathcal{C}}(y, z). \quad (4.2.3)$$

Thus, a 2d TFT with fusion category symmetry \mathcal{C} gives us a functor $Z : \mathcal{C} \rightarrow \text{Vec}$ from \mathcal{C} to the category of vector spaces. This functor obeys various consistency conditions so that the assignment of Hilbert spaces and linear maps are well-defined. Specifically, a TFT with fusion category symmetry \mathcal{C} is a functor $Z : \mathcal{C} \rightarrow \text{Vec}$ equipped with a set of linear maps (X, η, M) that satisfies the following consistency conditions [11]:

1. Well-definedness of the change of the base point:

$$X_{x,y} = X_{y,x}^{-1}. \quad (4.2.4)$$

2. Naturality of the change of the base point:

$$\begin{aligned} Z(\text{id}_y \otimes f) \circ X_{x,y} &= X_{x',y} \circ Z(f \otimes \text{id}_y), \quad \forall f \in \text{Hom}(x, x'), \\ Z(g \otimes \text{id}_x) \circ X_{x,y} &= X_{x,y'} \circ Z(\text{id}_x \otimes g), \quad \forall g \in \text{Hom}(y, y'). \end{aligned} \quad (4.2.5)$$

3. Associativity of the change of the base point:

$$X_{y,z \otimes x} \circ Z(\alpha_{y,z,x}) \circ X_{x,y \otimes z} \circ Z(\alpha_{x,y,z}) = Z(\alpha_{x,y,z}^{-1}) \circ X_{x \otimes y,z}. \quad (4.2.6)$$

4. Non-degeneracy of the pairing:

$$\text{The pairing } \eta^\dagger \circ Z(\text{ev}_x^L) \circ M_{x^*,x} : Z(x^*) \otimes Z(x) \rightarrow \mathbb{C} \text{ is non-degenerate.} \quad (4.2.7)$$

5. Unit constraint:

$$M_{\mathbf{1}_{\mathcal{C}},x} \circ (\eta \otimes \text{id}_{Z(x)}) = \text{id}_{Z(x)} = M_{x,\mathbf{1}_{\mathcal{C}}} \circ (\text{id}_{Z(x)} \otimes \eta). \quad (4.2.8)$$

6. Associativity of the multiplication:

$$Z(\alpha_{x,y,z}) \circ M_{x \otimes y,z} \circ (M_{x,y} \otimes \text{id}_{Z(z)}) = M_{x,y \otimes z} \circ (\text{id}_{Z(x)} \otimes M_{y,z}). \quad (4.2.9)$$

7. Twisted commutativity:

$$M_{y,x}(\psi_y \otimes \psi_x) = X_{x,y} \circ M_{x,y}(\psi_x \otimes \psi_y), \quad \forall \psi_x \in Z(x), \forall \psi_y \in Z(y). \quad (4.2.10)$$

8. Naturality of the multiplication:

$$\begin{aligned} M_{x',y} \circ (Z(f) \otimes \text{id}_{Z(y)}) &= Z(f \otimes \text{id}_y) \circ M_{x,y}, \quad \forall f \in \text{Hom}_{\mathcal{C}}(x, x'), \\ M_{x,y'} \circ (\text{id}_{Z(x)} \otimes Z(g)) &= Z(\text{id}_x \otimes g) \circ M_{x,y}, \quad \forall g \in \text{Hom}_{\mathcal{C}}(y, y'). \end{aligned} \quad (4.2.11)$$

9. Uniqueness of the multiplication:

$$\begin{aligned} &Z((\text{id}_x \otimes \text{ev}_z^L) \otimes \text{id}_{y^*}) \circ \mathcal{A}_{(x \otimes z^*) \otimes (z \otimes y^*) \rightarrow (x \otimes (z^* \otimes z)) \otimes y^*} \circ M_{x \otimes z^*, z \otimes y^*} \\ &= Z(\text{id}_{x \otimes y^*} \otimes \text{ev}_z^R) \circ \mathcal{A}_{(z^* \otimes x) \otimes (y^* \otimes z) \rightarrow (x \otimes y^*) \otimes (z \otimes z^*)} \circ M_{z^* \otimes x, y^* \otimes z} \circ (X_{x,z^*} \otimes X_{z,y^*}), \end{aligned} \quad (4.2.12)$$

where \mathcal{A} is a generalized associator that we will define below.

10. Consistency on the torus:

$$\begin{aligned} &\mathcal{A}_{(x \otimes w) \otimes (z \otimes y) \rightarrow (y \otimes x) \otimes (w \otimes z)} \circ M_{x \otimes w, z \otimes y} \circ (X_{w,x} \otimes X_{y,z}) \circ M_{w \otimes x, y \otimes z}^\dagger \\ &= M_{y \otimes x, w \otimes z} \circ (X_{x,y} \otimes X_{z,w}) \circ M_{x \otimes y, z \otimes w}^\dagger \circ \mathcal{A}_{(w \otimes x) \otimes (y \otimes z) \rightarrow (x \otimes y) \otimes (z \otimes w)}. \end{aligned} \quad (4.2.13)$$

In the last two equations, the generalized associator $\mathcal{A}_{p \rightarrow q} : Z(p) \rightarrow Z(q)$ is defined by the composition of the change of the base point X and the associator $Z(\alpha)$. We note that the isomorphism $\mathcal{A}_{p \rightarrow q}$ is uniquely determined by p and q [11].

In summary, a 2d unitary TFT with fusion category symmetry \mathcal{C} is a functor $Z : \mathcal{C} \rightarrow \text{Vec}$ equipped with a triple (X, η, M) that satisfies the consistency conditions (4.2.4)–(4.2.13). As we discussed in Section 4.1, semisimple indecomposable 2d bosonic TFTs with fusion category symmetry \mathcal{C} are classified by indecomposable \mathcal{C} -module categories. Therefore, it should be possible in principle to write down a solution (Z, X, η, M) of the consistency conditions (4.2.4)–(4.2.13) in terms of the data of a \mathcal{C} -module category. However, the relation between the quadruple (Z, X, η, M) and the data of a \mathcal{C} -module category is not clear at this stage. In Section 4.2.3, using the pullback of a fusion category symmetry, we will give a concrete relation between a 2d TFT (Z, X, η, M) and the data of a \mathcal{C} -module category when the vacuum of the TFT is unique on a circle, i.e., when the TFT describes the low energy limit of an SPT phase.

4.2.2 Pullback of fusion category symmetries in TFTs

Let (Z', X', η', M') be a 2d TFT with symmetry \mathcal{C}' . Given a tensor functor $(F, J, \phi) : \mathcal{C} \rightarrow \mathcal{C}'$, we can construct a 2d TFT (Z, X, η, M) with symmetry \mathcal{C} by defining the functor $Z : \mathcal{C} \rightarrow \text{Vec}$ and the linear maps (X, η, M) as follows:

$$Z := Z' \circ F, \quad (4.2.14)$$

$$X_{x,y} := Z'(J_{y,x}) \circ X'_{F(x), F(y)} \circ Z'(J_{x,y}^{-1}), \quad (4.2.15)$$

$$\eta := Z'(\phi) \circ \eta', \quad (4.2.16)$$

$$M_{x,y} := Z'(J_{x,y}) \circ M'_{F(x), F(y)}. \quad (4.2.17)$$

We can show that the quadruple (Z, X, η, M) defined above becomes a 2d TFT, provided that (Z', X', η', M') satisfies the consistency conditions (4.2.4)–(4.2.13). We will explicitly check

some of the consistency conditions for (Z, X, η, M) below. The other equations can also be checked similarly.

Let us begin with eq. (4.2.4). This equation holds because the right-hand side can be written as

$$\text{RHS} = (Z'(J_{x,y}) \circ X'_{F(y),F(x)} \circ Z'(J_{y,x}^{-1}))^{-1} = Z'(J_{y,x}) \circ X'_{F(x),F(y)} \circ Z'(J_{x,y}^{-1}) = \text{LHS}, \quad (4.2.18)$$

where we used the fact that X' satisfies eq. (4.2.4). Equation (4.2.5) follows from the naturality of J :

$$F(g \otimes f) \circ J_{y,x} = J_{y',x'} \circ (F(g) \otimes F(f)), \quad \forall g \in \text{Hom}_{\mathcal{C}}(y, y'), \quad \forall f \in \text{Hom}_{\mathcal{C}}(x, x'). \quad (4.2.19)$$

Indeed, if we set either g or f to the identity morphism and use eq. (4.2.5) for X' , we obtain eq. (4.2.5) for X . To show eq. (4.2.6), we note that $F(\alpha_{xyz})$ can be written in terms of the associators $\alpha'_{F(x),F(y),F(z)}$ of \mathcal{C}' as

$$F(\alpha_{xyz}) = J_{x,y \otimes z} \circ (\text{id}_{F(x)} \otimes J_{y,z}) \circ \alpha'_{F(x),F(y),F(z)} \circ (J_{x,y}^{-1} \otimes \text{id}_{F(z)}) \circ J_{x \otimes y, z}^{-1}, \quad (4.2.20)$$

which immediately follows from the consistency condition (2.1.7) on the natural isomorphism J . We also notice that the naturality (4.2.5) of X' implies

$$\begin{aligned} X'_{F(x \otimes y), F(z)} &= Z'(\text{id}_{F(z)} \otimes J_{x,y}) \circ X'_{F(x) \otimes F(y), F(z)} \circ Z'(J_{x,y}^{-1} \otimes \text{id}_{F(z)}), \\ X'_{F(x), F(y \otimes z)} &= Z'(J_{y,z} \otimes \text{id}_{F(x)}) \circ X'_{F(x), F(y) \otimes F(z)} \circ Z'(\text{id}_{F(x)} \otimes J_{y,z}^{-1}). \end{aligned} \quad (4.2.21)$$

By plugging eqs. (4.2.20) and (4.2.21) into the left-hand side of eq. (4.2.6), we find

$$\begin{aligned} \text{LHS} &= Z'(J_{z \otimes x, y}) \circ Z'(J_{z,x} \otimes \text{id}_{F(y)}) \circ X'_{F(y), F(z) \otimes F(x)} \circ Z'(\alpha'_{F(y), F(z), F(x)}) \\ &\quad \circ X'_{F(x), F(y) \otimes F(z)} \circ Z'(\alpha'_{F(x), F(y), F(z)}) \circ Z'(J_{x,y}^{-1} \otimes \text{id}_{F(z)}) \circ Z'(J_{x \otimes y, z}^{-1}) \\ &= Z'(J_{z \otimes x, y}) \circ Z'(J_{z,x} \otimes \text{id}_{F(y)}) \circ Z'((\alpha'_{F(z), F(x), F(y)})^{-1}) \\ &\quad \circ X'_{F(x) \otimes F(y), F(z)} \circ Z'(J_{x,y}^{-1} \otimes \text{id}_{F(z)}) \circ Z'(J_{x \otimes y, z}^{-1}) \\ &= \text{RHS}. \end{aligned} \quad (4.2.22)$$

The non-degeneracy condition (4.2.7) for an object $x \in \mathcal{C}$ follows from that for $F(x) \in \mathcal{C}'$ because

$$\eta^\dagger \circ Z(\text{ev}_x^L) \circ M_{x^*, x} = (\eta')^\dagger \circ Z'(\text{ev}_{F(x)}^L) \circ M'_{F(x)^*, F(x)}, \quad (4.2.23)$$

where we used $F(x)^* = F(x^*)$ and $F(\text{ev}_x^L) = \phi \circ \text{ev}_{F(x)}^L \circ J_{x^*, x}^{-1}$, cf. Exercise 2.10.6. in [29]. The unit constraint (4.2.8) is an immediate consequence of the consistency condition (2.1.8) and eqs. (4.2.8) and (4.2.11) for (η', M') . We can also check the remaining equations similarly. Thus, we find that the quadruple (Z, X, η, M) defined by eqs. (4.2.14)–(4.2.17) becomes a 2d TFT with symmetry \mathcal{C} . We call a TFT (Z, X, η, M) the pullback of a TFT (Z', X', η', M') by a tensor functor $(F, J, \phi) : \mathcal{C} \rightarrow \mathcal{C}'$. It should be straightforward to generalize the above procedure to the case of multifusion category symmetries.

A general scheme to construct TFTs with fusion category symmetries. By using the pullback, we can construct all the 1+1d bosonic TFTs with general unitary fusion category symmetry \mathcal{C} . To see this, we first recall that every unitary fusion category \mathcal{C} is equivalent to the representation category $\text{Rep}(H)$ of a semisimple weak Hopf algebra H and bosonic TFTs with $\text{Rep}(H)$ symmetry are classified by $\text{Rep}(H)$ -module categories. Any indecomposable

module category over a unitary fusion category $\text{Rep}(H)$ is equivalent to the category ${}_K\mathcal{M}$ of left K -modules where K is an H -simple left H -comodule algebra [98].⁹ When K is a left H -comodule algebra, we have a tensor functor $F_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$ that represents the $\text{Rep}(H)$ -module category structure on ${}_K\mathcal{M}$. This functor enables us to construct a $\text{Rep}(H)$ symmetric TFT from a ${}_K\mathcal{M}_K$ symmetric one by pulling back the symmetry. In order to obtain a $\text{Rep}(H)$ symmetric TFT corresponding to a $\text{Rep}(H)$ -module category ${}_K\mathcal{M}$, we need to pull back a ${}_K\mathcal{M}_K$ symmetric TFT whose category of boundary condition is ${}_K\mathcal{M}$. Such a TFT can be systematically constructed from the semisimple algebra K by the state sum construction as we will describe in detail in Section 4.3. Thus, by pulling back the state sum TFT constructed from an H -simple left H -comodule algebra K by the tensor functor $F_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$, we obtain the $\text{Rep}(H)$ symmetric TFT whose category of boundary conditions is the $\text{Rep}(H)$ -module category ${}_K\mathcal{M}$, see Section 4.3.4 for more details. We emphasize that all bosonic TFTs with $\text{Rep}(H)$ symmetry can be obtained by the above procedure.

4.2.3 SPT phases with fusion category symmetries

As an example, we consider the pullback of a trivial TFT. Here, the trivial TFT is a topological field theory such that the vacuum on a circle is unique and the transition amplitude on every surface is the identity map. More specifically, the data $(Z^{\text{triv}}, X^{\text{triv}}, \eta^{\text{triv}}, M^{\text{triv}})$ of the trivial TFT are given by

$$Z^{\text{triv}} = 1_{\text{Vec}}, \quad X_{x,y} = c_{x,y}^{\text{triv}}, \quad \eta = \text{id}_{\mathbb{C}}, \quad M_{x,y} = \text{id}_{x \otimes y}, \quad (4.2.24)$$

where $c_{x,y}^{\text{triv}} : x \otimes y \rightarrow y \otimes x$ is the trivial braiding of \mathbb{C} -vector spaces. Since the trivial TFT has a trivial symmetry Vec , it can be pulled back to a TFT with fusion category symmetry \mathcal{C} if there exists a tensor functor from \mathcal{C} to Vec , i.e., a fiber functor. The TFT obtained by the pullback of a trivial TFT has a unique vacuum on a circle because the pullback does not affect the underlying TFT. In other words, this TFT describes an SPT phase with fusion category symmetry \mathcal{C} . Every SPT phase with symmetry \mathcal{C} can be obtained in this way because SPT phases with symmetry \mathcal{C} are classified by fiber functors as we reviewed in Section 4.1 [13].

The data (Z, X, η, M) of an SPT phase with symmetry \mathcal{C} are determined by the formula (4.2.14)–(4.2.17). Concretely, the SPT phase obtained by the pullback of the trivial TFT by a fiber functor $(F, J, \phi) : \mathcal{C} \rightarrow \text{Vec}$ consists of the following data:

$$Z = F, \quad X_{x,y} = J_{x,y} \circ c_{x,y}^{\text{triv}} \circ J_{x,y}^{-1}, \quad \eta = \phi, \quad M_{x,y} = J_{x,y}. \quad (4.2.25)$$

The above data can also be derived by directly solving the consistency conditions (4.2.4)–(4.2.13) under the condition that the state space $Z(1_{\mathcal{C}})$ on a circle without a topological line is one-dimensional [140].

Pullback of more general TFTs. Let \mathcal{T} be a \mathcal{D} -symmetric bosonic TFT whose category of boundary conditions is a \mathcal{D} -module category \mathcal{M} . The pullback of this TFT \mathcal{T} by a tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is denoted by $F^*\mathcal{T}$. Since the pullback does not change the underlying TFT, the category of boundary conditions of the \mathcal{C} -symmetric TFT $F^*\mathcal{T}$ should also be \mathcal{M} as a

⁹A left H -comodule algebra K with a comodule action $\lambda_K^L : K \rightarrow H \otimes K$ is said to be H -simple if it does not have a proper non-zero ideal $I \subset K$ such that $\lambda_K^L(I) \subset H \otimes I$. Since H and K are supposed to be semisimple, K is H -simple if and only if it cannot be decomposed into a direct sum of two non-zero H -comodule algebras.

category.¹⁰ The \mathcal{C} -module category structure on \mathcal{M} is induced by the \mathcal{D} -module category structure on it via eq. (2.1.21). Thus, the pullback $F^*\mathcal{T}$ of a \mathcal{D} -symmetric TFT \mathcal{T} whose category of boundary conditions is a \mathcal{D} -module category \mathcal{M} is a \mathcal{C} -symmetric TFT whose category of boundary conditions is a \mathcal{C} -module category \mathcal{M} . This fact will be used in Section 4.3.4 to construct general bosonic TFTs with fusion category symmetries. The above construction reduces to the construction of SPT phases when \mathcal{T} is the trivial TFT, i.e., when $\mathcal{D} = \mathcal{M} = \text{Vec}$.

4.3 State sum construction of bosonic TFTs

The state sum construction is a recipe to construct a 2d bosonic TFT from a semisimple algebra [80, 81]. The state sum TFT constructed from a semisimple algebra K will be denoted by \mathfrak{B}_K . The TFT \mathfrak{B}_K can be defined on two-dimensional oriented surfaces with topological line defects labeled by (K, K) -bimodules [141], which implies that this TFT has a symmetry described by the category ${}_K\mathcal{M}_K$ of (K, K) -bimodules. When the input algebra K is a left comodule algebra over a semisimple weak Hopf algebra H , we can pull back the symmetry ${}_K\mathcal{M}_K$ by the tensor functor $F_K : \text{Rep}(H) \rightarrow \text{End}({}_K\mathcal{M}) \cong {}_K\mathcal{M}_K$. This procedure changes the ${}_K\mathcal{M}_K$ symmetry of the state sum TFT \mathfrak{B}_K into $\text{Rep}(H)$ symmetry. By a slight abuse of notation, the TFT with $\text{Rep}(H)$ symmetry obtained in this way will also be denoted by \mathfrak{B}_K , which is justified because the pullback does not change the underlying TFT. In this section, we explicitly construct the TFT \mathfrak{B}_K on oriented surfaces in the presence of topological defects of $\text{Rep}(H)$ symmetry. We also show that topological boundary conditions of this TFT form a $\text{Rep}(H)$ -module category ${}_K\mathcal{M}$, namely, the TFT \mathfrak{B}_K describes the low energy limit of the $\text{Rep}(H)$ symmetric topological phase $\mathcal{T}_{{}_K\mathcal{M}}$.

4.3.1 Bosonic state sum TFTs with defects

Let us first review the state sum construction of 2d bosonic TFTs with defects following [141]. We slightly modify the description of topological junctions in [141] so that it fits into the context of TFTs with fusion category symmetries discussed in the previous section.

Let Σ be an oriented two-dimensional surface with boundary $\partial\Sigma$. Each connected component of the boundary $\partial\Sigma$ is equipped with a base point and has its own orientation. Depending on the orientation, each connected component of $\partial\Sigma$ belongs to either the in-boundary $\partial_{\text{in}}\Sigma$ or the out-boundary $\partial_{\text{out}}\Sigma$. More specifically, a connected component of $\partial\Sigma$ belongs to the in-boundary if its orientation agrees with the orientation induced by that of Σ , while it belongs to the out-boundary if its orientation is the opposite. The surface Σ is decorated by a network of topological lines that are labeled by objects of the category ${}_K\mathcal{M}_K$, where K is a semisimple algebra. We assume that the junctions of these topological lines are trivalent and labeled by morphisms of ${}_K\mathcal{M}_K$.

To define the transition amplitude on Σ , we first give a triangulation $T(\Sigma)$ of Σ such that every triangle t contains at most one trivalent junction and every edge e intersects at most one topological line. The base points on the boundary $\partial\Sigma$ are supposed to be vertices of the triangulation $T(\Sigma)$. We also equip the triangulation $T(\Sigma)$ with the following additional data:

- orientations of all edges
- a marked edge on the boundary of each triangle

¹⁰We recall that the data of the underlying TFT are completely encoded in the category of boundary conditions as we reviewed in Section 4.1.2.

The orientations of edges on the boundary $\partial\Sigma$ are chosen so that they agree with the orientation of $\partial\Sigma$. On the other hand, the orientations of internal edges and the positions of marked edges can be chosen arbitrarily.¹¹

One can enumerate possible configurations of topological defects on a triangle as follows:

$$(1) \quad , (2) \quad , (3) \quad , (4) \quad , (5) \quad (4.3.1)$$

Here, we assume that each triangle is equipped with an orientation that induces the counter-clockwise orientation on its boundary. The blue lines in eq. (4.3.1) represent topological lines labeled by (K, K) -bimodules $X, X_1, X_2, X_3 \in {}_K\mathcal{M}_K$, and trivalent junctions in configurations (4) and (5) are labeled by bimodule morphisms $f \in \text{Hom}_{{}_K\mathcal{M}_K}(X_3, X_1 \otimes_K X_2)$ and $g \in \text{Hom}_{{}_K\mathcal{M}_K}(X_1 \otimes_K X_2, X_3)$. Marked edges are those marked with a small red triangle. We note that every configuration listed above can be regarded as a special case of configuration (5) because configurations (1)–(3) are obtained by taking some of the topological lines to be trivial lines and configuration (4) is obtained by replacing one of the topological lines by its dual.¹² Therefore, we need not consider configurations (1)–(4) independently. However, it is convenient to deal with these configurations separately when we compute transition amplitudes.

The transition amplitude on the triangulated surface $T(\Sigma)$ is a linear map from the vector space $Z_T(\partial_{\text{in}}\Sigma)$ on the in-boundary to the vector space $Z_T(\partial_{\text{out}}\Sigma)$ on the out-boundary. Each of these vector spaces is defined by the tensor product of vector spaces assigned to the edges on the boundary. In particular, when the boundary $\partial_a\Sigma$ ($a = \text{in}, \text{out}$) consists of a single connected component, the vector space $Z_T(\partial_a\Sigma)$ is given by

$$Z_T(\partial_a\Sigma) := \bigotimes_{e \in \partial_a\Sigma} R_e, \quad (4.3.2)$$

where R_e is defined as follows:

$$R_e := \begin{cases} K & \text{when no topological line intersects } e, \\ X_e & \text{when a topological line } X_e \text{ intersects } e \text{ from the right of } e, \\ X_e^* & \text{when a topological line } X_e \text{ intersects } e \text{ from the left of } e. \end{cases} \quad (4.3.3)$$

Here, the left and right of e are defined with respect to the orientation of e . The order of the tensor product in eq. (4.3.2) is determined by the position of the base point and the orientation of the boundary, see Figure 4.7 for more details. When the boundary $\partial_a\Sigma$ has more than one connected component, the vector space $Z_T(\partial_a\Sigma)$ is given by the tensor product of vector spaces on all connected components.

For the triangulated surface $T(\Sigma)$, we define the transition amplitude $Z_T(\Sigma)$ by the compo-

¹¹The TFT obtained by the state sum construction does not depend on these choices.

¹²A topological line labeled by the dual object $X^* \in {}_K\mathcal{M}_K$ is regarded as the orientation-reversal of a topological line labeled by X .

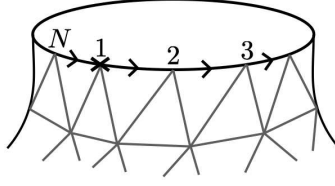


Figure 4.7: The vector space assigned to the above boundary is $R_{1,2} \otimes R_{2,3} \otimes \cdots \otimes R_{N,1}$, where $R_{i,i+1}$ is the vector space (4.3.3) assigned to the edge $[i, i+1]$ on the boundary and the vertex $[1]$ is the base point. The arrows on the boundary edges represent the orientation of the boundary.

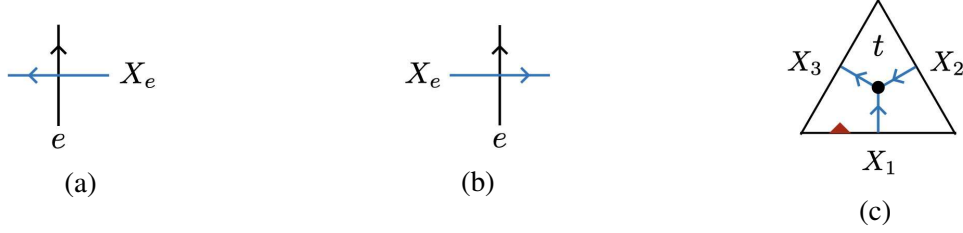


Figure 4.8: We assign vector spaces to the edges as follows: (a) $X_e^L = X_e$, $X_e^R = X_e^*$, (b) $X_e^L = X_e^*$, $X_e^R = X_e$, (c) $X_t^1 = X_1$, $X_t^2 = X_2$, and $X_t^3 = X_3^*$.

sition of the following linear maps [141]:

$$P(\Sigma) : Z_T(\partial_{\text{in}}\Sigma) \rightarrow Z_T(\partial_{\text{in}}\Sigma) \otimes \left(\bigotimes_{e \in \Sigma \setminus \partial_{\text{in}}\Sigma} X_e^L \otimes X_e^R \right), \quad (4.3.4)$$

$$c(\Sigma) : Z_T(\partial_{\text{in}}\Sigma) \otimes \left(\bigotimes_{e \in \Sigma \setminus \partial_{\text{in}}\Sigma} X_e^L \otimes X_e^R \right) \rightarrow \left(\bigotimes_{t \in \Sigma} X_t^1 \otimes X_t^2 \otimes X_t^3 \right) \otimes Z_T(\partial_{\text{out}}\Sigma), \quad (4.3.5)$$

$$E(\Sigma) : \left(\bigotimes_{t \in \Sigma} X_t^1 \otimes X_t^2 \otimes X_t^3 \right) \otimes Z_T(\partial_{\text{out}}\Sigma) \rightarrow Z_T(\partial_{\text{out}}\Sigma). \quad (4.3.6)$$

Detailed definitions of the above linear maps will be given shortly. In the above equations, X_e^L and X_e^R are vector spaces assigned to the left side and the right side of the edge e . Specifically, we define $X_e^L = X_e$ and $X_e^R = X_e^*$ when the edge e intersects a topological line X_e oriented from the right of e to the left of e . On the other hand, we define $X_e^L = X_e^*$ and $X_e^R = X_e$ when the topological line X_e intersecting the edge e is oriented in the opposite direction. When the edge e does not intersect a topological line, we define $X_e^L = X_e^R = K$. Similarly, X_t^i is the vector space assigned to the i th edge on the boundary of t , where the order of the edges is determined by the orientation of ∂t induced by that of t , with the marked edge being the first edge. Concretely, we have $X_t^i = X_e$ if the topological line X_e goes into t across its i th edge e , and we have $X_t^i = X_e^*$ otherwise. We note that the vector space X_t^i does not depend on the orientations of edges. See also Figure 4.8 for the assignment of vector spaces.

Let us now define the linear maps in eqs. (4.3.4), (4.3.5), and (4.3.6).

The linear map $P(\Sigma)$. The linear map $P(\Sigma)$ is defined by the tensor product of linear maps

$$P(\Sigma) := \text{id}_{Z_T(\partial_{\text{in}}\Sigma)} \otimes \left(\bigotimes_{e \in \Sigma \setminus \partial_{\text{in}}\Sigma} P_e \right), \quad (4.3.7)$$

where the tensor product is taken over all edges e of Σ except for those on the in-boundary. The linear map $P_e : \mathbb{C} \rightarrow X_e^L \otimes X_e^R$ for each edge $e \in \Sigma \setminus \partial_{\text{in}}\Sigma$ is given by

$$P_e := \begin{cases} \Delta_K \circ \eta_K & \text{when no topological line intersects } e, \\ \text{coev}_{X_e}^L & \text{when a topological line } X_e \text{ intersects } e \text{ from the right,} \\ \text{coev}_{X_e}^R & \text{when a topological line } X_e \text{ intersects } e \text{ from the left,} \end{cases} \quad (4.3.8)$$

where $\Delta_K : K \rightarrow K \otimes K$ is the comultiplication defined by eq. (2.2.7), $\eta_K : \mathbb{C} \rightarrow K$ is the unit of K , and coev_X^L and coev_X^R are the coevaluation maps defined by eq. (2.1.18). In eq. (4.3.8), the order of the tensor product over edges $e \in \Sigma \setminus \partial_{\text{in}}\Sigma$ is arbitrary.

The linear map $c(\Sigma)$. The linear map $c(\Sigma)$ is given by the composition of the trivial braiding c_{triv} , which only changes the order of the tensor product from $Z_T(\partial_{\text{in}}\Sigma) \otimes (\bigotimes_{e \in \Sigma \setminus \partial_{\text{in}}\Sigma} X_e^L \otimes X_e^R)$ to $(\bigotimes_{t \in \Sigma} X_t^1 \otimes X_t^2 \otimes X_t^3) \otimes Z_T(\partial_{\text{out}}\Sigma)$.

The linear map $E(\Sigma)$. The linear map $E(\Sigma)$ is again given by the tensor product

$$E(\Sigma) := \left(\bigotimes_{t \in \Sigma} E_t \right) \otimes \text{id}_{Z_T(\partial_{\text{out}}\Sigma)}, \quad (4.3.9)$$

where the order of the tensor product over triangles $t \in \Sigma$ is arbitrary. The linear map $E_t : X_t^1 \otimes X_t^2 \otimes X_t^3 \rightarrow \mathbb{C}$ for each triangle $t \in \Sigma$ shown in eq. (4.3.1) is defined as follows:

$$E_t := \begin{cases} (1) & \epsilon_K \circ m_K \circ (m_K \otimes \text{id}_K) : K \otimes K \otimes K \rightarrow \mathbb{C}, \\ (2) & \text{ev}_X^R \circ (\text{id}_X \otimes \rho_{X^*}^R) : X \otimes X^* \otimes K \rightarrow \mathbb{C}, \\ (3) & \text{ev}_X^R \circ (\rho_X^R \otimes \text{id}_{X^*}) : X \otimes K \otimes X^* \rightarrow \mathbb{C}, \\ (4) & \text{ev}_{X_1 \otimes X_2}^R \circ ((\iota_{X_1, X_2} \circ f) \otimes \text{id}_{X_2^* \otimes X_1^*}) : X_3 \otimes X_2^* \otimes X_1^* \rightarrow \mathbb{C}, \\ (5) & \text{ev}_{X_3}^R \circ ((g \circ \pi_{X_1, X_2}) \otimes \text{id}_{X_3^*}) : X_1 \otimes X_2 \otimes X_3^* \rightarrow \mathbb{C}, \end{cases} \quad (4.3.10)$$

Here, $\epsilon_K : K \rightarrow \mathbb{C}$ is the counit defined by eq. (2.2.8), $m_K : K \otimes K \rightarrow K$ is the multiplication of K , $\rho_X^R : X \otimes K \rightarrow X$ denotes the right K -module action on X , and π_{X_1, X_2} and ι_{X_1, X_2} are the splitting maps defined in Section 2.2.2. As we already mentioned, the linear maps for the configurations (1)–(4) are special cases of the linear map for the configuration (5). The above expression of the linear map E_t is based on a specific choice of a marked edge as in eq. (4.3.1). For other choices of a marked edge, we define the linear map E_t by the composition of eq. (4.3.10) and the trivial braiding c_{triv} that cyclically changes the order of the tensor product $X_t^1 \otimes X_t^2 \otimes X_t^3$.

The linear map $Z_T(\Sigma) := E(\Sigma) \circ c(\Sigma) \circ P(\Sigma)$ does not depend on the order of edges $e \in \Sigma \setminus \partial_{\text{in}}\Sigma$ in eq. (4.3.8) because both $E(\Sigma)$ and $c(\Sigma) \circ P(\Sigma)$ do not. Furthermore, $Z_T(\Sigma)$ does not depend on the order of triangles $t \in \Sigma$ in eq. (4.3.9) because for any vectors $v \in X_t^1 \otimes X_t^2 \otimes X_t^3$ and $v' \in X_{t'}^1 \otimes X_{t'}^2 \otimes X_{t'}^3$, we have $E_t \otimes E_{t'}(v \otimes v') = E_t(v) E_{t'}(v') = E_{t'} \otimes E_t(v' \otimes v)$. Therefore, the linear map $Z_T(\Sigma)$ is well-defined. In addition, $Z_T(\Sigma)$ does not depend on the additional data of the triangulation such as the orientations of edges and the positions of marked edges. We can verify this by a direct computation: the invariance of $Z_T(\Sigma)$ under the orientation flip of an edge follows from the symmetricity of the coevaluation morphism $\text{coev}_X^R = c_{\text{triv}} \circ \text{coev}_X^L$, and the invariance of $Z_T(\Sigma)$ under the change of a marked edge follows from the symmetricity of the evaluation morphism $\text{ev}_X^R = \text{ev}_X^L \circ c_{\text{triv}}$. Therefore, the linear map $Z_T(\Sigma)$ depends only on the triangulation $T(\Sigma)$.

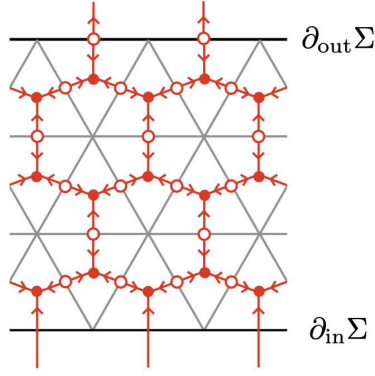


Figure 4.9: The transition amplitude of the state sum TFT \mathfrak{B}_K on an oriented surface without defects is represented by a string diagram that consists of oriented strands labeled by K . The string diagram is drawn on the Poincaré dual of the triangulation. A bivalent vertex (i.e., a white dot) represents $\Delta_K \circ \eta_K : \mathbb{C} \rightarrow K^{\otimes 2}$, while a trivalent vertex (i.e., a red dot) represents $\epsilon_K \circ m_K \circ (m_K \otimes \text{id}_K) : K^{\otimes 3} \rightarrow \mathbb{C}$. The order of the tensor product of K 's that appear in these linear maps is determined by the orientation of edges.

The linear map $Z_T(\Sigma)$ turns out to be topological, namely, $Z_T(\Sigma)$ is invariant under the change of a triangulation of Σ as long as it does not affect the triangulation of the boundary $\partial\Sigma$.¹³ However, $Z_T(\Sigma)$ is not yet the transition amplitude of a TFT on Σ because the linear map $Z_T(\partial_{\text{in}}\Sigma \times [0, 1])$ assigned to a cylinder $\partial_{\text{in}}\Sigma \times [0, 1]$ is not the identity map. Specifically, the linear map $Z_T(\partial_{\text{in}}\Sigma \times [0, 1])$ is an idempotent on $Z_T(\partial_{\text{in}}\Sigma)$, i.e., it satisfies $Z_T(\partial_{\text{in}}\Sigma \times [0, 1]) \circ Z_T(\partial_{\text{in}}\Sigma \times [0, 1]) = Z_T(\partial_{\text{in}}\Sigma \times [0, 1])$, which follows from the topological invariance of $Z_T(\Sigma)$. The transition amplitude of a TFT is obtained by restricting the source and target of $Z_T(\Sigma)$ to the images of the idempotents $Z_T(\partial_{\text{in}}\Sigma \times [0, 1])$ and $Z_T(\partial_{\text{out}}\Sigma \times [0, 1])$ respectively. We denote the bosonic state sum TFT defined in this way by \mathfrak{B}_K . The state space $Z(S^1)$ of this TFT on a circle S^1 is known to be isomorphic to the center of K [80].

It is instructive to see the transition amplitude on a surface without topological defects. When there are no topological defects, the transition amplitude $Z_T(\Sigma)$ on a triangulated surface $T(\Sigma)$ is a linear map from $K^{\otimes N_{\text{in}}}$ to $K^{\otimes N_{\text{out}}}$, where N_{in} and N_{out} denote the number of edges on the in-boundary and out-boundary. This transition amplitude is represented by a string diagram drawn on the triangulated surface $T(\Sigma)$ as in Figure 4.9. Topological invariance of $Z_T(\Sigma)$ follows from the Frobenius relation (2.2.4) and the Δ -separability (2.2.5).¹⁴ Moreover, the symmetricity (2.2.6) guarantees that the transition amplitude does not depend on the choice of orientations of internal edges.

4.3.2 The ${}_K\mathcal{M}_K$ symmetry of bosonic state sum TFTs

In the previous subsection, we constructed a bosonic TFT \mathfrak{B}_K on oriented surfaces with defects labeled by (K, K) -bimodules. This strongly suggests that the TFT \mathfrak{B}_K has a symmetry described by the category ${}_K\mathcal{M}_K$ of (K, K) -bimodules. However, the fact that the TFT \mathfrak{B}_K is defined on surfaces with defects labeled by (K, K) -bimodules is not sufficient to conclude that \mathfrak{B}_K has a symmetry ${}_K\mathcal{M}_K$. In order to show the ${}_K\mathcal{M}_K$ symmetry of \mathfrak{B}_K , we need to

¹³The proof of the topological invariance in [141] can still be applied even though the description of topological junctions is slightly modified.

¹⁴More specifically, the Frobenius relation implies the invariance under the Pachner 2-2 move, while the Δ -separability implies the invariance under the Pachner 3-1 move.

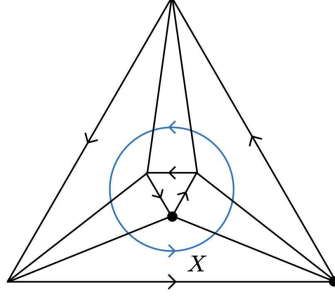


Figure 4.10: A triangulation of a cylinder. The blue circle represents a topological defect wrapping around a non-trivial cycle of the cylinder. The outermost triangle and the innermost triangle are the in-boundary and the out-boundary respectively, where the black dots represent the base points. The orientations of edges inside the cylinder and the positions of marked edges can be arbitrary.

show that the algebraic structure of the defects labeled by (K, K) -bimodule is described by the multifusion category ${}_K\mathcal{M}_K$. In particular, we need to show that the associator of the category of topological defects agrees with that of ${}_K\mathcal{M}_K$.

Based on the definition of the transition amplitude in the previous subsection, we can show that the two possible ways to resolve a 4-valent junction into two trivalent junctions are indeed related by the associator α_{X_1, X_2, X_3} of ${}_K\mathcal{M}_K$ defined by eq. (2.2.17). Namely, we have the following equality of transition amplitudes on a disk:

$$\begin{array}{c} X_1 \otimes_K (X_2 \otimes_K X_3) \\ \begin{array}{c} \text{Diagram 1: A square with vertices labeled } X_1, X_2, X_3. \text{ A central dot is labeled } \alpha. \text{ Arrows indicate a path from } X_1 \text{ to } X_2 \text{ and } X_3 \text{ to } X_2. \text{ The central dot } \alpha \text{ is connected to } X_2. \end{array} \\ X_2 \end{array} = \begin{array}{c} X_1 \otimes_K (X_2 \otimes_K X_3) \\ \begin{array}{c} \text{Diagram 2: A square with vertices labeled } X_1, X_2, X_3. \text{ A central dot is labeled } \text{id}. \text{ Arrows indicate a path from } X_1 \text{ to } X_2 \text{ and } X_3 \text{ to } X_2. \text{ The central dot } \text{id} \text{ is connected to } X_2. \end{array} \\ X_2 \end{array} \cdot \quad (4.3.11)$$

Here, the square in the above equation represents a local patch (i.e., a region isomorphic to a disk) of an arbitrary triangulated surface. Equation (4.3.11) implies that the symmetry of the state sum TFT is precisely described by ${}_K\mathcal{M}_K$.

We can compute the action of the ${}_K\mathcal{M}_K$ symmetry by evaluating the transition amplitude on a cylinder with a topological line inserted along the spatial direction. In order to write down the action of a symmetry defect $X \in {}_K\mathcal{M}_K$ on the state space on a circle, we take the triangulation of a cylinder $S^1 \times [0, 1]$ as shown in Figure 4.10. The transition amplitude on this triangulated cylinder can be computed by the state sum construction described in the previous subsection. A straightforward calculation shows that the symmetry action U_X is represented by the following string diagram:

$$U_X = \begin{array}{c} \text{Diagram: A string diagram representing the symmetry action } U_X. \text{ It shows a vertical line on the left, a vertical line on the right, and a horizontal line in the middle. A blue circle is drawn around the horizontal line, labeled } \lambda_X^L. \text{ The circle is also labeled } \rho_X^R \text{ and } \text{coev}_X^R. \text{ The horizontal line is labeled } K. \end{array} \quad (4.3.12)$$

Here, $\lambda_X^L : X \rightarrow K \otimes X$ denotes the left K -coaction on X defined by eq. (2.2.15). The composition of the symmetry operators U_X and U_Y becomes another symmetry operator $U_{X \otimes_K Y}$ as

it should be. Furthermore, the symmetry operator associated with the direct sum of topological lines X and Y is equal to the sum of the symmetry operators U_X and U_Y . This implies that the symmetry operators obey the fusion rules of ${}_K\mathcal{M}_K$, i.e., if $X \otimes_K Y \cong \bigoplus_Z N_{XY}^Z Z$, the symmetry operators satisfy $U_X \circ U_Y = \sum_Z N_{XY}^Z U_Z$. In particular, the vacua of the TFT \mathfrak{B}_K form a representation of the above fusion rules.

State sum construction as generalized gauging. The ${}_K\mathcal{M}_K$ symmetry of the state sum TFT \mathfrak{B}_K can also be understood from the point of view of generalized gauging. Here, the generalized gauging is a procedure to obtain a QFT \mathcal{T}' with some symmetry \mathcal{C}' from another QFT \mathcal{T} with a different symmetry \mathcal{C} [11, 15, 17, 18]. More specifically, when a fusion category \mathcal{C} has a Δ -separable symmetric Frobenius algebra object $A \in \mathcal{C}$, we can condense it by inserting a fine mesh of topological defects labeled by A . Condensing an algebra object $A \in \mathcal{C}$ produces a new QFT \mathcal{T}' , whose partition function is defined by the partition function of the original QFT \mathcal{T} in the presence of a fine mesh of A . This procedure is called generalized gauging because it generalizes the ordinary gauging of a finite group symmetry. The new QFT obtained by gauging the algebra object $A \in \mathcal{C}$ is denoted by \mathcal{T}/A . In general, when the symmetry of the original QFT \mathcal{T} is described by \mathcal{C} , the symmetry of the gauged QFT \mathcal{T}/A is described by the category ${}_A\mathcal{C}_A$ of (A, A) -bimodules in \mathcal{C} [11]. This is because a topological defect $X \in \mathcal{C}$ of the original QFT \mathcal{T} remains topological after gauging if and only if X is an (A, A) -bimodule. In particular, when the symmetry of the original QFT is Vec , we can gauge a Δ -separable symmetric Frobenius algebra $K \in \text{Vec}$ to obtain a new QFT with ${}_K\mathcal{M}_K$ symmetry. This is what happens in the state sum construction. Indeed, the state sum TFT can be regarded as the gauging of a trivial TFT by an algebra object $K \in \text{Vec}$ [142].

4.3.3 The category of boundary conditions

In this subsection, we show that the category of boundary conditions of the state sum TFT \mathfrak{B}_K is described by the ${}_K\mathcal{M}_K$ -module category ${}_K\mathcal{M}$, which is the category of left K -modules and is equipped with the ${}_K\mathcal{M}_K$ -module category structure reviewed in Section 2.2.2. To this end, we first notice that a boundary of the state sum TFT is equivalent to an interface between the state sum TFT and the trivial TFT. In general, a topological interface between state sum TFTs \mathfrak{B}_K and $\mathfrak{B}_{K'}$ is labeled by a (K, K') -bimodule [141]. This is a slight generalization of the fact that a topological interface between the same state sum TFT \mathfrak{B}_K is labeled by a (K, K) -bimodule as we saw in Section 4.3.1.¹⁵ Since the trivial TFT is the state sum TFT constructed from the trivial one-dimensional algebra \mathbb{C} , topological interfaces between \mathfrak{B}_K and the trivial TFT are labeled by (K, \mathbb{C}) -bimodules, i.e., left K -modules. This implies that the category of boundary conditions of \mathfrak{B}_K is equivalent to ${}_K\mathcal{M}$ as a category.

In order to determine the ${}_K\mathcal{M}_K$ -module category structure on the category of boundary conditions, let us compute the action of the ${}_K\mathcal{M}_K$ symmetry on the boundary states. The boundary state $|M\rangle$ corresponding to a topological boundary condition $M \in {}_K\mathcal{M}$ is defined by the transition amplitude on a cylinder illustrated in Figure 4.11a. The transition amplitude on this cylinder is a linear map from the state space of the trivial TFT to the state space of \mathfrak{B}_K . Since the state space of the trivial TFT is canonically isomorphic to \mathbb{C} , we can canonically identify the boundary state $|M\rangle$ with an element of the state space of \mathfrak{B}_K . Thus, we can regard $|M\rangle$ as a state of the TFT \mathfrak{B}_K . If we choose the triangulation of the cylinder as shown in

¹⁵A topological line of \mathfrak{B}_K is a topological interface between \mathfrak{B}_K and itself.

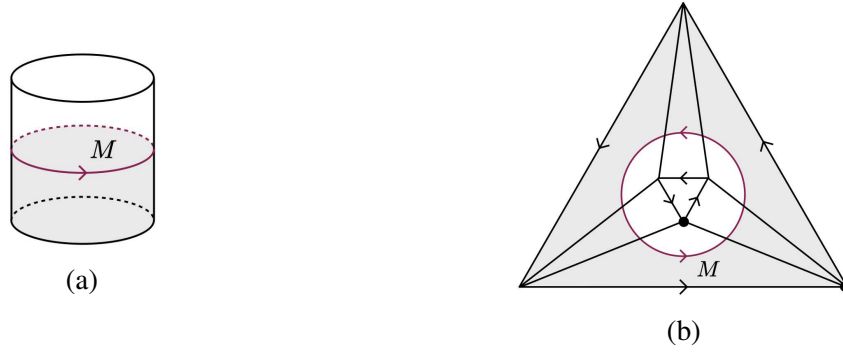


Figure 4.11: (a) The transition amplitude on the above cylinder defines a boundary state $|M\rangle$. The cylinder is separated into two regions by a topological interface $M \in {}_K\mathcal{M}$. The lower half (i.e., the shaded region) of the cylinder supports the trivial TFT, while the upper half (i.e., the unshaded region) supports the state sum TFT \mathfrak{B}_K . (b) A triangulation of the cylinder.

Figure 4.11b, the boundary state $|M\rangle$ can be computed as

$$|M\rangle = \begin{array}{c} K \quad K \quad K \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{ev}_M^L \\ \lambda_M^L \\ \text{coev}_M^R \end{array} \quad (4.3.13)$$

The symmetry operator U_X defined by eq. (4.3.12) acts on the above boundary state as

$$U_X |M\rangle = |X \otimes_K M\rangle. \quad (4.3.14)$$

Thus, the action of a topological line X on a topological boundary M is given by the tensor product over K , which is consistent with the ${}_K\mathcal{M}_K$ -module action on ${}_K\mathcal{M}$ that we reviewed in Section 2.2.2. Furthermore, we can show that the two different ways of resolving a 4-valent junction on the boundary are related by the module associativity isomorphism (2.2.18), namely, we have

$$M \otimes_K (X_1 \otimes_K X_2) = M \otimes_K (X_1 \otimes_K X_2), \quad (4.3.15)$$

where the shaded region represents the trivial TFT and the unshaded region represents the state sum TFT \mathfrak{B}_K . This shows that the category of boundary conditions of \mathfrak{B}_K is equivalent to ${}_K\mathcal{M}$ as a ${}_K\mathcal{M}_K$ -module category.

The category of boundary conditions of the TFT \mathfrak{B}_K can also be understood from the point of view of generalized gauging. As we discussed in the previous subsection, the state sum construction can be viewed as a generalized gauging of the trivial TFT [142]. In order to understand the category of boundary conditions of \mathfrak{B}_K , we consider how the generalized gauging affects the category of boundary conditions. Let \mathcal{T} be a 2d bosonic TFT with a fusion category symmetry \mathcal{C} and let \mathcal{B} be its category of boundary conditions. Since \mathcal{B} is the category

of boundary conditions of a \mathcal{C} -symmetric TFT, it is a left \mathcal{C} -module category as we reviewed in Section 4.1.2. Such a category can always be written as the category of right B -modules in \mathcal{C} for some Δ -separable symmetric Frobenius algebra object $B \in \mathcal{C}$ [95, 143]. When $\mathcal{B} \cong \mathcal{C}_B$ is the category of right B -modules in \mathcal{C} , the category of boundary conditions of the gauged TFT \mathcal{T}/A should be the category of (A, B) -bimodules in \mathcal{B} [43]. This is because the algebra object A is condensed in the gauged theory \mathcal{T}/A and hence a boundary condition in \mathcal{B} survives after gauging only when it is a left A -module.¹⁶ In the case of the state sum TFT \mathfrak{B}_K , the condensed algebra object is $K \in \text{Vec}$ and the category of boundary conditions of the original TFT is Vec . Therefore, the category of boundary conditions of the state sum TFT should be the category ${}_K\mathcal{M}$ of left K -modules.

4.3.4 Bosonic TFTs with fusion category symmetries

Let H be a semisimple weak Hopf algebra H whose representation category $\text{Rep}(H)$ is a unitary fusion category. When the input algebra K of the state sum TFT is a left H -comodule algebra, the ${}_K\mathcal{M}_K$ symmetry of the state sum TFT \mathfrak{B}_K can be pulled back to $\text{Rep}(H)$ symmetry by the tensor functor $F_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$ that represents the $\text{Rep}(H)$ -module category structure on ${}_K\mathcal{M}$. The pullback of the state sum TFT \mathfrak{B}_K by F_K is a $\text{Rep}(H)$ -symmetric TFT. The category of boundary conditions of this TFT is the $\text{Rep}(H)$ -module category ${}_K\mathcal{M}$ because the category of boundary conditions of the original ${}_K\mathcal{M}_K$ -symmetric TFT \mathfrak{B}_K is the ${}_K\mathcal{M}_K$ -module category ${}_K\mathcal{M}$. The TFT obtained by the pullback of \mathfrak{B}_K is indecomposable as a $\text{Rep}(H)$ -symmetric TFT if the left H -comodule algebra K is H -simple because in this case the category ${}_K\mathcal{M}$ of boundary conditions is indecomposable as a $\text{Rep}(H)$ -module category. Any semisimple indecomposable TFTs with a unitary fusion category symmetry $\text{Rep}(H)$ can be obtained in this way because any indecomposable $\text{Rep}(H)$ -module category is equivalent to the category of left modules over an H -simple left H -comodule algebra [98].¹⁷

The above construction generalizes the state sum construction of 2d bosonic TFTs with an ordinary finite group symmetry G [82, 83, 144]. We can reproduce the TFTs with a finite group symmetry G by taking the weak Hopf algebra H to be the coopposite dual group algebra $(\mathbb{C}[G]^*)^{\text{cop}}$ because a finite group G is described by $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ as a fusion category. We note that a left $(\mathbb{C}[G]^*)^{\text{cop}}$ -comodule algebra K is a G -equivariant algebra, which agrees with the input datum of the state sum construction of G -symmetric TFTs [82, 83, 144]. In particular, the bosonic TFT \mathfrak{B}_K has a non-anomalous \mathbb{Z}_2 symmetry when K is a \mathbb{Z}_2 -graded algebra, i.e., a superalgebra. This fact will be used in Chapter 6 when we discuss the fermionization of the bosonic state sum TFTs.

In the rest of this subsection, we describe the state sum construction of bosonic TFTs with $\text{Rep}(H)$ symmetry in more detail. To this end, we consider a two-dimensional oriented surface Σ with a topological defect network whose edges and vertices are labeled by objects and morphisms of $\text{Rep}(H)$ respectively. In order to define the transition amplitude on Σ , we first map the topological defects labeled by objects and morphisms of $\text{Rep}(H)$ to those labeled by objects and morphisms of ${}_K\mathcal{M}_K$ by the tensor functor $F_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$. More specifically, a topological line $V \in \text{Rep}(H)$ is mapped to $F_K(V) \in {}_K\mathcal{M}_K$, and topological junctions $f \in \text{Hom}_{\text{Rep}(H)}(V_3, V_1 \boxtimes V_2)$ and $g \in \text{Hom}_{\text{Rep}(H)}(V_1 \boxtimes V_2, V_3)$ are mapped to $J_{V_1, V_2}^{-1} \circ F_K(f) \in \text{Hom}_{{}_K\mathcal{M}_K}(F_K(V_3), F_K(V_1) \otimes_K F_K(V_2))$ and $F(g) \circ J_{V_1, V_2} \in$

¹⁶The reason why we use left A -modules instead of right A -modules is that the category \mathcal{B} of boundary conditions is supposed to be a left module category over \mathcal{C} .

¹⁷An H -simple left H -comodule algebra K is always semisimple when $\text{Rep}(H)$ is unitary [98], and hence it works as an input of the state sum construction.

$\text{Hom}_{\mathcal{K}\mathcal{M}_K}(F_K(V_1) \otimes_K F_K(V_2), F_K(V_3))$, where $J_{V_1, V_2} : F_K(V_1) \otimes_K F_K(V_2) \rightarrow F_K(V_1 \boxtimes V_2)$ is the natural isomorphism associated with the tensor functor F_K .¹⁸ Since the topological defects on Σ are now labeled by objects and morphisms of $\mathcal{K}\mathcal{M}_K$, we can apply the state sum construction described in Section 4.3.1 to define the transition amplitude on Σ . The transition amplitudes defined in this way are clearly topological. Furthermore, the monoidal structure axiom (2.1.7) guarantees that the associativity of the fusion of topological lines is captured by the associator of $\text{Rep}(H)$. Therefore, the above procedure gives us a bosonic TFT with $\text{Rep}(H)$ symmetry.

Precisely, in order to make the above construction consistent with the pullback of TFTs described in Section 4.2, the state space on the circle with two topological lines V_1 and V_2 should be identified with the state space on a circle with a single topological line $V_1 \boxtimes V_2$. This identification is achieved by using an isomorphism $Z(J_{V_1, V_2}) : Z(F_K(V_1) \otimes_K F_K(V_2)) \rightarrow Z(F_K(V_1 \boxtimes V_2))$, where the domain $Z(F_K(V_1) \otimes_K F_K(V_2))$ is the state space on a circle with two topological lines V_1 and V_2 , while the codomain $Z(F_K(V_1 \boxtimes V_2))$ is the state space on a circle with a single topological line $V_1 \boxtimes V_2$. Similarly, the state space on a circle without a topological defect should be identified with the state space on a circle with a trivial defect $H_t \in \text{Rep}(H)$. This identification is achieved by using an isomorphism $Z(\phi) : Z(K) \rightarrow Z(F_K(H_t))$, where $\phi : K \rightarrow F_K(H_t)$ is the isomorphism associated with the tensor functor F_K . We note, however, that whether or not we identify these state spaces is a matter of convention.

Let us now compute the $\text{Rep}(H)$ -module category structure on the category of boundary conditions. We first compute the fusion of topological line $V \in \text{Rep}(H)$ with a topological boundary $M \in \mathcal{K}\mathcal{M}$ by computing the action of the corresponding symmetry operator \mathcal{U}_V on the boundary state $|M\rangle$. By the definition of the pullback, we have $\mathcal{U}_V = U_{F_K(V)}$ and therefore the action of \mathcal{U}_V on $|M\rangle$ can be computed as

$$\mathcal{U}_V |M\rangle = U_{F_K(V)} |M\rangle = |F_K(V) \otimes_K M\rangle = |V \overline{\otimes} M\rangle, \quad (4.3.16)$$

where the second equality follows from eq. (4.3.14) and the last equality follows from the definition of the module action $\overline{\otimes}$, see Section 2.4. The above equation indicates that the fusion of $V \in \text{Rep}(H)$ and $M \in \mathcal{K}\mathcal{M}$ is given by $V \overline{\otimes} M \in \mathcal{K}\mathcal{M}$, which is consistent with the $\text{Rep}(H)$ -module category structure on $\mathcal{K}\mathcal{M}$. Furthermore, it readily follows from eq. (4.3.15) that the two possible resolutions of a 4-valent junction on the boundary are related to each other by the module associativity isomorphism of the $\text{Rep}(H)$ -module category $\mathcal{K}\mathcal{M}$. Here, the module associativity isomorphism of the $\text{Rep}(H)$ -module category $\mathcal{K}\mathcal{M}$ is determined by the natural isomorphism associated with the tensor functor $F_K : \text{Rep}(H) \rightarrow \mathcal{K}\mathcal{M}_K$ and the module associativity isomorphism (2.2.18) of the $\mathcal{K}\mathcal{M}_K$ -module category $\mathcal{K}\mathcal{M}$, see eq. (2.1.21). Thus, we find that topological boundary conditions of the $\text{Rep}(H)$ symmetric TFT obtained by the pullback of the state sum TFT \mathfrak{B}_K form the $\text{Rep}(H)$ -module category $\mathcal{K}\mathcal{M}$.

4.4 Lattice models with fusion category symmetries

Let H be a semisimple Hopf algebra. In this section, we construct $\text{Rep}(H)$ symmetric commuting projector Hamiltonians whose ground states are described by the state sum TFTs constructed in the previous section. These commuting projector Hamiltonians realize all gapped phases including all SPT phases with non-anomalous fusion category symmetries. We also discuss the relation between our models and other lattice models with fusion category symmetries,

¹⁸We recall that the tensor product in $\text{Rep}(H)$ is denoted by \boxtimes when H is a weak Hopf algebra, see Section 2.4. This tensor product \boxtimes reduces to the ordinary tensor product \otimes when H is a Hopf algebra.

which are known as anyon chain models [73, 74]. The content of Section 4.4.1 is not limited to gapped phases.

4.4.1 Fusion category symmetries on the lattice

We first define the action of a non-anomalous fusion category symmetry $\text{Rep}(H)$ on a general 1+1d lattice model whose state space admits a tensor product decomposition. We suppose that the state space \mathcal{H} of the model is given by $\mathcal{H} = \bigotimes_i \mathcal{H}_i$, where each on-site state space \mathcal{H}_i is a left H -comodule.¹⁹ The H -coaction on \mathcal{H}_i will be denoted by $\lambda_{\mathcal{H}_i}^L : \mathcal{H}_i \rightarrow H \otimes \mathcal{H}_i$. The action of the $\text{Rep}(H)$ symmetry on this state space is defined via the left H -comodule actions on the on-site state spaces. More specifically, for each object $V \in \text{Rep}(H)$, we define a symmetry operator $\hat{\mathcal{U}}_V$ by the following string diagram:²⁰

$$\hat{\mathcal{U}}_V := \text{ev}_V^L \circ \rho_V^R \circ \text{coev}_V^R \circ \lambda_{\mathcal{H}_1}^H \circ \lambda_{\mathcal{H}_2}^H \cdots \lambda_{\mathcal{H}_N}^H. \quad (4.4.1)$$

Here, $\rho_V^R : V \otimes H \rightarrow V$ is the right H -action on V , which is induced by the left H -action $\rho_{V^*}^L$ on the dual representation $V^* \in \text{Rep}(H)$, i.e.,

$$\rho_V^R := (\text{ev}_V^R \otimes \text{id}_{V^*}) \circ (\text{id}_{V^*} \otimes \rho_{V^*}^L \otimes \text{id}_{V^*}) \circ (\text{id}_{V^*} \otimes \text{id}_H \otimes \text{coev}_V^R). \quad (4.4.2)$$

Recalling that the left H -action on V^* is related to the left H -action on V as in eq. (2.3.6), we can express the right H -action ρ_V^R in terms of the left H -action ρ_V^L as

$$\rho_V^R = \rho_V^L \circ c_{\text{triv}} \circ (\text{id}_V \otimes S), \quad (4.4.3)$$

where c_{triv} denotes the trivial braiding between V and H . The symmetry operators defined by eq. (4.4.1) obey the fusion rules of $\text{Rep}(H)$. Namely, if the fusion rules of $\text{Rep}(H)$ are given by $V \otimes W \cong \bigoplus_X N_{VW}^X X$, the symmetry operators satisfy $\hat{\mathcal{U}}_V \hat{\mathcal{U}}_W = \sum_X N_{VW}^X \hat{\mathcal{U}}_X$. This is a consequence of the additivity $\hat{\mathcal{U}}_{V \oplus W} = \hat{\mathcal{U}}_V + \hat{\mathcal{U}}_W$ and the monoidality $\hat{\mathcal{U}}_{V \otimes W} = \hat{\mathcal{U}}_V \hat{\mathcal{U}}_W$ of the symmetry operators. The additivity immediately follows from the definition (4.4.1) and the monoidality can also be checked by a direct computation. We note that the action of the symmetry operator (4.4.1) is symmetric under cyclic permutation of the lattice sites and therefore is well-defined on a periodic chain as well as on an open chain. Practically, each on-site state space \mathcal{H}_i has to be sufficiently large so that the $\text{Rep}(H)$ symmetry acts faithfully.

A (1+1)-dimensional lattice model has the $\text{Rep}(H)$ symmetry if the Hamiltonian commutes with the symmetry operators (4.4.1), or in other words, if H is an H -comodule map on the state space $\mathcal{H} = \bigotimes_i \mathcal{H}_i$. In particular, when the Hamiltonian is given by the sum of local operators, each local operator contained in the Hamiltonian has to be an H -comodule map. For example, a $\text{Rep}(H)$ symmetric Hamiltonian only with nearest-neighbor interactions can generally be written as $\sum_i h_{i,i+1}$, where $h_{i,i+1} : \mathcal{H}_i \otimes \mathcal{H}_{i+1} \rightarrow \mathcal{H}_i \otimes \mathcal{H}_{i+1}$ is an H -comodule map for all i .

¹⁹A similar idea of constructing lattice models with symmetries beyond groups is presented in [145].

²⁰The symmetry operator (4.4.1) is an example of a matrix product operator (MPO). The MPO representations of symmetry operators for general fusion category symmetries are also studied in [78, 79, 146, 147].

Example: finite group symmetry G . As the simplest example, we consider the action of a finite group symmetry G on the lattice. It is well-known that we can define the action of G if each on-site state space \mathcal{H}_i is a representation of G , or equivalently, a left module over a group algebra $\mathbb{C}[G]$. Since a left $\mathbb{C}[G]$ -module naturally has a structure of a left $(\mathbb{C}[G]^*)^{\text{cop}}$ -comodule (see eq. (2.3.7)), we can also define the action of $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ via eq. (4.4.1) when \mathcal{H}_i is a representation of G . As we will see below, the action of $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ agrees with the ordinary action of a finite group G .²¹

For concreteness, we choose the on-site state space \mathcal{H}_i to be the regular representation V_{reg} of G . The action of $\mathbb{C}[G]$ on the basis $\{v_g \mid g \in G\}$ of V_{reg} is given by

$$g \cdot v_h = v_{gh}, \quad g \in \mathbb{C}[G]. \quad (4.4.4)$$

The corresponding $(\mathbb{C}[G]^*)^{\text{cop}}$ -comodule structure $\lambda_{\text{reg}} : V_{\text{reg}} \rightarrow (\mathbb{C}[G]^*)^{\text{cop}} \otimes V_{\text{reg}}$ is given by

$$\lambda_{\text{reg}}(v_g) = \sum_{h \in G} \hat{h} \otimes v_{h^{-1}g}, \quad (4.4.5)$$

where $\{\hat{g} \mid g \in G\}$ is the dual basis of $(\mathbb{C}[G]^*)^{\text{cop}}$ defined by $\hat{g}(h) = \delta_{g,h}$. Therefore, the action of a symmetry operator $\hat{\mathcal{U}}_g$ labeled by a simple $(\mathbb{C}[G]^*)^{\text{cop}}$ -module $V_g = \text{Span}\{\hat{g}\}$ can be computed as

$$\hat{\mathcal{U}}_g(v_{g_1} \otimes v_{g_2} \otimes \cdots \otimes v_{g_N}) = v_{gg_1} \otimes v_{gg_2} \otimes \cdots \otimes v_{gg_N}. \quad (4.4.6)$$

This agrees with the ordinary G -action defined via the left $\mathbb{C}[G]$ -module structure on V_{reg} .

Example: $\text{Rep}(G)$ symmetry. As another example related to a finite group, we consider the action of $\text{Rep}(G)$, which physically describes the symmetry of finite group gauge theories. For concreteness, we choose the on-site state space \mathcal{H}_i to be the regular representation V'_{reg} of the dual group algebra $\mathbb{C}[G]^*$. The action of $\mathbb{C}[G]^*$ on the basis $\{v'_g \mid g \in G\}$ of V'_{reg} is given by

$$\hat{g} \cdot v'_h = \delta_{g,h} v'_h, \quad \hat{g} \in \mathbb{C}[G]^*. \quad (4.4.7)$$

The corresponding $\mathbb{C}[G]^{\text{cop}} (= \mathbb{C}[G])$ -comodule structure $\lambda'_{\text{reg}} : V'_{\text{reg}} \rightarrow \mathbb{C}[G] \otimes V'_{\text{reg}}$ can be written as

$$\lambda'_{\text{reg}}(v'_g) = g^{-1} \otimes v'_g, \quad (4.4.8)$$

where $\{g \mid g \in G\}$ is the dual basis of $\mathbb{C}[G]$ that satisfies $\hat{g}(h) = \delta_{g,h}$. We can compute the action of a left $\mathbb{C}[G]$ -module $V \in \text{Rep}(G)$ on a basis state of $(V'_{\text{reg}})^{\otimes N}$ as

$$\hat{\mathcal{U}}_V(v'_{g_1} \otimes v'_{g_2} \otimes \cdots \otimes v'_{g_N}) = \chi_V(g_N \cdots g_2 g_1) v'_{g_1} \otimes v'_{g_2} \otimes \cdots \otimes v'_{g_N}, \quad (4.4.9)$$

where $\chi_V : \mathbb{C}[G] \rightarrow \mathbb{C}$ is the character of V . The above action agrees with the standard action of a Wilson line operator of finite group gauge theories.

4.4.2 Commuting projector Hamiltonians

In this subsection, we write down a $\text{Rep}(H)$ symmetric commuting projector Hamiltonian whose ground states are described by the $\text{Rep}(H)$ symmetric state sum TFT constructed in Section 4.3. We consider a lattice model on a periodic chain consisting of N sites. The state

²¹We recall that $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ is equivalent to Vec_G , which describes a finite group symmetry G .

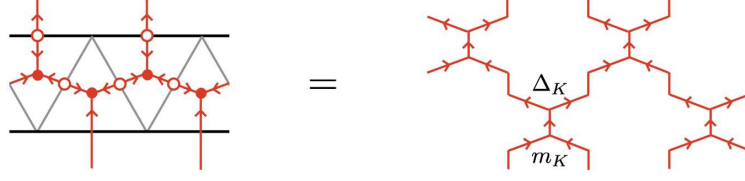


Figure 4.12: The transition amplitude of the state sum TFT on a triangulated cylinder (the left-hand side) is equal to the product of commuting projectors $h_{i,i+1}$ (the right-hand side). The equality follows from the fact that K is a Δ -separable symmetric Frobenius algebra.

space \mathcal{H} of the model is given by the tensor product $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$, where the state space \mathcal{H}_i on each site is a left H -comodule algebra K . Since each on-site state space \mathcal{H}_i is equipped with an H -comodule structure, we can define the action of $\text{Rep}(H)$ on the total state space \mathcal{H} as discussed in the previous subsection. The Hamiltonian of the model is defined by

$$H := \sum_{i=1}^N (1 - h_{i,i+1}), \quad h_{i,i+1} := \Delta_K \circ m_K : \mathcal{H}_i \otimes \mathcal{H}_{i+1} \rightarrow \mathcal{H}_i \otimes \mathcal{H}_{i+1}, \quad (4.4.10)$$

where the comultiplication Δ_K is defined by eq. (2.2.7) and \mathcal{H}_{N+1} is identified with \mathcal{H}_1 because the lattice is periodic. The linear map $h_{i,i+1}$ turns out to be a local commuting projector, i.e., it satisfies $h_{i,i+1}h_{j,j+1} = h_{j,j+1}h_{i,i+1}$ and $h_{i,i+1}^2 = h_{i,i+1}$. The first equality follows from the Frobenius relation (2.2.4), and the second equality follows from the Δ -separability (2.2.5). This implies that all terms in the Hamiltonian can be diagonalized simultaneously and their eigenvalues are either 0 or 1. In particular, the ground states of the Hamiltonian (4.4.10) are simultaneous eigenstates of all $h_{i,i+1}$'s with eigenvalues $h_{i,i+1} = 1$.

The relation between the above model and the state sum TFT \mathfrak{B}_K becomes clear if we compare the ground state subspace of the lattice model on a periodic chain and the state space of \mathfrak{B}_K on a circle. On one hand, the ground state subspace \mathcal{H}_{GS} of the lattice model is the image of the projectors $h_{i,i+1}$ for all $i \in \{1, 2, \dots, N\}$, i.e., $\mathcal{H}_{\text{GS}} = P\mathcal{H}$, where $P := \prod_i h_{i,i+1}$ is the product of the local projectors $h_{i,i+1}$ on all sites. On the other hand, the state space $Z(S^1)$ of the TFT \mathfrak{B}_K on a circle is the image of the transition amplitude $Z_T(S^1 \times [0, 1])$ on a triangulated cylinder, i.e., $Z(S^1) = Z_T(S^1 \times [0, 1])Z_T(S^1)$, where $Z_T(S^1)$ is the vector space assigned to the triangulated circle. When both the in-boundary and out-boundary of the triangulated cylinder consist of N segments, the transition amplitude $Z_T(\Sigma \times [0, 1])$ turns out to be the product of the local projectors $h_{i,i+1}$, i.e., we have $Z_T(\Sigma \times [0, 1]) = P$, see Figure 4.12. Thus, the ground state subspace \mathcal{H}_{GS} of the lattice model agrees with the state space $Z(S^1)$ of the state sum TFT \mathfrak{B}_K . This shows that the low-energy limit of the above lattice model is described by the topological field theory \mathfrak{B}_K .²²

We can show that the lattice model defined above has a fusion category symmetry $\text{Rep}(H)$. As we discussed in Section 4.4.1, it suffices to show that each term $h_{i,i+1} = \Delta_K \circ m_K$ in the Hamiltonian (4.4.10) is an H -comodule map. Each term $h_{i,i+1}$ is clearly an H -comodule map if Δ_K is because m_K is an H -comodule map by definition. Conversely, Δ_K is an H -comodule map if $h_{i,i+1}$ is because we have $\Delta_K = h_{i,i+1} \circ (\eta_K \otimes \text{id}_K)$ and $(\eta_K \otimes \text{id}_K)$ is an H -comodule map. Therefore, the local Hamiltonian $h_{i,i+1}$ is an H -comodule map if and only if the comultiplication Δ_K is an H -comodule map. Namely, the Hamiltonian (4.4.10) has a

²²Even in higher dimensions, we can write down commuting projector Hamiltonians that realize bosonic state sum TFTs in a similar way [148, 149].

fusion category symmetry $\text{Rep}(H)$ if and only if the following equality holds:

$$\begin{array}{c} \text{---} H \text{---} \\ \diagup \quad \diagdown \\ \text{---} K \text{---} \end{array} = \begin{array}{c} \text{---} H \text{---} \\ \diagdown \\ \text{---} K \text{---} \end{array} . \quad (4.4.11)$$

In order to show the above equation, we first notice that the counit ϵ_K given by eq. (2.2.8) is an H -comodule map, i.e., it satisfies $(\text{id}_H \otimes \epsilon_K) \circ \lambda_K^L = \eta \circ \epsilon_K$, where η is the unit of H . This equality can be derived as follows:

$$\begin{array}{c} \epsilon_K \\ \circ \\ | \\ \text{red line} \end{array} = \begin{array}{c} \text{red line} \\ \circlearrowleft \\ \text{red line} \\ \bullet S \\ | \\ \text{red line} \end{array} = \begin{array}{c} \text{red line} \\ \circlearrowleft \\ \text{red line} \\ | \\ \text{red line} \end{array} = \begin{array}{c} \text{red line} \\ \circlearrowleft \\ \text{red line} \\ | \\ \text{red line} \end{array} = \begin{array}{c} \text{red line} \\ \circlearrowleft \\ \text{red line} \\ | \\ \text{red line} \end{array} = \begin{array}{c} \text{red line} \\ \circlearrowleft \\ \text{red line} \\ \bullet S \\ | \\ \text{red line} \end{array} = \begin{array}{c} \text{red line} \\ \circlearrowleft \\ \text{red line} \\ \bullet S \\ | \\ \text{red line} \end{array} = \begin{array}{c} \epsilon_K \\ \circ \\ | \\ \text{red line} \end{array}, \quad (4.4.12)$$

In the above equation, the left H -comodule action $\lambda_{K^*}^L : K^* \rightarrow H \otimes K^*$ on the dual algebra K^* is defined by

$$\lambda_{K^*}^L = (S \otimes \text{ev}_K^L \otimes \text{id}_{K^*}) \circ (c_{\text{triv}} \otimes \text{id}_K \otimes \text{id}_{K^*}) \circ (\text{id}_{K^*} \otimes \lambda_K^L \otimes \text{id}_{K^*}) \circ (\text{id}_{K^*} \otimes \text{coev}_K^L), \quad (4.4.13)$$

which is analogous to the definition of the left H -module action (2.3.6) on the dual of a left H -module. The fact that the comultiplication ϵ_K is an H -comodule map implies that the algebra isomorphism $\Phi : K \rightarrow K^*$ is also an H -comodule map because

$$S \bullet \Phi = \text{diagram 2} = \text{diagram 3} = \text{diagram 4} = S \bullet \Phi. \quad (4.4.14)$$

The inverse $\Phi^{-1} : K^* \rightarrow K$ is an H -comodule map as well. Therefore, we have

$$\begin{array}{c} \text{diagram} \end{array} = \begin{array}{c} \text{diagram} \end{array} = \begin{array}{c} \text{diagram} \end{array} = \begin{array}{c} \text{diagram} \end{array}, \quad (4.4.15)$$

which shows eq. (4.4.11). Thus, we find that the commuting projector Hamiltonian (4.4.10) has the $\text{Rep}(H)$ symmetry.

Let us compute the action of the $\text{Rep}(H)$ symmetry on the ground states of the Hamiltonian (4.4.10). As we discussed above, the ground state subspace of the model agrees with the state space of the TFT \mathfrak{B}_K . Therefore, the ground states of the Hamiltonian (4.4.10) are in one-to-one correspondence with the boundary states (4.3.13) of the state sum TFT \mathfrak{B}_K . The action of the symmetry operator $\hat{\mathcal{U}}_V$ on the ground state $|M\rangle$ can be computed as

$$\hat{\mathcal{U}}_V |M\rangle = \text{diagram} = \text{diagram} = |V \otimes M\rangle. \quad (4.4.16)$$

The second equality follows from the fact that the left K -comodule structure $\lambda_{V \overline{\otimes} M}^L$ on $V \overline{\otimes} M$ that is induced by the left K -module structure (2.3.9) via eq. (2.2.15) satisfies

$$\lambda_{V \otimes M}^L = (c_{\text{triv}} \otimes \text{id}_M) \circ (\rho_V^R \otimes \text{id}_K \otimes \text{id}_M) \circ (\text{id}_V \otimes \lambda_K^L \otimes \text{id}_M) \circ (\text{id}_V \otimes \lambda_M^L), \quad (4.4.17)$$

where c_{triv} is the trivial braiding between V and K , ρ_V^R is the right H -action on V defined by eq. (4.4.3), λ_K^L is the left H -comodule action on K , and λ_M^L is the left K -comodule action on M defined by eq. (2.2.15). Equation (4.4.16) shows that the $\text{Rep}(H)$ symmetry acts on the ground states of the lattice model in the same way as it acts on the vacua of the state sum TFT, cf. eq. (4.3.16). This implies that symmetry operators on the lattice reduce to those of TFTs in the low energy limit. We emphasize that every topological phase with non-anomalous fusion category symmetry $\text{Rep}(H)$ can be realized by the above lattice model if we choose the algebra K appropriately.

A comment on unitarity. So far, we have not imposed the unitarity both on the TFT and lattice model. The state sum TFT \mathfrak{B}_K becomes unitary when the input algebra K is equipped with a structure of a Hilbert space such that the transition amplitude on an oriented surface is the Hermitian conjugate of the transition amplitude on its orientation reversal. We expect that if the state sum TFT is unitary, the transition amplitude on a triangulated surface also becomes its Hermitian conjugate when we reverse the orientation of the surface.²³ If this is the case, the transition amplitude on a triangulated cylinder whose in-boundary and out-boundary have the same triangulation should be Hermitian because a triangulated cylinder should give rise to the same projector regardless of its orientation.²⁴ As we discussed in Section 4.4.2, the transition amplitude on such a triangulated cylinder is given by the product of the local projectors $h_{i,i+1}$. Therefore, if the transition amplitude on this triangulated cylinder is Hermitian, so is $h_{i,i+1}$. This argument implies that the Hamiltonian (4.4.10) is Hermitian if the corresponding state sum TFT \mathfrak{B}_K is unitary.

Example: gapped Hamiltonians with $\text{Rep}(G)$ symmetry. As a simple example, let us consider the case where the Hopf algebra H is a group algebra $\mathbb{C}[G]$. In this case, a left H -comodule algebra K is a G -graded algebra, i.e., an algebra equipped with a G -grading $K = \bigoplus_{g \in G} K_g$ that is compatible with the algebra structure in the sense that $m_K(k_g \otimes k_h) \in K_{gh}$ for all $k_g \in K_g$ and $k_h \in K_h$. The lattice model (4.4.10) constructed from a G -graded algebra K has a $\text{Rep}(G)$ symmetry. The action of a representation $V \in \text{Rep}(G)$ on a state $\bigotimes_{i=1}^N k_i \in K^{\otimes N}$ is given by the character $\chi_V(g_N \cdots g_2 g_1)$, where g_i denotes the grading of $k_i \in K_{g_i} \subset K$.

Gapped phases with $\text{Rep}(G)$ symmetry are classified by pairs $(H, [\omega])$ of a subgroup $H \subset G$ and a second cohomology class $[\omega] \in H^2(H, \text{U}(1))$ [13]. Correspondingly, G -graded algebras are also classified by the same pairs $(H, [\omega])$ up to Morita equivalence.²⁵ A representative of the equivalence class labeled by $(H, [\omega])$ is given by a twisted group algebra $\mathbb{C}[H]^\omega$, which is a group algebra $\mathbb{C}[H]$ whose multiplication is twisted by a second cocycle ω :

$$m_{\mathbb{C}[H]^\omega}(v_h \otimes v_l) = \omega(h, l)v_{hl}. \quad (4.4.18)$$

We suppose that ω is normalized so that $\omega(g, h) = 1$ when either g or h is the unit element of G . This normalization is always possible without changing the cohomology class of ω . The comultiplication on $\mathbb{C}[H]^\omega$ is determined by the multiplication (4.4.18) due to eq. (2.2.7) as

$$\Delta_{\mathbb{C}[H]^\omega}(v_h) = \frac{1}{|H|} \sum_{l \in H} \frac{\omega(l^{-1}, h)}{\omega(l^{-1}, l)} v_l \otimes v_{l^{-1}h}. \quad (4.4.19)$$

²³The state sum TFT is clearly unitary if the transition amplitudes of triangulated surfaces satisfy this property. However, the converse is not obvious.

²⁴A cylinder equipped with the opposite orientation coincides with the original cylinder by turning it inside out topologically. Thus, the transition amplitude on a triangulated cylinder does not depend on its orientation.

²⁵Morita equivalent algebras give rise to the same gapped phase.

The gapped phase labeled by the pair $(H, [\omega])$ is realized by choosing the input algebra K to be the twisted group algebra $\mathbb{C}[H]^\omega$. For this choice, each term $h_{i,i+1} = \Delta_K \circ m_K$ of the Hamiltonian (4.4.10) can be written down explicitly in terms of ω as follows:

$$h_{i,i+1}(v_h \otimes v_l) = \frac{1}{|H|} \omega(h, l) \sum_{k \in H} \frac{\omega(k^{-1}, hl)}{\omega(k^{-1}, k)} v_k \otimes v_{k^{-1}hl}. \quad (4.4.20)$$

Although the on-site state space of this model is an $|H|$ -dimensional vector space $\mathbb{C}[H]^\omega$, one can always embed this vector space into a $|G|$ -dimensional one $\mathbb{C}[G]$. More specifically, when the on-site state space is $\mathbb{C}[G]$, the gapped phase labeled by the pair $(H, [\omega])$ is realized by the Hamiltonian $H = -\sum_i (\iota_i \otimes \iota_{i+1}) \circ h_{i,i+1} \circ (\pi_i \otimes \pi_{i+1})$, where $\iota_i : \mathbb{C}[H]^\omega \rightarrow \mathbb{C}[G]$ and $\pi_i : \mathbb{C}[G] \rightarrow \mathbb{C}[H]^\omega$ are the obvious inclusion and projection on the i th site.²⁶ We note that ι and π preserve the $\text{Rep}(G)$ symmetry of the model.

When $G = \mathbb{Z}_2$, the $\text{Rep}(G)$ symmetry reduces to an ordinary \mathbb{Z}_2 symmetry because $\text{Rep}(\mathbb{Z}_2)$ is equivalent to $\text{Vec}_{\mathbb{Z}_2}$ as a fusion category.²⁷ There are two different gapped phases with \mathbb{Z}_2 symmetry, i.e., the symmetric phase and the symmetry broken phase. These gapped phases correspond to \mathbb{Z}_2 -graded algebras \mathbb{C} and $\mathbb{C}[\mathbb{Z}_2]$ respectively. More specifically, if we identify the basis $\{v_+, v_-\}$ of the on-site state space $\mathbb{C}[\mathbb{Z}_2]$ with the X basis $\{|+\rangle, |-\rangle\}$ of a qubit, the Hamiltonian for each of these \mathbb{Z}_2 -graded algebras can be written as

$$\begin{aligned} H &= -\sum_i \frac{1 + \sigma_x^{(i)}}{2} \otimes \frac{1 + \sigma_x^{(i+1)}}{2} \quad \text{for } K = \mathbb{C}, \\ H &= -\sum_i \frac{1 + \sigma_z^{(i)} \otimes \sigma_z^{(i+1)}}{2} \quad \text{for } K = \mathbb{C}[\mathbb{Z}_2], \end{aligned} \quad (4.4.21)$$

where $\sigma_x^{(i)}$ and $\sigma_z^{(i)}$ denote the Pauli X and Pauli Z operators acting on the site i . The Hamiltonian on the first line has a unique symmetric ground state, while the Hamiltonian on the second line has two-fold degenerate symmetry-breaking ground states.

4.4.3 Edge modes of SPT phases with fusion category symmetries

Symmetry protected topological (SPT) phases with fusion category symmetry \mathcal{C} are gapped phases with unique ground states preserving the symmetry \mathcal{C} . Since anomalous fusion category symmetries do not admit SPT phases by definition, the symmetry \mathcal{C} has to be non-anomalous, i.e., it can be written as $\mathcal{C} = \text{Rep}(H)$ where H is a Hopf algebra. SPT phases with $\text{Rep}(H)$ symmetry are realized by the commuting projector Hamiltonians (4.4.10) when the input algebra K is simple.²⁸ A simple example of an SPT phase with a fusion category symmetry is the complete Higgs phase of a finite group gauge theory, which is protected by $\text{Rep}(G)$ symmetry generated by Wilson lines. Here, the complete Higgs phase is the gapped phase obtained by gauging a finite group symmetry G of a symmetry broken phase where the symmetry G is spontaneously broken down to the trivial group. The complete Higgs phase has a unique ground state because (1) all ground states of the untwisted sector of the G -symmetry broken phase are identified after gauging and (2) the ground states of twisted sectors have energies

²⁶The same trick is used in [150] to obtain commuting projector Hamiltonians for all gapped phases with finite group symmetries.

²⁷In general, $\text{Rep}(G)$ is equivalent to Vec_G if and only if G is abelian.

²⁸More generally, the number of the ground states of the Hamiltonian (4.4.10) on a circle is equal to the number of simple left K -modules.

higher than those of the untwisted sector. From the point of view of the classification of SPT phases reviewed in Section 4.1, the complete Higgs phase corresponds to the forgetful functor from $\text{Rep}(G)$ to Vec .

The Hamiltonians of SPT phases can have degenerate ground states on an interval even though they have unique ground states on a circle. In particular, if we define the Hamiltonian on an interval by eq. (4.4.10), the ground states on the interval form the input algebra K itself rather than its center [141, 151]. Since K is simple, it is isomorphic to $\text{End}(M) \cong M^* \otimes M$ where M is a simple left K -module, which is unique up to isomorphism. We can interpret M^* and M as the edge modes localized to the left and right boundaries because the bulk is uniquely gapped. Indeed, if we choose a basis of the on-site state space on an edge e as $\{|v^i\rangle_e \otimes |v_j\rangle_e \in M^* \otimes M \mid i, j = 1, 2, \dots, \dim M\}$, we can write the ground states of the commuting projector Hamiltonian (4.4.10) on an interval $[1, N]$ as $|v^i\rangle_1 \otimes |\Omega\rangle_{1,2} \otimes |\Omega\rangle_{2,3} \otimes \dots \otimes |\Omega\rangle_{N-1,N} \otimes |v_j\rangle_N$, where $|\Omega\rangle_{e,e+1} := \sum_k |v_k\rangle_e \otimes |v^k\rangle_{e+1}$ is the maximally entangled state. This expression indicates that the degrees of freedom that take values in M^* and M remain on the left and right boundaries respectively. Therefore, the edge modes of the Hamiltonian (4.4.10) for a $\text{Rep}(H)$ -symmetric SPT phase $\mathcal{T}_{K\mathcal{M}}$ are described by a right K -module M^* and a left K -module M . We note that these edge modes are not necessarily minimal in the sense that it should be possible to partially lift the degeneracy by adding symmetric perturbations around the boundaries. A more detailed analysis of the edge modes is left for future work.

It is instructive to consider the case of an ordinary finite group symmetry G . As we discussed in Section 2.3, a finite group symmetry G is described by the category $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ of representations of the coopposite dual group algebra $(\mathbb{C}[G]^*)^{\text{cop}}$. SPT phases with this symmetry are classified by the second group cohomology $H^2(G, \text{U}(1))$ [67, 137, 144, 152–156]. An SPT phase labeled by $\omega \in H^2(G, \text{U}(1))$ is realized by the commuting projector Hamiltonian (4.4.10) when K is the smash product $\mathbb{C}[G]^\omega \# \mathbb{C}[G]^*$ of a twisted group algebra $\mathbb{C}[G]^\omega$ and the dual group algebra $\mathbb{C}[G]^*$.²⁹ Since $\mathbb{C}[G]^\omega$ is a subalgebra of K , a left K -module M is a left $\mathbb{C}[G]^\omega$ -module in particular. This implies that a finite group G acts projectively on the edge mode M with the projective phase given by ω . As is well known, this projectivity of the G -action indicates an anomaly of G symmetry on the boundary of the 1+1d SPT phase.

4.4.4 Relation to anyon chain models

The anyon chain model is a 1+1 dimensional lattice model with a general fusion category symmetry \mathcal{C} [73, 74]. In this subsection, we will briefly discuss the relation between the anyon chain models and the lattice models that we constructed in Section 4.4.2.

The state space of the anyon chain model is spanned by the fusion trees shown in Figure 4.13. Objects on the horizontal edges and morphisms on the trivalent vertices are dynamical variables of the model, while objects on the vertical edges are not. We choose the non-dynamical object on every vertical edge to be $\tau \in \mathcal{C}$. We note that τ is not necessarily simple. The fusion trees whose horizontal edges and trivalent vertices are labeled by simple objects and basis morphisms form an orthonormal basis of the state space. This state space does not admit a tensor product decomposition in general because a configuration of dynamical variables is constrained by the fusion rules.

The Hamiltonian of the model is given by the sum of the next-nearest-neighbor interactions.

²⁹The smash product $A \# H$ of an H -module algebra A and a Hopf algebra H is a vector space $A \otimes H$ equipped with the following multiplication: $(a \# h) \cdot (a' \# h') = a(h_{(1)} \cdot a') \# h_{(2)} h'$ where $a, a' \in A$ and $h, h' \in H$.

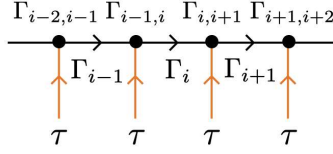


Figure 4.13: A fusion tree that defines a state of the anyon chain model. Horizontal edges and trivalent vertices of the fusion tree are labeled by objects $\Gamma_i \in \mathcal{C}$ and morphisms $\Gamma_{i,i+1} \in \text{Hom}_{\mathcal{C}}(\Gamma_{i-1} \otimes \tau, \Gamma_i)$ respectively, which are dynamical variables of the model. On the other hand, the vertical edges are labeled by a fixed object $\tau \in \mathcal{C}$, which is not dynamical.

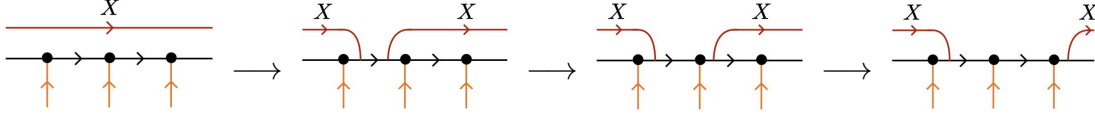


Figure 4.14: The symmetry action is defined by the fusion of topological lines. We fuse a topological line $X \in \mathcal{C}$ into the fusion tree by applying the F -move sequentially.

Each interaction term $h_{i-1,i,i+1}$ can be represented diagrammatically as

$$h_{i-1,i,i+1} \begin{array}{c} \Gamma_{i-1} \quad \Gamma_i \quad \Gamma_{i+1} \\ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\ \uparrow \quad \uparrow \\ \tau \quad \tau \end{array} = \begin{array}{c} \Gamma_{i-1} \quad \Gamma_i \quad \Gamma_{i+1} \\ \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \\ \boxed{\phi} \\ \uparrow \quad \uparrow \\ \tau \quad \tau \end{array}, \quad (4.4.22)$$

where $\phi \in \text{End}(\tau \otimes \tau)$ is an endomorphism of $\tau \otimes \tau$. The action of the Hamiltonian defined by the above equation can also be expressed in terms of F -symbols more explicitly. We note that interactions involving more than three sites can also be represented by similar endomorphisms. For example, a next-next-nearest-neighbor interaction is represented by an endomorphism of $\tau \otimes \tau \otimes \tau$. We will not consider such a long-range interaction in the following.

The anyon chain model defined above has a fusion category symmetry \mathcal{C} [74]. The action of \mathcal{C} on a state is defined by fusing topological lines labeled by objects of \mathcal{C} into the horizontal edges of the fusion tree. We can write down this action explicitly by using F -symbols as illustrated in Figure 4.14. The symmetry action defined in this way automatically commutes with the Hamiltonian (4.4.22) due to the pentagon equation for the F -symbols. More intuitively, the commutativity between the symmetry action and the Hamiltonian follows from the fact that the former acts on a fusion tree from above, while the latter acts on a fusion tree from below. This shows that the anyon chain model has a fusion category symmetry \mathcal{C} whatever $\tau \in \mathcal{C}$ and $\phi \in \text{End}_{\mathcal{C}}(\tau \otimes \tau)$ we choose.

The next-nearest neighbor Hamiltonian (4.4.22) becomes a commuting projector when τ is a Δ -separable symmetric Frobenius algebra object $A \in \mathcal{C}$ and $\phi := \Delta \circ m$ is the composition of the comultiplication morphism $\Delta \in \text{Hom}_{\mathcal{C}}(A, A \otimes A)$ and the multiplication morphism $m \in \text{Hom}_{\mathcal{C}}(A \otimes A, A)$. Accordingly, the anyon chain model in this case realizes a gapped phase with symmetry \mathcal{C} , whose low energy limit is described by a \mathcal{C} -symmetric bosonic TFT. In order to identify the category of boundary conditions of this TFT, we recall that the anyon chain model is the anisotropic limit of a two-dimensional classical statistical mechanical model known as the height model [75]. The 2d height model is obtained by putting the 3d Turaev-Viro TFT on a slab as shown in Figure 4.1, where the dynamical boundary on the right is chosen to be the Dirichlet boundary decorated by a specific topological defect network [75]. In particular, when this defect network is given by a fine mesh of $A \in \mathcal{C}$ as shown in Figure 4.15,

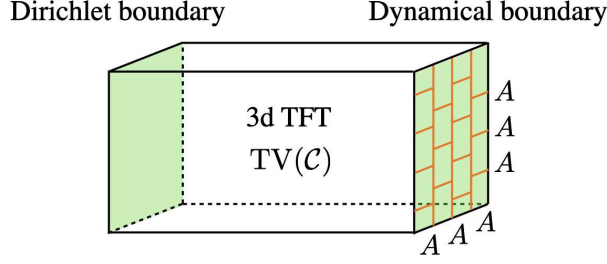


Figure 4.15: The dynamical boundary condition defined by the fine mesh of an algebra object $A \in \mathcal{C}$ gives rise to a topological 2d height model. The trivalent vertices of the mesh are labeled by the multiplication m or the comultiplication Δ depending on the orientations of the edges.

the dynamical boundary becomes topological and therefore the corresponding height model reduces to a 2d bosonic TFT. The category of boundary conditions of this TFT is the category \mathcal{C}_A of right A -modules in \mathcal{C} [43]. It turns out that the transfer matrix of this height model agrees with the projector onto the ground state subspace of the anyon chain model with $\tau = A$ and $\phi = \Delta \circ m$.³⁰ Therefore, the low energy limit of this anyon chain model is described by the 2d bosonic TFT whose category of boundary conditions is a \mathcal{C} -module category \mathcal{C}_A . In particular, the ground states of the model are labeled by simple objects of \mathcal{C}_A , i.e., irreducible right A -modules in \mathcal{C} . More concretely, the ground state $|M\rangle$ labeled by a right A -module $M \in \mathcal{C}_A$ is the fusion tree whose horizontal edges and trivalent vertices are labeled by M and the right A -action $\rho_M^R : M \otimes A \rightarrow M$ respectively:

$$|M\rangle = \begin{array}{c} \rho_M^R \quad \rho_M^R \quad \rho_M^R \quad \rho_M^R \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \xrightarrow{M} \xrightarrow{M} \xrightarrow{M} \xrightarrow{M} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ A \quad A \quad A \quad A \end{array} . \quad (4.4.23)$$

The associativity of the right A -action on M and the Δ -separability of A implies $h_{i-1,i,i+1} |M\rangle = |M\rangle$ for all i , which shows that $|M\rangle$ is indeed a ground state.

In order to compare the anyon chain model with the lattice model that we defined in Section 4.4.2, we restrict our attention to non-anomalous fusion category symmetries. When \mathcal{C} is a non-anomalous fusion category symmetry $\text{Rep}(H)$, a Δ -separable symmetric Frobenius algebra object $A \in \mathcal{C}$ is a left H -module algebra. The anyon chain model constructed from a left H -module algebra A realizes a gapped phase described by a $\text{Rep}(H)$ -symmetric TFT whose topological boundary conditions form a $\text{Rep}(H)$ -module category $\text{Rep}(H)_A$. On the other hand, the lattice model (4.4.10) with $\text{Rep}(H)$ symmetry is constructed from a left H -comodule algebra K rather than a left H -module algebra A . The topological boundary conditions of the corresponding TFT form a $\text{Rep}(H)$ -module category ${}_K\mathcal{M}$. The lattice model (4.4.10) constructed from K is in the same phase as the anyon chain model constructed from A when there is an equivalence of $\text{Rep}(H)$ -module categories $\text{Rep}(H)_A \cong {}_K\mathcal{M}$.³¹ Although these models are similar in that they can realize the same gapped phases by commuting projector Hamiltonians, there is also a big difference in the structure of the state space: the lattice model (4.4.10) admits a tensor product decomposition of the state space, while the anyon chain model does not.

³⁰The (2+1)-dimensional generalization of this fact is discussed in detail in [157].

³¹The $\text{Rep}(H)$ -module category $\text{Rep}(H)_A$ is equivalent to ${}_K\mathcal{M}$ when $K = A^{\text{op}} \# H^{\text{cop}}$ is the smash product of A^{op} and H^{cop} [91].

It would be an interesting problem to study which fusion category symmetries are compatible with the tensor product structure on the state space. As we demonstrated in Section 4.4.2, every non-anomalous fusion category symmetry can be realized on a tensor product state space. Furthermore, any anomalous finite group (i.e., invertible) symmetry can also be realized on a tensor product state space because the state space of the anyon chain model with this symmetry reduces to the tensor product of on-site state spaces if we choose ρ to be the direct sum of all simple objects. However, it remains unclear whether anomalous non-invertible symmetries can be represented on a tensor product space in general. We leave this problem to a future investigation.

Chapter 5

Fermionization of fusion category symmetries

In this chapter, we propose a general formula to determine the symmetry of the 1+1d fermionic system obtained by the fermionization of a 1+1d bosonic system with a fusion category symmetry. The derivation of the formula will be given in Chapter 6 in the context of topological field theories. Based on the formula, we explicitly compute the fermionization of fusion category symmetries for several concrete examples such as finite group symmetries and duality symmetries. We expect that our fermionization formula is applicable to any 1+1d systems with fusion category symmetries regardless of whether they are topological or not. The content of this chapter, except for Section 5.1, is based on the author's original paper [45].

5.1 Review: bosonization and fermionization in 1+1 dimensions

In 1+1 dimensions, bosonic systems with non-anomalous \mathbb{Z}_2 symmetry are in bijective correspondence with fermionic systems with non-anomalous fermion parity symmetry \mathbb{Z}_2^F . Here, fermionic systems refer to those that can be defined only on spacetimes equipped with spin structures.¹ The map from bosonic systems to fermionic systems is called fermionization, while its inverse is called bosonization. In particular, every fermionic system with \mathbb{Z}_2^F symmetry can be obtained by the fermionization of its bosonization. The bosonization and fermionization in 1+1d can be summarized as shown in Figure 5.1. In this section, we review the procedures involved in Figure 5.1 following [69, 70, 84, 85, 161, 162].

Gauging \mathbb{Z}_2 symmetry. Gauging a non-anomalous \mathbb{Z}_2 symmetry of a bosonic system B in 1+1d produces another bosonic system \tilde{B} with the dual \mathbb{Z}_2 symmetry [163]. At the level of partition functions, the gauging of the \mathbb{Z}_2 symmetry can be expressed as

$$Z_{\tilde{B}}(a) = \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{b \in H^1(\Sigma, \mathbb{Z}_2)} Z_B(b) (-1)^{\int_{\Sigma} a \cup b}, \quad (5.1.1)$$

where $Z_B(b)$ is the partition function of the original bosonic system B on an oriented two-dimensional surface Σ equipped with a background \mathbb{Z}_2 gauge field $b \in H^1(\Sigma, \mathbb{Z}_2)$, while $Z_{\tilde{B}}(a)$

¹In the context of relativistic quantum field theories, this is due to the spin-statistics relation [158]. Even on the lattice, we expect that a spin structure is necessary to define fermionic systems on spacetimes with non-trivial topology, see, e.g., [159].

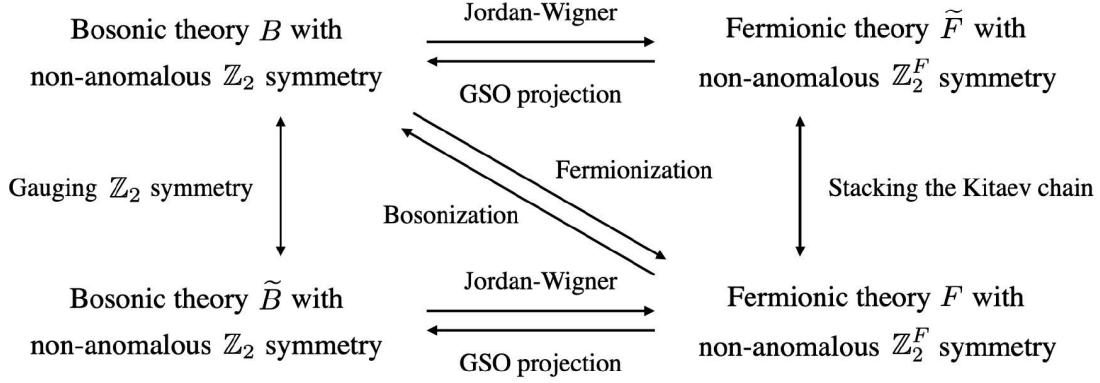


Figure 5.1: Bosonization and fermionization in 1+1 dimensions can be summarized by the above diagram. The Jordan-Wigner transformation is the gauging of a \mathbb{Z}_2 symmetry coupled to a spin structure [84, 113]. The dual of the gauged \mathbb{Z}_2 symmetry is the fermion parity symmetry. The inverse of the Jordan-Wigner transformation is the summation over spin structures, which is called the Gliozzi-Scherk-Olive (GSO) projection [160]. In this paper, fermionization refers to the map from bosonic theory B to fermionic theory F . For example, the fermionization of a trivial TFT is a trivial TFT, while the fermionization of a \mathbb{Z}_2 symmetry broken TFT is the Kitaev chain. The bosonization is the inverse map of the fermionization. We note that the fermionization in this dissertation is called the Jordan-Wigner transformation in [161], whereas the Jordan-Wigner transformation in this paper is called the fermionization in [69, 70].

is the partition function of the gauged bosonic system \tilde{B} on Σ equipped with the background gauge field a of the dual \mathbb{Z}_2 symmetry. The gauged theory \tilde{B} goes back to the original theory B if we gauge the dual \mathbb{Z}_2 symmetry:

$$Z_B(a) = \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{b \in H^1(\Sigma, \mathbb{Z}_2)} Z_{\tilde{B}}(b) (-1)^{\int_{\Sigma} a \cup b}. \quad (5.1.2)$$

The relation between the partition functions Z_B and $Z_{\tilde{B}}$ on a torus implies that gauging the \mathbb{Z}_2 symmetry permutes four sectors, i.e., the untwisted even sector $\mathcal{H}^{u,+}$, the untwisted odd sector $\mathcal{H}^{u,-}$, the twisted even sector $\mathcal{H}^{t,+}$, and the twisted odd sector $\mathcal{H}^{t,-}$, as follows:

$$\mathcal{H}_{\tilde{B}}^{u,+} = \mathcal{H}_B^{u,+}, \quad \mathcal{H}_{\tilde{B}}^{u,-} = \mathcal{H}_B^{t,+}, \quad \mathcal{H}_{\tilde{B}}^{t,+} = \mathcal{H}_B^{u,-}, \quad \mathcal{H}_{\tilde{B}}^{t,-} = \mathcal{H}_B^{t,-}. \quad (5.1.3)$$

The above equation shows that the untwisted sector of the original theory becomes the \mathbb{Z}_2 -even sector of the gauged theory, while the twisted sector of the original theory becomes the \mathbb{Z}_2 -odd sector of the gauged theory. Similarly, the \mathbb{Z}_2 even sector of the original theory becomes the untwisted sector of the gauged theory, while the \mathbb{Z}_2 odd sector of the original theory becomes the twisted sector of the gauged theory.

Stacking the Kitaev chain. The Kitaev chain is a 1+1d fermionic SPT phase whose ground state is fermionic in the Ramond sector and is bosonic in the Neveu-Schwarz sector [164].² This is a unique non-trivial SPT phase protected by the fermion parity \mathbb{Z}_2^F symmetry [153, 165]. The low energy limit of the Kitaev chain is described by an invertible fermionic topological field theory, which we also call the Kitaev chain by abuse of terminology. Stacking the Kitaev chain

²The Ramond sector is the state space on a circle with the periodic boundary condition. On the other hand, the Neveu-Schwarz sector is the state space on a circle with the anti-periodic boundary condition.

on top of a fermionic theory F with a non-anomalous \mathbb{Z}_2^F symmetry gives us another fermionic theory \tilde{F} that also has a non-anomalous \mathbb{Z}_2^F symmetry. At the level of partition functions, the relation between F and \tilde{F} can be expressed as

$$Z_{\tilde{F}}(\eta) = Z_F(\eta) \text{Arf}(\eta), \quad (5.1.4)$$

where $Z_F(\eta)$ and $Z_{\tilde{F}}(\eta)$ are the partition functions of F and \tilde{F} on an oriented surface Σ equipped with a spin structure η , and $\text{Arf}(\eta) = \pm 1$ is the partition function of the Kitaev chain known as the Arf invariant of a spin structure η . Stacking the Kitaev chain twice maps the original theory to itself because the stack of two copies of the Kitaev chain is the trivial SPT phase [153, 165]. As in the case of the \mathbb{Z}_2 gauging, the relation between the torus partition functions of F and \tilde{F} tells us that the four sectors of a fermionic theory, i.e., the Neveu-Schwarz bosonic sector, the Neveu-Schwarz fermionic sector, the Ramond bosonic sector, and the Ramond fermionic sector, are permuted by stacking the Kitaev chain as follows:

$$\mathcal{H}_{\tilde{F}}^{\text{NS},b} = \mathcal{H}_F^{\text{NS},b}, \quad \mathcal{H}_{\tilde{F}}^{\text{NS},f} = \mathcal{H}_F^{\text{NS},f}, \quad \mathcal{H}_{\tilde{F}}^{\text{R},b} = \mathcal{H}_F^{\text{R},f}, \quad \mathcal{H}_{\tilde{F}}^{\text{R},f} = \mathcal{H}_F^{\text{R},b}. \quad (5.1.5)$$

The superscripts NS and R represent Neveu-Schwarz and Ramond sectors, while b and f represent bosonic and fermionic sectors. The above equation shows that stacking the Kitaev chain flips the fermion parity of the Ramond sector while keeping the Neveu-Schwarz sector unaffected.

Jordan-Wigner transformation. Given a bosonic system B with a non-anomalous \mathbb{Z}_2 symmetry, we can construct a fermionic system \tilde{F} by coupling the \mathbb{Z}_2 symmetry to a spin structure and then gauging the symmetry. This procedure is called the Jordan-Wigner transformation, which is a generalization of the Jordan-Wigner transformation of quantum spin chains originally discussed in [166]. The fermionic theory \tilde{F} has a non-anomalous fermion parity symmetry \mathbb{Z}_2^F , which is the dual symmetry of the gauged \mathbb{Z}_2 symmetry. One can write down the Jordan-Wigner transformation explicitly at the level of partition functions as follows:

$$Z_{\tilde{F}}(\eta) = \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{a \in H^1(\Sigma, \mathbb{Z}_2)} Z_B(a) (-1)^{q_\eta(a)}. \quad (5.1.6)$$

Here, $q_\eta(a) = 0$ if η induces the NS spin structure on the Poincaré dual of a and $q_\eta(a) = 1$ if η induces the R spin structure on the Poincaré dual of a [159, 167]. The \mathbb{Z}_2 -valued function q_η is related to the Arf invariant as $(-1)^{q_\eta(a)} = \text{Arf}(\eta) \text{Arf}(\eta+a)$. Equivalently, the Arf invariant can be written in terms of q_η as $\text{Arf}(\eta) = \sum_{a \in H^1(\Sigma, \mathbb{Z}_2)} (-1)^{q_\eta(a)}$. We note that $q_\eta(a)$ is a quadratic refinement of the intersection form, i.e., it satisfies $q_\eta(a) + q_\eta(b) = q_\eta(a+b) + \int_\Sigma a \cup b$ modulo 2 [159].³ Physically, the factor $(-1)^{q_\eta(a)}$ is the partition function of a fermionic SPT phase known as the Gu-Wen phase with $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry [159, 169]. Therefore, eq. (5.1.6) means that the Jordan-Wigner transformation is achieved by stacking the Gu-Wen SPT phase on top of a bosonic theory with \mathbb{Z}_2 symmetry and then gauging the common \mathbb{Z}_2 symmetry. We can see how the Jordan-Wigner transformation permutes the four sectors of the theory by comparing the torus partition function of the fermionic theory \tilde{F} with that of the original bosonic theory B :

$$\mathcal{H}_{\tilde{F}}^{\text{NS},b} = \mathcal{H}_B^{u,+}, \quad \mathcal{H}_{\tilde{F}}^{\text{NS},f} = \mathcal{H}_B^{t,-}, \quad \mathcal{H}_{\tilde{F}}^{\text{R},b} = \mathcal{H}_B^{u,-}, \quad \mathcal{H}_{\tilde{F}}^{\text{R},f} = \mathcal{H}_B^{t,+}. \quad (5.1.7)$$

³Quadratic refinements of the intersection form on an oriented surface Σ are in one-to-one correspondence with spin structures on Σ [168]. The function q_η is the quadratic refinement corresponding to the spin structure η .

When B is a trivial bosonic TFT with \mathbb{Z}_2 symmetry (i.e., the low energy limit of the disordered phase of the Ising model), both the untwisted sector and twisted sector consists of a single \mathbb{Z}_2 -even state, i.e., $\mathcal{H}_B^{u,+} \cong \mathcal{H}_B^{t,+} \cong \mathbb{C}$ and $\mathcal{H}_B^{t,-} = \mathcal{H}_B^{u,-} = \emptyset$. Correspondingly, the fermionic theory \tilde{F} has a single bosonic state in the NS sector and a single fermionic state in the R sector. This implies that the Jordan-Wigner transformation of the trivial TFT with \mathbb{Z}_2 symmetry gives rise to the Kitaev chain.⁴

GSO projection. Given a fermionic theory F with a non-anomalous \mathbb{Z}_2^F symmetry, we can obtain a bosonic theory \tilde{B} by summing over spin structures on a two-dimensional spacetime. This procedure is known as the Gliozzi-Scherk-Olive (GSO) projection [160]. The bosonic theory B obtained by the GSO projection has a non-anomalous \mathbb{Z}_2 symmetry, which is dual to the fermion parity symmetry of the original theory F . At the level of partition functions, the GSO projection can be expressed as

$$Z_{\tilde{B}}(a) = \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{\eta} Z_F(\eta) (-1)^{q_{\eta}(a)}. \quad (5.1.8)$$

We note that the GSO projection followed by the Jordan-Wigner transformation does not change the partition function of a fermionic theory F , i.e.,

$$\frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_a Z_{\tilde{B}}(a) (-1)^{q_{\eta}(a)} = \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|} \sum_a \sum_{\xi} Z_F(\xi) (-1)^{q_{\eta}(a) + q_{\xi}(a)} = Z_F(\eta), \quad (5.1.9)$$

where the last equality follows from $\sum_a (-1)^{q_{\eta}(a) + q_{\xi}(a)} = \delta_{\eta, \xi} |H^1(\Sigma, \mathbb{Z}_2)|$. Similarly, the Jordan-Wigner transformation followed by the GSO projection also leaves the partition function invariant. This implies that the GSO projection is the inverse of the Jordan-Wigner transformation.

Fermionization. The fermionic theory F is obtained by applying the Jordan-Wigner transformation to a bosonic theory B and stacking the Kitaev chain on top of the resulting fermionic theory \tilde{F} . Equivalently, F is obtained by first gauging the \mathbb{Z}_2 symmetry of B and then applying the Jordan-Wigner transformation to the resulting bosonic theory \tilde{B} . The procedure that maps the bosonic theory B to the fermionic theory F is called the fermionization. Due to equations (5.1.4) and (5.1.6), the partition function of B and F are related by

$$\begin{aligned} Z_F(\eta) &= \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{a \in H^1(\Sigma, \mathbb{Z}_2)} Z_B(a) (-1)^{q_{\eta}(a)} \text{Arf}(\eta) \\ &= \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{a \in H^1(\Sigma, \mathbb{Z}_2)} Z_B(a) \text{Arf}(\eta + a), \end{aligned} \quad (5.1.10)$$

where $\eta + a$ is the spin structure obtained by shifting η by a \mathbb{Z}_2 gauge field $a \in H^1(\Sigma, \mathbb{Z}_2)$. The second equality in the above equation follows from $\text{Arf}(\eta + a) = (-1)^{q_{\eta}(a)} \text{Arf}(\eta)$. Equation (5.1.10) implies that the fermionization can also be done by stacking the Kitaev chain on top of a bosonic theory B and gauging the diagonal \mathbb{Z}_2 subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry.

⁴This result slightly differs from the traditional Jordan-Wigner transformation, which maps the disordered phase of the Ising model to the trivial fermionic phase. The traditional Jordan-Wigner transformation is more like the fermionization discussed below.

Bosonization. The bosonic theory B is obtained by stacking the Kitaev chain on top of a fermionic theory F and applying the GSO projection to the resulting fermionic theory \tilde{F} . Equivalently, B is obtained by applying the GSO projection to F and gauging the \mathbb{Z}_2 symmetry of the resulting bosonic theory \tilde{B} . The procedure that maps the fermionic theory F to the bosonic theory B is known as the bosonization, which is the inverse of the fermionization.

5.2 Fermionization of fusion category symmetries

Let B be a bosonic theory with a fusion category symmetry \mathcal{C}_B . When \mathcal{C}_B has a non-anomalous \mathbb{Z}_2 subgroup symmetry, we can fermionize B to obtain a fermionic theory F with a superfusion category symmetry \mathcal{C}_F . In this section, we write down the general formula to determine the symmetry \mathcal{C}_F from the original symmetry \mathcal{C}_B and its non-anomalous \mathbb{Z}_2 subgroup. We call this formula the fermionization formula of fusion category symmetries. The formula will be derived in Section 6.4 when the bosonic theory B is a topological field theory. Although the derivation only applies to the case of topological field theories, the concrete examples discussed in Section 5.3 suggest that our fermionization formula is also applicable to the symmetries of non-topological theories.

5.2.1 Non-anomalous symmetries

We first consider the fermionization of non-anomalous fusion category symmetries. As we mentioned in Section 2.3, every non-anomalous symmetry of a 1+1d bosonic system is described by the representation category $\text{Rep}(H)$ of a semisimple Hopf algebra H . The $\text{Rep}(H)$ symmetry can be fermionized if it has a non-anomalous \mathbb{Z}_2 subgroup symmetry.⁵ When $\text{Rep}(H)$ has more than one \mathbb{Z}_2 subgroups, there are multiple ways to fermionize the $\text{Rep}(H)$ symmetry depending on the choice of its \mathbb{Z}_2 subgroup. A different choice of a \mathbb{Z}_2 subgroup generally gives rise to a different fermionized symmetry.

A \mathbb{Z}_2 subgroup of $\text{Rep}(H)$ symmetry is related to a group-like element in the dual Hopf algebra H^* . In general, a subgroup of $\text{Rep}(H)$ symmetry consists of one-dimensional representations of H , which are denoted by V_g . Since V_g is a one-dimensional representation, the action of $a \in H$ on $v \in V_g$ can be written as $a \cdot v = g(a)v$ by using a linear functional $g \in H^*$ that satisfies

$$g(ab) = g(a)g(b), \quad g(\eta(1)) = 1, \quad \forall a, b \in H. \quad (5.2.1)$$

The first equality of the above equation is equivalent to

$$\Delta_{H^*}(g) = g \otimes g, \quad (5.2.2)$$

where $\Delta_{H^*} : H^* \rightarrow H^* \otimes H^*$ is the comultiplication on H^* , cf. equation (2.3.4). A non-zero element $g \in H^*$ satisfying eq. (5.2.2) is called a group-like element in H^* . When $g \in H^*$ is a group-like element, the second equality of eq. (5.2.1) is automatically satisfied due to the defining property (2.3.1) of a Hopf algebra. An immediate consequence of eq. (5.2.2) is that group-like elements of H^* form a group $G(H^*)$, whose multiplication is defined by the multiplication on H^* and whose unit element is the counit $\epsilon : H \rightarrow \mathbb{C}$.⁶ Accordingly, one-dimensional representations of H also form a group, which is isomorphic to $G(H^*)$. The

⁵Henceforth, a non-anomalous \mathbb{Z}_2 subgroup will be simply called a \mathbb{Z}_2 subgroup because any \mathbb{Z}_2 subgroup of a non-anomalous symmetry $\text{Rep}(H)$ is non-anomalous.

⁶The inverse of $g \in G(H^*)$ is given by $S_{H^*}(g)$, where $S_{H^*} : H^* \rightarrow H^*$ is the antipode of H^* .

multiplication of one-dimensional representations V_g and V_h is given by the tensor product of representations $V_g \otimes V_h \cong V_{gh}$. In particular, if $G(H^*)$ has a \mathbb{Z}_2 subgroup generated by $u \in G(H^*)$, a fusion category $\text{Rep}(H)$ has a \mathbb{Z}_2 subgroup generated by a one-dimensional representation $V_u \in \text{Rep}(H)$. The generator u of a \mathbb{Z}_2 subgroup of $G(H^*)$ is called a \mathbb{Z}_2 group-like element in H^* , which satisfies $u \otimes u(\Delta(a)) = \epsilon(a)$ for all $a \in H$ in addition to eq. (5.2.1).

Let $u \in G(H^*)$ be a \mathbb{Z}_2 group-like element in H^* and let \mathbb{Z}_2^u be the \mathbb{Z}_2 subgroup of $\text{Rep}(H)$ generated by $V_u \in \text{Rep}(H)$. We propose that the fermionization of $\text{Rep}(H)$ symmetry with respect to \mathbb{Z}_2^u subgroup is a superfusion category symmetry $\text{sRep}(\mathcal{H}^u)$, where \mathcal{H}^u is a Hopf superalgebra defined as follows:

- The underlying vector space of \mathcal{H}^u is H .
- The \mathbb{Z}_2 -grading automorphism $p_u : \mathcal{H}^u \rightarrow \mathcal{H}^u$ is defined by the adjoint action of u , i.e.,

$$p_u := (u \otimes \text{id} \otimes u) \circ (\text{id} \otimes \Delta) \circ \Delta. \quad (5.2.3)$$

The \mathbb{Z}_2 -grading of a homogeneous element $a \in \mathcal{H}^u$ is denoted by $p_u(a) = (-1)^{|a|}a$.

- The multiplication $m_u : \mathcal{H}^u \otimes \mathcal{H}^u \rightarrow \mathcal{H}^u$ of homogeneous elements a and b is given by

$$m_u(a \otimes b) = m \circ (\text{id} \otimes u^{|b|} \otimes \text{id}) \circ (\Delta \otimes \text{id})(a \otimes b), \quad (5.2.4)$$

where m on the right-hand side is the multiplication on the original Hopf algebra H . The above multiplication is extended linearly to the multiplication of inhomogeneous elements.

- The antipode $S_u : \mathcal{H}^u \rightarrow \mathcal{H}^u$ of a homogeneous element a is given by

$$S_u(a) = (u^{|a|} \otimes S) \circ \Delta(a), \quad (5.2.5)$$

where S on the right-hand side is the antipode of the original Hopf algebra H . The above antipode is extended linearly to inhomogeneous elements.

- The comultiplication $\Delta : \mathcal{H}^u \rightarrow \mathcal{H}^u \otimes \mathcal{H}^u$, the unit $\eta : \mathbb{C} \rightarrow \mathcal{H}^u$, and the counit $\epsilon : \mathcal{H}^u \rightarrow \mathbb{C}$ are the same as those of H .

As expected, the superfusion category $\text{sRep}(\mathcal{H}^u)$ has a fermion parity symmetry \mathbb{Z}_2^F , which is generated by a one-dimensional super representation $V_u \in \text{sRep}(\mathcal{H}^u)$ defined by $a \cdot v = u(a)v$ for all $a \in \mathcal{H}^u$ and $v \in V_u$. We refer the reader to Section 6.4 for the derivation of the above fermionization formula in the context of topological field theory.

In the rest of this subsection, we show that the super vector space \mathcal{H}^u equipped with the structure maps $(m_u, \Delta, \eta, \epsilon, S_u)$ turns out to be a Hopf superalgebra [109]. To this end, we first notice that the structure maps defined above preserve the \mathbb{Z}_2 -grading because the \mathbb{Z}_2 group-like element $u \in G(H^*)$ is \mathbb{Z}_2 -even and the structure maps of the original Hopf algebra H preserve the \mathbb{Z}_2 -grading defined by eq. (5.2.3). Moreover, a straightforward computation shows that the multiplication m_u defined by eq. (5.2.4) is associative and the multiplicative unit with respect to m_u is given by η . This shows that $(\mathcal{H}^u, m_u, \eta)$ is an associative unital superalgebra. It is also clear that $(\mathcal{H}^u, \Delta, \epsilon)$ is a coassociative counital supercoalgebra. Therefore, it remains to show eqs. (2.5.10), (2.5.11), and (2.5.12). Among these equations, the second equality of eq. (2.5.10) and the second equality of eq. (2.5.11) are obvious because they do not involve the

modified structure maps m_u and S_u . We can show the first equality of eq. (2.5.10) by using string diagrams as

$$\text{LHS} = m_u = \begin{array}{c} \text{diagram with two inputs } a, b \text{ and one output, with a dot on the line} \end{array} = \begin{array}{c} \text{diagram with two inputs } a, b \text{ and one output, with two dots and labels } u^{|b_{(1)}|}, u^{|b_{(2)}|}, u^{|b|} \text{ and } c_{\text{triv}} \end{array} = \begin{array}{c} \text{diagram with two inputs } a, b \text{ and one output, with two dots and labels } m_u, m_u \text{ and } c_{\text{super}} \end{array} = \text{RHS}, \quad (5.2.6)$$

where we used the Sweedler notation $\Delta(b) = b_{(1)} \otimes b_{(2)}$ and the identity $c_{\text{triv}}(a_{(2)} \otimes b_{(1)}) = c_{\text{super}} \circ (p_u^{|b_{(1)}|} \otimes \text{id})(a_{(2)} \otimes b_{(1)})$. When b is homogeneous, we can assume without loss of generality that $b_{(1)}$ and $b_{(2)}$ are homogeneous because the comultiplication Δ is \mathbb{Z}_2 -even, and therefore the notations $|b_{(1)}|$ and $|b_{(2)}|$ in eq. (5.2.6) make sense. The first equality of eq. (2.5.11) also holds because

$$\text{LHS} = m_u = \begin{array}{c} \text{diagram with two inputs } a, b \text{ and one output, with a dot and label } \epsilon \end{array} = \begin{array}{c} \text{diagram with two inputs } a, b \text{ and one output, with a dot and label } u^{|b|} \end{array} \epsilon = \begin{array}{c} \text{diagram with two inputs } a, b \text{ and one output, with two dots and labels } \epsilon, \epsilon \end{array} = \text{RHS}, \quad (5.2.7)$$

where the third equality follows from the fact that $\epsilon(b) = 0$ unless $|b| = 0$. Equation (2.5.12) can also be checked similarly. Thus, we find that the super vector space \mathcal{H}^u equipped with the structure maps $(m_u, \eta, \Delta, \epsilon, S_u)$ is a Hopf superalgebra.

Bosonization of superfusion category symmetries. As we discussed above, a non-anomalous fusion category symmetry $\text{Rep}(H)$ can be fermionized if the dual Hopf algebra H^* has a \mathbb{Z}_2 group-like element $u \in H^*$. Conversely, the superfusion category symmetry $\text{sRep}(\mathcal{H}^u)$ obtained by the fermionization can be bosonized back to the original symmetry $\text{Rep}(H)$. The bosonization of the structure maps of \mathcal{H}^u is given by

$$\begin{aligned} m(a \otimes b) &= m_u \circ (\text{id} \otimes u^{|b|} \otimes \text{id}) \circ (\Delta \otimes \text{id})(a \otimes b), \\ S(a) &= (u^{|a|} \otimes S_u) \circ \Delta(a). \end{aligned} \quad (5.2.8)$$

5.2.2 Anomalous symmetries

A fusion category symmetry is anomalous if it cannot be described by the representation category of a semisimple Hopf algebra. An anomalous fusion category symmetry can always be described by the representation category of a semisimple weak Hopf algebra. In this subsection, we discuss the fermionization of an anomalous symmetry $\text{Rep}(H)$ where H is a weak Hopf algebra.

In order to perform the fermionization of $\text{Rep}(H)$ symmetry, we first specify a non-anomalous \mathbb{Z}_2 subgroup symmetry of $\text{Rep}(H)$. As in the case of non-anomalous symmetries, the existence of a \mathbb{Z}_2 group-like element in the dual weak Hopf algebra H^* implies that $\text{Rep}(H)$ symmetry has a non-anomalous \mathbb{Z}_2 subgroup symmetry. Here, a group-like element $g \in H^*$ is defined as an invertible element that satisfies

$$\Delta_{H^*}(g) = \Delta_{H^*}(\epsilon) \cdot g \otimes g = g \otimes g \cdot \Delta_{H^*}(\epsilon), \quad (5.2.9)$$

where $\epsilon \in H^*$ is the counit of H . In particular, we call $u \in H^*$ a \mathbb{Z}_2 group-like element if it satisfies $u \cdot u = \epsilon$ in addition to eq. (5.2.9). The representation $V_u \in \text{Rep}(H)$ associated with a \mathbb{Z}_2 group-like element $u \in H^*$ is defined as follows [170]: the underlying vector space of V_u is the target counital subalgebra H_t and the H -action $\rho_{V_u} : H \otimes V_u \rightarrow V_u$ is given by $\rho_{V_u} = (\epsilon_t \otimes u) \circ \Delta \circ m$, where $m : H \otimes V_u \rightarrow H$ and $\Delta : V_u \rightarrow H \otimes H$ is the multiplication and the comultiplication on H restricted to $H \otimes V_u$ and V_u respectively. The representation V_u generates a \mathbb{Z}_2 subgroup of $\text{Rep}(H)$ [171], which is denoted by \mathbb{Z}_2^u .

We propose that the fermionization of an anomalous fusion category symmetry $\text{Rep}(H)$ with respect to \mathbb{Z}_2^u subgroup symmetry is a superfusion category symmetry $\text{sRep}(\mathcal{H}^u)$, where \mathcal{H}^u is a weak Hopf superalgebra defined as follows:

- The underlying vector space of \mathcal{H}^u is H .
- The \mathbb{Z}_2 -grading on \mathcal{H}^u is defined by eq. (5.2.3).
- The multiplication $m_u : \mathcal{H}^u \otimes \mathcal{H}^u \rightarrow \mathcal{H}^u$ and the antipode $S_u : \mathcal{H}^u \rightarrow \mathcal{H}^u$ are given by eqs. (5.2.4) and (5.2.5) respectively.
- The other structure maps are the same as those of H .

We again refer the reader to Section 6.4 for the derivation of the above formula. As in the case of non-anomalous symmetries, the bosonization of the superfusion category symmetry $\text{sRep}(\mathcal{H}^u)$ is given by eq. (5.2.8). In the rest of this subsection, we show that super vector space \mathcal{H}^u equipped with the above structure maps becomes a weak Hopf superalgebra.

We first show that the structure maps of \mathcal{H}^u are even with respect to the \mathbb{Z}_2 -grading (5.2.3). Since the structure maps of \mathcal{H}_u consists of the \mathbb{Z}_2 group-like element $u \in H^*$ and the structure maps of H , it suffices to show that these constituents are all \mathbb{Z}_2 -even. The \mathbb{Z}_2 group-like element u is even because $u \circ p_u = (\epsilon \otimes u) \circ \Delta = u$, where the first equality follows from $u \cdot u = \epsilon$. The multiplication m is even because

$$(5.2.10)$$

The comultiplication Δ is even due to the coassociativity of Δ and the equality $u \cdot u = \epsilon$:

$$(5.2.11)$$

The antipode S is also even due to the fact that S is an algebra and coalgebra homomorphism:

$$(5.2.12)$$

The unit η and counit ϵ are automatically even because the multiplication m and comultiplication Δ are even. Therefore, the structure maps of \mathcal{H}^u are all even.

Next, we show that $(\mathcal{H}^u, m_u, \eta)$ is an associative unital superalgebra and $(\mathcal{H}^u, \Delta, \epsilon)$ is a coassociative counital supercoalgebra. The latter is clear because Δ and ϵ are \mathbb{Z}_2 -even as shown above and (H, Δ, ϵ) is a coassociative counital coalgebra. Thus, it suffices to show that $(\mathcal{H}^u, m_u, \eta)$ is an associative unital superalgebra. The associativity of the multiplication m_u is an immediate consequence of the definition of m_u :

(5.2.13)

Moreover, the unit of \mathcal{H}^u is given by the unit of the original Hopf algebra H because

(5.2.14)

where $1 := \eta(1) \in \mathcal{H}^u$. The second equality follows from the relation $(\text{id} \otimes u) \circ \Delta \circ \eta = \eta$, which can be derived as follows:

(5.2.15)

The third and last equalities of eq. (5.2.15) follow from the fact that the unit η is even. Equations (5.2.13) and (5.2.14) show that $(\mathcal{H}^u, m_u, \eta)$ is an associative unital superalgebra.

Finally, we show that the structure maps $(m_u, \eta, \Delta, \epsilon, S_u)$ satisfy the defining properties (2.5.13)–(2.5.16) of a weak Hopf superalgebra. Equation (2.5.13) follows from the same computation as in the case of non-anomalous symmetries, cf. eq. (5.2.6). Equation (2.5.14) also follows from a direct computation as follows:

(2.5.14)

The last equality of the first line follows from the fact that $\epsilon \circ m_u(b_{(2)} \otimes c)$ vanishes unless $|b_{(2)}c| = 0$. Similarly, the last equality of the second line is because $\epsilon \circ m_u(b_{(1)} \otimes c)$ vanishes unless $|b_{(1)}c| = 0$. We can also show eq. (2.5.15) analogously. Equation (2.5.16) can also be

derived as

$$\begin{aligned}
& m_u \circ S_u = \text{diagram} = \text{diagram} = \text{diagram} = c_{\text{triv}} = \text{diagram} = c_{\text{super}} = \text{diagram}, \\
& S_u \circ m_u = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = c_{\text{super}} \circ m_u = c_{\text{super}} \circ m_u, \\
& S_u \circ S_u = \text{diagram} = \text{diagram} = \text{diagram} = \text{diagram} = S_u.
\end{aligned}$$

This completes the proof of the weak Hopf superalgebra structure on \mathcal{H}^u .

5.3 Examples

In this section, we explicitly compute the fermionization of fusion category symmetries for several examples including both non-anomalous and anomalous symmetries.

5.3.1 Finite group symmetries

The first example is the fermionization of a non-anomalous finite group symmetry G . As a fusion category symmetry, a finite group symmetry G is described by the representation category $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ of the coopposite dual group algebra $(\mathbb{C}[G]^*)^{\text{cop}}$. The basis of $(\mathbb{C}[G]^*)^{\text{cop}}$ is denoted by $\{\hat{g} \mid g \in G\}$, which is dual to the standard basis $\{g \in G\}$ of $\mathbb{C}[G]$, i.e. we have $\hat{g}(h) = \delta_{g,h}$. The Hopf algebra structure on $(\mathbb{C}[G]^*)^{\text{cop}}$ is given by

$$m(\hat{g} \otimes \hat{h}) = \delta_{g,h} \hat{g}, \quad \eta(1) = \sum_{g \in G} \hat{g}, \quad \Delta(\hat{g}) = \sum_{h \in G} \widehat{gh^{-1}} \otimes \hat{h}, \quad \epsilon(\hat{g}) = \delta_{g,e}, \quad S(\hat{g}) = \widehat{g^{-1}}. \quad (5.3.1)$$

We note that the basis \hat{g} is an idempotent and thus $(\mathbb{C}[G]^*)^{\text{cop}}$ is isomorphic to the direct sum of one-dimensional algebras $\mathbb{C}\hat{g}$ as a semisimple algebra:

$$(\mathbb{C}[G]^*)^{\text{cop}} = \bigoplus_{g \in G} \mathbb{C}\hat{g} \cong \bigoplus_{g \in G} \text{End}(V_g). \quad (5.3.2)$$

Here, V_g is a one-dimensional representation of $(\mathbb{C}[G]^*)^{\text{cop}}$ defined by

$$\hat{h} \cdot v_g = \delta_{h,g} v_g, \quad \forall v_g \in V_g. \quad (5.3.3)$$

The above decomposition (5.3.2) indicates that simple objects of $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ are one-dimensional representations V_g . These simple objects form a group under the tensor product of representations because we have an isomorphism $V_g \otimes V_h \cong V_{gh}$. Thus, the representation category $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$ describes an ordinary finite group symmetry G .

To perform the fermionization of a finite group symmetry, we specify a \mathbb{Z}_2 subgroup of a fusion category $\text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$. A \mathbb{Z}_2 subgroup associated with a \mathbb{Z}_2 group-like element $u \in ((\mathbb{C}[G]^*)^{\text{cop}})^* \cong \mathbb{C}[G]^{\text{op}}$ is denoted by \mathbb{Z}_2^u , which is generated by $V_u \in \text{Rep}((\mathbb{C}[G]^*)^{\text{cop}})$. If we use the \mathbb{Z}_2^u subgroup for fermionization, the fermionized symmetry becomes $\text{sRep}(\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u)$ where $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ is a Hopf superalgebra obtained by the formula in Section 5.2.1.

Hopf superalgebra $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$. Let us investigate the Hopf superalgebra structure on $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$. The \mathbb{Z}_2 -grading automorphism $p_u : \mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u \rightarrow \mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ defined by the adjoint action (5.2.3) of the \mathbb{Z}_2 group-like element $u \in \mathbb{C}[G]^{\text{op}}$ is computed as

$$p_u(\widehat{g}) = \widehat{ug}. \quad (5.3.4)$$

This equation implies that $\widehat{g} \in (\mathbb{C}[G]^*)^{\text{cop}}$ is a \mathbb{Z}_2 -even element when $g \in G$ commutes with u . On the other hand, when $g \in G$ does not commute with u , \widehat{g} is not a homogeneous element, but the linear combinations $\widehat{g}_{\pm} = \widehat{g} \pm \widehat{ug}$ are homogeneous because $p(\widehat{g}_{\pm}) = \pm \widehat{g}_{\pm}$. Therefore, the \mathbb{Z}_2 -even sector and the \mathbb{Z}_2 -odd sector of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ are given by

$$(\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u)_0 = \left(\bigoplus_{g \in C_G(u)} \mathbb{C}\widehat{g} \right) \oplus \left(\bigoplus_{\substack{[g] \in G/\sim \\ \text{s.t. } g \notin C_G(u)}} \mathbb{C}\widehat{g}_+ \right), \quad (\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u)_1 = \bigoplus_{\substack{[g] \in G/\sim \\ \text{s.t. } g \notin C_G(u)}} \mathbb{C}\widehat{g}_-. \quad (5.3.5)$$

Here, $C_G(u) := \{g \in G \mid gu = ug\}$ is the centralizer of $u \in G$ and G/\sim is the quotient of G by the equivalence relation $g \sim ugu$. Given the above \mathbb{Z}_2 -grading on $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$, we can compute the multiplication m_u and the antipode S_u defined by eqs. (5.2.4) and (5.2.5) as follows:

$$m_u(\widehat{g} \otimes \widehat{h}) = \frac{1}{2}(\delta_{g,h} + \delta_{g,uhu})\widehat{g} + \frac{1}{2}(\delta_{g,hu} - \delta_{g,uh})\widehat{gu}, \quad (5.3.6)$$

$$S_u(\widehat{g}) = \frac{1}{2}(\widehat{g^{-1}} + \widehat{ug^{-1}u} + \widehat{g^{-1}u} - \widehat{ug^{-1}}). \quad (5.3.7)$$

It turns out that the superalgebra $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ with multiplication (5.3.6) can be decomposed into a direct sum of simple superalgebras as follows:

$$\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u \cong \left(\bigoplus_{g \in C_G(u)} \mathbb{C}\widehat{g} \right) \oplus \left(\bigoplus_{\substack{[g] \in G/\sim' \\ \text{s.t. } g \notin C_G(u)}} \text{Span}\{\widehat{g}_{\pm}, \widehat{gu}_{\pm}\} \right), \quad (5.3.8)$$

where G/\sim' is the quotient of G by the equivalence relation $g \sim' ug \sim' gu \sim' ugu$. The direct summand $\mathbb{C}\widehat{g}$ is isomorphic to the endomorphism superalgebra $\text{End}(V_g)$ of a one-dimensional super representation $V_g \cong \mathbb{C}^{1|0}$ on which $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ acts by eq. (5.3.3). On the other hand, the direct summand $\text{Span}\{\widehat{g}_{\pm}, \widehat{gu}_{\pm}\}$ is isomorphic to the complex Clifford algebra $\text{Cl}(2)$ with two odd generators. This isomorphism becomes clear if we define the basis

$$1_g := \widehat{g}_+ + \widehat{gu}_+, \quad \Gamma_g := \widehat{g}_- + \widehat{gu}_-, \quad \Gamma'_g := \widehat{g}_- - \widehat{gu}_-, \quad \Gamma_g \Gamma'_g = \widehat{g}_+ - \widehat{gu}_+, \quad (5.3.9)$$

which satisfy the same algebra as $\text{Cl}(2)$, i.e. $(\Gamma_g)^2 = -(\Gamma'_g)^2 = 1_g$ and $\Gamma_g \Gamma'_g = -\Gamma'_g \Gamma_g$. Since the Clifford algebra $\text{Cl}(2)$ is isomorphic to $\text{End}(\mathbb{C}^{1|1})$ [107], we have an isomorphism

$$\text{Span}\{\widehat{g}_{\pm}, \widehat{gu}_{\pm}\} \cong \text{End}(W_g) \quad (5.3.10)$$

for a two-dimensional super representation $W_g \cong \mathbb{C}^{1|1}$. The action of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ on W_g is given by

$$\Gamma_g \cdot (w_g)_+ = \Gamma'_g \cdot (w_g)_+ = (w_g)_-, \quad \Gamma_g \cdot (w_g)_- = -\Gamma'_g \cdot (w_g)_- = (w_g)_+, \quad (5.3.11)$$

where $(w_g)_+$ and $(w_g)_-$ are a \mathbb{Z}_2 -even basis and a \mathbb{Z}_2 -odd basis of W_g respectively.⁷ The direct sum components other than $\text{Span}\{\widehat{g}_\pm, \widehat{g}u_\pm\}$ act trivially on W_g , namely we have $a \cdot (w_g)_\pm = 0$ for any $a \notin \text{Span}\{\widehat{g}_\pm, \widehat{g}u_\pm\}$. We note that W_g is evenly isomorphic to W_{ugu} and oddly isomorphic to W_{gu} and W_{ug} . In particular, the isomorphism class of a two-dimensional super representation W_g is labeled by $[g] \in G/\sim'$ where $g \notin C_G(u)$. In terms of irreducible representations V_g and W_g , we can write the direct sum decomposition (5.3.8) as

$$\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u \cong \left(\bigoplus_{g \in C_G(u)} \text{End}(V_g) \right) \oplus \left(\bigoplus_{\substack{[g] \in G/\sim' \\ \text{s.t. } g \notin C_G(u)}} \text{End}(W_g) \right). \quad (5.3.12)$$

Superfusion category $\text{sRep}(\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u)$. The direct sum decomposition (5.3.12) indicates that irreducible super representations of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ are isomorphic to either one-dimensional super representations V_g for $g \in C_G(u)$ or two-dimensional super representations W_g for $g \notin C_G(u)$. These irreducible super representations are simple objects of the superfusion category $\text{sRep}(\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u)$. All of these simple objects are m-type objects because irreducible super representations of superalgebras $\text{End}(\mathbb{C}^{p|q})$ do not have odd automorphisms for any p and q [107]. We can also compute the fusion rules as follows:

$$\begin{aligned} V_g \otimes V_h &\cong V_{gh}, & V_g \otimes W_h &\cong W_{gh}, & W_g \otimes V_h &\cong W_{gh}, \\ W_g \otimes W_h &\cong \begin{cases} V_{gh} \oplus V_{uguh} \oplus \Pi V_{ghu} \oplus \Pi V_{guh}, & \text{if } gh, uguh \in C_G(u), \\ V_{gh} \oplus \Pi V_{ghu} \oplus W_{uguh}, & \text{if } gh \in C_G(u), uguh \notin C_G(u), \\ W_{gh} \oplus V_{uguh} \oplus \Pi V_{guh}, & \text{if } gh \notin C_G(u), uguh \in C_G(u), \\ W_{gh} \oplus W_{uguh}, & \text{if } gh, uguh \notin C_G(u). \end{cases} \end{aligned} \quad (5.3.13)$$

These fusion rules will be derived in Appendix A. The isomorphisms in the above equation are even isomorphisms and ΠV_g denotes a one-dimensional super representation oddly isomorphic to V_g . We note that the fermionization of a finite group symmetry G is also G if \mathbb{Z}_2^u is a central subgroup of G . In particular, if G is a non-trivial central extension of G_b by \mathbb{Z}_2^u , the fermionized symmetry is a non-trivial central extension of G_b by the fermion parity symmetry \mathbb{Z}_2^F . On the other hand, if \mathbb{Z}_2^u is not a central subgroup of G , the symmetry of the fermionized theory is no longer a group. This is reminiscent of the gauging of a non-central \mathbb{Z}_2 subgroup, which turns a finite group symmetry into a non-invertible symmetry [11]. However, the fermionization is not quite the same as the \mathbb{Z}_2 gauging. For example, the fermionization of a non-anomalous finite group symmetry is always non-anomalous, while the \mathbb{Z}_2 gauging of a non-anomalous finite group symmetry can become anomalous [11, 118].

5.3.2 Rep(G) symmetry

As another example related to a finite group, let us consider the fermionization of the representation category $\text{Rep}(G)$ of a group algebra $\mathbb{C}[G]$. We denote the fermionization of $\text{Rep}(G)$

⁷We can equally define the action of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ on W_g as $\Gamma_g \cdot (w_g)_+ = -\Gamma'_g \cdot (w_g)_+ = (w_g)_-$ and $\Gamma_g \cdot (w_g)_- = \Gamma'_g \cdot (w_g)_- = (w_g)_+$. The super representation defined by this action is oddly isomorphic to the super representation defined by eq. (5.3.11)

by $\text{sRep}(\mathcal{H}_G^u)$, where $u \in \mathbb{C}[G]^*$ is a \mathbb{Z}_2 group-like element that specifies a \mathbb{Z}_2 subgroup of $\text{Rep}(G)$. We note that a \mathbb{Z}_2 group-like element $u \in \mathbb{C}[G]^*$ is an algebra homomorphism from $\mathbb{C}[G]$ to \mathbb{C} that satisfies $u(g)^2 = 1$ for all $g \in G$.

For any choice of $u \in \mathbb{C}[G]^*$, the \mathbb{Z}_2 -grading p_u on the Hopf superalgebra \mathcal{H}_G^u is trivial because we have

$$p_u(g) = (u \otimes \text{id} \otimes u) \circ (\text{id} \otimes \Delta) \circ \Delta(g) = u(g)^2 g = g, \quad (5.3.14)$$

where we used the equality $\Delta(g) = g \otimes g$ for $g \in \mathbb{C}[G]$. Since the \mathbb{Z}_2 -grading is trivial, the Hopf superalgebra \mathcal{H}_G^u is an ordinary Hopf algebra. The structure maps of \mathcal{H}_G^u are given by those of the group algebra $\mathbb{C}[G]$ because the trivial \mathbb{Z}_2 -grading p_u does not modify the structure maps at all, cf. eqs. (5.2.4) and (5.2.5). Therefore, the Hopf superalgebra \mathcal{H}_G^u is a group algebra $\mathbb{C}[G]$ equipped with a trivial \mathbb{Z}_2 -grading.

Simple objects of the superfusion category $\text{sRep}(\mathcal{H}_G^u)$ are irreducible $\mathbb{C}[G]$ -modules equipped with purely even or purely odd \mathbb{Z}_2 -gradings. In particular, $\text{sRep}(\mathcal{H}_G^u)$ has the same set of (isomorphism classes of) simple objects as $\text{Rep}(G)$. The fusion rules of $\text{sRep}(\mathcal{H}_G^u)$ are also the same as those of $\text{Rep}(G)$. Thus, the fermionization of $\text{Rep}(G)$ is essentially the same as $\text{Rep}(G)$ whatever \mathbb{Z}_2 subgroup is used for the fermionization. More precisely, the fermionization of $\text{Rep}(G)$ is a superfusion category $\text{sVec} \boxtimes \text{Rep}(G)$, where \boxtimes denotes the Deligne tensor product of superfusion categories.

In general, the Hopf superalgebra \mathcal{H}^u becomes purely even if the \mathbb{Z}_2 group-like element u is in the center of H^* . In this case, \mathcal{H}^u is a Hopf algebra H equipped with a trivial \mathbb{Z}_2 -grading, and the superfusion category $\text{sRep}(\mathcal{H}^u)$ is equivalent to $\text{sVec} \boxtimes \text{Rep}(H)$. The fermionization of $\text{Rep}(G)$ is a special example of this. We note that all simple objects of a superfusion category $\text{sVec} \boxtimes \text{Rep}(H)$ are m-type objects.

5.3.3 $\text{Rep}(H_8)$ symmetry

The next example is the fermionization of the representation category $\text{Rep}(H_8)$ of a Hopf algebra H_8 known as the eight-dimensional Kac-Paljutkin algebra, which is the smallest non-commutative and non-cocommutative semisimple Hopf algebra [172]. The Kac-Paljutkin algebra H_8 has four one-dimensional irreducible representations V_i ($i = 1, 2, 3, 4$) and a single two-dimensional irreducible representation W . The one-dimensional representations obey the fusion rules of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and therefore they generate the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\text{Rep}(H_8)$. On the other hand, the two-dimensional representation W obeys the following fusion rule:

$$W \otimes W \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4. \quad (5.3.15)$$

This fusion rule physically implies that a physical system with $\text{Rep}(H_8)$ symmetry is invariant under condensing a topological line $V_1 \oplus V_2 \oplus V_3 \oplus V_4$ on a two-dimensional spacetime.⁸ Equivalently, a physical system with $\text{Rep}(H_8)$ symmetry is self-dual under gauging the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry because the condensation of $V_1 \oplus V_2 \oplus V_3 \oplus V_4$ implements the gauging of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by V_i 's [11]. Thus, the topological line W that obeys the fusion rule (5.3.15) is called a duality defect. A simple example of a physical system with $\text{Rep}(H_8)$ symmetry is the stacking of two Ising CFTs [13]. More generally, the $\text{Rep}(H_8)$ symmetry is realized on the entire orbifold branch of the $c = 1$ bosonic CFTs [28].

⁸Any physical system with a fusion category symmetry \mathcal{C} is invariant under condensing $X \otimes X^*$ for arbitrary $X \in \mathcal{C}$ because a fine mesh of $X \otimes X^*$ is nothing but a bunch of small bubbles of X [52, 173]. Mathematically, this is because the algebra object $X \otimes X^* \in \mathcal{C}$ is Morita equivalent to the trivial algebra $1 \in \mathcal{C}$.

The category $\text{Rep}(H_8)$ is equivalent to one of the three non-anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami categories [174],⁹ all of which describe self-dualities under gauging $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The other two non-anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami categories are $\text{Rep}(D_8)$ and $\text{Rep}(Q_8)$, where D_8 and Q_8 are the dihedral group of order eight and the quaternion group respectively. Although these three fusion categories have the same set of simple objects and the same fusion rules, the fermionization of $\text{Rep}(H_8)$ turns out to be qualitatively different from the fermionization of $\text{Rep}(D_8)$ and $\text{Rep}(Q_8)$. In particular, we will see that the fermionization of $\text{Rep}(H_8)$ has q-type objects, whereas the fermionization of $\text{Rep}(D_8)$ and $\text{Rep}(Q_8)$ does not have q-type objects as we saw in Section 5.3.2.

In order to compute the fermionization of $\text{Rep}(H_8)$, let us first recall the definition of the Kac-Paljutkin algebra H_8 . The Kac-Paljutkin algebra H_8 is generated by three elements $x, y, z \in H_8$ that obey the following multiplication law:

$$x^2 = y^2 = 1, \quad xy = yx, \quad xz = zy, \quad zx = yz, \quad z^2 = \frac{1}{2}(1 + x + y - xy). \quad (5.3.16)$$

The other structure maps of H_8 are given by

$$\begin{aligned} \Delta(x) &= x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z) \\ \epsilon(x) &= \epsilon(y) = \epsilon(z) = 1, \quad S(x) = x, \quad S(y) = y, \quad S(z) = z. \end{aligned} \quad (5.3.17)$$

These structure maps are extended to the entire Hopf algebra H_8 by demanding that Δ and ϵ are algebra homomorphisms and S is an algebra anti-homomorphism.

A \mathbb{Z}_2 subgroup of $\text{Rep}(H_8)$ is associated with a \mathbb{Z}_2 group-like element $u \in H_8^*$. Since a \mathbb{Z}_2 group-like element $u \in H_8^*$ is multiplicative, i.e., it satisfies $u(ab) = u(a)u(b)$ for any $a, b \in H_8$, the element u is uniquely determined by the values that it assigns to the generators $x, y, z \in H_8$. A straightforward calculation shows that there are three \mathbb{Z}_2 group-like elements in H_8^* except for the trivial one $u = \epsilon$:

$$(u(x), u(y), u(z)) = (1, 1, -1), (-1, -1, i), (-1, -1, -i). \quad (5.3.18)$$

These correspond to three different \mathbb{Z}_2 subgroups of the group-like symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \text{Rep}(H_8)$. When u is given by $(u(x), u(y), u(z)) = (1, 1, -1)$, the \mathbb{Z}_2 -grading defined by the adjoint action of u becomes purely even, which means that the fermionization of $\text{Rep}(H_8)$ symmetry is the Deligne tensor product $\text{sVec} \boxtimes \text{Rep}(H_8)$. In particular, the fermionized symmetry does not have q-type objects in this case. On the other hand, when u is given by $(u(x), u(y), u(z)) = (-1, -1, \pm i)$, the adjoint action of u endows H_8 with a non-trivial \mathbb{Z}_2 -grading. Consequently, the fermionization of $\text{Rep}(H_8)$ becomes qualitatively different from the original fusion category $\text{Rep}(H_8)$. For this reason, we will focus on the latter case below. The fermionization of $\text{Rep}(H_8)$ symmetry with this choice of a \mathbb{Z}_2 subgroup will be denoted by $\text{sRep}(\mathcal{H}_8^u)$. Examples of physical systems with this symmetry include the stacking of the Ising CFT and a single massless Majorana fermion.

Hopf superalgebra \mathcal{H}_8^u . The \mathbb{Z}_2 -grading automorphism $p_u : \mathcal{H}_8^u \rightarrow \mathcal{H}_8^u$ defined by eq. (5.2.3) is computed as

$$p_u(x) = x, \quad p_u(y) = y, \quad p_u(z) = xyz. \quad (5.3.19)$$

⁹There are four $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami categories [174], one of which is anomalous and the others are non-anomalous [13, 175].

Since the \mathbb{Z}_2 -grading p_u is multiplicative, we also have $p_u(xy) = xy$ and $p_u(xz) = zx$. The above equation implies that the \mathbb{Z}_2 -even part and the \mathbb{Z}_2 -odd part of \mathcal{H}_8^u are given by

$$(\mathcal{H}_8^u)_0 = \text{Span}\{1, x, y, xy, z + xyz, zx + xz\}, \quad (\mathcal{H}_8^u)_1 = \text{Span}\{z - xyz, zx - xz\}. \quad (5.3.20)$$

Based on this \mathbb{Z}_2 -grading, we can compute the Hopf superalgebra structure on \mathcal{H}_8^u following the definitions (5.2.4) and (5.2.5). Specifically, if we define the \mathbb{Z}_2 -even basis $\{e_i \mid 1 \leq i \leq 6\}$ and the \mathbb{Z}_2 -odd basis $\{e_7, e_8\}$ of \mathcal{H}_8^u as

$$\begin{aligned} e_1 &= \frac{1}{8}[1 + x + y + xy + (z + xyz) + (zx + xz)], \\ e_2 &= \frac{1}{8}[1 + x + y + xy - (z + xyz) - (zx + xz)], \\ e_3 &= \frac{1}{8}[1 - x - y + xy + i(z + xyz) - i(zx + xz)], \\ e_4 &= \frac{1}{8}[1 - x - y + xy - i(z + xyz) + i(zx + xz)], \\ e_5 &= \frac{1}{4}(1 + x - y - xy), \quad e_6 = \frac{1}{4}(1 - x + y - xy), \\ e_7 &= \frac{i}{4\sqrt{u(z)}}[(z - xyz) + (zx - xz)], \quad e_8 = \frac{1}{4\sqrt{u(z)}}[(z - xyz) - (zx - xz)], \end{aligned} \quad (5.3.21)$$

the multiplication m_u and the antipode S_u can be computed as follows:

$$m_u(e_i \otimes e_j) = \begin{cases} e_i & \text{for } 1 \leq i = j \leq 6, \\ e_5 & \text{for } i = j = 7, \\ e_6 & \text{for } i = j = 8, \\ e_7 & \text{for } (i, j) = (5, 7), (7, 5), \\ e_8 & \text{for } (i, j) = (6, 8), (8, 6), \\ 0 & \text{otherwise,} \end{cases} \quad S_u(e_i) = \begin{cases} e_i & \text{for } 1 \leq i \leq 6, \\ u(z)e_7 & \text{for } i = 7, \\ -u(z)e_8 & \text{for } i = 8. \end{cases} \quad (5.3.22)$$

This implies that the Hopf superalgebra \mathcal{H}_8^u is decomposed into a direct sum of simple superalgebras as

$$\mathcal{H}_8^u \cong \left(\bigoplus_{1 \leq i \leq 4} \mathbb{C}e_i \right) \oplus \text{Span}\{e_5, e_7\} \oplus \text{Span}\{e_6, e_8\}. \quad (5.3.23)$$

The direct summand $\mathbb{C}e_i$ for $1 \leq i \leq 4$ is isomorphic to the endomorphism superalgebra $\text{End}(\mathbb{C}^{1|0})$, whereas the other two direct summands $\text{Span}\{e_5, e_7\}$ and $\text{Span}\{e_6, e_8\}$ are isomorphic to the Clifford algebra $\text{Cl}(1)$ with one odd generator.

Superfusion category $\text{sRep}(\mathcal{H}_8^u)$. The direct sum decomposition (5.3.23) indicates that the superfusion category $\text{sRep}(\mathcal{H}_8^u)$ consists of the following simple objects:

- a one-dimensional super representation $V_i \cong \mathbb{C}^{1|0}$ for each $i \in \{1, 2, 3, 4\}$, which is a unique (up to isomorphism) irreducible super representation of $\mathbb{C}e_i \cong \text{End}(\mathbb{C}^{1|0})$,
- a two-dimensional super representation $W_1 \cong \mathbb{C}^{1|1}$, which is a unique (up to isomorphism) irreducible super representation of $\text{Span}\{e_5, e_7\} \cong \text{Cl}(1)$,
- a two-dimensional super representation $W_2 \cong \mathbb{C}^{1|1}$, which is a unique (up to isomorphism) irreducible super representation of $\text{Span}\{e_6, e_8\} \cong \text{Cl}(1)$.

We note that the two-dimensional super representations W_1 and W_2 are q-type objects because they are irreducible super representations of $\text{Cl}(1)$ [107]. Therefore, the superfusion category $\text{sRep}(\mathcal{H}_8^u)$ has q-type objects as opposed to the fermionization of the other two non-anomalous $\mathbb{Z}_2 \times \mathbb{Z}_2$ Tambara-Yamagami categories $\text{Rep}(D_8)$ and $\text{Rep}(Q_8)$. This demonstrates that the existence of q-type objects depends not only on the fusion rules but also on the F -symbols of the original fusion category symmetry.

The fusion rules of $\text{sRep}(\mathcal{H}_8^u)$ can be computed explicitly as described in detail in Appendix A. The one-dimensional super representations V_i for $1 \leq i \leq 4$ obey the group-like fusion rules of $\mathbb{Z}_2 \times \mathbb{Z}_2^F$, whose unit element is given by V_1 :

$$V_1 \otimes V_i \cong V_i \otimes V_1 \cong V_i, \quad V_2 \otimes V_2 \cong V_3 \otimes V_3 \cong V_4 \otimes V_4 \cong V_1, \quad V_2 \otimes V_3 \cong V_3 \otimes V_2 \cong V_4. \quad (5.3.24)$$

As we discussed in Section 5.2.1, the generator of the fermion parity symmetry \mathbb{Z}_2^F is a one-dimensional super representation on which $e_i \in \mathcal{H}_8^u$ acts as a scalar multiplication by $u(e_i)$. If we choose a \mathbb{Z}_2 group-like element u as $(u(x), u(y), u(z)) = (-1, -1, i)$, the generator of \mathbb{Z}_2^F is given by V_4 because $u(e_i) = \delta_{i,4}$. On the other hand, if we choose u as $(u(x), u(y), u(z)) = (-1, -1, -i)$, the generator of \mathbb{Z}_2^F is given by V_3 because $u(e_i) = \delta_{i,3}$. The fusion rules of the one-dimensional super representations V_i and the two-dimensional super representations W_j are given by

$$\begin{aligned} V_1 \otimes W_j &\cong V_2 \otimes W_j \cong W_j \cong W_j \otimes V_1 \cong W_j \otimes V_2, \\ V_3 \otimes W_1 &\cong V_4 \otimes W_1 \cong W_2 \cong W_1 \otimes V_3 \cong W_1 \otimes V_4. \end{aligned} \quad (5.3.25)$$

In particular, the second line implies that the generator of the fermion parity symmetry \mathbb{Z}_2^F exchanges two q-type objects W_1 and W_2 by the fusion. Finally, the fusion rules of the two-dimensional super representations are given by

$$W_1 \otimes W_1 \cong V_1 \oplus V_2 \oplus \Pi V_1 \oplus \Pi V_2, \quad (5.3.26)$$

where ΠV_i is a one-dimensional super representation oddly isomorphic to V_i . The other fusion rules are determined by the associativity of the fusion rules.

Self-duality. Equation (5.3.26) implies that W_1 is a duality defect for the condensation of $V_1 \oplus V_2 \oplus \Pi V_1 \oplus \Pi V_2$. In particular, a fermionic system with $\text{sRep}(\mathcal{H}_8^u)$ symmetry should be self-dual under condensing the sum of topological lines $V_1 \oplus V_2 \oplus \Pi V_1 \oplus \Pi V_2$. Since we have an isomorphism $V_1 \oplus V_2 \oplus \Pi V_1 \oplus \Pi V_2 \cong (V_1 \oplus V_2) \otimes (V_1 \oplus \Pi V_1)$, the condensation of $V_1 \oplus V_2 \oplus \Pi V_1 \oplus \Pi V_2$ is implemented by condensing $V_1 \oplus V_2$ and $V_1 \oplus \Pi V_1$ successively. The condensation of $V_1 \oplus V_2$ is equivalent to gauging a \mathbb{Z}_2 symmetry generated by V_2 , while the condensation of $V_1 \oplus \Pi V_1$ is equivalent to stacking the Kitaev chain.¹⁰ Therefore, $\text{sRep}(\mathcal{H}_8^u)$ symmetry can be understood as a self-duality under gauging a \mathbb{Z}_2 subgroup and stacking the Kitaev chain. We note that the relation between the fusion category symmetry $\text{Rep}(H_8)$ and its fermionization $\text{sRep}(\mathcal{H}_8^u)$ is a non-anomalous analogue of the relation between the symmetry of the Ising CFT and that of a single massless Majorana fermion, which will be discussed in Section 5.3.4.

We can verify that the partition function of a fermionic theory is indeed invariant under gauging \mathbb{Z}_2 symmetry and stacking the Kitaev chain if it is obtained by the fermionization of a bosonic theory with $\text{Rep}(H_8)$ symmetry. Let B be a bosonic theory with $\text{Rep}(H_8)$ symmetry and let F be its fermionization. We denote the partition function of B and F on a closed

¹⁰In particular, condensing $V_1 \oplus \Pi V_1$ on a closed spin surface amounts to multiplying the partition function by the Arf invariant, which we can show by a direct computation.

oriented surface Σ by $Z_B(\alpha_1, \alpha_2)$ and $Z_F(\alpha_1, \eta)$ respectively, where α_1 and α_2 are background \mathbb{Z}_2 gauge fields and η is a spin structure on Σ . Since F is the fermionization of B , the partition functions Z_B and Z_F are related by

$$Z_F(\alpha_1, \eta) = \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{\alpha_2 \in H^1(\Sigma, \mathbb{Z}_2)} Z_B(\alpha_1, \alpha_2) (-1)^{q_\eta(\alpha_2)} \text{Arf}(\eta), \quad (5.3.27)$$

cf. equation (5.1.10). When a bosonic theory B has $\text{Rep}(H_8)$ symmetry, the partition function Z_B is invariant under gauging $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the diagonal pairing of background gauge fields [13]:

$$Z_B(\alpha_1, \alpha_2) = \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|} \sum_{\beta_1, \beta_2} Z_B(\beta_1, \beta_2) (-1)^{\int \alpha_1 \cup \beta_1 + \alpha_2 \cup \beta_2}. \quad (5.3.28)$$

This equation implies that the fermionic partition function (5.3.27) satisfies

$$Z_F(\alpha_1, \eta) = \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{\beta_1} Z_F(\beta_1, \eta) (-1)^{\int \alpha_1 \cup \beta_1} \text{Arf}(\eta), \quad (5.3.29)$$

which shows that the fermionized theory F is self-dual under gauging \mathbb{Z}_2 symmetry and stacking the Kitaev chain.

Let us compare this self-duality with the fermionization of $\text{Rep}(D_8)$ and $\text{Rep}(Q_8)$ symmetries. If the original bosonic theory B has $\text{Rep}(D_8)$ or $\text{Rep}(Q_8)$ symmetry, the bosonic partition function Z_B is invariant under gauging $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with the off-diagonal pairing of background gauge fields [13].¹¹

$$Z_B(\alpha_1, \alpha_2) = \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|} \sum_{\beta_1, \beta_2} Z_B(\beta_1, \beta_2) (-1)^{\int \alpha_1 \cup \beta_2 + \alpha_2 \cup \beta_1}. \quad (5.3.30)$$

Correspondingly, the fermionic partition function Z_F is invariant under doing the GSO projection and the Jordan-Wigner transformation simultaneously:

$$Z_F(\alpha_1, \eta) = \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|} \sum_{\beta_1} \sum_{\xi} Z_F(\beta_1, \xi) (-1)^{q_\xi(\alpha_1)} (-1)^{q_\eta(\beta_1)}. \quad (5.3.31)$$

The difference between (5.3.29) and (5.3.31) is attributed to the different fusion rules of superfusion categories $\text{sRep}(\mathcal{H}_8^u)$ and $\text{sRep}(\mathcal{H}_{D_8 \text{ or } Q_8}^u)$. More specifically, eq. (5.3.31) is a consequence of the following fusion rule of a duality line $D \in \text{sRep}(\mathcal{H}_{D_8 \text{ or } Q_8}^u)$:

$$D \otimes D \cong 1 \oplus V \oplus (-1)^F \oplus V(-1)^F \cong (1 \oplus V) \otimes (1 \oplus (-1)^F), \quad (5.3.32)$$

where V and $(-1)^F$ are the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry. The above fusion rule implies that a fermionic system with $\text{sRep}(\mathcal{H}_{D_8 \text{ or } Q_8}^u)$ symmetry is invariant under condensing $1 \oplus V$ and $1 \oplus (-1)^F$ successively. Here, the condensation of $1 \oplus V$ should be understood as the gauging of a \mathbb{Z}_2 symmetry with a coupling $(-1)^{q_\eta(\alpha)}$ between a \mathbb{Z}_2 gauge field α and a spin structure η . Therefore, the condensation of $1 \oplus V$ is the Jordan-Wigner transformation. On the other hand, the condensation of $1 \oplus (-1)^F$ is the GSO projection.

¹¹The only difference between $\text{Rep}(D_8)$ and $\text{Rep}(Q_8)$ is the Frobenius-Schur indicator of the duality line, which does not affect eq. (5.3.30).

5.3.4 Self-dualities under gauging \mathbb{Z}_2 symmetry

As an example of an anomalous superfusion category symmetry, we consider the fermionization of self-dualities under gauging \mathbb{Z}_2 symmetry. This example was studied in detail in [70] by using a different method from ours. These self-dualities are described by \mathbb{Z}_2 Tambara-Yamagami categories $\text{TY}(\mathbb{Z}_2, \chi, \pm 1)$, where $\chi : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \text{U}(1)$ is the unique symmetric non-degenerate bicharacter on \mathbb{Z}_2 [174]. Fusion category symmetries $\text{TY}(\mathbb{Z}_2, \chi, +1)$ and $\text{TY}(\mathbb{Z}_2, \chi, -1)$ are realized by, e.g., the Ising CFT and the $\text{SU}(2)_2$ WZW model respectively [13, 70]. The fermionization of the Ising CFT is a single massless Majorana fermion, which has $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry with an 't Hooft anomaly $1 \in \mathbb{Z}_8$ [69, 70, 176]. Similarly, the fermionization of the $\text{SU}(2)_2$ WZW model is three massless Majorana fermions, which have $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry with an 't Hooft anomaly $3 \in \mathbb{Z}_8$ [70]. We note that the generator of the chiral \mathbb{Z}_2 symmetry of a massless Majorana fermion is a q-type object [43]. Therefore, the above CFT examples suggest that the fermionization of $\text{TY}(\mathbb{Z}_2, \chi, \pm 1)$ has two q-type objects, one of which is obtained by fusing the fermion parity defect with the other. Furthermore, since \mathbb{Z}_2 Tambara-Yamagami categories describe self-dualities under gauging \mathbb{Z}_2 symmetry, their fermionization should imply the invariance under stacking the Kitaev chain, cf. Figure 5.1. In the following, we will see that the fermionization of $\text{TY}(\mathbb{Z}_2, \chi, \pm 1)$ indeed has these properties. This suggests that the fermionization formula of fusion category symmetries is also applicable to non-topological systems.

We first recall the definition of the Tambara-Yamagami category $\text{TY}(A, \chi, \epsilon)$, where A is a finite abelian group, $\chi : A \times A \rightarrow \text{U}(1)$ is a symmetric non-degenerate bicharacter on A , and $\epsilon \in \{\pm 1\}$ is a sign. The fusion category $\text{TY}(A, \chi, \epsilon)$ consists of simple objects labeled by group elements $g \in A$ and an additional simple object m called a duality object. The set of simple objects of $\text{TY}(A, \chi, \epsilon)$ will be denoted by $\Omega := A \sqcup \{m\}$. The fusion rules are given by $g \otimes h \cong gh$, $g \otimes m \cong m \otimes g \cong m$, $m \otimes m \cong \bigoplus_{g \in A} g$ for $g, h \in A$. In particular, A is the group-like symmetry of $\text{TY}(A, \chi, \epsilon)$. The group-like symmetry $A \subset \text{TY}(A, \chi, \epsilon)$ is always non-anomalous even if the Tambara-Yamagami category $\text{TY}(A, \chi, \epsilon)$ is anomalous. The complete data of F -symbols can be found in [174].

The Tambara-Yamagami category $\text{TY}(A, \chi, \epsilon)$ is equivalent to the representation category of a weak Hopf algebra $H_{A, \chi, \epsilon}$, whose structure maps are spelled out in [177]. As a semisimple algebra, $H_{A, \chi, \epsilon}$ is isomorphic to the direct sum of endomorphism algebras $\text{End}(H^x)$ for $x \in \Omega$, i.e.,

$$H_{A, \chi, \epsilon} \cong \bigoplus_{x \in \Omega} \text{End}(H^x) = \left(\bigoplus_{g \in A} \text{End}(H^g) \right) \oplus \text{End}(H^m), \quad (5.3.33)$$

where $H^g \cong \mathbb{C}^{|A|+1}$ and $H^m \cong \mathbb{C}^{2|A|}$. The bases of the subalgebras $\text{End}(H^g)$ and $\text{End}(H^m)$ are denoted by $\{e_{\alpha, \beta}^g \mid \alpha, \beta \in \Omega\}$ and $\{e_{\alpha, \beta}^m \mid \alpha, \beta \in A \sqcup \bar{A}\}$ where $\bar{A} := \{\bar{a} \mid a \in A\}$ is a copy of A .¹² The multiplication on $H_{A, \chi, \epsilon}$ is given by the usual multiplication of matrices:

¹²The bar is just a notation, which allows us to distinguish, e.g., $e_{a, b}^m$ and $e_{\bar{a}, \bar{b}}^m$ for $a, b \in A$.

$m(e_{\alpha,\beta}^x \otimes e_{\gamma,\delta}^y) = \delta_{x,y} \delta_{\beta,\gamma} e_{\alpha,\delta}^x$. The comultiplication of each basis element $e_{\alpha,\beta}^x$ is given by

$$\begin{aligned}
\Delta(e_{a,b}^g) &= \sum_{h \in A} e_{h^{-1}a, h^{-1}b}^{gh^{-1}} \otimes e_{a,b}^h + e_{g^{-1}a, g^{-1}b}^m \otimes e_{\bar{a}, \bar{b}}^m, \\
\Delta(e_{a,m}^g) &= \sum_{h \in A} e_{h^{-1}a, m}^{gh^{-1}} \otimes e_{a,m}^h + \frac{\epsilon}{\sqrt{|A|}} \sum_{b \in A} \chi(g, b^{-1}) e_{g^{-1}a, \bar{b}}^m \otimes e_{\bar{a}, b}^m, \\
\Delta(e_{m,a}^g) &= \sum_{h \in A} e_{m, h^{-1}a}^{gh^{-1}} \otimes e_{m,a}^h + \frac{\epsilon}{\sqrt{|A|}} \sum_{b \in A} \chi(g, b) e_{\bar{b}, g^{-1}a}^m \otimes e_{b, \bar{a}}^m, \\
\Delta(e_{m,m}^g) &= \sum_{h \in A} e_{m,m}^{gh^{-1}} \otimes e_{m,m}^h + \frac{1}{|A|} \sum_{a,b \in A} \chi(g, ab^{-1}) e_{\bar{a}, \bar{b}}^m \otimes e_{a,b}^m, \\
\Delta(e_{a,b}^m) &= \sum_{g \in A} \chi(g, a^{-1}b) e_{a,b}^m \otimes e_{m,m}^g + \sum_{g \in A} e_{ag, bg}^g \otimes e_{ag, bg}^m, \\
\Delta(e_{a, \bar{b}}^m) &= \sum_{g \in A} \chi(g, a^{-1}) e_{a, g^{-1}\bar{b}}^m \otimes e_{m,b}^g + \sum_{g \in A} \chi(g, b) e_{ag, m}^g \otimes e_{ag, \bar{b}}^m, \\
\Delta(e_{\bar{a}, b}^m) &= \sum_{g \in A} \chi(g, b) e_{g^{-1}\bar{a}, b}^m \otimes e_{a,m}^g + \sum_{g \in A} \chi(g, a^{-1}) e_{m, bg}^g \otimes e_{\bar{a}, bg}^m, \\
\Delta(e_{\bar{a}, \bar{b}}^m) &= \sum_{g \in A} e_{g^{-1}\bar{a}, g^{-1}\bar{b}}^m \otimes e_{a,b}^g + \sum_{g \in A} \chi(g, a^{-1}b) e_{m,m}^g \otimes e_{\bar{a}, \bar{b}}^m.
\end{aligned}$$

The unit and the counit of $H_{A,\chi,\epsilon}$ are defined by $\eta(1) = \sum_{x \in \Omega} \sum_{\alpha} e_{\alpha,\alpha}^x$ and $\epsilon(e_{\alpha,\beta}^x) = \delta_{x,1}$ respectively. The antipode is also given in [177], although we will not use it in the following discussion.

A non-anomalous \mathbb{Z}_2 subgroup symmetry of $\text{TY}(A, \chi, \epsilon)$ generated by $u \in A$ is associated with a \mathbb{Z}_2 group-like element $\eta_u \in H_{A,\chi,\epsilon}^*$ defined by $\eta_u(e_{\alpha,\beta}^x) = \delta_{x,u}$. The adjoint action of η_u defines a \mathbb{Z}_2 -grading p_u on $H_{A,\chi,\epsilon}$ as follows:

$$\begin{aligned}
p_u(e_{a,b}^g) &= e_{ua,ub}^g, & p_u(e_{a,m}^g) &= e_{ua,m}^g, & p_u(e_{m,a}^g) &= e_{m,ua}^g, & p_u(e_{m,m}^g) &= e_{m,m}^g, \\
p_u(e_{a,b}^m) &= \chi(u, a^{-1}b) e_{ua,ub}^m, & p_u(e_{a,\bar{b}}^m) &= \chi(u, u) \chi(u, a^{-1}b) e_{ua,\bar{u}\bar{b}}^m, \\
p_u(e_{\bar{a},b}^m) &= \chi(u, u) \chi(u, a^{-1}b) e_{\bar{u}\bar{a},ub}^m, & p_u(e_{\bar{a},\bar{b}}^m) &= \chi(u, a^{-1}b) e_{\bar{u}\bar{a},\bar{u}\bar{b}}^m.
\end{aligned} \tag{5.3.34}$$

A weak Hopf algebra $H_{A,\chi,\epsilon}$ equipped with the above \mathbb{Z}_2 -grading becomes a weak Hopf superalgebra $\mathcal{H}_{A,\chi,\epsilon}^u$ if we modify the multiplication and the antipode as discussed in Section 5.2.2.

Weak Hopf superalgebra $\mathcal{H}_{\mathbb{Z}_2,\chi,\epsilon}^u$. We now restrict our attention to the case $A = \mathbb{Z}_2$. In this case, the symmetric non-degenerate bicharacter χ is uniquely given by

$$\chi(1,1) = \chi(1,u) = \chi(u,1) = 1, \quad \chi(u,u) = -1. \tag{5.3.35}$$

Thus, eq. (5.3.34) implies that the \mathbb{Z}_2 -even part and the \mathbb{Z}_2 -odd part of $\mathcal{H}_{\mathbb{Z}_2,\chi,\epsilon}^u$ are given by

$$\begin{aligned}
(\mathcal{H}_{\mathbb{Z}_2,\chi,\epsilon}^u)_0 &= \left(\bigoplus_{g \in \mathbb{Z}_2} \text{Span}\{(e_{11}^g)_+, (e_{1u}^g)_+, (e_{1m}^g)_+, (e_{m1}^g)_+, e_{mm}^g\} \right) \\
&\quad \oplus \left(\bigoplus_{a \in \mathbb{Z}_2} \text{Span}\{(e_{1a}^m)_+, (e_{1\bar{a}}^m)_+, (e_{\bar{1}a}^m)_+, (e_{\bar{1}\bar{a}}^m)_+\} \right), \\
(\mathcal{H}_{\mathbb{Z}_2,\chi,\epsilon}^u)_1 &= \left(\bigoplus_{g \in \mathbb{Z}_2} \text{Span}\{(e_{11}^g)_-, (e_{1u}^g)_-, (e_{1m}^g)_-, (e_{m1}^g)_-\} \right) \\
&\quad \oplus \left(\bigoplus_{a \in \mathbb{Z}_2} \text{Span}\{(e_{1a}^m)_-, (e_{1\bar{a}}^m)_-, (e_{\bar{1}a}^m)_-, (e_{\bar{1}\bar{a}}^m)_-\} \right),
\end{aligned} \tag{5.3.36}$$

where the homogeneous elements $(e_{\alpha,\beta}^x)_\pm$ are defined as follows:

$$\begin{aligned} (e_{11}^g)_\pm &= e_{11}^g \pm e_{uu}^g, & (e_{1u}^g)_\pm &= e_{1u}^g \pm e_{u1}^g, & (e_{1m}^g)_\pm &= e_{1m}^g \pm e_{um}^g, & (e_{m1}^g)_\pm &= e_{m1}^g \pm e_{mu}^g, \\ (e_{11}^m)_\pm &= e_{11}^m \pm e_{uu}^m, & (e_{1u}^m)_\pm &= e_{1u}^m \mp e_{u1}^m, & (e_{1\bar{1}}^m)_\pm &= e_{1\bar{1}}^m \mp e_{u\bar{u}}^m, & (e_{1\bar{u}}^m)_\pm &= e_{1\bar{u}}^m \pm e_{u\bar{1}}^m, \\ (e_{\bar{1}\bar{1}}^m)_\pm &= e_{\bar{1}\bar{1}}^m \mp e_{\bar{u}\bar{u}}^m, & (e_{\bar{1}u}^m)_\pm &= e_{\bar{1}u}^m \pm e_{u\bar{1}}^m, & (e_{\bar{1}\bar{1}}^m)_\pm &= e_{\bar{1}\bar{1}}^m \pm e_{\bar{u}\bar{u}}^m, & (e_{\bar{1}\bar{u}}^m)_\pm &= e_{\bar{1}\bar{u}}^m \mp e_{\bar{u}\bar{1}}^m. \end{aligned} \quad (5.3.37)$$

Let us compute the multiplication on $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ based on the above \mathbb{Z}_2 -grading and determine the direct sum decomposition of $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$. To this end, we first notice that $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ is decomposed into the direct sum of a subalgebra spanned by $\{e_{\alpha,\beta}^g \mid g \in \mathbb{Z}_2, \alpha, \beta \in \Omega\}$ and its complement spanned by $\{e_{\gamma,\delta}^m \mid \gamma, \delta \in \mathbb{Z}_2 \sqcup \overline{\mathbb{Z}_2}\}$. This is because the multiplication of $e_{\alpha,\beta}^g$ and $e_{\gamma,\delta}^m$ vanishes for any choice of α, β, γ , and δ . As we will see below, each of these subalgebras is further decomposed into the direct sum of two simple superalgebras. Specifically, the former subalgebra is isomorphic to the direct sum of two copies of the endomorphism superalgebra $\text{End}(\mathbb{C}^{2|1})$, whereas the latter subalgebra is isomorphic to the direct sum of two copies of $\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)$. In order to see an isomorphism $\text{Span}\{e_{\alpha,\beta}^g \mid g \in \mathbb{Z}_2, \alpha, \beta \in \Omega\} \cong \text{End}(\mathbb{C}^{2|1}) \oplus \text{End}(\mathbb{C}^{2|1})$, we define a new basis of the algebra spanned by $\{e_{\alpha,\beta}^g \mid g \in \mathbb{Z}_2, \alpha, \beta \in \Omega\}$ as follows:

$$\begin{aligned} x_{11}^g &= \frac{1}{2}[(e_{11}^g)_+ + (e_{1u}^g)_+], & x_{1u}^g &= -\frac{1}{2}[(e_{11}^{ug})_- - (e_{1u}^{ug})_-], & x_{1m}^g &= \frac{1}{2}(e_{1m}^g)_+, \\ x_{u1}^g &= \frac{1}{2}[(e_{11}^g)_- + (e_{1u}^g)_-], & x_{uu}^g &= \frac{1}{2}[(e_{11}^{ug})_+ - (e_{1u}^{ug})_+], & x_{um}^g &= \frac{1}{2}(e_{1m}^g)_-, \\ x_{m1}^g &= (e_{m1}^g)_+, & x_{mu}^g &= -(e_{m1}^{ug})_-, & x_{mm}^g &= e_{mm}^g. \end{aligned} \quad (5.3.38)$$

For this basis, the multiplication on $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ can be expressed as

$$m_u(x_{\alpha,\beta}^g \otimes x_{\gamma,\delta}^h) = \delta_{g,h} \delta_{\beta,\gamma} x_{\alpha,\delta}^g, \quad (5.3.39)$$

which shows that the subalgebra spanned by $\{x_{\alpha,\beta}^g \mid \alpha, \beta \in \Omega\}$ is a full matrix algebra for each $g \in \mathbb{Z}_2$. Since this subalgebra has superdimension $(5, 4)$ as can be seen from eq. (5.3.38), there is an even isomorphism between $\text{Span}\{x_{\alpha,\beta}^g \mid \alpha, \beta \in \Omega\}$ and $\text{End}(\mathbb{C}^{2|1})$. Therefore, we have the following direct sum decomposition of a superalgebra:

$$\text{Span}\{e_{\alpha,\beta}^g \mid g \in \mathbb{Z}_2, \alpha, \beta \in \Omega\} \cong \text{End}(\mathbb{C}^{2|1}) \oplus \text{End}(\mathbb{C}^{2|1}). \quad (5.3.40)$$

Similarly, an isomorphism $\text{Span}\{e_{\alpha,\beta}^m \mid \alpha, \beta \in \mathbb{Z}_2 \sqcup \overline{\mathbb{Z}_2}\} \cong (\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)) \oplus (\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1))$ also becomes clear if we define a new basis of the algebra spanned by $\{e_{\alpha,\beta}^m \mid \alpha, \beta \in \mathbb{Z}_2 \sqcup \overline{\mathbb{Z}_2}\}$ as

$$\begin{aligned} (x_{11}^{m,s})_+ &= \frac{1}{2}[(e_{11}^m)_+ + is(e_{1u}^m)_+], & (x_{11}^{m,s})_- &= \frac{1}{2}[(e_{11}^m)_- + is(e_{1u}^m)_-], \\ (x_{1\bar{1}}^{m,s})_+ &= \frac{1}{2}[(e_{1\bar{1}}^m)_+ - is(e_{1\bar{u}}^m)_+], & (x_{1\bar{1}}^{m,s})_- &= \frac{1}{2}[(e_{1\bar{1}}^m)_- - is(e_{1\bar{u}}^m)_-], \\ (x_{\bar{1}\bar{1}}^{m,s})_+ &= \frac{1}{2}[(e_{\bar{1}\bar{1}}^m)_+ + is(e_{\bar{1}u}^m)_+], & (x_{\bar{1}\bar{1}}^{m,s})_- &= \frac{is}{2}[(e_{\bar{1}\bar{1}}^m)_- + is(e_{\bar{1}u}^m)_-], \\ (x_{\bar{1}\bar{1}}^{m,s})_+ &= \frac{1}{2}[(e_{\bar{1}\bar{1}}^m)_+ - is(e_{\bar{1}\bar{u}}^m)_+], & (x_{\bar{1}\bar{1}}^{m,s})_- &= \frac{is}{2}[(e_{\bar{1}\bar{1}}^m)_- - is(e_{\bar{1}\bar{u}}^m)_-], \end{aligned} \quad (5.3.41)$$

where the superscript s takes values in $\{\pm 1\}$. The multiplication on $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ for this basis can be written as

$$m_u((x_{\alpha,\beta}^{m,s})_p \otimes (x_{\gamma,\delta}^{m,t})_q) = \delta_{s,t} \delta_{\beta,\gamma} (x_{\alpha,\delta}^{m,s})_{pq}. \quad (5.3.42)$$

This implies that the subalgebra spanned by $\{(x_{\alpha,\beta}^{m,s})_{\pm} \mid \alpha, \beta = 1, \bar{1}\}$ for each $s \in \{\pm 1\}$ is isomorphic to a simple superalgebra $\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)$, where the odd generator of the Clifford algebra $\text{Cl}(1)$ corresponds to $(x_{11}^{m,s})_- + (x_{\bar{1}\bar{1}}^{m,s})_-$. Thus, we find the following direct sum decomposition:

$$\text{Span}\{e_{\alpha,\beta}^m \mid \alpha, \beta \in \mathbb{Z}_2 \sqcup \overline{\mathbb{Z}_2}\} \cong (\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)) \oplus (\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)). \quad (5.3.43)$$

Equations (5.3.40) and (5.3.43) show that the weak Hopf superalgebra $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ is decomposed into the direct sum of two copies of $\text{End}(\mathbb{C}^{2|1})$ and two copies of $\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)$.

Superfusion category $\text{sRep}(\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u)$. The direct sum decomposition of $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ indicates that $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ has three-dimensional irreducible super representations $V_g \cong \mathbb{C}^{2|1}$ for $g \in \mathbb{Z}_2$ and four-dimensional irreducible super representations $W_s \cong \mathbb{C}^{2|2}$ for $s = \pm 1$. The actions of $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ on V_g and W_s are given by the standard actions of the direct summands $\text{End}(\mathbb{C}^{2|1})$ and $\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)$. More specifically, if we write the bases of V_g and W_s as $\{v_{\alpha}^g \mid \alpha \in \Omega\}$ and $\{(w_{\alpha}^s)_{\pm} \mid \alpha = 1, \bar{1}\}$ respectively, the actions of $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ on V_g and W_s are given by

$$x_{\alpha,\beta}^h \cdot v_{\gamma}^g = \delta_{g,h} \delta_{\beta,\gamma} v_{\alpha}^g, \quad (x_{\alpha,\beta}^{m,t})_q \cdot v_{\gamma}^g = 0, \quad (5.3.44)$$

$$x_{\alpha,\beta}^h \cdot (w_{\gamma}^s)_p = 0, \quad (x_{\alpha,\beta}^{m,t})_q \cdot (w_{\gamma}^s)_p = \delta_{s,t} \delta_{\beta,\gamma} (w_{\alpha}^s)_{pq}, \quad (5.3.45)$$

where $v_1^g, v_{\bar{1}}^g \in V_g$ and $(w_1^s)_+, (w_{\bar{1}}^s)_+ \in W_s$ are \mathbb{Z}_2 -even elements and the others are \mathbb{Z}_2 -odd elements. We note that W_+ and W_- are q-type objects because they are irreducible super representations of $\text{End}(\mathbb{C}^{2|0}) \otimes \text{Cl}(1)$ [107].

As we will see in Appendix A, simple objects $V_1, V_u, W_+, W_- \in \text{sRep}(\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u)$ obey the following fusion rules:

$$V_u \boxtimes V_u \cong V_1, \quad V_u \boxtimes W_+ \cong W_- \cong W_+ \boxtimes V_u, \quad W_+ \boxtimes W_+ \cong V_1 \oplus \Pi V_1, \quad (5.3.46)$$

where ΠV_1 is oddly isomorphic to the trivial defect V_1 . The associativity of the fusion rules uniquely determines the other fusion rules. Equation (5.3.46) implies that a q-type object W_+ and an m-type object V_u generate $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry with an odd 't Hooft anomaly.¹³ In particular, a fermionic system with $\text{sRep}(\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u)$ symmetry is invariant under stacking the Kitaev chain [70], which is a consequence of the fusion rule $W_+ \boxtimes W_+ \cong V_1 \oplus \Pi V_1$. This is consistent with the fact that the fermionization of the Ising CFT and the $\text{SU}(2)_2$ WZW model has $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetries with 't Hooft anomalies 1 and 3 modulo 8 respectively. The determination of the anomaly would require further analysis on the F -symbols of $\text{sRep}(\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u)$.

¹³In general, when $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry has an odd 't Hooft anomaly, the generator η of the \mathbb{Z}_2 subgroup does not satisfy the ordinary \mathbb{Z}_2 group-like fusion rule because η is a q-type object and hence $\eta \otimes \eta$ has an odd automorphism. More specifically, $\eta \otimes \eta$ is not a trivial defect 1 but rather the direct sum of a trivial defect 1 and another defect Π that is oddly isomorphic to 1. If normalized appropriately, the action of η on the NS sector satisfies the ordinary \mathbb{Z}_2 fusion rule because both 1 and Π act as the identity operator on the NS sector, cf. eq. (6.3.5).

Chapter 6

Fermionic topological phases with superfusion category symmetries

In this chapter, we explicitly construct fermionic state sum TFTs with superfusion category symmetries. These TFTs can be thought of as the fermionization of bosonic state sum TFTs with fusion category symmetries. Based on the construction, we derive the fermionization formula of fusion category symmetries, which we proposed in the previous chapter, by comparing the symmetries of the fermionic TFTs with those of the bosonic TFTs. We will also write down gapped Hamiltonians with non-anomalous superfusion category symmetries on the lattice. The content of this chapter is a generalization of the bosonic case discussed in Chapter 4. The string diagrams appearing in this chapter are supposed to be those in sVec unless otherwise stated. In particular, the braiding in string diagrams always represents the symmetric braiding c_{super} . This chapter is based on the author's original paper [45].

6.1 Review: combinatorial description of spin structures

To set the stage for the state sum construction of fermionic TFTs, we first recall the combinatorial description of spin structures on triangulated surfaces following [39]. Let Σ be an oriented surface and $T(\Sigma)$ be its triangulation. The triangulated surface $T(\Sigma)$ is also denoted by Σ when no confusion can arise. A spin structure on a triangulated surface $T(\Sigma)$ is specified by the following set of data, which is called a marking:

- an orientation of each edge of Σ
- an edge index $s(e) \in \{0, 1\}$ of each edge e
- a choice of a marked edge on the boundary of each triangle
- a choice of a base point on each connected component of the boundary $\partial\Sigma$

The base points on the boundary are supposed to be vertices of the triangulated surface $T(\Sigma)$. A connected component of the boundary $\partial\Sigma$ belongs to either the in-boundary $\partial_{\text{in}}\Sigma$ or the out-boundary $\partial_{\text{out}}\Sigma$. The orientations of edges on $\partial_{\text{in}}\Sigma$ are induced by the orientation of Σ . On the other hand, the orientations of edges on $\partial_{\text{out}}\Sigma$ are opposite to the induced orientation.

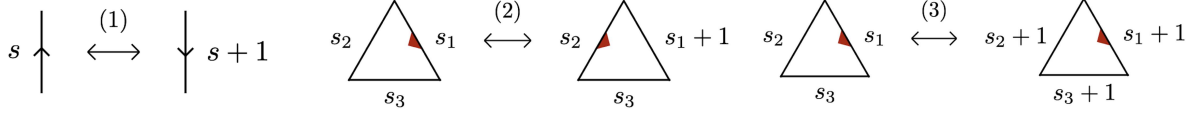


Figure 6.1: Markings related by the above local moves give the same spin structure. (1) Change of an edge orientation: we reverse the orientation of an edge and shift the edge index by 1. (2) Change of a marked edge: we choose the edge next to the original marked edge as the new marked edge and shift the edge index accordingly. The marked edge is represented by an edge with a small red triangle attached. (3) Leaf exchange on a triangle: we shift the edge indices of all edges on the boundary of a triangle simultaneously. We note that the leaf exchange is equivalent to changing the marked edge three times.

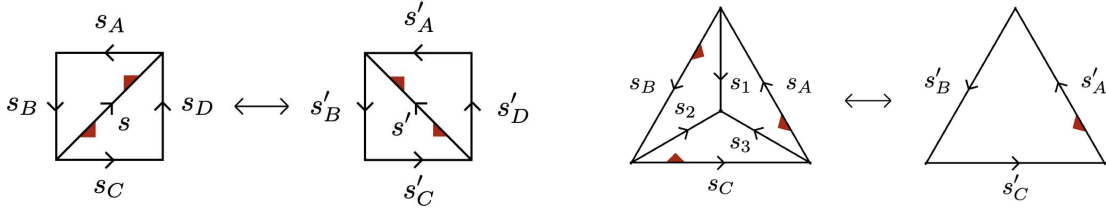


Figure 6.2: The Pachner 2-2 move (left) and the Pachner 3-1 move (right). The edge indices are determined uniquely up to local moves. For the Pachner 2-2 move, we have $s' = s$, $s'_A = s_A$, $s'_B = s_B + s + 1$, $s'_C = s_C + 1$, $s'_D = s_D + s + 1$. For the Pachner 3-1 move, we have $s'_A = s_A$, $s'_B = s_B + s_1$, $s'_C = s_C + s_1 + s_2$, $s_3 = s_1 + s_2 + 1$.

A marking on a triangulated surface $T(\Sigma)$ encodes a spin structure on Σ if it satisfies the following admissibility condition for each vertex v :

$$\sum_{\substack{e: \text{edges} \\ \text{s.t. } v \in \partial e}} s(e) = \begin{cases} D_v + E_v + \lambda \bmod 2 & \text{when } v \text{ is a base point on } \partial \Sigma, \\ D_v + E_v + 1 \bmod 2 & \text{otherwise.} \end{cases} \quad (6.1.1)$$

Here, D_v is the number of triangles t such that a small counterclockwise loop around v enters t through the marked edge of t , E_v is the number of edges whose initial vertex is v , and λ is an index defined by

$$\lambda = \begin{cases} 0 & \text{if } v \text{ is on an NS boundary,} \\ 1 & \text{if } v \text{ is on an R boundary.} \end{cases} \quad (6.1.2)$$

Here, NS and R denote the Neveu-Schwarz (i.e., bounding) and Ramond (i.e., non-bounding) spin structures on a circle, which correspond to the anti-periodic and periodic boundary conditions on fermions. Two markings on Σ give rise to the same spin structure if they are related by a sequence of the following local moves: (1) change of an edge orientation, (2) change of a marked edge, (3) leaf exchange on a triangle, see Figure 6.1. These local moves define an equivalence relation between markings. The quotient of the set of admissible markings on Σ by this equivalence relation is in bijective correspondence with the set of spin structures on Σ [39].

Any two triangulations of an oriented surface are related by a finite sequence of Pachner moves. For a triangulated spin surface Σ equipped with a marking, the Pachner moves also change the marking on affected triangles, see Figure 6.2. The change of a marking is determined by the admissibility condition (6.1.1) uniquely up to local moves shown in Figure 6.1.

Based on the above combinatorial description of spin structures, we can construct fermionic TFTs from semisimple superalgebras by the state sum construction [39, 159]. The fermionic TFT obtained from a semisimple superalgebra K will be denoted by \mathfrak{F}_K , which turns out to be the fermionization of bosonic TFT \mathfrak{B}_K [159, 167].¹ We will describe the state sum construction of fermionic TFTs in detail in the subsequent section.

6.2 Fermionic TFTs with ${}_K\mathcal{SM}_K$ symmetry

In this section, we define the transition amplitudes of a fermionic TFT \mathfrak{F}_K on spin surfaces with defects labeled by (K, K) -superbimodules. The construction of fermionic TFTs with defects that we will describe below is parallel to the construction of bosonic TFTs discussed in Section 4.3.1. The content of this section generalizes the fermionic state sum construction in the absence of defects [39, 159].

We begin with the triangulation of spin surfaces with defects. We assume that every junction of topological lines is trivalent. A triangulation of a spin surface with defects is a triangulation of the underlying spin surface such that each edge intersects a topological defect at most once and each triangle contains at most one trivalent junction. Specifically, possible configurations of topological defects on a single triangle up to local moves are listed as follows:

$$(1) \quad (2) \quad (3) \quad (4) \quad (5) \quad (6.2.1)$$

Here, topological lines (i.e., blue lines) are labeled by (K, K) -superbimodules $X, X_1, X_2, X_3 \in {}_K\mathcal{SM}_K$, and topological junctions (i.e., black dots) are labeled by (K, K) -superbimodule morphisms $f : X_3 \rightarrow X_1 \otimes_K X_2$ and $g : X_1 \otimes_K X_2 \rightarrow X_3$. As in Section 4.3.1, we assume that the above triangles are oriented so that the orientations induced on their boundaries are counterclockwise.

In order to define the transition amplitude on Σ , we first define the vector spaces $Z_T(\partial_{\text{in}}\Sigma)$ and $Z_T(\partial_{\text{out}}\Sigma)$ on the in-boundary and the out-boundary. To this end, we fix the order of the connected components of $\partial_a\Sigma$ for $a = \text{in}, \text{out}$, and denote the i th component of $\partial_a\Sigma$ by $(\partial_a\Sigma)_i$. The vector space $Z_T(\partial_a\Sigma)$ is given by the tensor product of the vector spaces $Z_T((\partial_a\Sigma)_i)$ on the connected components $(\partial_a\Sigma)_i$:

$$Z_T(\partial_a\Sigma) = Z_T((\partial_a\Sigma)_1) \otimes Z_T((\partial_a\Sigma)_2) \otimes \cdots \otimes Z_T((\partial_a\Sigma)_{n_a}), \quad (6.2.2)$$

where n_a denotes the number of connected components of $\partial_a\Sigma$. The vector space $Z_T((\partial_a\Sigma)_i)$ on each connected component is defined by the tensor product of vector spaces R_e assigned to edges $e \in (\partial_a\Sigma)_i$:

$$Z_T((\partial_a\Sigma)_i) = \bigotimes_{e \in (\partial_a\Sigma)_i} R_e. \quad (6.2.3)$$

The order of the tensor product on the right-hand side is determined as in the bosonic case, cf. Figure 4.7. The vector space R_e assigned to an edge e is a (K, K) -superbimodule that labels the topological line intersecting the edge e . More specifically, when a topological line

¹We recall that the bosonic TFT \mathfrak{B}_K has a non-anomalous \mathbb{Z}_2 symmetry and is therefore fermionizable when K is a superalgebra.

X_e intersecting the edge e is oriented from the right of e to the left of e , we assign the vector space X_e to the edge e , i.e. $R_e = X_e$. On the other hand, when a topological line X_e is oriented in the opposite direction, we assign the dual vector space X_e^* to the edge e , i.e. $R_e = X_e^*$. When a boundary edge e does not intersect a topological line, we assign a regular (K, K) -superbimodule K to e , i.e. $R_e = K$.

Let us now define the transition amplitude $Z_T(\Sigma) : Z_T(\partial_{\text{in}}\Sigma) \rightarrow Z_T(\partial_{\text{out}}\Sigma)$ on a triangulated spin surface Σ with defects. We first consider the case where all trivalent junctions on Σ are bosonic. In this case, the transition amplitude is defined by

$$Z_T(\Sigma) = E(\Sigma) \circ c(\Sigma) \circ P(\Sigma) : Z_T(\partial_{\text{in}}\Sigma) \rightarrow Z_T(\partial_{\text{out}}\Sigma), \quad (6.2.4)$$

where the linear maps $P(\Sigma)$, $c(\Sigma)$, and $E(\Sigma)$ are defined below. Although each of the linear maps $P(\Sigma)$, $c(\Sigma)$, and $E(\Sigma)$ depends on the orders of internal edges and triangles, their composition $Z_T(\Sigma)$ does not depend on these data. Thus, we can fix the orders of these edges and triangles arbitrarily in what follows. As in the bosonic case, the transition amplitude $Z(\Sigma)$ of the fermionic TFT on Σ is obtained by restricting the domain and codomain of eq. (6.2.4) to the images of the cylinder amplitudes.

The linear map $P(\Sigma) : Z_T(\partial_{\text{in}}\Sigma) \rightarrow Z_T(\partial_{\text{in}}\Sigma) \otimes (\bigotimes_{e \in \Sigma \setminus \partial_{\text{in}}\Sigma} X_e^L \otimes X_e^R)$. The vector spaces X_e^L and X_e^R are (K, K) -superbimodules assigned to the left and the right of an edge $e \in \Sigma \setminus \partial_{\text{in}}\Sigma$. Specifically, when a topological line labeled by X_e goes across e from the right, we set $X_e^L = X_e$ and $X_e^R = X_e^*$. On the other hand, when a topological line labeled by X_e goes across e from the left, we set $X_e^L = X_e^*$ and $X_e^R = X_e$. When e does not intersect a topological line, we assign a regular (K, K) -superbimodule K to both the left and right of e . The linear map $P(\Sigma)$ is given by the tensor product

$$P(\Sigma) = \left(\bigotimes_{e \in \partial_{\text{in}}\Sigma} p_{R_e}^{s(e)+1} \right) \otimes \left(\bigotimes_{e \in \Sigma \setminus \partial_{\text{in}}\Sigma} (\text{id}_{X_e^L} \otimes p_{X_e^R}^{s(e)+1}) \circ P_e \right), \quad (6.2.5)$$

where $p_X : X \rightarrow X$ is the \mathbb{Z}_2 -grading automorphism of a (K, K) -superbimodule X and $P_e : \mathbb{C} \rightarrow X_e^L \otimes X_e^R$ is either the left coevaluation morphism $\text{coev}_{X_e^L}^L$ or the right coevaluation morphism $\text{coev}_{X_e^R}^R$ depending on the orientation of the topological line X_e . Specifically, we define $P_e = \text{coev}_{X_e^L}^L$ when $X_e^L = X_e$ and $X_e^R = X_e^*$, whereas we define $P_e = \text{coev}_{X_e^R}^R$ when $X_e^L = X_e^*$ and $X_e^R = X_e$. When e does not intersect a topological line, the linear map P_e is given by $\Delta_K \circ \eta_K : \mathbb{C} \rightarrow K \otimes K$, where $\Delta_K : K \rightarrow K \otimes K$ and $\eta_K : \mathbb{C} \rightarrow K$ are the comultiplication (2.2.7) and the unit of K . In the following, we will omit the subscript X of the \mathbb{Z}_2 -grading automorphism p_X when it is clear from the context.

The linear map $c(\Sigma) : Z_T(\partial_{\text{in}}\Sigma) \otimes (\bigotimes_{e \in \Sigma \setminus \partial_{\text{in}}\Sigma} X_e^L \otimes X_e^R) \rightarrow (\bigotimes_{t \in \Sigma} X_t^1 \otimes X_t^2 \otimes X_t^3) \otimes Z_T(\partial_{\text{out}}\Sigma)$. The linear map $c(\Sigma)$, which only changes the order of the tensor product, is given by the composition of the symmetric braiding c_{super} of super vector spaces. Here, the vector space X_t^i that appears in the domain of $c(\Sigma)$ is a (K, K) -superbimodule assigned to the i th edge on ∂t .² Specifically, we have $X_t^i = X_e^L$ when a triangle t is on the left side of the edge $e \in \partial t$, and we have $X_t^i = X_e^R$ otherwise.

²As in the bosonic case, the order of edges on ∂t is determined by the orientation of ∂t induced by that of t . The first edge on ∂t is the marked edge.

The linear map $E(\Sigma) : (\bigotimes_{t \in \Sigma} X_t^1 \otimes X_t^2 \otimes X_t^3) \otimes Z_T(\partial_{\text{out}}\Sigma) \rightarrow Z_T(\partial_{\text{out}}\Sigma)$. The linear map $E(\Sigma)$ is given by the tensor product of linear maps $E_t : X_t^1 \otimes X_t^2 \otimes X_t^3 \rightarrow \mathbb{C}$ for all triangles $t \in \Sigma$:

$$E(\Sigma) = \left(\bigotimes_{t \in \Sigma} E_t \right) \otimes \text{id}_{Z_T(\partial_{\text{out}}\Sigma)}. \quad (6.2.6)$$

The explicit form of E_t depends on the configuration of topological defects on t . Specifically, the linear map E_t for each configuration in eq. (6.2.1) is given by

$$E_t = \begin{cases} (1) & \epsilon_K \circ m_K \circ (m_K \otimes \text{id}_K) : K \otimes K \otimes K \rightarrow \mathbb{C}, \\ (2) & \text{ev}_X^R \circ (\text{id}_X \otimes \rho_{X^*}^R) : X \otimes X^* \otimes K \rightarrow \mathbb{C}, \\ (3) & \text{ev}_X^R \circ (\rho_X^R \otimes \text{id}_{X^*}) : X \otimes K \otimes X^* \rightarrow \mathbb{C}, \\ (4) & \text{ev}_{X_1 \otimes X_2}^R \circ ((\iota_{X_1, X_2} \circ f) \otimes \text{id}_{X_2^* \otimes X_1^*}) : X_3 \otimes X_2^* \otimes X_1^* \rightarrow \mathbb{C}, \\ (5) & \text{ev}_{X_3}^R \circ ((g \circ \pi_{X_1, X_2}) \otimes \text{id}_{X_3^*}) : X_1 \otimes X_2 \otimes X_3^* \rightarrow \mathbb{C}, \end{cases} \quad (6.2.7)$$

where ϵ_K and m_K are the counit (2.2.8) and the multiplication on K , $\rho_X^R : X \otimes K \rightarrow X$ is the right K -supermodule action on X , $\iota_{X_1, X_2} : X_1 \otimes_K X_2 \rightarrow X_1 \otimes X_2$ and $\pi_{X_1, X_2} : X_1 \otimes X_2 \rightarrow X_1 \otimes_K X_2$ are the splitting maps, and $f : X_3 \rightarrow X_1 \otimes_K X_2$ and $g : X_1 \otimes_K X_2 \rightarrow X_3$ are \mathbb{Z}_2 -even (K, K) -superbimodule morphisms that represent bosonic trivalent junctions. When the marked edge on a triangle t is not the bottom edge in eq. (6.2.1), we first move the marked edge to the bottom by the local move depicted in Figure 6.1 and define the transition amplitude as in eq. (6.2.7). Since applying the change of a marked edge three times gives rise to the leaf exchange, the above definition makes sense only when the transition amplitude is invariant under the leaf exchange on each triangle. This invariance is guaranteed by the assumption that the trivalent junctions f and g are \mathbb{Z}_2 -even, i.e., bosonic.

A straightforward calculation shows that the transition amplitude $Z_T(\Sigma)$ is invariant under local moves in Figure 6.1 and Pachner moves of a triangulated spin surface $T(\Sigma)$. Thus, the transition amplitude $Z_T(\Sigma)$ is a topological invariant of a spin surface Σ with defects. We note that $Z_T(\Sigma)$ reduces to the transition amplitude given in [39] when there are no topological defects on Σ .

We now incorporate fermionic trivalent junctions. Let us suppose that topological defects on a triangulated spin surface Σ have fermionic junctions, which are labeled by \mathbb{Z}_2 -odd morphisms of (K, K) -superbimodules. We fix the order of the fermionic junctions and denote the i th fermionic junction by f_i . In order to define the transition amplitude $Z_T(\Sigma)$, we first consider the transition amplitude on the punctured spin surface $\Sigma \setminus \bigsqcup_i D_i$, where D_i is a small disk around f_i . We note that the boundary of a disk D_i is equipped with the NS spin structure. This boundary will be regarded as an out-boundary of $\Sigma \setminus \bigsqcup_i D_i$. Since the punctured surface $\Sigma \setminus \bigsqcup_i D_i$ only contains bosonic trivalent junctions, the transition amplitude on $\Sigma \setminus \bigsqcup_i D_i$ is defined by eq. (6.2.4):

$$Z_T(\Sigma \setminus \bigsqcup_i D_i) : Z_T(\partial_{\text{in}}\Sigma) \rightarrow \left(\bigotimes_i Z_T(\partial D_i) \right) \otimes Z_T(\partial_{\text{out}}\Sigma). \quad (6.2.8)$$

The order of the tensor product on the right-hand side is determined by the order of fermionic junctions. If we choose a triangulation of $\Sigma \setminus \bigsqcup_i D_i$ so that the boundary of each disk D_i consists of three edges, the topological defect network on a disk D_i looks like configurations (4) or (5) in eq. (6.2.1), where the edges of a triangle are boundary edges on ∂D_i and the left bottom vertex is chosen as the base point. With this choice of a triangulation, we define the transition amplitude $Z_T(\Sigma)$ as follows:

$$Z_T(\Sigma) = \left(\left(\bigotimes_i E_{D_i} \right) \otimes \text{id}_{Z_T(\partial_{\text{out}}\Sigma)} \right) \circ Z_T(\Sigma \setminus \bigsqcup_i D_i). \quad (6.2.9)$$

Here, the linear map E_{D_i} is defined by the last two equations of eq. (6.2.7) in which bosonic junctions f and g are replaced by a fermionic junction f_i . The topological invariance of the transition amplitude (6.2.9) follows from the topological invariance of the transition amplitude on the punctured surface. We note that $Z_T(\Sigma)$ depends on the order of fermionic junctions because of the anti-commutation relation (3.1.2). We also note that $Z_T(\Sigma)$ depends on a spin structure on the punctured surface $\Sigma \setminus \bigsqcup_i D_i$ rather than a spin structure on Σ . More specifically, the transition amplitude acquires an extra minus sign if we apply the leaf exchange on a disk D_i , which does not change a spin structure on Σ but changes a spin structure on the punctured surface $\Sigma \setminus \bigsqcup_i D_i$. This extra minus sign is attributed to the fact that the leaf exchange on D_i is equivalent to winding a fermion parity line around a fermionic junction f_i .

This concludes the construction of fermionic TFT \mathfrak{F}_K on spin surfaces with defects. We note that \mathfrak{F}_K reduces to the bosonic TFT \mathfrak{B}_K if the \mathbb{Z}_2 -grading on the input algebra K is trivial. By construction, the symmetry of this TFT is described by the supercategory ${}_K\mathcal{SM}_K$ of (K, K) -superbimodules. We can also define the transition amplitudes on spin surfaces with interfaces between different state sum TFTs. In general, a topological interface between state sum TFTs \mathfrak{F}_K and $\mathfrak{F}_{K'}$ is labeled by a (K, K') -superbimodule. In particular, a topological line labeled by a (K, K) -superbimodule can be regarded as a topological interface between the same TFTs \mathfrak{F}_K . The transition amplitudes on spin surfaces with interfaces are defined just by replacing (K, K) -superbimodules by (K, K') -superbimodules in the above definition.

Relation to the fermionization. Before proceeding, let us discuss the relation between the above state sum construction and the fermionization of bosonic TFTs [159, 167]. To this end, we consider the partition function $Z_{\mathfrak{F}_K}(\Sigma, \eta)$ of the fermionic state sum TFT \mathfrak{F}_K on a closed spin surface Σ equipped with a spin structure η . We suppose that Σ does not have any topological defects. As in the bosonic case, the partition function $Z_{\mathfrak{F}_K}(\Sigma, \eta)$ is represented by a string diagram drawn on the Poincaré dual of the triangulated spin surface Σ , cf. Figure 4.9. One can compute this partition function by first evaluating the string diagram with a fixed \mathbb{Z}_2 -grading on each strand and then taking the sum over all possible \mathbb{Z}_2 -gradings. Since every vertex of the string diagram preserves the \mathbb{Z}_2 -grading, the \mathbb{Z}_2 -grading $a(e) \in \{0, 1\}$ on the strand dual to an edge e should satisfy $\sum_{e \in \partial t} a(e) = 0 \bmod 2$ for all triangles $t \in \Sigma$. This means that a defines a \mathbb{Z}_2 -valued 1-cocycle on Σ , which can be regarded as a \mathbb{Z}_2 gauge field. Thus, the partition function on Σ can be written as

$$Z_{\mathfrak{F}_K}(\Sigma, \eta) = \sum_a Z(a) (-1)^{\tilde{q}_\eta(a)}, \quad (6.2.10)$$

where $(-1)^{\tilde{q}_\eta(a)}$ is the sign that arises from the \mathbb{Z}_2 -grading automorphism on each strand and the symmetric braiding c_{super} , and $Z(a)$ is the remaining numerical factor. We note that $Z(a)$ is related to the partition function of the bosonic state sum TFT \mathfrak{B}_K via

$$Z_{\mathfrak{B}_K}(\Sigma) = \sum_a Z(a). \quad (6.2.11)$$

Namely, $Z(a)$ is the partition function of a bosonic TFT which upon gauging its \mathbb{Z}_2 symmetry becomes the state sum TFT \mathfrak{B}_K . The sign factor $(-1)^{\tilde{q}_\eta(a)}$ in eq. (6.2.10) can be computed explicitly by choosing a spin triangulation of a collar neighborhood of the dual 1-cycle of a . In particular, when a is the dual of a single loop, its collar neighborhood is a cylinder $S^1 \times [0, 1]$, whose triangulation can be chosen as shown in Figure 6.3. Based on this triangulation and the admissible edge indices on it,³ one can compute the sign factor as $(-1)^{\tilde{q}_\eta(a)} = 1$ when the spin

³See eq. (6.3.1) for more details of the edge indices.

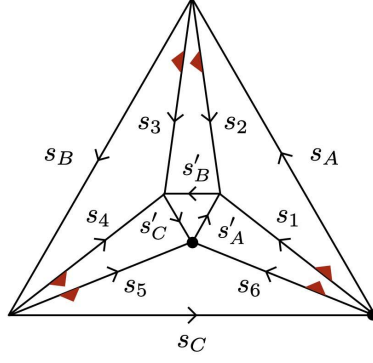


Figure 6.3: A triangulation of a cylinder equipped with a spin structure. The outer triangle is the in-boundary, while the inner triangle is the out-boundary. The black dots represent base points. The edge indices are determined by the admissibility condition up to local moves as in eq. (6.3.1).

structure on the dual of a is NS and $(-1)^{\tilde{q}_\eta(a)} = -1$ when the spin structure on the dual of a is R. This shows that $\tilde{q}_\eta(a) = q_\eta(a)$, where $q_\eta(a)$ is the quadratic function defined in Section 5.1. Therefore, eqs. (6.2.10) and (6.2.11) implies that the fermionic TFT \mathfrak{F}_K and the bosonic TFT \mathfrak{B}_K are related by the bosonization and fermionization.

6.3 Action of ${}_K\mathcal{SM}_K$ symmetry

In this section, we will explicitly compute the action of ${}_K\mathcal{SM}_K$ symmetry on the state space of the fermionic TFT \mathfrak{F}_K . The action of a simple object $X \in {}_K\mathcal{SM}_K$ on the state space on a circle is defined by the transition amplitude on a cylinder with a topological defect X wrapping around its circumference. When $X \in {}_K\mathcal{SM}_K$ is a q-type object, we can modify this action by putting a fermionic point-like defect f on X . These unmodified and modified symmetry operators are denoted by $U_{X;\text{id}}^\lambda$ and $U_{X;f}^\lambda$ respectively, where $\lambda \in \{0, 1\}$ specifies the spin structure on a circle. We recall that the NS spin structure corresponds to $\lambda = 0$, while the R spin structure corresponds to $\lambda = 1$, see eq. (6.1.2).

Before we compute the symmetry action, we first choose a triangulation of a spin cylinder and a marking on it as shown in Figure 6.3. The admissibility condition (6.1.1) determines the edge indices up to local moves as

$$s_1 = s_2 = s_3 = s_4 = s_5 = s'_A = s'_B = s'_C = 0, \quad s_6 = 1 + \lambda, \quad s_A = s_B = s_C =: s. \quad (6.3.1)$$

The two solutions labeled by $s \in \{0, 1\}$ correspond to spin cylinders with and without a sheet exchange in the longitudinal direction. To identify the solution corresponding to a spin cylinder without a sheet exchange, we consider the transition amplitude P_λ^s on the triangulated spin cylinder with edge indices given by eq. (6.3.1). The string diagram representation of the linear map P_λ^s is given by [39]

$$P_\lambda^s = \begin{array}{c} \begin{array}{c} m_K \\ \bullet \\ p^{\lambda+1} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ p^s \end{array} \begin{array}{c} \bullet \\ p^s \end{array} \begin{array}{c} \bullet \\ p^s \end{array} \begin{array}{c} \Delta_K \end{array} \\ \begin{array}{c} K \quad K \quad K \end{array} \end{array}, \quad (6.3.2)$$

where the trivalent junctions represent the multiplication m_K and the comultiplication Δ_K , and the braiding of two strands represents the symmetric braiding c_{super} . We note that the superscript s is additive under the composition of linear maps, i.e. we have $P_\lambda^{s_2} \circ P_\lambda^{s_1} = P_\lambda^{s_1+s_2}$. In particular, the linear map P_λ^0 is an idempotent, which indicates that the solution $s = 0$ corresponds to a spin cylinder without a sheet exchange.

We can compute the unmodified action $U_{X;\text{id}}^\lambda$ of a topological defect $X \in {}_K\mathcal{SM}_K$ by wrapping it around a spatial circle of a spin cylinder without a sheet exchange. As in the bosonic case, the linear map $U_{X;\text{id}}^\lambda$ is expressed by the following string diagram:

$$U_{X;\text{id}}^\lambda = \text{Diagram} \quad (6.3.3)$$

The left K -supercomodule action on X in the above equation is defined as in eq. (2.2.15). We note that eq. (6.3.3) reduces to the cylinder amplitude P_λ^0 when X is a trivial defect K . In particular, the trivial defect K acts as the identity operator on the state space of the fermionic TFT \mathfrak{F}_K .

For any (K, K) -superbimodule X , we can define another (K, K) -superbimodule ${}_pX$ by twisting the left K -action on X using the \mathbb{Z}_2 -grading automorphism p of K . Specifically, the left K -action on ${}_pX$ is given by $\rho_{{}_pX}^L(a \otimes x) := \rho_X^L(p(a) \otimes x)$ for $a \in K$ and $x \in X$. It follows from eq. (6.3.3) that the action of ${}_pX$ is related to that of X as follows:

$$U_{{}_pX;\text{id}}^\lambda = (-1)^F \circ U_{X;\text{id}}^\lambda = U_{X;\text{id}}^\lambda \circ (-1)^F. \quad (6.3.4)$$

Here, $(-1)^F := P_\lambda^1$ is the transition amplitude on a spin cylinder with a sheet exchange, which physically describes the action of the fermion parity symmetry. When X is a trivial defect K , eq. (6.3.4) reduces to $U_{{}_pK;\text{id}}^\lambda = (-1)^F$. This implies that ${}_pK$ is the generator of the fermion parity symmetry \mathbb{Z}_2^F . Equation (6.3.4) shows that the fermion parity symmetry is central, i.e., it commutes with the unmodified action $U_{X;\text{id}}^\lambda$ for any $X \in {}_K\mathcal{SM}_K$.

The (K, K) -superbimodule ${}_pX$ equipped with the opposite \mathbb{Z}_2 -grading is another (K, K) -superbimodule ΠX , which is oddly isomorphic to X . The (K, K) -superbimodule isomorphism $\zeta_X : \Pi X \rightarrow X$ is given by the identity map of the underlying vector space. Equations (6.3.3) and (3.1.6) imply that the actions of ΠX and X differ by a scalar $(-1)^\lambda$:

$$U_{\Pi X;\text{id}}^\lambda = (-1)^\lambda U_{X;\text{id}}^\lambda. \quad (6.3.5)$$

In particular, when X is a trivial defect K , the oddly isomorphic defect ΠK acts as $+1$ on the NS sector and -1 on the R sector. This suggests that the topological defect ΠK measures the spin structure on the circle that it winds around. Equation (6.3.5) implies that the action of a q-type object X on the R sector vanishes because $U_{X;\text{id}}^\lambda = U_{\Pi X;\text{id}}^\lambda = (-1)^\lambda U_{X;\text{id}}^\lambda$, where the first equality follows from the existence of an even isomorphism $\Pi X \cong X$ for a q-type object X . This particularly means that the torus partition function vanishes if a q-type object X is winding around a spatial circle equipped with an R spin structure. Thus, due to the modular invariance, the X -twisted sector has the same number of bosonic states and fermionic states

when $X \in {}_K\mathcal{SM}_K$ is a q-type object. This can also be understood from the fact that the X -twisted sector has an odd automorphism when X is a q-type object.

We next compute the modified action $U_{X,f}^\lambda$ of a q-type object X . Since we have a fermionic point-like defect f on X , the transition amplitude on a cylinder with X wrapping around a spatial circle depends on the spin structure on the punctured cylinder. We can choose a spin structure so that the action $U_{X,f}^\lambda$ is represented by the string diagram in eq. (6.3.3) where p^λ is replaced by $f \circ p^\lambda$.⁴ This modified action $U_{X,f}^\lambda$ satisfies $U_{X,f}^\lambda = (-1)^{\lambda+1} U_{X,f}^\lambda$ because moving a fermionic point-like defect f around a topological defect X by using eq. (3.1.4) produces a sign $(-1)^{\lambda+1}$. Therefore, the modified action $U_{X,f}^\lambda$ on the NS sector vanishes. Since the unmodified action $U_{X,\text{id}}^\lambda$ vanishes on the R sector, the action of a q-type object is given by $U_{X,\text{id}}^\lambda$ on the NS sector and $U_{X,f}^\lambda$ on the R sector. We note that the action of a q-type object on the R sector anti-commutes with the fermion parity symmetry,⁵ i.e.

$$U_{X,f}^\lambda \circ (-1)^F = -(-1)^F \circ U_{X,f}^\lambda, \quad (6.3.6)$$

which implies that the R sector is at least two-fold degenerate if the ${}_K\mathcal{SM}_K$ symmetry has a q-type object whose action on the R sector is non-zero. However, we emphasize that the existence of a q-type object in a general superfusion category symmetry does not imply degenerate ground states. Indeed, the fermionization $\text{sRep}(\mathcal{H}_8^u)$ of a non-anomalous fusion category symmetry $\text{Rep}(H_8)$ is an example of a superfusion category symmetry with q-type objects that admits a non-degenerate ground state.

Let us apply the above linear maps $U_{X,\text{id}}^\lambda$ and $U_{X,f}^\lambda$ to the ground states of TFT \mathfrak{F}_K . We first write down the ground states of \mathfrak{F}_K , which are in one-to-one correspondence with simple left K -supermodules [167]. The ground state corresponding to a simple left K -supermodule M is obtained by evaluating the transition amplitude on a cylinder with a topological boundary condition M imposed on one end of the cylinder. The other end of the cylinder is regarded as an out-boundary. Since a topological boundary is an interface between the state sum TFT \mathfrak{F}_K and the trivial TFT, this transition amplitude is a linear map from the state space of the trivial TFT to the state space of \mathfrak{F}_K . This linear map can be canonically identified with a state of \mathfrak{F}_K because the state space of the trivial TFT is \mathbb{C} . The state obtained by this canonical identification is denoted by $|M; \text{id}\rangle_\lambda$, where λ labels the spin structure on a circle. When a simple K -supermodule M has an odd automorphism $f : M \rightarrow M$, which is unique up to rescaling, we can modify the boundary condition by putting the fermionic point-like defect f on the topological boundary M . The state corresponding to this modified boundary condition is denoted by $|M; f\rangle_\lambda$. The string diagram representations of these states are given by

$$|M; \beta\rangle_\lambda = \begin{array}{c} \begin{array}{c} K \quad K \quad K \\ \text{---} \end{array} \\ \text{---} \end{array} \begin{array}{c} \text{ev}_M^L \\ \beta \\ p^\lambda \\ \text{coev}_M^R \end{array}, \quad (6.3.7)$$

where $\beta : M \rightarrow M$ is either the identity morphism id or the fermionic point-like defect f . The trivalent junctions in the above diagram represent the left K -supercomodule action on M ,

⁴The other choice of a spin structure gives rise to the symmetry operator with the opposite sign.

⁵The anti-commutation relation between the fermion parity defect and a q-type object is observed in the example of massless Majorana fermions [69, 85, 178].

which is defined as in eq. (2.2.15). We note that the fermion parity of $|M; \beta\rangle_\lambda$ agrees with the \mathbb{Z}_2 -grading of β , i.e. we have $(-1)^F |M; \beta\rangle_\lambda = (-1)^{|\beta|} |M; \beta\rangle_\lambda$. When M has an odd automorphism, the NS sector state $|M; f\rangle_{\lambda=0}$ and the R sector state $|M; \text{id}\rangle_{\lambda=1}$ vanish due to the equalities $|M; f\rangle_\lambda = (-1)^{\lambda+1} |M; f\rangle_\lambda$ and $|M; \text{id}\rangle_\lambda = (-1)^\lambda |M; \text{id}\rangle_\lambda$.⁶ Therefore, states in the NS and R sectors are both in one-to-one correspondence with simple K -supermodules. Although we can also define a state $|N; \beta\rangle$ for a non-simple K -supermodule N and a general supermodule morphism $\beta : N \rightarrow N$ in the same way, such a state is a linear combination of the ground states labeled by simple K -supermodules.

The action of a simple topological defect $X \in {}_K\mathcal{SM}_K$ on the ground states (6.3.7) can be computed as

$$U_{X;\alpha}^\lambda |M; \beta\rangle_\lambda = |X \otimes_K M; \alpha \otimes_K \beta\rangle_\lambda. \quad (6.3.8)$$

Here, $\alpha : X \rightarrow X$ is a point-like defect on X and $\beta : M \rightarrow M$ is a point-like defect on M . Since X and M are a simple (K, K) -superbimodule and a simple K -supermodule respectively, the point-like defects α and β are either the identity morphism or an odd automorphism unique up to scalar multiplication. The expression (6.3.8) is also valid for non-simple topological defects $X \in {}_K\mathcal{SM}_K$ and a general point-like defect $\alpha : X \rightarrow X$.

6.4 Fermionic TFTs with superfusion category symmetries

In this section, we show that the ${}_K\mathcal{SM}_K$ symmetry of the fermionic TFT \mathfrak{F}_K can be pulled back to $\text{sRep}(\mathcal{H}^u)$ symmetry if the bosonic TFT \mathfrak{B}_K constructed from the same algebra K has $\text{Rep}(H)$ symmetry. Here, H is a weak Hopf algebra and \mathcal{H}^u is a weak Hopf superalgebra defined in Section 5.2. Since the fermionic TFT \mathfrak{F}_K is the fermionization of the bosonic TFT \mathfrak{B}_K [159, 167], the above result verifies the fermionization formula of fusion category symmetries that we proposed in Section 5.2.

To derive the fermionization formula of fusion category symmetries, let us first recall the state sum construction of bosonic TFTs with $\text{Rep}(H)$ symmetry. As we discussed in Section 4.3, the bosonic state sum TFT \mathfrak{B}_K constructed from a semisimple algebra K has the symmetry ${}_K\mathcal{M}_K$ described by the category of (K, K) -bimodules. When the input algebra K is a left H -comodule algebra, the ${}_K\mathcal{M}_K$ symmetry can be pulled back to $\text{Rep}(H)$ by a tensor functor $F_K : \text{Rep}(H) \rightarrow {}_K\mathcal{M}_K$. Thus, the state sum TFT \mathfrak{B}_K has $\text{Rep}(H)$ symmetry if K is a left H -comodule algebra. Any indecomposable semisimple bosonic TFTs with $\text{Rep}(H)$ symmetry can be constructed in this way [64].

The bosonic TFT \mathfrak{B}_K with $\text{Rep}(H)$ symmetry can be fermionized if it has a non-anomalous \mathbb{Z}_2 subgroup symmetry. As we discussed in Section 5.2, a fusion category symmetry $\text{Rep}(H)$ has a non-anomalous \mathbb{Z}_2^u subgroup generated by $V_u \in \text{Rep}(H)$ if the dual weak Hopf algebra H^* has a \mathbb{Z}_2 group-like element $u \in H^*$. This \mathbb{Z}_2 group-like element u endows an H -comodule algebra K with a \mathbb{Z}_2 -grading

$$p := (u \otimes \text{id}_K) \circ \lambda_K^L, \quad (6.4.1)$$

where $\lambda_K^L : K \rightarrow H \otimes K$ is the H -comodule action on K . A semisimple algebra K equipped with the above \mathbb{Z}_2 -grading is denoted by K^u . We note that K^u is a superalgebra because the multiplication and unit of K are even with respect to the \mathbb{Z}_2 -grading (6.4.1). Hence, we can use K^u as an input of the fermionic state sum construction. The fermionic TFT \mathfrak{F}_{K^u} constructed from K^u is the fermionization of the bosonic TFT \mathfrak{B}_K with respect to the \mathbb{Z}_2^u symmetry.

⁶In particular, the NS sector does not have fermionic states [137, 167].

Let us now determine the symmetry of \mathfrak{F}_{K^u} by using the pullback of superfusion category symmetries. To this end, we first show that a semisimple superalgebra K^u is a left \mathcal{H}^u -supercomodule algebra when K is a left H -comodule algebra. The left \mathcal{H}^u -supercomodule action on the superalgebra K^u is given by the left H -comodule action λ_K^L on the underlying algebra K . This comodule action is \mathbb{Z}_2 -even with respect to the \mathbb{Z}_2 -gradings on \mathcal{H}^u and K^u . Furthermore, the comodule action λ_K^L is compatible with the algebra structure on K^u , i.e., it satisfies

$$\begin{array}{c} \mathcal{H}^u \\ \text{---} \lambda_K^L \\ \text{---} m_K \\ \text{---} K^u \quad K^u \end{array} = \begin{array}{c} \text{---} m \\ \text{---} c_{\text{triv}} \\ \text{---} \end{array} = \begin{array}{c} \text{---} m_u \\ \text{---} c_{\text{super}} \\ \text{---} \end{array}, \quad (6.4.2)$$

$$\begin{array}{c} \text{---} \eta_K \circ \\ \text{---} \end{array} = \begin{array}{c} \text{---} c_{\text{triv}} \\ \text{---} m \\ \text{---} \end{array} = \begin{array}{c} u |1_{(1)}| \\ \text{---} c_{\text{super}} \\ \text{---} m_u \\ \text{---} \end{array} = \begin{array}{c} \text{---} \epsilon_s \\ \text{---} \eta_K \circ \end{array}. \quad (6.4.3)$$

In the second line, $1 := \eta(1)$ denotes the unit element of \mathcal{H}^u and ϵ_s denotes the source counital map of \mathcal{H}_u . The above equations show that K^u is a left \mathcal{H}^u -supercomodule algebra. Therefore, the supercategory ${}_{K^u}\mathcal{SM}$ of left K^u -supermodules is an $\text{sRep}(\mathcal{H}_u)$ -supermodule category. Equivalently, we have a supertensor functor $\mathcal{F}_{K^u} : \text{sRep}(\mathcal{H}^u) \rightarrow {}_{K^u}\mathcal{SM}_{K^u}$, which enables us to pull back the ${}_{K^u}\mathcal{SM}_{K^u}$ symmetry of the fermionic TFT \mathfrak{F}_{K^u} . Since the pullback of the symmetry leaves the underlying TFT unchanged, we conclude that the fermionic TFT \mathfrak{F}_{K^u} has a superfusion category symmetry $\text{sRep}(\mathcal{H}^u)$.

Action of superfusion category symmetry We can explicitly compute the action of a topological line $V \in \text{sRep}(\mathcal{H}^u)$ on the state space on a circle. The action of V is denoted by $\mathcal{U}_{V;\alpha}^\lambda$, where $\alpha : V \rightarrow V$ is a point-like defect on V and λ specifies a spin structure on a circle. Since the $\text{sRep}(\mathcal{H}^u)$ symmetry of the fermionic TFT \mathfrak{F}_{K^u} is the pullback of the ${}_{K^u}\mathcal{SM}_{K^u}$ symmetry by a supertensor functor \mathcal{F}_{K^u} , the action of $V \in \text{sRep}(\mathcal{H}^u)$ reduces to the action of $\mathcal{F}_{K^u}(V) \in {}_{K^u}\mathcal{SM}_{K^u}$, which was computed in Section 6.3. Therefore, a topological line V acts on the ground states (6.3.7) as

$$\mathcal{U}_{V;\alpha}^\lambda |M; \beta\rangle_\lambda = |\mathcal{F}_{K^u}(V) \otimes_{K^u} M; \mathcal{F}_{K^u}(\alpha) \otimes_{K^u} \beta\rangle_\lambda = |V \bar{\otimes} M; \alpha \bar{\otimes} \beta\rangle_\lambda, \quad (6.4.4)$$

where $\bar{\otimes} : \text{sRep}(\mathcal{H}^u) \times {}_{K^u}\mathcal{SM} \rightarrow {}_{K^u}\mathcal{SM}$ is the $\text{sRep}(\mathcal{H}^u)$ -supermodule action on ${}_{K^u}\mathcal{SM}$.

Equation (6.4.4) implies that the category of boundary conditions of \mathfrak{F}_{K^u} is an $\text{sRep}(\mathcal{H}^u)$ -supermodule category ${}_{K^u}\mathcal{SM}$. On the other hand, the category of boundary conditions of the original bosonic TFT \mathfrak{B}_K is a $\text{Rep}(H)$ -module category ${}_K\mathcal{M}$ [64]. Therefore, we find that the fermionization of a $\text{Rep}(H)$ -symmetric bosonic TFT whose category of boundary conditions is a $\text{Rep}(H)$ -module category ${}_K\mathcal{M}$ is an $\text{sRep}(\mathcal{H}^u)$ -symmetric fermionic TFT whose category of boundary conditions is an $\text{sRep}(\mathcal{H}^u)$ -supermodule category ${}_{K^u}\mathcal{SM}$.

6.5 Lattice models with superfusion category symmetries

We can construct lattice models of gapped phases described by fermionic TFTs \mathfrak{F}_{K^u} in the low-energy limit. In this subsection, we explicitly write down the Hamiltonians of these lattice models and discuss the superfusion category symmetries on the lattice. For simplicity, we will

restrict our attention to non-anomalous symmetries $\text{sRep}(\mathcal{H}^u)$ where \mathcal{H}^u is a Hopf superalgebra.

Let N be the number of lattice sites. The state space K_i^u on the i th site is given by a left \mathcal{H}^u -supercomodule algebra K^u defined in the previous section. We suppose that K^u is chosen so that the action of $\text{sRep}(\mathcal{H}^u)$ symmetry defined below becomes faithful on the lattice. The Hamiltonian H_λ on a circle with a spin structure λ is of the form

$$H_\lambda = \sum_i (1 - h_{i,i+1}^\lambda), \quad (6.5.1)$$

where $\lambda = 0$ for an NS circle and $\lambda = 1$ for an R circle. The interaction term $h_{i,i+1}^\lambda$ between the neighboring sites is given by

$$h_{i,i+1}^\lambda = \begin{cases} \Delta_K \circ m_K : K_i^u \otimes K_{i+1}^u \rightarrow K_i^u \otimes K_{i+1}^u & \text{for } i \neq N, \\ (\text{id}_K \otimes p^{\lambda+1}) \circ \Delta_K \circ m_K \circ (\text{id}_K \otimes p^{\lambda+1}) : K_N^u \otimes K_1^u \rightarrow K_N^u \otimes K_1^u & \text{for } i = N, \end{cases} \quad (6.5.2)$$

where $p : K^u \rightarrow K^u$ is the \mathbb{Z}_2 -grading automorphism (6.4.1). The above Hamiltonian is a fermionic analogue of the bosonic Hamiltonian given in Section 4.4.2. Each term $h_{i,i+1}^\lambda$ is a commuting projector because a semisimple (super)algebra K^u is a Δ -separable symmetric Frobenius algebra. Thus, the space of the ground states is the image of the projector $P = \prod_i h_{i,i+1}^\lambda$. This ground state subspace agrees with the state space of the fermionic TFT \mathfrak{F}_{K^u} on a circle because the projector P is the same as the transition amplitude (6.3.2) on a triangulated cylinder. Therefore, the ground states of the Hamiltonian (6.5.1) are given by the boundary states (6.3.7) of \mathfrak{F}_{K^u} .

We define the symmetry operator $\hat{\mathcal{U}}_{V;\alpha}^\lambda : \bigotimes_i K_i^u \rightarrow \bigotimes_i K_i^u$ for $V \in \text{sRep}(\mathcal{H}^u)$ and $\alpha : V \rightarrow V$ by the following string diagram:

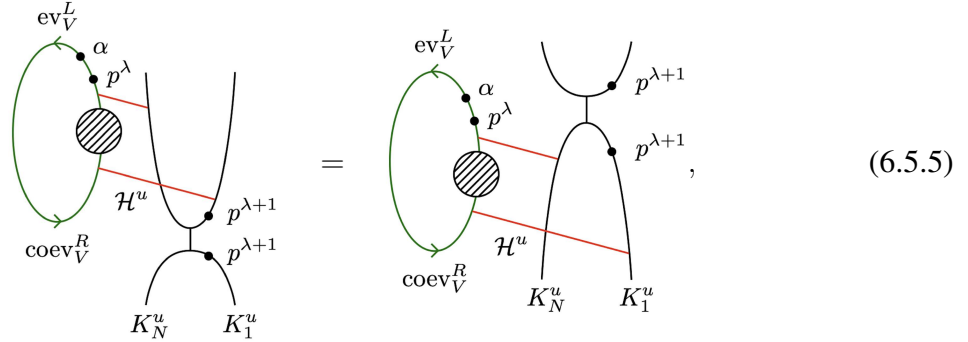
$$\hat{\mathcal{U}}_{V;\alpha}^\lambda = \begin{array}{c} \text{ev}_V^L \\ \uparrow \\ \text{green oval} \\ \downarrow \\ \text{coev}_V^R \end{array} \begin{array}{c} \alpha \\ \downarrow \\ p^\lambda \end{array} \begin{array}{c} \text{red line} \\ \downarrow \\ \mathcal{H}^u \end{array} \begin{array}{c} \lambda_K^L \\ \downarrow \\ K_1^u \quad K_2^u \quad \dots \quad K_N^u \end{array}. \quad (6.5.3)$$

The right \mathcal{H}^u -action on V in the above equation is induced by the left \mathcal{H}^u -action on V^* as in eq. (4.4.2). We note that a morphism $\alpha : V \rightarrow V$ commutes with the right \mathcal{H}^u -action because the dual of α commutes with the left \mathcal{H}^u -action on V^* . For a simple object $V \in \text{sRep}(\mathcal{H}^u)$, a point-like defect α in eq. (6.5.3) is either the identity morphism id or the unique \mathbb{Z}_2 -odd automorphism f . When V is a q-type object, the symmetry action on the lattice satisfies $\hat{\mathcal{U}}_{V;f}^{\lambda=0} = \hat{\mathcal{U}}_{V;\text{id}}^{\lambda=1} = 0$, which is analogous to the equality $\mathcal{U}_{V;f}^{\lambda=0} = \mathcal{U}_{V;\text{id}}^{\lambda=1} = 0$ for the corresponding operators of the fermionic TFT \mathfrak{F}_{K^u} . The composition of operators defined by eq. (6.5.3) are compatible with the monoidal structure on $\text{sRep}(\mathcal{H}^u)$, i.e., $\hat{\mathcal{U}}_{V;\alpha}^\lambda \circ \hat{\mathcal{U}}_{V';\alpha'}^\lambda = \hat{\mathcal{U}}_{V \otimes V'; \alpha \otimes \alpha'}^\lambda$.

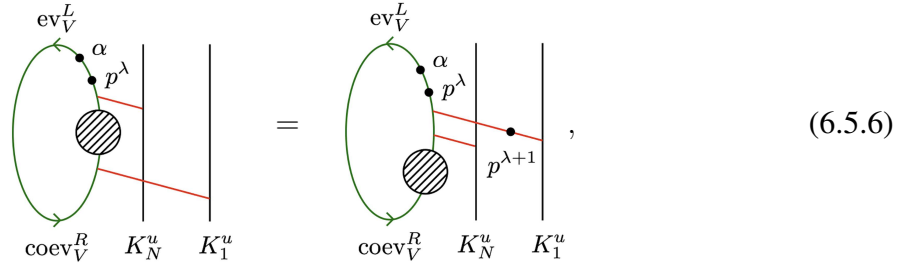
The symmetry action (6.5.3) commutes with the Hamiltonian (6.5.1). The commutativity of the symmetry action $\hat{\mathcal{U}}_{V;\alpha}^\lambda$ and the commuting projector $h_{i,i+1}^\lambda$ for $i \neq N$ follows from the equality

$$\begin{array}{c} \mathcal{H}^u \\ \downarrow \\ m_u \end{array} \begin{array}{c} \text{cup} \\ \downarrow \\ K_i^u \quad K_{i+1}^u \end{array} = \begin{array}{c} \mathcal{H}^u \\ \downarrow \\ m_u \end{array} \begin{array}{c} \text{cup} \\ \downarrow \\ K_i^u \quad K_{i+1}^u \end{array}, \quad (6.5.4)$$

which we can show in the same way as in the bosonic case. In order to show that the symmetry operator $\widehat{\mathcal{U}}_{V;\alpha}^\lambda$ also commutes with $h_{N,1}^\lambda$, we write the equation $\widehat{\mathcal{U}}_{V;\alpha}^\lambda h_{N,1}^\lambda = h_{N,1}^\lambda \widehat{\mathcal{U}}_{V;\alpha}^\lambda$ in terms of string diagrams as

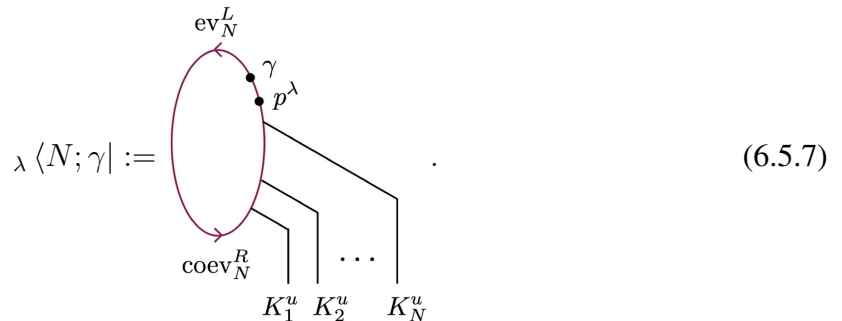


where the blob represents the action of \mathcal{H}^u coacting on $K_{i \neq 1, N}^u$. This equation reduces to eq. (6.5.4) due to the equality



which we can show by a direct computation. Therefore, the symmetry action $\widehat{\mathcal{U}}_{V;\alpha}^\lambda$ also commutes with $h_{N,1}^\lambda$. This shows that the Hamiltonian (6.5.1) has a superfusion category symmetry $\text{sRep}(\mathcal{H}^u)$ on the lattice.

Finally, we argue that the action of $\widehat{\mathcal{U}}_{V;\alpha}^\lambda$ on the ground states of the Hamiltonian (6.5.1) agrees with the action of the corresponding operator $\mathcal{U}_{V;\alpha}^\lambda$ of the fermionic TFT \mathfrak{F}_{K^u} . To this end, we compute the actions of $\widehat{\mathcal{U}}_{V;\alpha}^\lambda$ and $\mathcal{U}_{V;\alpha}^\lambda$ on the dual vector space of the ground state subspace.⁷ States in the dual vector space are denoted by ${}_\lambda \langle N; \gamma |$, where N is a simple right K^u -supermodule and $\gamma : N \rightarrow N$ is a right K^u -supermodule morphism. A dual state ${}_\lambda \langle N; \gamma |$ is the transition amplitude on a cylinder whose boundary consists of an in-boundary and a topological boundary labeled by N , where γ is a point-like defect on the topological boundary. In the string diagram notation, a dual state ${}_\lambda \langle N; \gamma |$ can be written as



Since N is simple, γ is the identity morphism id or the unique \mathbb{Z}_2 -odd automorphism f . We note that ${}_{\lambda=0} \langle N; f | = {}_{\lambda=1} \langle N; \text{id} | = 0$ if N has an odd automorphism. The action of $\widehat{\mathcal{U}}_{V;\alpha}^\lambda$

⁷As in the bosonic case (cf. eq. (4.4.16)), it should also be possible to directly compute the action of $\widehat{\mathcal{U}}_{V;\alpha}^\lambda$ on the ground states without using the dual vector space.

on a dual state (6.5.7) is computed as ${}_{\lambda} \langle N; \gamma | \widehat{\mathcal{U}}_{V;\alpha}^{\lambda} = {}_{\lambda} \langle N \otimes V; \gamma \otimes \alpha |$, where the right K^u -supermodule structure on $N \otimes V$ is defined via the left \mathcal{H}^u -supercomodule action on K^u . On the other hand, the symmetry operator $\mathcal{U}_{V;\alpha}^{\lambda}$ of a fermionic TFT \mathfrak{F}_{K^u} acts on a dual state ${}_{\lambda} \langle N; \gamma |$ as

$${}_{\lambda} \langle N; \gamma | \mathcal{U}_{V;\alpha}^{\lambda} = {}_{\lambda} \langle N \otimes_{K^u} \mathcal{F}_{K^u}(V); \gamma \otimes_{K^u} \mathcal{F}_{K^u}(\alpha) | = {}_{\lambda} \langle N \otimes V; \gamma \otimes \alpha |, \quad (6.5.8)$$

where we used the isomorphism of right K^u -supermodules $N \otimes_{K^u} \mathcal{F}_{K^u}(V) \cong N \otimes_{K^u} (V \otimes K^u) \cong N \otimes_{K^u} (K^u \otimes V) \cong N \otimes V$.⁸ Therefore, we find that symmetry operators of the lattice models and those of the TFTs act in the same way on the dual of the ground state subspace. In particular, we have ${}_{\lambda} \langle N; \gamma | \widehat{\mathcal{U}}_{V;\alpha}^{\lambda} |M; \beta\rangle_{\lambda} = {}_{\lambda} \langle N; \gamma | \mathcal{U}_{V;\alpha}^{\lambda} |M; \beta\rangle_{\lambda}$ for any ${}_{\lambda} \langle N; \gamma |$ and $|M; \beta\rangle_{\lambda}$. This shows that $\widehat{\mathcal{U}}_{V;\alpha}^{\lambda}$ reduces to $\mathcal{U}_{V;\alpha}^{\lambda}$ when it acts on the ground states.

⁸The second isomorphism follows from $V^* \otimes K^u \cong \mathcal{F}_{K^u}(V^*) \cong \mathcal{F}_{K^u}(V)^* \cong (K^u)^* \otimes V^* \cong K^u \otimes V^*$, where the first and the third isomorphisms follow from the definition of \mathcal{F}_{K^u} , the second isomorphism is due to the fact that \mathcal{F}_{K^u} is a supertensor functor, and the last isomorphism is given by the superalgebra isomorphism (2.2.9).

Chapter 7

Conclusion

In this dissertation, we constructed all (1+1)-dimensional bosonic and fermionic topological field theories with general finite symmetries beyond ordinary groups. These bosonic and fermionic TFTs were obtained by the state sum construction based on semisimple algebras and semisimple superalgebras respectively. We found that the symmetries of these TFTs originate from additional structures on the input algebras of the state sum construction. Specifically, the bosonic TFT \mathfrak{B}_K constructed from a semisimple algebra K has a fusion category symmetry $\text{Rep}(H)$ when K is a comodule algebra over a weak Hopf algebra H . Similarly, the fermionic TFT \mathfrak{F}_K constructed from a semisimple superalgebra K has a superfusion category symmetry $\text{sRep}(\mathcal{H})$ when K is a supercomodule algebra over a weak Hopf superalgebra \mathcal{H} .

The bosonic TFT \mathfrak{B}_K and fermionic TFT \mathfrak{F}_K constructed from the same superalgebra K are related by the bosonization and fermionization. Based on this fact, we derived the fermionization formula of fusion category symmetries by comparing the symmetry of the bosonic TFT \mathfrak{B}_K and that of the fermionic TFT \mathfrak{F}_K . This fermionization formula greatly generalizes the relation between the Kramers-Wannier self-duality of the critical Ising model and the anomalous chiral fermion parity symmetry of the massless Majorana fermion. As concrete examples, we computed the fermionization of finite group symmetries, symmetries of finite group gauge theories, and self-dualities. In particular, we found that the fermionized symmetry becomes non-group-like (i.e., non-invertible) if we fermionize a non-central \mathbb{Z}_2 subgroup of a finite group symmetry.

We also constructed concrete lattice models with non-anomalous (super)fusion category symmetries in 1+1d. The state space of our model admits a tensor product decomposition as opposed to the state space of the anyon chain model. The Hamiltonian is given by the sum of local commuting projectors and hence is exactly solvable. These models turn out to be described in the low energy limit by the topological field theories that we constructed.

There are several interesting directions that we did not investigate in this dissertation. For example, we did not construct lattice models with anomalous fusion category symmetries. When the symmetries are anomalous, a similar construction of lattice models works with a slight modification of the state space [179]. Due to this modification, the state space of the model with an anomalous symmetry no longer admits a tensor product decomposition in general. However, it might be possible to come up with other lattice models in which anomalous fusion category symmetries are realized on tensor product state spaces. It would be interesting to figure out which anomalous fusion category symmetries are compatible with the tensor product decomposition of the state space.

Another interesting direction is to investigate phase transitions between gapped phases with fusion category symmetries. Since we have concrete gapped Hamiltonians with fusion category

symmetries on the lattice, it should be possible to numerically study the phase diagram of the model perturbed by symmetry-preserving interactions, which drive a gapped phase into other ones. The phase transition points of such a model may exhibit novel symmetry-enriched quantum criticality.

It would also be fascinating to generalize the construction of topological field theories and lattice models to higher dimensions. In higher dimensions, general finite symmetries are expected to be described by higher categories. For example, in 2+1 dimensions, finite symmetries of bosonic systems are described by fusion 2-categories, which are realized by 2+1d analogues of the anyon chain models [157]. However, in spacetime dimensions $d \geq 4$, a precise description of higher categorical symmetries is still under development. The general structures of symmetries in fermionic systems also remain unclear when the spacetime dimension is greater than or equal to three. We leave the construction of physical systems with such symmetries to future studies.

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Appendix A

Fusion rules of superfusion category symmetries

In this appendix, we compute the fusion rules of the superfusion categories discussed in Section 5.3.

A.1 Fusion rules of $\text{sRep}(\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u)$

We recall that the Hopf superalgebra $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ has one-dimensional super representations $V_g \cong \mathbb{C}^{1|0}$ for $g \in C_G(u)$ and two-dimensional super representations W_g for $g \notin C_G(u)$. The two-dimensional super representation W_g is evenly isomorphic to W_{ugu} and oddly isomorphic to W_{gu} and W_{ug} , namely, we have even isomorphisms $W_g \cong W_{ugu} \cong \Pi W_{gu} \cong \Pi W_{ug}$. In the following, we will compute the fusion rules of these irreducible super representations. We will follow the definitions and the notations used in Section 5.3.1.

The fusion rule of V_g and V_h . Let us first compute the fusion rule of one-dimensional super representations V_g and V_h . The action of $\widehat{l} \in \mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ on the tensor product representation $V_g \otimes V_h$ is computed as

$$\Delta(\widehat{l}) \cdot v_g \otimes v_h = \delta_{l,gh} v_g \otimes v_h. \quad (\text{A.1.1})$$

This implies that $V_g \otimes V_h$ is evenly isomorphic to V_{gh} , i.e. we have $V_g \otimes V_h \cong V_{gh}$.

The fusion rule of V_g and W_h . The action of $\widehat{l} \in \mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ on $V_g \otimes W_h$ is given by

$$\Delta(\widehat{l}) \cdot v_g \otimes w_h = \delta_{l,gh} v_g \otimes \widehat{h} w_h + \delta_{l,ughu} v_g \otimes \widehat{u} \widehat{h} u w_h + \delta_{l,ghu} v_g \otimes \widehat{h} u w_h + \delta_{l,ugh} v_g \otimes \widehat{u} \widehat{h} w_h, \quad (\text{A.1.2})$$

where we used the fact that $g \in C_G(u)$ commutes with u . This equation implies that the linear map $v_g \otimes w_h \mapsto w_{gh}$ gives an even isomorphism between super representations $V_g \otimes W_h$ and W_{gh} . Therefore, we have $V_g \otimes W_h \cong W_{gh}$. Similarly, the action of $\widehat{l} \in \mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ on $W_g \otimes V_h$ is given by

$$\Delta(\widehat{l}) \cdot w_g \otimes v_h = \delta_{l,gh} \widehat{g} w_g \otimes v_h + \delta_{l,ughu} \widehat{u} \widehat{g} u w_g \otimes v_h + \delta_{l,ghu} \widehat{g} u w_g \otimes v_h + \delta_{l,ugh} \widehat{u} \widehat{g} w_g \otimes v_h, \quad (\text{A.1.3})$$

which implies that the linear map $w_g \otimes v_h \mapsto w_{gh}$ gives an even isomorphism $W_g \otimes V_h \cong W_{gh}$.

The fusion rule of W_g and W_h . The action of $\widehat{l} \in \mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ on $W_g \otimes W_h$ is given by

$$\begin{aligned} \Delta(\widehat{l}) \cdot (w_g)_s \otimes (w_h)_t &= \frac{1}{4}(\widehat{lh^{-1}} + \widehat{luh^{-1}u} + \widehat{luh^{-1}} + \widehat{lh^{-1}u})(w_g)_s \otimes (w_h)_t \\ &\quad + \frac{t}{4}(\widehat{lh^{-1}} + \widehat{luh^{-1}u} - \widehat{luh^{-1}} - \widehat{lh^{-1}u})(w_g)_s \otimes (w_h)_t \\ &\quad + \frac{s}{4}(\widehat{lh^{-1}} - \widehat{luh^{-1}u} + \widehat{luh^{-1}} - \widehat{lh^{-1}u})(w_g)_s \otimes (w_h)_{-t} \\ &\quad + \frac{st}{4}(\widehat{lh^{-1}} - \widehat{luh^{-1}u} - \widehat{luh^{-1}} + \widehat{lh^{-1}u})(w_g)_s \otimes (w_h)_{-t}, \end{aligned} \quad (\text{A.1.4})$$

where $s, t = \pm 1$ represents the \mathbb{Z}_2 -grading of $(w_g)_s$ and $(w_h)_t$. We note that an element \widehat{l} acts non-trivially on $W_g \otimes W_h$ only when l is in a set $\{gh, ughu, gh, ugh\} \sqcup \{uguh, guhu, uguhu, guh\}$. This set depends on whether gh and $uguh$ commute with u . Accordingly, the fusion rule of W_g and W_h is divided into four cases. Let us investigate these cases one by one.

When both gh and $uguh$ commute with u , the subalgebra of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ that acts non-trivially on $W_g \otimes W_h$ is spanned by $\{\widehat{gh}, \widehat{ghu}\} \sqcup \{\widehat{uguh}, \widehat{guh}\}$. Since this subalgebra only has one-dimensional irreducible super representations, the four-dimensional super representation $W_g \otimes W_h$ can be decomposed into the direct sum of four one-dimensional super representations associated with \widehat{gh} , \widehat{ghu} , \widehat{uguh} , and \widehat{guh} . More specifically, the non-trivial action of $\widehat{l} \in \mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ is summarized as

$$\begin{aligned} \Delta(\widehat{gh}) \cdot [(w_g)_+ \otimes (w_h)_+ + (w_g)_- \otimes (w_h)_-] &= (w_g)_+ \otimes (w_h)_+ + (w_g)_- \otimes (w_h)_-, \\ \Delta(\widehat{ghu}) \cdot [(w_g)_+ \otimes (w_h)_- + (w_g)_- \otimes (w_h)_+] &= (w_g)_+ \otimes (w_h)_- + (w_g)_- \otimes (w_h)_+, \\ \Delta(\widehat{uguh}) \cdot [(w_g)_+ \otimes (w_h)_+ - (w_g)_- \otimes (w_h)_-] &= (w_g)_+ \otimes (w_h)_+ - (w_g)_- \otimes (w_h)_-, \\ \Delta(\widehat{guh}) \cdot [(w_g)_+ \otimes (w_h)_- - (w_g)_- \otimes (w_h)_+] &= (w_g)_+ \otimes (w_h)_- - (w_g)_- \otimes (w_h)_+. \end{aligned} \quad (\text{A.1.5})$$

This shows that we have an even isomorphism of super representations

$$W_g \otimes W_h \cong V_{gh} \oplus V_{uguh} \oplus \Pi V_{ghu} \oplus \Pi V_{guh}. \quad (\text{A.1.6})$$

When only gh commutes with u , the subalgebra of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ that acts non-trivially on $W_g \otimes W_h$ is spanned by $\{\widehat{gh}, \widehat{ghu}\} \sqcup \{\widehat{uguh}, \widehat{guh}, \widehat{uguhu}, \widehat{guh}\}$. The action of \widehat{gh} and \widehat{ghu} are given by the first two equalities in eq. (A.1.5), which implies that the tensor product representation $W_g \otimes W_h$ contains one-dimensional super representations V_{gh} and ΠV_{ghu} . The remaining two-dimensional subspace spanned by $(w_g)_+ \otimes (w_h)_+ - (w_g)_- \otimes (w_h)_-$ and $(w_g)_+ \otimes (w_h)_- - (w_g)_- \otimes (w_h)_+$ becomes an irreducible super representation W_{uguh} , on which $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ acts as

$$\begin{aligned} \Gamma_{uguh}[(w_g)_+ \otimes (w_h)_+ - (w_g)_- \otimes (w_h)_-] &= (w_g)_+ \otimes (w_h)_- - (w_g)_- \otimes (w_h)_+, \\ \Gamma_{uguh}[(w_g)_+ \otimes (w_h)_- - (w_g)_- \otimes (w_h)_+] &= (w_g)_+ \otimes (w_h)_+ - (w_g)_- \otimes (w_h)_-, \\ \Gamma'_{uguh}[(w_g)_+ \otimes (w_h)_+ - (w_g)_- \otimes (w_h)_-] &= (w_g)_+ \otimes (w_h)_- - (w_g)_- \otimes (w_h)_+, \\ \Gamma'_{uguh}[(w_g)_+ \otimes (w_h)_- - (w_g)_- \otimes (w_h)_+] &= -[(w_g)_+ \otimes (w_h)_+ - (w_g)_- \otimes (w_h)_-]. \end{aligned} \quad (\text{A.1.7})$$

Therefore, we have an even isomorphism of super representations

$$W_g \otimes W_h \cong V_{gh} \oplus \Pi V_{ghu} \oplus W_{uguh}. \quad (\text{A.1.8})$$

When only $uguh$ commutes with u , the subalgebra of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ that acts non-trivially on $W_g \otimes W_h$ is spanned by $\{\widehat{gh}, \widehat{ughu}, \widehat{ghu}, \widehat{ugh}\} \sqcup \{\widehat{uguh}, \widehat{guh}\}$. The action of \widehat{uguh} and \widehat{guh} are given by the last two equalities in eq. (A.1.5), which implies that the tensor product representation $W_g \otimes W_h$ contains one-dimensional super representations V_{uguh} and ΠV_{guh} . The remaining two-dimensional subspace spanned by $(w_g)_+ \otimes (w_h)_+ + (w_g)_- \otimes (w_h)_-$ and $(w_g)_+ \otimes (w_h)_- + (w_g)_- \otimes (w_h)_+$ is an irreducible super representation W_{gh} , on which $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ acts as

$$\begin{aligned}\Gamma_{gh}[(w_g)_+ \otimes (w_h)_+ + (w_g)_- \otimes (w_h)_-] &= (w_g)_+ \otimes (w_h)_- + (w_g)_- \otimes (w_h)_+, \\ \Gamma_{gh}[(w_g)_+ \otimes (w_h)_- + (w_g)_- \otimes (w_h)_+] &= (w_g)_+ \otimes (w_h)_+ + (w_g)_- \otimes (w_h)_-, \\ \Gamma'_{gh}[(w_g)_+ \otimes (w_h)_+ + (w_g)_- \otimes (w_h)_-] &= (w_g)_+ \otimes (w_h)_- + (w_g)_- \otimes (w_h)_+, \\ \Gamma'_{gh}[(w_g)_+ \otimes (w_h)_- + (w_g)_- \otimes (w_h)_+] &= -[(w_g)_+ \otimes (w_h)_+ + (w_g)_- \otimes (w_h)_-].\end{aligned}\tag{A.1.9}$$

Therefore, we have an even isomorphism of super representations

$$W_g \otimes W_h \cong W_{gh} \oplus V_{uguh} \oplus \Pi V_{guh}.\tag{A.1.10}$$

Finally, when both gh and $uguh$ do not commute with u , the subalgebra of $\mathcal{H}_{(\mathbb{C}[G]^*)^{\text{cop}}}^u$ that acts non-trivially on $W_g \otimes W_h$ is spanned by $\{\widehat{gh}, \widehat{ughu}, \widehat{ghu}, \widehat{ugh}\} \sqcup \{\widehat{uguh}, \widehat{guh}, \widehat{uguhu}, \widehat{guh}\}$. The action of this subalgebra on $W_g \otimes W_h$ is given by eqs. (A.1.7) and (A.1.9). Therefore, we have an even isomorphism of super representations

$$W_g \otimes W_h \cong W_{gh} \oplus W_{uguh}.\tag{A.1.11}$$

A.2 Fusion rules of $\text{sRep}(\mathcal{H}_8^u)$

The Hopf superalgebra \mathcal{H}_8^u has four one-dimensional super representations V_i labeled by $1 \leq i \leq 4$ and two two-dimensional super representations W_1 and W_2 . The one-dimensional super representation V_i is a super vector space $\mathbb{C}^{1|0}$ on which the idempotent $e_i \in \mathcal{H}_8^u$ acts as the identity and $e_j \in \mathcal{H}_8^u$ acts as zero if $j \neq i$. The two-dimensional super representation W_1 is a super vector space $\mathbb{C}^{1|1}$ on which the subalgebra of \mathcal{H}_8^u spanned by e_5 and e_7 acts non-trivially and $e_{j \neq 5,7}$ acts as zero. The action of e_5 and e_7 on W_1 is given by

$$e_5 \cdot w_{\pm} = w_{\pm}, \quad e_7 \cdot w_{\pm} = w_{\mp},\tag{A.2.1}$$

where w_+ and w_- are \mathbb{Z}_2 -even and \mathbb{Z}_2 -odd elements of W_1 . Similarly, the two-dimensional super representation W_2 is a super vector space $\mathbb{C}^{1|1}$ on which the subalgebra of \mathcal{H}_8^u spanned by e_6 and e_8 acts analogously to eq. (A.2.1) and $e_{j \neq 6,8}$ acts as zero.

In order to derive the fusion rules, we first compute the comultiplication of \mathcal{H}_8^u . A direct

computation shows that the comultiplication is given by

$$\begin{aligned}
\Delta(e_1) &= e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4 + \frac{1}{2}(e_5 \otimes e_5 - u(z)e_7 \otimes e_7) + \frac{1}{2}(e_6 \otimes e_6 + u(z)e_8 \otimes e_8), \\
\Delta(e_2) &= e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3 + \frac{1}{2}(e_5 \otimes e_5 + u(z)e_7 \otimes e_7) + \frac{1}{2}(e_6 \otimes e_6 - u(z)e_8 \otimes e_8), \\
\Delta(e_3) &= e_1 \otimes e_3 + e_2 \otimes e_4 + e_3 \otimes e_1 + e_4 \otimes e_2 + \frac{1}{2}(e_5 \otimes e_6 + e_6 \otimes e_5 + u(z)e_7 \otimes e_8 - u(z)e_8 \otimes e_7), \\
\Delta(e_4) &= e_1 \otimes e_4 + e_2 \otimes e_3 + e_3 \otimes e_2 + e_4 \otimes e_1 + \frac{1}{2}(e_5 \otimes e_6 + e_6 \otimes e_5 - u(z)e_7 \otimes e_8 + u(z)e_8 \otimes e_7), \\
\Delta(e_5) &= (e_1 + e_2) \otimes e_5 + (e_3 + e_4) \otimes e_6 + e_5 \otimes (e_1 + e_2) + e_6 \otimes (e_3 + e_4), \\
\Delta(e_6) &= (e_1 + e_2) \otimes e_6 + (e_3 + e_4) \otimes e_5 + e_5 \otimes (e_3 + e_4) + e_6 \otimes (e_1 + e_2), \\
\Delta(e_7) &= (e_1 - e_2) \otimes e_7 + (e_3 - e_4) \otimes e_8 + e_7 \otimes (e_1 - e_2) - e_8 \otimes (e_3 - e_4), \\
\Delta(e_8) &= (e_1 - e_2) \otimes e_8 + (e_3 - e_4) \otimes e_7 - e_7 \otimes (e_3 - e_4) + e_8 \otimes (e_1 - e_2).
\end{aligned} \tag{A.2.2}$$

Let us derive the fusion rules of irreducible super representations of \mathcal{H}_8^u based on the above expression of the comultiplication.

The fusion rule of V_i and V_j . An element $e_k \in \mathcal{H}_8^u$ acts non-trivially on the tensor product representation $V_i \otimes V_j$ only if the comultiplication $\Delta(e_k)$ contains $e_i \otimes e_j$. When e_k acts non-trivially on $V_i \otimes V_j$, this is the unique element (up to scalar multiplication) that acts non-trivially on $V_i \otimes V_j$ because the tensor product of one-dimensional super representations V_i and V_j is also one-dimensional. This gives an even isomorphism of super representations $V_i \otimes V_j \cong V_k$. More specifically, we have the following fusion rules:

$$V_1 \otimes V_i \cong V_i \otimes V_1 \cong V_i, \quad V_2 \otimes V_2 \cong V_3 \otimes V_3 \cong V_4 \otimes V_4 \cong V_1, \quad V_2 \otimes V_3 \cong V_3 \otimes V_2 \cong V_4. \tag{A.2.3}$$

The fusion rule of V_i and W_j . An element $e_k \in \mathcal{H}_8^u$ acts non-trivially on $V_i \otimes W_1$ only if the comultiplication $\Delta(e_k)$ contains $e_i \otimes e_5$ or $e_i \otimes e_7$. Similarly, an element $e_k \in \mathcal{H}_8^u$ acts non-trivially on $W_1 \otimes V_i$ only if the comultiplication $\Delta(e_k)$ contains $e_5 \otimes e_i$ or $e_7 \otimes e_i$. These facts completely determine the fusion rules of V_i and W_1 as follows:

$$\begin{aligned}
V_1 \otimes W_1 &\cong V_2 \otimes W_1 \cong W_1 \cong W_1 \otimes V_1 \cong W_1 \otimes V_2, \\
V_3 \otimes W_1 &\cong V_4 \otimes W_1 \cong W_2 \cong W_1 \otimes V_3 \cong W_1 \otimes V_4.
\end{aligned} \tag{A.2.4}$$

The fusion rules of V_i and W_2 are uniquely determined by the associativity of the fusion rules.

The fusion rule of W_i and W_j . We only need to consider the fusion of W_1 with itself because the associativity of the fusion rules uniquely determines the other fusion rules. An element $e_k \in \mathcal{H}_8^u$ acts non-trivially on $W_1 \otimes W_1$ only if the comultiplication $\Delta(e_k)$ contains at least one of $e_5 \otimes e_5$, $e_5 \otimes e_7$, $e_7 \otimes e_5$, and $e_7 \otimes e_7$. As we can see from eq. (A.2.2), the elements that satisfy this condition are e_1 and e_2 . Therefore, the tensor product representation $W_1 \otimes W_1$ only contains one-dimensional super representations V_1 , V_2 , and their oddly isomorphic variants ΠV_1 and ΠV_2 . More specifically, the non-trivial action of e_1 and e_2 on $W_1 \otimes W_1$ is given by

$$\begin{aligned}
\Delta(e_1) \cdot [w_+ \otimes w_+ - u(z)w_- \otimes w_-] &= w_+ \otimes w_+ - u(z)w_- \otimes w_-, \\
\Delta(e_1) \cdot [w_+ \otimes w_- - u(z)w_- \otimes w_+] &= w_+ \otimes w_- - u(z)w_- \otimes w_+, \\
\Delta(e_2) \cdot [w_+ \otimes w_+ + u(z)w_- \otimes w_-] &= w_+ \otimes w_+ + u(z)w_- \otimes w_-, \\
\Delta(e_2) \cdot [w_+ \otimes w_- + u(z)w_- \otimes w_+] &= w_+ \otimes w_- + u(z)w_- \otimes w_+.
\end{aligned} \tag{A.2.5}$$

This implies that we have an even isomorphism of super representations

$$W_1 \otimes W_1 \cong V_1 \oplus V_2 \oplus \Pi V_1 \oplus \Pi V_2. \quad (\text{A.2.6})$$

A.3 Fusion rules of $\text{sRep}(\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u)$

A weak Hopf superalgebra $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ has two three-dimensional super representations $V_g \cong \mathbb{C}^{2|1}$ labeled by $g \in \mathbb{Z}_2$ and two four-dimensional super representations $W_s \cong \mathbb{C}^{2|2}$ labeled by $s = \pm 1$. The actions of $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ on these super representations are defined by eqs. (5.3.44) and (5.3.45). In the following, we compute the fusion rules of these super representations.

The fusion rule of V_g and V_h The tensor product representation $V_g \boxtimes V_h$ is obtained by projecting the vector space $V_g \otimes V_h$ to the image of the action of the unit element $\eta(1) \in \mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$. Based on the definitions of the unit and the comultiplication of $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ given in Section 5.3.4, we can compute the action of $\eta(1)$ on $V_g \otimes V_h$ as

$$\begin{aligned} & \Delta(\eta(1)) \cdot \sum_{\alpha, \beta} c_{\alpha\beta} v_\alpha^g \otimes v_\beta^h \\ &= \sum_{a \in \mathbb{Z}_2} \frac{c_{1a} + (-1)^{\delta_{h,u}} c_{u,au}}{2} [v_1^g \otimes v_a^h + (-1)^{\delta_{h,u}} v_u^g \otimes v_{au}^h] + c_{mm} v_m^g \otimes v_m^h, \end{aligned} \quad (\text{A.3.1})$$

where $c_{\alpha\beta}$ is an arbitrary complex number. The above equation implies that the tensor product representation $V_g \boxtimes V_h$ is spanned by two \mathbb{Z}_2 -even elements $v_1^g \otimes v_1^h + (-1)^{\delta_{h,u}} v_u^g \otimes v_u^h$ and $v_m^g \otimes v_m^h$ and a \mathbb{Z}_2 -odd element $v_1^g \otimes v_u^h + (-1)^{\delta_{h,u}} v_u^g \otimes v_1^h$. Therefore, $V_g \boxtimes V_h \cong \mathbb{C}^{2|1}$ is a three-dimensional super representation, which is evenly isomorphic to either V_1 or V_u . In order to identify this super representation, we compute the action of $x_{mm}^l \in \mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ on $v_m^g \otimes v_m^h \in V_g \boxtimes V_h$:

$$\Delta(x_{mm}^l) \cdot v_m^g \otimes v_m^h = \delta_{l,gh} v_m^g \otimes v_m^h. \quad (\text{A.3.2})$$

This indicates that $V_g \boxtimes V_h$ contains a three-dimensional super representation V_{gh} . Since $V_g \boxtimes V_h$ itself is three-dimensional, we have an even isomorphism of super representations

$$V_g \boxtimes V_h \cong V_{gh}. \quad (\text{A.3.3})$$

The fusion rule of V_g and W_s We first consider the tensor product representation $V_g \boxtimes W_s$. The action of the unit $\eta(1)$ on a general element of $V_g \otimes W_s$ is computed as

$$\begin{aligned} & \Delta(\eta(1)) \cdot \sum_{\alpha, \beta, p} c_{\alpha\beta}^p v_\alpha^g \otimes (w_\beta^s)_p \\ &= \sum_p \left[\frac{c_{11}^p + c_{u1}^{-p}}{2} [v_1^g \otimes (w_1^s)_p + v_u^g \otimes (w_1^s)_{-p}] + c_{m1}^p v_m^g \otimes (w_1^s)_p \right], \end{aligned} \quad (\text{A.3.4})$$

where $c_{\alpha\beta}^p$ is an arbitrary complex number. The above equation implies that $V_g \boxtimes W_s$ is a four-dimensional super representation spanned by $\{v_1^g \otimes (w_1^s)_p + v_u^g \otimes (w_1^s)_{-p}, v_m^g \otimes (w_1^s)_p \mid p = \pm 1\}$. Hence, $V_g \boxtimes W_s$ is isomorphic to either W_+ or W_- . Furthermore, the action of $(x_{11}^{m,t})_+ \in \mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ on $v_m^g \otimes (w_1^s)_p \in V_g \boxtimes W_s$ is computed as

$$\Delta((x_{11}^{m,t})_+) \cdot v_m^g \otimes (w_1^s)_p = \frac{1 + st(-1)^{\delta_{g,u}}}{2} v_m^g \otimes (w_1^s)_p, \quad (\text{A.3.5})$$

which indicates that $V_g \boxtimes W_s$ contains a four-dimensional super representation $W_{s(-1)^{\delta_{g,u}}}$. Therefore, we have isomorphisms of super representations

$$V_1 \boxtimes W_s \cong W_s, \quad V_u \boxtimes W_s \cong W_{-s}. \quad (\text{A.3.6})$$

We can also compute the tensor product representation $W_s \boxtimes V_g$ similarly. The action of the unit $\eta(1)$ on a general element of $W_s \otimes V_g$ is given by

$$\begin{aligned} & \Delta(\eta(1)) \cdot \sum_{\alpha, \beta, p} c_{\alpha\beta}^p (w_\alpha^s)_p \otimes v_\beta^g \\ &= \sum_p \left[\frac{c_{11}^p + isp(-1)^{\delta_{g,u}} c_{1u}^{-p}}{2} [(w_1^s)_p \otimes v_1^g - isp(-1)^{\delta_{g,u}} (w_1^s)_{-p} \otimes v_u^g] + c_{1m}^p (w_1^s)_p \otimes v_m^g \right], \end{aligned} \quad (\text{A.3.7})$$

which implies that $W_s \boxtimes V_g$ is a four-dimensional super representation spanned by $\{(w_1^s)_p \otimes v_1^g - isp(-1)^{\delta_{g,u}} (w_1^s)_{-p} \otimes v_u^g, (w_1^s)_p \otimes v_m^g \mid p = \pm 1\}$. Furthermore, $W_s \boxtimes V_g$ contains a four-dimensional super representation $W_{s(-1)^{\delta_{g,u}}}$ because $(x_{11}^{m,t})_+$ acts on $(w_1^s)_p \otimes v_m^g \in W_s \boxtimes V_g$ as

$$\Delta((x_{11}^{m,t})_+) \cdot (w_1^s)_p \otimes v_m^g = \frac{1 + st(-1)^{\delta_{g,u}}}{2} (w_1^s)_p \otimes v_m^g. \quad (\text{A.3.8})$$

Therefore, we have isomorphisms of super representations

$$W_s \boxtimes V_1 \cong W_s, \quad W_s \boxtimes V_u \cong W_{-s}. \quad (\text{A.3.9})$$

The fusion rule of W_s and W_t The unit element $\eta(1)$ acts on a general element of $W_s \otimes W_t$ as

$$\begin{aligned} & \Delta(\eta(1)) \cdot \sum_{\alpha, \beta, p, q} c_{\alpha\beta}^{pq} (w_\alpha^s)_p \otimes (w_\beta^t)_q \\ &= \sum_{pq} c_{11}^{pq} (w_1^s)_p \otimes (w_1^t)_q + \sum_q \frac{c_{11}^{+,q} + isc_{11}^{-,-q}}{2} [(w_1^s)_+ \otimes (w_1^t)_q - is(w_1^s)_- \otimes (w_1^t)_{-q}], \end{aligned} \quad (\text{A.3.10})$$

This implies that $W_s \boxtimes W_t$ is a six-dimensional super representation spanned by $(w_1^s)_p \otimes (w_1^t)_q$ and $(w_1^s)_+ \otimes (w_1^t)_q - is(w_1^s)_- \otimes (w_1^t)_{-q}$ for $p, q = \pm 1$. Therefore, $W_s \boxtimes W_t$ can be decomposed into the direct sum of two three-dimensional super representations. Since $W_s \boxtimes W_t$ has an odd automorphism due to the fact that W_s and W_t are q-type objects, a six-dimensional super representation $W_s \boxtimes W_t$ is isomorphic to either $V_1 \oplus \Pi V_1$ or $V_u \oplus \Pi V_u$. This isomorphism is determined by computing the action of $\mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ on $W_s \boxtimes W_t$. Specifically, it turns out that $x_{11}^g \in \mathcal{H}_{\mathbb{Z}_2, \chi, \epsilon}^u$ acts non-trivially on $(w_1^s)_p \otimes (w_1^t)_q \in W_s \boxtimes W_t$ if $st = (-1)^{\delta_{g,u}}$, which indicates that $W_s \boxtimes W_t$ contains V_1 if $s = t$ and V_u if $s = -t$. Therefore, we find the following isomorphisms of super representations:

$$W_s \boxtimes W_s \cong V_1 \oplus \Pi V_1, \quad W_s \boxtimes W_{-s} \cong V_u \oplus \Pi V_u. \quad (\text{A.3.11})$$

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