

# 博士論文

## 論文題目

Geometric Structure of Affine Deligne-Lusztig Varieties for  $GL_n$   
( $GL_n$  のアファイン Deligne-Lusztig 多様体の幾何構造)

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# Geometric Structure of Affine Deligne-Lusztig Varieties for $\mathrm{GL}_n$

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## Abstract

The affine Deligne-Lusztig variety  $X_{\preceq\mu}(b)$ , attached to a reductive group  $G$  with a cocharacter  $\mu$  of  $G$ , is an arithmetic geometric object closely related to Shimura varieties. In this paper we study  $X_{\preceq\mu}(b)$  for  $G = \mathrm{GL}_n$  and basic  $b$ . We first study explicit construction of irreducible components of  $X_{\preceq\mu}(b)$  from crystal bases via the Chen-Zhu conjecture. As an application, we compare the  $\mathbb{J}$ -stratification (or the semi-module stratification) and the Ekedahl-Oort stratification of  $X_{\preceq\mu}(b)$  in the superbasic case. Interestingly, it turns out that the  $\mathbb{J}$ -stratification is finer than the Ekedahl-Oort stratification if and only if  $(\mathrm{GL}_n, \mu)$  satisfies a certain condition generalizing the cases of Coxeter type introduced by Görtz-He. We also show that the latter condition induces a certain simple geometric structure of  $X_{\preceq\mu}(b)$  for minuscule  $\mu$ .

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# 1 Introduction

The affine Deligne-Lusztig variety was introduced by Rapoport in [42], which plays an important role in understanding geometric and arithmetic properties of Shimura varieties. The uniformization theorem by Rapoport and Zink [41] allows us to describe the Newton strata of Shimura varieties in terms of Rapoport-Zink spaces, whose underlying spaces are special cases of affine Deligne-Lusztig varieties.

Let  $F$  be a non-archimedean local field with finite residue field  $\mathbb{F}_q$  of prime characteristic  $p$ , and let  $L$  be the completion of the maximal unramified extension of  $F$ . Let  $\sigma$  denote the Frobenius automorphism of  $L/F$ . Further, we write  $\mathcal{O}$  (resp.  $\mathcal{O}_F$ ) for the valuation ring of  $L$  (resp.  $F$ ). Finally, we denote by  $\varpi$  a uniformizer of  $F$  (and  $L$ ) and by  $v_L$  the valuation of  $L$  such that  $v_L(\varpi) = 1$ .

Let  $G$  be an unramified connected reductive group over  $\mathcal{O}_F$ . Let  $B \subset G$  be a Borel subgroup and  $T \subset B$  a maximal torus in  $B$ , both defined over  $\mathcal{O}_F$ . For  $\mu, \mu' \in X_*(T)$  (resp.  $X_*(T)_{\mathbb{Q}}$ ), we write  $\mu' \preceq \mu$  if  $\mu - \mu'$  is a non-negative integral (resp. rational) linear combination of positive coroots. For a cocharacter  $\mu \in X_*(T)$ , let  $\varpi^\mu$  be the image of  $\varpi \in \mathbb{G}_m(F)$  under the homomorphism  $\mu: \mathbb{G}_m \rightarrow T$ .

Set  $K = G(\mathcal{O})$ . We fix a dominant cocharacter  $\mu \in X_*(T)_+$  and  $b \in G(L)$ . Then the affine Deligne-Lusztig variety  $X_\mu(b)$  is the locally closed reduced  $\overline{\mathbb{F}}_q$ -subscheme of the affine Grassmannian  $\mathcal{G}r = G(L)/K$  defined as

$$X_\mu(b) = \{xK \in \mathcal{G}r \mid x^{-1}b\sigma(x) \in K\varpi^\mu K\}.$$

The closed affine Deligne-Lusztig variety is the closed reduced  $\overline{\mathbb{F}}_q$ -subscheme of  $\mathcal{G}r$  defined as

$$X_{\preceq \mu}(b) = \bigcup_{\mu' \preceq \mu} X_{\mu'}(b).$$

Both  $X_\mu(b)$  and  $X_{\preceq \mu}(b)$  are locally of finite type in the equal characteristic case and locally perfectly of finite type in the mixed characteristic case (cf. [24, Corollary 6.5], [23, Lemma 1.1]). Also, the affine Deligne-Lusztig varieties  $X_\mu(b)$  and  $X_{\preceq \mu}(b)$  carry a natural action (by left multiplication) by the  $\sigma$ -centralizer of  $b$

$$\mathbb{J} = \mathbb{J}_b = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}.$$

(Since  $b$  is usually fixed in the discussion, we mostly omit it from the notation.)

The geometric properties of affine Deligne-Lusztig varieties have been studied by many people. For example, the non-emptiness criterion and the dimension formula are already known for the affine Deligne-Lusztig varieties in the affine Grassmannian (see [11], [50] and [22]). Let  $B(G)$  denote the set of  $\sigma$ -conjugacy classes of  $G(L)$ . Thanks to Kottwitz [35], a  $\sigma$ -conjugacy class  $[b] \in B(G)$  is uniquely determined by two invariants: the Kottwitz point  $\kappa(b) \in \pi_1(G)/((1 - \sigma)\pi_1(G))$  and the Newton

point  $\nu_b \in X_*(T)_{\mathbb{Q},+}$ . Set  $B(G, \mu) = \{[b] \in B(G) \mid \kappa(b) = \kappa(\varpi^\mu), \nu_b \preceq \mu^\diamond\}$ , where  $\mu^\diamond$  denotes the  $\sigma$ -average of  $\mu$ . Then  $X_\mu(b) \neq \emptyset$  if and only if  $[b] \in B(G, \mu)$ . If this is the case, then we have

$$\dim X_\mu(b) = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2} \text{def}(b),$$

where  $\rho$  is the half sum of positive roots and  $\text{def}(b)$  is the defect of  $b$ . It is also known that (closed) affine Deligne-Lusztig varieties are equidimensional (cf. [25], [48]). Note that the dimension formula and the equidimensionality imply that  $X_{\preceq \mu}(b)$  is actually the closure of  $X_\mu(b)$  in  $\mathcal{G}r$  (cf. [24, Corollary 1.3]).

Besides there exists a description of  $\mathbb{J} \backslash \text{Irr } X_\mu(b)$  in terms of crystal bases, where  $\text{Irr } X_\mu(b)$  denotes the set of irreducible components of  $X_\mu(b)$ . Let  $\widehat{G}$  be the Langlands dual of  $G$  defined over  $\overline{\mathbb{Q}}_l$  with  $l \neq p$ . Let  $V_\mu$  denote the irreducible  $\widehat{G}$ -module of highest weight  $\mu$ . The crystal basis  $\mathbb{B}_\mu$  of  $V_\mu$  was first constructed by Kashiwara and Lusztig (cf. [31]). In  $X_*(T)$ , there is a distinguished element  $\lambda_b$  determined by  $b$ . It is the “best integral approximation” of the Newton vector of  $b$ , but we omit the precise definition. For this, see [23, §2.1] (in fact, [23, Example 2.3] is enough for our purpose). In [40], Nie proved that there exists a natural bijection between  $\mathbb{J} \backslash \text{Irr } X_\mu(b)$  and the  $\lambda_b$ -weight space  $\mathbb{B}_\mu(\lambda_b)$ . In particular,  $|\mathbb{J} \backslash \text{Irr } X_\mu(b)| = \dim V_\mu(\lambda_b)$ . The proof is reduced to the case where  $G = \text{GL}_n$  and  $b$  is superbasic. So this case is particularly important. This theorem is first conjectured by Miaofen Chen and Xinwen Zhu. Before the work by Nie, Xiao-Zhu [53] proved the conjecture under the assumption that  $b$  is unramified (or equivalently,  $\text{def}(b) = 0$ ), and Hamacher-Viehmann [23] proved the minuscule case. The numerical version is also proved by Rong Zhou and Yihang Zhu in [55].

In Xiao-Zhu’s proof, they constructed irreducible components of  $X_\mu(b)$  from crystal elements of  $\mathbb{B}_\mu(\lambda_b)$  using the crystal structure of  $\mathbb{B}_\mu$ . On the other hand, Nie constructed a map from  $\mathbb{J} \backslash \text{Irr } X_\mu(b)$  to  $\mathbb{B}_\mu(\lambda_b)$  and proved that it is bijective. The first main theorem of this paper states that such explicit construction of irreducible components as Xiao-Zhu [53] is also possible in the case where  $G = \text{GL}_n$  and  $b$  is superbasic (i.e.,  $\text{def}(b) = n - 1$ ). From now on, we fix (a representative in  $G(L)$  of a) length 0 element  $\tau_\mu$  in the Iwahori-Weyl group  $\tilde{W}$  of  $T$  whose  $\sigma$ -conjugacy class in  $G(L)$  is the unique basic element in  $B(G, \mu)$ .

**Theorem A** (Corollary 5.5). Let  $G = \text{GL}_n$ . Assume that  $\tau_\mu$  is superbasic. Then, using the crystal structure of  $\mathbb{B}_\mu$ , we can construct each irreducible component of  $X_\mu(\tau_\mu)$  from the corresponding crystal element. In other words, we construct the inverse map of the natural bijection by Nie.

A crystal is a finite set with a weight map  $\text{wt}$  and Kashiwara operators satisfying certain conditions; see §4. For more details on the construction, see §5.2. It is known

that in the superbasic case,  $\mathbb{J} \backslash \text{Irr } X_\mu(\tau_\mu)$  can be also parametrized by semi-modules for  $\mu$  in an explicit way (see Theorem 5.1). In [40, Remark 0.10], Nie pointed out that it would be interesting to give a direct correspondence between (extended) semi-modules and crystal elements. We will prove Theorem A answering this question.

It is natural to expect that the same as [53] and Theorem A holds in general. We shall explore the generalization as a future work by reducing the general case to Theorem A following the strategy in [40]. This would suggest that the crystal structure of  $\mathbb{B}_\mu$ , introduced by Kashiwara in the representation theory of quantum groups, knows the geometry of affine Deligne-Lusztig varieties  $X_\mu(b)$ . This is not only interesting, but also useful. For example, Fox-Imai [10] applied the construction in [53] to study the geometric structure of irreducible components of  $X_\mu(\tau_\mu)$  for an unramified unitary group of signature  $(2, n-2)$  and  $\mu = (1, 1, 0, \dots, 0)$ . They essentially made use of some information from corresponding crystal elements. Also, Theorem A would be also useful to determine the type of the stabilizer in  $\mathbb{J}$  of each irreducible component for general  $G$  and  $b$  (this is pointed out to the author by Nie).

In this paper, we also have another application of Theorem A, which we will explain from now. Via the relationship to Shimura varieties, or more directly to Rapoport-Zink spaces, the results on the geometry of affine Deligne-Lusztig varieties have numerous applications to number theory (e.g., the Kudla-Rapoport program [36], Zhang’s Arithmetic Fundamental Lemma [54], . . .). Many of these applications make use of the special cases where  $X_{\leq \mu}(b)$  admits a simple description. The fully Hodge-Newton decomposable case, introduced by Görtz, He and Nie in [18], is one of such cases. They proved that if  $(G, \mu)$  is fully Hodge-Newton decomposable, then  $X_{\leq \mu}(\tau_\mu)$  is naturally a union of (classical) Deligne-Lusztig varieties (in fact, they studied the cases with arbitrary parahoric level). This stratification is the so-called weak Bruhat-Tits stratification, a stratification indexed in terms of the Bruhat-Tits building of  $\mathbb{J}$  (which exists only in the fully Hodge-Newton decomposable case). The case of Coxeter type is a special case of this case such that each Deligne-Lusztig variety appearing in this stratification is of Coxeter type (cf. [19, §2.3]). In this case, we drop the “weak” above. For example, the cases of Coxeter type include the case for certain unitary groups of signature  $(1, n-1)$  studied in [52] by Vollaard and Wedhorn, which has been used in [36] and [54].

To give a conceptual way to explain the relationship between the geometry of affine Deligne-Lusztig varieties and the Bruhat-Tits building of  $\mathbb{J}$  indicated by above examples, Chen and Viehmann [4] introduced the  $\mathbb{J}$ -stratification. The  $\mathbb{J}$ -strata are locally closed subsets of  $\mathcal{G}r$ . By intersecting each  $\mathbb{J}$ -stratum with  $X_{\leq \mu}(b)$ , we obtain the  $\mathbb{J}$ -stratification of  $X_{\leq \mu}(b)$  (see §2.3 for details). In [14], Görtz showed that the Bruhat-Tits stratification coincides with the  $\mathbb{J}$ -stratification. In fact the Bruhat-Tits stratification is a refinement of the Ekedahl-Oort stratification (see §2.2 for the latter). So the  $\mathbb{J}$ -stratification is also a refinement of the Ekedahl-Oort stratification

when  $(G, \mu)$  is of Coxeter type. This does not hold in general even if  $\mu$  is minuscule. See [4, Example 4.1] for a counterexample in the case  $G = \mathrm{GL}_9$ . Therefore the cases when  $\mathbb{J}$ -stratification is a refinement of the Ekedahl-Oort stratification should be special cases, which are of particular interest.

Usually it seems very difficult to study the  $\mathbb{J}$ -stratification. However, in the case that  $G = \mathrm{GL}_n$  and  $b$  is superbasic, the  $\mathbb{J}$ -stratification coincides with a stratification by semi-modules ([4, Proposition 3.4]). As an application of Theorem A, we compare the Ekedahl-Oort stratification and the semi-module stratification. To state the main results, we need some notation. Let  $W_0$  be the (finite) Weyl group of  $T$  in  $G$  and let  $\tilde{W}$  be the Iwahori-Weyl group of  $T$  in  $G$ . Then  $\tilde{W} = X_*(T) \rtimes W_0$ . We denote the projection  $\tilde{W} \rightarrow W_0$  by  $p$ . For  $\mu \in X_*(T)_+$ , we denote by  $\mathrm{Adm}(\mu)$  the admissible subset of  $\tilde{W}$ . Let  ${}^S\mathrm{Adm}(\mu)$  be a certain subset of  $\mathrm{Adm}(\mu)$ , which is the index set of the Ekedahl-Oort stratification of  $X_{\leq \mu}(\tau_\mu)$  (see §2.2). Finally, let  $\mathrm{LP}(w) \subseteq W_0$  be the length positive elements for  $w$  (see §2.5).

**Theorem B** (Theorem 9.2). Let  $G = \mathrm{GL}_n$  and let  $\mu \in X_*(T)_+$ . Assume that  $\tau_\mu$  is superbasic. Then the following assertions are equivalent.

- (i) For any  $w \in {}^S\mathrm{Adm}(\mu)$  whose corresponding Ekedahl-Oort stratum is non-empty, there exists  $v \in \mathrm{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element.
- (ii) The  $\mathbb{J}$ -stratification (or the semi-module stratification) of  $X_{\leq \mu}(\tau_\mu)$  gives a refinement of the Ekedahl-Oort stratification in the sense that every Ekedahl-Oort stratum is a union of some  $\mathbb{J}$ -strata.

In a joint work [45] with Schremmer and Yu, we proved that (i) implies a simple geometric structure on each Ekedahl-Oort stratum of  $X_{\leq \mu}(\tau_\mu)$  (for general  $G$ ). Since the case of Coxeter type satisfies (ii) as explained above, (i) is also a generalization of Coxeter type. In fact, this follows directly from [45, Theorem 4.12] (if  $G = \mathrm{GL}_n$ ). So Theorem B tells us that these two conditions which contain the cases of Coxeter type are actually equivalent at least in the superbasic case.

It is hard to study the condition (ii) in general because the  $\mathbb{J}$ -stratification is quite complicated outside the superbasic case. On the other hand, the condition (i) is much easier to study. In this paper, we classify  $\mu$  satisfying (i) for  $G = \mathrm{GL}_n$  (including the non-superbasic case):

**Theorem C** (Theorem 10.11). Let  $G = \mathrm{GL}_n$  and let  $\mu \in X_*(T)_+$ . Then the following assertions are equivalent.

- (i) For any  $w \in {}^S\mathrm{Adm}(\mu)$  whose corresponding Ekedahl-Oort stratum is non-empty, there exists  $v \in \mathrm{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element.

(ii) The cocharacter  $\mu$  is central or one of the following forms modulo  $\mathbb{Z}\omega_n$ :

$$\begin{aligned}
& \omega_1, \quad \omega_{n-1}, & (n \geq 1), \\
& \omega_1 + \omega_{n-1}, \quad \omega_2, \quad 2\omega_1, \quad \omega_{n-2}, \quad 2\omega_{n-1}, \\
& \omega_2 + \omega_{n-1}, \quad 2\omega_1 + \omega_{n-1} \quad \omega_1 + \omega_{n-2}, \quad \omega_1 + 2\omega_{n-1}, & (n \geq 3), \\
& \omega_3, \quad \omega_{n-3}, & (n = 6, 7, 8), \\
& 3\omega_1, \quad 3\omega_{n-1}, & (n = 4, 5), \\
& \omega_1 + \omega_2, \quad \omega_3 + \omega_4, & (n = 5), \\
& 4\omega_1, \quad \omega_1 + 3\omega_2, \quad 4\omega_2, \quad 3\omega_1 + \omega_2, & (n = 3), \\
& m\omega_1 \text{ with } m \in \mathbb{Z}_{>0}, & (n = 2).
\end{aligned}$$

Here  $\omega_k$  denotes the cocharacter of the form  $(1, \dots, 1, 0, \dots, 0)$  in which 1 is repeated  $k$  times.

In [46], the author studied a similar condition to (i) for  $G = \mathrm{GL}_n$ . The condition requires that for any  $w \in {}^S\mathrm{Adm}(\mu)$  whose corresponding Ekedahl-Oort stratum is non-empty,  $p(w)$  is a Coxeter element. After this work, Schremmer [43] defined a notion of length positive elements in his thesis. The condition (i) is a natural generalization of the condition considered in [46] by length positive elements.

Recently, Chen-Tong [3] introduced the weak full Hodge-Newton decomposability in the context of  $p$ -adic Hodge theory and studied it under the minuscule condition. The weakly admissible locus  $\mathcal{F}(G, \mu, \tau_\mu)^{wa}$  inside the flag variety  $\mathcal{F}(G, \mu)$ , attached to  $G$  with a minuscule cocharacter  $\mu$ , is a vast generalization of the Drinfeld upper half plane. The admissible locus  $\mathcal{F}(G, \mu, \tau_\mu)^a \subseteq \mathcal{F}(G, \mu, \tau_\mu)^{wa}$  is a  $p$ -adic analogue of the complex analytic period spaces. Surprisingly, by Chen-Fargues-Shen [2],  $(G, \mu)$  is fully Hodge-Newton decomposable if and only if  $\mathcal{F}(G, \mu, \tau_\mu)^a = \mathcal{F}(G, \mu, \tau_\mu)^{wa}$ . If this is the case, then the Newton stratification of  $\mathcal{F}(G, \mu)$  gives a refinement of the Harder-Narashimhan stratification (see [3, §1.4.3] for these stratifications). The main result of [3] states that the weak full Hodge-Newton decomposability, which is a generalization of the full Hodge-Newton decomposability by definition, is equivalent to the condition that the Newton stratification is finer than the Harder-Narashimhan stratification. They also classified the weakly fully Hodge-Newton decomposable cases. The classification tells us that for minuscule  $\mu$ ,  $(\mathrm{GL}_n, \mu)$  is weakly fully Hodge-Newton decomposable if and only if  $\tau_\mu$  is superbasic or  $(\mathrm{GL}_n, \mu)$  satisfies the equivalent conditions in Theorem C. In [3, Remark 2.16], they pointed out that it will be an interesting question to investigate the basic affine Deligne-Lusztig varieties associated to a weakly fully Hodge-Newton decomposable pair. For the superbasic case, the geometry of  $X_\mu(\tau_\mu)$  for  $\mathrm{GL}_n$  is already studied in [51] to some extent. In this paper, we answer Chen-Tong's question for cocharacters in Theorem C.



**Theorem D** (Theorem 10.12). Let  $G = \mathrm{GL}_n$ . Let  $\mu$  be a minuscule cocharacter satisfying the equivalent conditions in Theorem C. Then the  $\mathbb{J}$ -stratification of  $X_\mu(\tau_\mu)$  gives a refinement of the Ekedahl-Oort stratification. Each  $\mathbb{J}$ -stratum is universally homeomorphic to the product of a Deligne-Lusztig variety of Coxeter type and a finite-dimensional affine space. Moreover, the closure relation can be described in terms of the rational Bruhat-Tits building of  $\mathbb{J}$ .

If  $\mu$  is minuscule, then the affine Deligne-Lusztig variety  $X_\mu(b)(= X_{\preceq\mu}(b))$  for  $\mathrm{GL}_n$  can be considered as the underlying space of the Rapoport-Zink space for  $\mathrm{GL}_n$  (cf. [13, §4.9]). So this case is particularly important for the application towards number theory. In general, the  $\mathbb{J}$ -stratification is a stratification only in the loose sense that it is a decomposition into disjoint locally closed subsets. On the other hand, the (weak) Bruhat-Tits stratification satisfies the closure relation, i.e., the closure of each stratum is a union of strata. Moreover, this union can be described explicitly in terms of the rational Bruhat-Tits building of  $\mathbb{J}$ . The closure relation in Theorem D is a natural generalization of this description (see §2.3).

After we finished this work, Schremmer informed the author that there is an upcoming work by Schremmer, He and Viehmann which also aims at generalizing the fully Hodge-Newton decomposable case. For a pair  $(G, \mu)$ , they define a non-negative rational number  $\mathrm{depth}(G, \mu)$ . Then it is known that  $(G, \mu)$  is fully Hodge-Newton decomposable if and only if  $\mathrm{depth}(G, \mu) \leq 1$  (cf. [18, Definition 3.2]). They classified the cases where  $1 < \mathrm{depth}(G, \mu) < 2$ . The classification can be reduced to the case where  $G$  is simple. If  $G$  is a simple group with  $1 < \mathrm{depth}(G, \mu) < 2$  for some  $\mu$ , then  $G$  is a split group over  $F$  of Dynkin type  $A_{n-1}$ ,  $C_3$  or  $D_5$ . If  $G = \mathrm{GL}_n$ , then their case is contained in our case (both cases completely coincide if  $n \geq 6$ ). They also studied some geometric properties of the corresponding Ekedahl-Oort strata in a different point of view from this paper.

It is natural to expect that a similar description of affine Deligne-Lusztig varieties as in Theorem D also exists for  $(G, \mu)$  in the classification by Chen-Tong and Schremmer-He-Viehmann. Also, it is worth mentioning that there are some other  $(G, \mu)$  such that the corresponding basic affine Deligne-Lusztig variety admits a certain simple description (which is less explicit than Theorem D, though). For example, the works by Fox-Imai [10] (see also [9]) and Trentin [49] are such cases. Interestingly, both cases have  $\mathrm{depth}(G, \mu) = 2$ . We hope to generalize our approach for  $\mathrm{GL}_n$  in the future to study these cases (more) explicitly.

This paper is organized as follows. In §2 we introduce the affine Deligne-Lusztig variety and stratifications of it. We also recall some geometric properties of affine Deligne-Lusztig varieties, which will be frequently used throughout this paper. In §3 and §4, we recollect known results on semi-modules and crystal bases respectively. In §5, we first recall a known result on the relationship between semi-modules and

crystal bases for the minuscule case. After that, we state a precise way of constructing top (extended) semi-modules from crystal elements. Also in §5, we prove that there exists a “non-cyclic” crystal elements in many cases using combinatorics on Young tableaux (Theorem 5.14). This is based on a realization of  $\mathbb{B}_\mu$  in terms of Young tableaux by Kashiwara-Nakashima (see §4.1). In §6, we prove Theorem A in a combinatorial way again. In §7 and §8, we examine the semi-module stratification and the Ekedahl-Oort stratification respectively by an explicit calculation of semi-modules and elements in  ${}^S\mathrm{Adm}(\mu)$ . In §9, we prove Theorem B, combining Theorem A, Theorem 5.14 and the results in §7 and §8. Finally in §10, we finish the classification in Theorem C by calculating the non-superbasic case. Based on this calculation, we prove Theorem D using the results in §2.

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## 2 Preliminaries

Keep the notation in §1. From now on, we sometimes drop the adjective “perfect” for notational convenience even in the mixed characteristic case. Also  $\cong$  always means a universal homeomorphism.

### 2.1 Notation

Let  $\Phi = \Phi(G, T)$  denote the set of roots of  $T$  in  $G$ . We denote by  $\Phi_+$  (resp.  $\Phi_-$ ) the set of positive (resp. negative) roots distinguished by  $B$ . Let  $\Delta$  be the set of simple roots and  $\Delta^\vee$  be the corresponding set of simple coroots. Let  $X_*(T)$  be the set of cocharacters, and let  $X_*(T)_+$  be the set of dominant cocharacters.

The Iwahori-Weyl group  $\tilde{W}$  is defined as the quotient  $N_{G(L)}T(L)/T(\mathcal{O})$ . This can be identified with the semi-direct product  $W_0 \ltimes X_*(T)$ , where  $W_0$  is the finite Weyl group of  $G$ . We denote the projection  $\tilde{W} \rightarrow W_0$  by  $p$ . Let  $S \subset W_0$  denote the subset of simple reflections, and let  $\tilde{S} \subset \tilde{W}$  denote the subset of simple affine reflections. We often identify  $\Delta$  and  $S$ . The affine Weyl group  $W_a$  is the subgroup

of  $\tilde{W}$  generated by  $\tilde{S}$ . Then we can write the Iwahori-Weyl group as a semi-direct product  $\tilde{W} = W_a \rtimes \Omega$ , where  $\Omega \subset \tilde{W}$  is the subgroup of length 0 elements. Moreover,  $(W_a, \tilde{S})$  is a Coxeter system. We denote by  $\leq$  the Bruhat order on  $\tilde{W}$ . For any  $J \subseteq \tilde{S}$ , let  ${}^J\tilde{W}$  be the set of minimal length elements for the cosets in  $W_J \backslash \tilde{W}$ , where  $W_J$  denotes the subgroup of  $\tilde{W}$  generated by  $J$ . We also have a length function  $\ell: \tilde{W} \rightarrow \mathbb{Z}_{\geq 0}$  given as

$$\ell(w_0 \varpi^\lambda) = \sum_{\alpha \in \Phi_+, w_0 \alpha \in \Phi_-} |\langle \alpha, \lambda \rangle + 1| + \sum_{\alpha \in \Phi_+, w_0 \alpha \in \Phi_+} |\langle \alpha, \lambda \rangle|,$$

where  $w_0 \in W_0$  and  $\lambda \in X_*(T)$ .

Let  $w \in \tilde{W}$ . There exists a positive integer  $k$  such that  $w^k = \varpi^\lambda$  for some  $\lambda \in X_*(T)$ . We set  $\nu_w = \lambda/k \in X_*(T)_{\mathbb{Q}}$ . This is independent of the choice of  $k$ .

For  $w \in W_a$ , we denote by  $\text{supp}(w) \subseteq \tilde{S}$  the set of simple affine reflections occurring in every (equivalently, some) reduced expression of  $w$ . Note that  $\tau \in \Omega$  acts on  $\tilde{S}$  by conjugation. We define the  $\sigma$ -support  $\text{supp}_\sigma(w\tau)$  of  $w\tau$  as the smallest  $\tau\sigma$ -stable subset of  $\tilde{S}$  which contains  $\text{supp}(w)$ . We call an element  $w\tau \in W_a\tau$  a  $\sigma$ -Coxeter element if exactly one simple reflection from each  $\tau\sigma$ -orbit on  $\text{supp}_\sigma(w\tau)$  occurs in every (equivalently, any) reduced expression of  $w$ .

For  $\tau \in \Omega$ , let  $\Sigma$  be an orbit of  $\tau\sigma$  on  $\tilde{S}$  and suppose that  $W_\Sigma$  is finite. We denote by  $s_\Sigma$  the unique longest element of  $W_\Sigma$ . Then  $s_\Sigma$  is fixed by  $\tau\sigma$ . The fixed point group  $W_a^{\tau\sigma} := \{w \in W_a \mid \tau\sigma(w)\tau^{-1} = w\}$  is the Weyl group whose simple reflections are the elements  $s_\Sigma$  such that  $\Sigma$  is a  $\tau\sigma$ -orbit on  $\tilde{S}$  with  $W_\Sigma$  finite. For any reduced decomposition  $w = s_{\Sigma_1} \cdots s_{\Sigma_r}$  as an element of  $W_a^{\tau\sigma}$ , we have

$$\ell(w) = \ell(s_{\Sigma_1}) + \cdots + \ell(s_{\Sigma_r}).$$

See [47] or [34, §2] for these facts.

For  $w, w' \in \tilde{W}$  and  $s \in \tilde{S}$ , we write  $w \xrightarrow{s}_\sigma w'$  if  $w' = sw\sigma(s)$  and  $\ell(w') \leq \ell(w)$ . We write  $w \rightarrow_\sigma w'$  if there is a sequence  $w = w_0, w_1, \dots, w_k = w'$  of elements in  $\tilde{W}$  such that for any  $i$ ,  $w_{i-1} \xrightarrow{s_i}_\sigma w_i$  for some  $s_i \in \tilde{S}$ . If  $w \rightarrow_\sigma w'$  and  $w' \rightarrow_\sigma w$ , we write  $w \approx_\sigma w'$ .

For  $\alpha \in \Phi$ , let  $U_\alpha \subseteq G$  denote the corresponding root subgroup. We set

$$I = T(\mathcal{O}) \prod_{\alpha \in \Phi_+} U_\alpha(\varpi \mathcal{O}) \prod_{\beta \in \Phi_-} U_\beta(\mathcal{O}) \subseteq G(L),$$

which is called the standard Iwahori subgroup associated to the triple  $T \subset B \subset G$ .

In the case  $G = \text{GL}_n$ , we will use the following description. Let  $\chi_{ij}$  be the character  $T \rightarrow \mathbb{G}_m$  defined by  $\text{diag}(t_1, t_2, \dots, t_n) \mapsto t_i t_j^{-1}$ . Then we have  $\Phi = \{\chi_{ij} \mid i \neq j\}$ ,  $\Phi_+ = \{\chi_{ij} \mid i < j\}$ ,  $\Phi_- = \{\chi_{ij} \mid i > j\}$  and  $\Delta = \{\chi_{i, i+1} \mid 1 \leq i < n\}$ . Through the isomorphism  $X_*(T) \cong \mathbb{Z}^n$ ,  $X_*(T)_+$  can be identified with

the set  $\{(m_1, \dots, m_n) \in \mathbb{Z}^n \mid m_1 \geq \dots \geq m_n\}$ . Let us write  $s_1 = (1 \ 2), s_2 = (2 \ 3), \dots, s_{n-1} = (n-1 \ n)$ . Set  $s_0 = \varpi^{\chi_{1,n}}(1 \ n)$ , where  $\chi_{1,n}$  is the unique highest root. Then  $S = \{s_1, s_2, \dots, s_{n-1}\}$  and  $\tilde{S} = S \cup \{s_0\}$ . The Iwahori subgroup  $I \subset K$  is the inverse image of the lower triangular matrices under the projection  $G(\mathcal{O}) \rightarrow G(\overline{\mathbb{F}}_q)$ ,  $\varpi \mapsto 0$ . Set  $\tau = \begin{pmatrix} 0 & \varpi \\ 1_{n-1} & 0 \end{pmatrix}$ . We often regard  $\tau$  as an element of  $\tilde{W}$ , which is a generator of  $\Omega \cong \mathbb{Z}$ . Note that  $b \in \mathrm{GL}_n(L)$  is superbasic if and only if  $[b] = [\tau^m]$  in  $B(\mathrm{GL}_n)$  for some  $m$  coprime to  $n$ .

## 2.2 Affine Deligne-Lusztig Varieties

For  $w \in \tilde{W}$  and  $b \in G(L)$ , the affine Deligne-Lusztig variety  $X_w(b)$  in the affine flag variety  $\mathcal{Fl} = G(L)/I$  is defined as

$$X_w(b) = \{xI \in G(L)/I \mid x^{-1}b\sigma(x) \in IwI\}.$$

For  $\mu \in X_*(T)_+$  and  $b \in G(L)$ , the affine Deligne-Lusztig variety  $X_\mu(b)$  in the affine Grassmannian  $\mathcal{Gr} = G(L)/K$  is defined as

$$X_\mu(b) = \{xK \in \mathcal{Gr} \mid x^{-1}b\sigma(x) \in K\varpi^\mu K\}.$$

The closed affine Deligne-Lusztig variety is the closed reduced  $\overline{\mathbb{F}}_q$ -subscheme of  $\mathcal{Gr}$  defined as

$$X_{\preceq \mu}(b) = \bigcup_{\mu' \preceq \mu} X_{\mu'}(b).$$

Left multiplication by  $g^{-1} \in G(L)$  induces an isomorphism between  $X_\mu(b)$  and  $X_\mu(g^{-1}b\sigma(g))$ . Thus the isomorphism class of the affine Deligne-Lusztig variety only depends on the  $\sigma$ -conjugacy class of  $b$ . Moreover, we have  $X_\mu(b) = X_{\mu+\lambda}(\varpi^\lambda b)$  for each central  $\lambda \in X_*(T)$ .

The admissible subset of  $\tilde{W}$  associated to  $\mu$  is defined as

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W} \mid w \leq \varpi^{w_0\mu} \text{ for some } w_0 \in W_0\}.$$

Note that  $\mathrm{Adm}(\mu') \subseteq \mathrm{Adm}(\mu)$  if  $\mu' \preceq \mu$  (see [21, Lemma 4.5]). Set  ${}^S\mathrm{Adm}(\mu) = \mathrm{Adm}(\mu) \cap {}^S\tilde{W}$ . Then, by [16, Theorem 3.2.1] (see also [20, §2.5]), we have

$$X_{\preceq \mu}(b) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)} \pi(X_w(b)),$$

where  $\pi: G(L)/I \rightarrow G(L)/K$  is the projection. This is the so-called Ekedahl-Oort stratification. In the sequel, we set  ${}^S\mathrm{Adm}(\mu)_0 := \{w \in {}^S\mathrm{Adm}(\mu) \mid X_w(\tau_\mu) \neq \emptyset\}$ , where  $\tau_\mu \in \Omega$  such that  $[\tau_\mu] \in B(G, \mu)$  is the unique basic element.

For any  $w \in {}^S\tilde{W}$ , set

$$Z(w) := \{w_0 \in W_0 \mid w_0 w = w w_0\}.$$

**Lemma 2.1.** Let  $\varpi^\mu y \in {}^S\tilde{W}$  with  $\mu$  dominant and  $y \in W_0$ . Assume that  $Z(\varpi^\mu y) = \{1\}$ . Then the projection map  $\pi: X_{\varpi^\mu y}(b) \rightarrow X_\mu(b)$  is injective.

*Proof.* The proof is similar to [29, Lemma 5.4]. We may assume that  $X_{\varpi^\mu y}(b) \neq \emptyset$ . Let  $gI, g'I \in X_{\varpi^\mu y}(b)$  such that  $\pi(gI) = \pi(g'I)$ . Then  $g'^{-1}g \in K$  and hence  $g'^{-1}g \in IxI$  for some  $x \in W_0$ . Since  $(g'^{-1}g)(g^{-1}b\sigma(g)) = (g'^{-1}b\sigma(g'))(\sigma(g'^{-1}g))$ , we have  $(IxI)(I\varpi^\mu yI) \cap (I\varpi^\mu yI)(IxI) \neq \emptyset$ . Note that  $(IxI)(I\varpi^\mu yI) = Ix\varpi^\mu yI$  because  $\varpi^\mu y \in {}^S\tilde{W}$ . This implies that  $x\varpi^\mu y = \varpi^\mu yx$ . By our assumption, we must have  $x = 1$  and hence  $g'^{-1}g \in I$  as desired.  $\square$

**Example 2.2.** Let  $G = \mathrm{GL}_n$  and let  $\varpi^\mu y \in {}^S\tilde{W}$  with  $\mu$  dominant and  $y \in W_0$ . If  $y$  is an  $n$ -cycle and  $\{s_1, s_{n-1}\} \not\subseteq Z(\varpi^\mu)$ , then we have  $Z(\varpi^\mu y) = \{1\}$ . Indeed, for any  $x \in W_0$ ,  $x\varpi^\mu y = \varpi^\mu yx$  implies that  $xyx^{-1} = y$  and  $x \in Z(\varpi^\mu)$ . Thus  $x = y^k$  for some  $0 \leq k \leq n-1$  and  $y^k\mu = \mu$ . Since  $\{s_1, s_{n-1}\} \not\subseteq Z(\varpi^\mu)$ , we must have  $k = 0$ .

The following proposition will be used in §10.

**Proposition 2.3.** Let  $\tau \in \Omega$ . Let  $w \in W_a\tau$  such that  $W_{\mathrm{supp}_\sigma(w)}$  is finite. Then

$$X_w(\tau) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_{\mathrm{supp}_\sigma(w)}} jY(w),$$

where  $P_{\mathrm{supp}_\sigma(w)}$  is the parahoric subgroup corresponding to  $\mathrm{supp}_\sigma(w)$  and  $Y(w) = \{gI \in P_{\mathrm{supp}_\sigma(w)}/I \mid g^{-1}\tau\sigma(g) \in IwI\}$  is a classical Deligne-Lusztig variety in the finite-dimensional flag variety  $P_{\mathrm{supp}_\sigma(w)}/I$ .

*Proof.* See [16, Proposition 2.2.1].  $\square$

## 2.3 The $\mathbb{J}$ -stratification

For any  $g, h \in G(L)$ , let  $\mathrm{inv}(g, h)$  (resp.  $\mathrm{inv}_K(g, h)$ ) denote the relative position, i.e., the unique element in  $\tilde{W}$  (resp.  $X_*(T)_+$ ) such that  $g^{-1}h \in I\mathrm{inv}(g, h)I$  (resp.  $K\varpi^{\mathrm{inv}_K(g, h)}K$ ). By definition, two elements  $gI, hI \in \mathcal{F}l$  (resp.  $gK, hK \in \mathcal{G}r$ ) lie in the same  $\mathbb{J}$ -stratum if and only if for all  $j \in \mathbb{J}$ ,  $\mathrm{inv}(j, g) = \mathrm{inv}(j, h)$  (resp.  $\mathrm{inv}_K(j, g) = \mathrm{inv}_K(j, h)$ ). Clearly, this does not depend on the choice of  $g, h$ . By [14, Theorem 2.10], the  $\mathbb{J}$ -strata are locally closed in  $\mathcal{F}l$  (resp.  $\mathcal{G}r$ ). By intersecting each  $\mathbb{J}$ -stratum with (closed) affine Deligne-Lusztig varieties, we obtain the  $\mathbb{J}$ -stratification of them.

As explained in [4, Remark 2.1], the  $\mathbb{J}$ -stratification heavily depends on the choice of  $b$  in its  $\sigma$ -conjugacy class. So we need to fix a specific representative to compare the  $\mathbb{J}$ -stratification on  $X_\mu(b)$  (or  $X_{\leq \mu}(b)$ ) to another stratification. It is pointed out loc. cit that if  $b$  is basic, then a reasonable choice is the unique length 0 element in  $B(G, \mu)$ . Also, for any  $w \in \tilde{W}$ , the  $\mathbb{J}_w$ -stratification is independent of the choice of a lift in  $G(L)$ . See [14, Lemma 2.5].

In general, the  $\mathbb{J}$ -stratification is too complicated to study. However, there are several cases such that the  $\mathbb{J}$ -stratification coincides with other well-known stratifications. Here we briefly recall two of such cases.

In the case where  $G = \mathrm{GL}_n$  and  $b = \tau^m$  with  $m$  coprime to  $n$ , there is a group-theoretic way to describe the  $\mathbb{J}$ -stratification, which we will call the semi-module stratification. Indeed, by [4, Remark 3.1 & Proposition 3.4], the  $\mathbb{J}$ -stratification on  $\mathcal{G}r$  coincides with the stratification

$$G(L)/K = \bigsqcup_{\lambda \in X_*(T)} I\varpi^\lambda K/K.$$

So in this case, each  $\mathbb{J}$ -stratum of  $X_\mu(b)$  (resp.  $X_{\preceq \mu}(b)$ ) coincides with  $X_\mu^\lambda(b)$  (resp.  $X_{\preceq \mu}^\lambda(b)$ ) for some  $\lambda \in X_*(T)$ , where  $X_\mu^\lambda(b) = X_\mu(b) \cap I\varpi^\lambda K/K$  (resp.  $X_{\preceq \mu}^\lambda(b) = X_{\preceq \mu}(b) \cap I\varpi^\lambda K/K$ ). Set  $\mathbb{J}^0 = \mathbb{J} \cap K = \mathbb{J} \cap I$ . Note that  $\tau X_\mu^\lambda(b) = X_\mu^{\tau\lambda}(b)$  and  $\mathbb{J}/\mathbb{J}^0 = \{\tau^k \mathbb{J}^0 \mid k \in \mathbb{Z}\}$ . Thus

$$\mathbb{J}X_\mu^\lambda(b) = \bigsqcup_{k \in \mathbb{Z}} X_\mu^{\tau^k \lambda}(b) \quad \text{and} \quad \mathbb{J}X_{\preceq \mu}^\lambda(b) = \bigsqcup_{k \in \mathbb{Z}} X_{\preceq \mu}^{\tau^k \lambda}(b).$$

See §3.1 for the precise definition of (extended) semi-modules. As we will explain in §3.2, the set  $\{\lambda \in X_*(T) \mid X_\mu^\lambda(b) \neq \emptyset\}$  can be regarded as semi-modules for  $\mu$ . Let  $w_{\max}$  be the longest element in  $W_0$ . Then we have

$$\{\lambda \in X_*(T) \mid X_{-w_{\max}\mu}^\lambda(b^{-1}) \neq \emptyset\} = \{-w_{\max}\lambda \in X_*(T) \mid X_\mu^\lambda(b) \neq \emptyset\}.$$

Indeed it is easy to check that the image of  $X_\mu^\lambda(b)$  under the automorphism of  $\mathcal{G}r$  by  $gK \mapsto w_{\max}^{-1}g^{-1}K$  is  $X_{-w_{\max}\mu}^{-w_{\max}\lambda}(b^{-1})$ . This gives the description of “dual” semi-modules for  $\mu$ .

Following [19, §2.3], we say that  $(G, \mu)$  is of Coxeter type if

$${}^S\mathrm{Adm}(\mu)_0 = \{w \in {}^S\mathrm{Adm}(\mu) \mid W_{\mathrm{supp}_\sigma(w)} \text{ is finite, and } w \text{ is } \sigma\text{-Coxeter in } W_{\mathrm{supp}_\sigma(w)}\}.$$

If  $(G, \mu)$  is of Coxeter type, then for each  $w \in {}^S\mathrm{Adm}(\mu)_0$ , we have

$$\pi(X_w(\tau_\mu)) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_w} j\pi(Y(w)),$$

where  $Y(w)$  is the same as in Proposition 2.3, and  $P_w$  is the parahoric subgroup corresponding to  $\mathrm{supp}(w) \cup \max\{J \subseteq S \mid \mathrm{Ad}(w)(\sigma(J)) = J\}$  (cf. [18, Proposition 5.7]). Moreover, we have  $Y(w) \cong \pi(Y(w))$ . Thus if  $(G, \mu)$  is of Coxeter type, we obtain the decomposition  $X_{\preceq \mu}(\tau_\mu)$  as a union of classical Deligne-Lusztig varieties of Coxeter type in a natural way. We call this stratification the *Bruhat-Tits stratification*. Also, this is a stratification in the strong sense, i.e., the closure of a stratum is a union of strata. The closure of a stratum  $j\pi(Y(w))$  contains a stratum  $j'\pi(Y(w'))$  if and only if the following two conditions are both satisfied:

- (1)  $w \geq_{S,\sigma} w'$ , which means by definition that there exists  $u \in W_0$  such that  $w \geq u^{-1}w'\sigma(u)$ .
- (2)  $j(\mathbb{J} \cap P_w) \cap j'(\mathbb{J} \cap P_{w'}) \neq \emptyset$ .

Let  $\mathcal{B}(\mathbb{J}, F)$  denote the rational Bruhat-Tits building of  $\mathbb{J}$ . Then (2) above is equivalent to requiring that  $\kappa(j) = \kappa(j')$  and that the simplices in  $\mathcal{B}(\mathbb{J}, F)$  corresponding to  $j(\mathbb{J} \cap P_w)j^{-1}$  and  $j'(\mathbb{J} \cap P_{w'})j'^{-1}$  are neighbors (i.e., there exists an alcove which contains both of them). In [14], Görtz proved that the Bruhat-Tits stratification coincides with the  $\mathbb{J}$ -stratification.

Unlike the Bruhat-Tits stratification, the  $\mathbb{J}$ -stratification always exists. So the  $\mathbb{J}$ -stratification is expected to play an important role to go beyond the cases of Coxeter type. In this paper, we will treat the case such that the  $\mathbb{J}$ -stratification of  $X_{\preceq \mu}(b)$  is a refinement of the Ekedahl-Oort stratification. More precisely, we treat the case satisfying all of the following conditions:

- For  $w \in {}^S\text{Adm}(\mu)_0$ ,  $\mathbb{J}$  acts transitively on the set of irreducible components of  $X_w(\tau_\mu)$ .
- For  $w \in {}^S\text{Adm}(\mu)_0$ , there exist a parahoric subgroup  $P_w \subset G(L)$  and an irreducible component  $Y(w)$  of  $X_w(\tau_\mu)$  such that  $\pi(X_w(\tau_\mu)) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_w} j\pi(Y(w))$ .
- For  $w \in {}^S\text{Adm}(\mu)_0$ ,  $Y(w) \cong \pi(Y(w))$  and each  $j\pi(Y(w))$  is a  $\mathbb{J}$ -stratum of  $X_{\preceq \mu}(b)$ .

In this case, we say that the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$  if the  $\mathbb{J}$ -stratification of  $X_{\preceq \mu}(b)$  is a stratification in the strong sense and  $j\pi(Y(w)) \supseteq j'\pi(Y(w'))$  is equivalent to the following condition:

There exist sequences  $w = w_0 \geq_{S,\sigma} w_1 \geq_{S,\sigma} \cdots \geq_{S,\sigma} w_k = w'$  in  ${}^S\text{Adm}(\mu)_0$  and  $j = j_0, j_1, \dots, j_k = j'$  in  $\mathbb{J}$  such that  $j_{i-1}(\mathbb{J} \cap P_{w_{i-1}}) \cap j_i(\mathbb{J} \cap P_{w_i}) \neq \emptyset$  for  $1 \leq i \leq k$ .

## 2.4 Deligne-Lusztig Reduction Method

The following Deligne-Lusztig reduction method was established in [15, Corollary 2.5.3].

**Proposition 2.4.** Let  $w \in \tilde{W}$  and let  $s \in \tilde{S}$  be a simple affine reflection. If  $\text{ch}(F) > 0$ , then the following two statements hold for any  $b \in G(L)$ .

- (i) If  $\ell(sw\sigma(s)) = \ell(w)$ , then there exists a  $\mathbb{J}$ -equivariant universal homeomorphism  $X_w(b) \rightarrow X_{sw\sigma(s)}(b)$ .

(ii) If  $\ell(sw\sigma(s)) = \ell(w) - 2$ , then there exists a decomposition  $X_w(b) = X_1 \sqcup X_2$  such that

- $X_1$  is open and there exists a  $\mathbb{J}$ -equivariant morphism  $X_1 \rightarrow X_{sw}(b)$ , which is the composition of a Zariski-locally trivial  $\mathbb{G}_m$ -bundle and a universal homeomorphism.
- $X_2$  is closed and there exists a  $\mathbb{J}$ -equivariant morphism  $X_2 \rightarrow X_{sw\sigma(s)}(b)$ , which is the composition of a Zariski-locally trivial  $\mathbb{A}^1$ -bundle and a universal homeomorphism.

Let  $gI \in X_w(b)$ . If  $\ell(sw) < \ell(w)$  (even in the case (i), we can reduce to this case by exchanging  $s$  and  $sw\sigma(s)$ ), then let  $g_1I$  denote the unique element in  $\mathcal{F}l$  such that  $\text{inv}(g, g_1) = s$  and  $\text{inv}(g_1, b\sigma(g)) = sw$ . The set  $X_1$  (resp.  $X_2$ ) above consists of the elements  $gI \in X_w(b)$  satisfying  $\text{inv}(g_1, b\sigma(g_1)) = sw$  (resp.  $sw\sigma(s)$ ). All of the maps in the proposition are given as the map sending  $gI$  to  $g_1I$ .

**Remark 2.5.** Assume that  $G$  is split over  $F$ . Let  $\text{pr}_I: \mathcal{F}l \times \mathcal{F}l \rightarrow \mathcal{F}l$  be the projection to the first factor. We denote by  $O(s) \subset \mathcal{F}l \times \mathcal{F}l$  the locally closed subvariety of pairs  $(gI, hI)$  such that  $\text{inv}(g, h) = s$ . Then the restriction  $\text{pr}_I: O(s) \rightarrow \mathcal{F}l$  is a Zariski-locally trivial  $\mathbb{A}^1$ -bundle. More precisely, this is trivial over (any translation of) the “big open cell” (cf. [7, pp. 45–48]). In particular, this is trivial over any Schubert cell  $IvI/I, v \in \tilde{W}$ . This implies that the morphism  $X_1 \rightarrow X_{sw}(b)$  (resp.  $X_2 \rightarrow X_{sw\sigma(s)}(b)$ ) in (ii) is trivial over  $X_{sw}(b) \cap IvI/I$  (resp.  $X_{sw\sigma(s)}(b) \cap IvI/I$ ).

The following result is proved in [28, Theorem 2.10], which allows us to reduce the study of  $X_w(b)$  for any  $w$ , via the Deligne-Lusztig reduction method, to the study of  $X_w(b)$  for  $w$  of minimal length in its  $\sigma$ -conjugacy class.

**Theorem 2.6.** For each  $w \in \tilde{W}$ , there exists an element  $w'$  which is of minimal length inside its  $\sigma$ -conjugacy class such that  $w \rightarrow_\sigma w'$ .

Following [29, §3.4], we construct the reduction trees for  $w$  by induction on  $\ell(w)$ .

The vertices of the trees are elements of  $\tilde{W}$ . We write  $x \rightarrow y$  if  $x, y \in \tilde{W}$  and there exists  $x' \in \tilde{W}$  and  $s \in \tilde{S}$  such that  $x \approx_\sigma x'$ ,  $\ell(sx'\sigma(s)) = \ell(x') - 2$  and  $y \in \{sx', sx'\sigma(s)\}$ . These are the (oriented) edges of the trees.

If  $w$  is of minimal length in its  $\sigma$ -conjugacy class of  $\tilde{W}$ , then the reduction tree for  $w$  consists of a single vertex  $w$  and no edges. Assume that  $w$  is not of minimal length and that a reduction tree is given for any  $z \in \tilde{W}$  with  $\ell(z) < \ell(w)$ . By Theorem 2.6, there exist  $w'$  and  $s \in \tilde{S}$  with  $w \approx_\sigma w'$  and  $\ell(sw'\sigma(s)) = \ell(w') - 2$ . Then a reduction tree of  $w$  consists of the given reduction trees of  $sw'$  and  $sw'\sigma(s)$  and the edges  $w \rightarrow sw'$  and  $w \rightarrow sw'\sigma(s)$ .



Let  $\mathcal{T}$  be a reduction tree of  $w$ . An end point of  $\mathcal{T}$  is a vertex in  $\mathcal{T}$  of minimal length. A reduction path in  $\mathcal{T}$  is a path  $\underline{p}: w \rightarrow w_1 \rightarrow \cdots \rightarrow w_n$ , where  $w_n$  is an end point of  $\mathcal{T}$ . Set  $\text{end}(\underline{p}) = w_n$ . We say that  $x \rightarrow y$  is of type I (resp. II) if  $\ell(x) - \ell(y) = 1$  (resp.  $\ell(x) - \ell(y) = 2$ ). For any reduction path  $\underline{p}$ , we denote by  $\ell_I(\underline{p})$  (resp.  $\ell_{II}(\underline{p})$ ) the number of type I (resp. II) edges in  $\underline{p}$ . We write  $X_{\underline{p}}$  for a locally closed subscheme of  $X_w(b)$  which is  $\mathbb{J}$ -equivariant universally homeomorphic to an iterated fibration of type  $(\ell_I(\underline{p}), \ell_{II}(\underline{p}))$  over  $X_{\text{end}(\underline{p})}(b)$ .

Let  $B(\tilde{W}, \sigma)$  be the set of  $\sigma$ -conjugacy classes in  $\tilde{W}$ . Let  $\Psi: B(\tilde{W}, \sigma) \rightarrow B(G)$  be the map sending  $[w] \in B(\tilde{W}, \sigma)$  to  $[w] \in B(G)$ . It is known that this map is well-defined and surjective; see [27, Theorem 3.7]. By [29, Proposition 3.9], we have the following description of  $X_w(b)$ .

**Proposition 2.7.** Let  $w \in \tilde{W}$  and  $\mathcal{T}$  be a reduction tree of  $w$ . For any  $b \in G(L)$ , there exists a decomposition

$$X_w(b) = \bigsqcup_{\substack{\underline{p} \text{ is a reduction path in } \mathcal{T}; \\ \Psi(\text{end}(\underline{p})) = [b]}} X_{\underline{p}}.$$

In the case that  $G = \text{GL}_n$  and  $b = \tau^m$  with  $m$  coprime to  $n$ , we can count the number of top irreducible components and rational points of  $X_w(b)^0 = \{gI \in X_w(b) \mid \kappa(g) = v_L(\det(g)) = 0\}$  using the reduction tree for  $w$ . By [28, Proposition 3.5], the  $\sigma$ -conjugacy class of  $\tau^m$  in  $\tilde{W}$  is the unique element in  $B(\tilde{W}, \sigma)$  which maps to  $[\tau^m] \in B(G)$  under  $\Psi$ . Note also that  $\tau^m$  is the unique minimal length element in its  $\sigma$ -conjugacy class. We define a polynomial as

$$F_{w,b} := \sum_{\underline{p}} (\mathbf{q} - 1)^{\ell_I(\underline{p})} \mathbf{q}^{\ell_{II}(\underline{p})} \in \mathbb{N}[\mathbf{q} - 1],$$

where  $\underline{p}$  runs over all the reduction paths in  $\mathcal{T}$  with  $\text{end}(\underline{p}) = \tau^m$ .

**Proposition 2.8.** Assume that  $G = \text{GL}_n$  and  $b = \tau^m$  with  $m$  coprime to  $n$ . Let  $w \in \tilde{W}$  and let  $\mathcal{T}$  be a reduction tree of  $w$ . Then the number of top irreducible components of  $X_w(b)^0$  is equal to the leading coefficient of  $F_{w,b}$ . Moreover, we have

$$|X_w(b)^{0,\sigma}| = F_{w,b}|_{\mathbf{q}=q}.$$

*Proof.* Note that each  $\mathbb{J}$ -orbit of an irreducible component of  $X_w(b)$  can be represented by an irreducible component of  $X_w(b)^0$ . Moreover, it is known that the stabilizer in  $\mathbb{J}$  is a parahoric subgroup (cf. [55, Proposition 3.1.4]), i.e.,  $\mathbb{J} \cap I = \{g \in \mathbb{J} \mid \kappa(g) = 0\}$ . Then the statement follows from [29, Theorem 3.4 & Proposition 3.5] and [30, Corollary 4.4].  $\square$

**Remark 2.9.** The polynomials  $F_{w,b}$  are called *class polynomials*. However, the definition above is an ad hoc one. See [29, §3] for the definition in general and the connection to reduction trees.

## 2.5 Length Positive Elements

We denote by  $\delta^+$  the indicator function of the set of positive roots, i.e.,

$$\delta^+ : \Phi \rightarrow \{0, 1\}, \quad \alpha \mapsto \begin{cases} 1 & (\alpha \in \Phi_+) \\ 0 & (\alpha \in \Phi_-). \end{cases}$$

Note that any element  $w \in \tilde{W}$  can be written in a unique way as  $w = x\varpi^\mu y$  with  $\mu$  dominant,  $x, y \in W_0$  such that  $\varpi^\mu y \in {}^S\tilde{W}$ . We have  $p(w) = xy$  and  $\ell(w) = \ell(x) + \langle \mu, 2\rho \rangle - \ell(y)$ . We define the set of *length positive* elements by

$$\text{LP}(w) = \{v \in W_0 \mid \langle v\alpha, y^{-1}\mu \rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha) \geq 0 \text{ for all } \alpha \in \Phi_+\}.$$

Then we always have  $y^{-1} \in \text{LP}(w)$ . Indeed,  $y$  is uniquely determined by the condition that  $\langle \alpha, \mu \rangle \geq \delta^+(-y^{-1}\alpha)$  for all  $\alpha \in \Phi_+$ . Since  $\delta^+(\alpha) + \delta^+(-\alpha) = 1$ , we have

$$\langle y^{-1}\alpha, y^{-1}\mu \rangle + \delta^+(y^{-1}\alpha) - \delta^+(x\alpha) = \langle \alpha, \mu \rangle - \delta^+(-y^{-1}\alpha) + \delta^+(-x\alpha) \geq 0.$$

**Lemma 2.10.** For any  $w = x\varpi^\mu y \in \tilde{W}$  as above, we define

$$\Phi_w := \{\alpha \in \Phi_+ \mid \langle \alpha, \mu \rangle - \delta^-(y^{-1}\alpha) + \delta^-(x\alpha) = 0\}.$$

Here  $\delta^-$  denotes the indicator function of the set of negative roots. Then we have

$$y \text{LP}(w) = \{r^{-1} \in W_0 \mid r(\Phi_+ \setminus \Phi_w) \subset \Phi_+ \text{ or equivalently, } r^{-1}\Phi_+ \subset \Phi_+ \cup -\Phi_w\}.$$

*Proof.* Let  $r \in W_0$  such that  $r(\Phi_+ \setminus \Phi_w) \subset \Phi_+$ . Let  $\alpha \in \Phi_+$ . If  $r^{-1}\alpha \in \Phi_+$ , then we can check that  $y^{-1}r^{-1} \in \text{LP}(w)$  similarly as the case  $r = 1$  above. If  $r^{-1}\alpha \in \Phi_-$ , then we must have  $r^{-1}\alpha \in -\Phi_w$ . Since  $\delta^-(-\alpha) = \delta^+(\alpha)$ , it follows that

$$\begin{aligned} & \langle y^{-1}r^{-1}\alpha, y^{-1}\mu \rangle + \delta^+(y^{-1}r^{-1}\alpha) - \delta^+(xr^{-1}\alpha) \\ &= -(\langle -r^{-1}\alpha, \mu \rangle - \delta^-(-y^{-1}r^{-1}\alpha) + \delta^-(-xr^{-1}\alpha)) = 0. \end{aligned}$$

Thus  $y^{-1}r^{-1} \in \text{LP}(w)$ . This shows  $\{r^{-1} \in W_0 \mid r(\Phi_+ \setminus \Phi_w) \subset \Phi_+\} \subseteq y \text{LP}(w)$ .

Let  $v \in \text{LP}(w)$  and let  $\alpha \in \Phi_+$ . If  $yv\alpha \in \Phi_-$ , then

$$\langle -yv\alpha, \mu \rangle - \delta^-(-v\alpha) + \delta^-(-xyv\alpha) = -(\langle v\alpha, y^{-1}\mu \rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha)) \leq 0.$$

On the other hand, by the characterization of  $y$  above, we have

$$\langle -yv\alpha, \mu \rangle - \delta^-(-v\alpha) + \delta^-(-xyv\alpha) = \langle -yv\alpha, \mu \rangle - \delta^+(v\alpha) + \delta^+(xyv\alpha) \geq 0.$$

Thus  $\langle -yv\alpha, \mu \rangle - \delta^-(-v\alpha) + \delta^-(-xyv\alpha) = 0$  and hence  $yv\alpha \in -\Phi_w$ . This shows  $y \text{LP}(w) \subseteq \{r^{-1} \in W_0 \mid r(\Phi_+ \setminus \Phi_w) \subset \Phi_+\}$ . The proof is finished.  $\square$

The notion of length positive elements is defined by Schremmer in [43]. The description of  $\text{LP}(w)$  in Lemma 2.10 is due to Lim [37].

We say that the Dynkin diagram of  $G$  is  $\sigma$ -connected if it cannot be written as a union of two proper  $\sigma$ -stable subdiagrams that are not connected to each other. The following theorem is a refinement of the non-emptiness criterion in [17], which is conjectured by Lim in [37] and proved by Schremmer in [44, Proposition 5].

**Theorem 2.11.** Assume that the Dynkin diagram of  $G$  is  $\sigma$ -connected. Let  $b \in G(L)$  be a basic element with  $\kappa(b) = \kappa(w)$ . Then  $X_w(b) = \emptyset$  if and only if both of the following two conditions are satisfied:

- (i)  $|W_{\text{supp}_\sigma(w)}|$  is not finite.
- (ii) There exists  $v \in \text{LP}(w)$  such that  $\text{supp}_\sigma(\sigma^{-1}(v)^{-1}p(w)v) \subsetneq S$ .

**Remark 2.12.** If  $\kappa(b) \neq \kappa(w)$ , then  $X_w(b) = \emptyset$ .

**Remark 2.13.** Let  $w \in \tilde{W}$ ,  $w_0 \in W_0$  and let  $J \subseteq \Delta$  such that  $J = \sigma(J)$ . Then we say that  $w$  is a  $(J, w_0, \sigma)$ -alcove element if the following conditions are both satisfied:

- 1.  $w_0^{-1}w\sigma(w_0) \in \tilde{W}_J := X_*(T) \rtimes W_J$ , and
- 2. For any  $\alpha \in w_0(\Phi_+ \setminus \Phi_J)$ ,  $U_\alpha \cap wIw^{-1} \subseteq U_\alpha \cap I$ , where  $\Phi_J$  denotes the root system generated by  $J$ .

In [44, Proposition 5], the condition (ii) in Theorem 2.11 is written as

- (ii)' There exist  $J \subsetneq \Delta$  and  $w_0 \in W_0$  such that  $w$  is a  $(J, w_0, \sigma)$ -alcove element.

The equivalence of (ii) and (ii)' follows from [37, Lemma 3.7 & Lemma 3.9] (see also [45, Definition 2.3] and the comment right after it).

In the case  $G = \text{GL}_n$ , there exists a length-preserving automorphism  $\varsigma$  of  $\tilde{W}$  defined as

$$w_0\varpi^\lambda \mapsto w_{\max}w_0w_{\max}^{-1}\varpi^{-w_{\max}\lambda}, \quad w_0 \in W_0, \lambda \in X_*(T).$$

Note that  $\varsigma(\tau^m) = \tau^{-m}$ ,  $\varsigma(s_0) = s_0$  and  $\varsigma(s_i) = s_{n-i}$  for  $1 \leq i \leq n-1$ . Let  $w = x\varpi^\mu y$  be as above. For any  $\alpha \in \Phi_+$  and  $v \in \text{LP}(w)$ , we have

$$\begin{aligned} & \langle \varsigma(v)(-w_{\max}\alpha), \varsigma(y^{-1})(-w_{\max}\mu) \rangle + \delta^+(\varsigma(v)(-w_{\max}\alpha)) - \delta^+(\varsigma(xy)\varsigma(v)(-w_{\max}\alpha)) \\ &= \langle v\alpha, y^{-1}\mu \rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha) \geq 0. \end{aligned}$$

Thus  $\text{LP}(\varsigma(w)) = \varsigma(\text{LP}(w)) = w_{\max}\text{LP}(w)w_{\max}^{-1}$ . In particular, there exists  $v \in \text{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element if and only if the same is true for  $\varsigma(w)$  and  $\text{LP}(\varsigma(w))$ .

### 3 Semi-Modules

From now and until the end of this paper, we set  $G = \mathrm{GL}_n$  and  $b = \tau^m$ . For  $\mu \in X_*(T)_+$ , let  $\mu(i)$  denotes the  $i$ -th entry of  $\mu$ . Then  $[\tau^m] \in B(G, \mu)$  if and only if  $m = \mu(1) + \cdots + \mu(n)$ . We assume this from now (i.e.,  $b = \tau^m = \tau_\mu$ ). Also, without loss of generality, we may and will assume that  $\mu(n) = 0$ . Finally, we assume that  $b$  is superbasic (i.e.,  $m$  is coprime to  $n$ ) except §10.

#### 3.1 Extended Semi-Modules

Here we recall the definition of extended semi-modules in a combinatorial way from [50]. Note that although we choose the subgroup of upper triangular matrices  $B$  as a Borel subgroup in this paper, the fixed Borel subgroup in [50] is the subgroup of lower triangular matrices.

**Definition 3.1.** A *semi-module* for  $m, n$  is a subset  $A \subset \mathbb{Z}$  that is bounded below and satisfies  $m + A \subset A$  and  $n + A \subset A$ . Set  $\bar{A} = A \setminus (n + A)$ . The semi-module  $A$  is called normalized if  $\sum_{a \in \bar{A}} a = \frac{n(n-1)}{2}$ .

For a semi-module  $A$ , there exists a unique  $\mu' \in \mathbb{N}^n$  satisfying the following condition: Let  $a_0 = \min \bar{A}$  and let inductively  $a_i = a_{i-1} + m - \mu'(i)n$  for  $i = 1, \dots, n$ . Then  $a_0 = a_n$  and  $\{a_0, a_1, \dots, a_{n-1}\} = \bar{A}$ . We call  $\mu'$  the *type* of  $A$ .

**Lemma 3.2.** There is a bijection between the set of normalized semi-modules for  $m, n$  and the set of possible types  $\mu' \in \mathbb{N}^n$  with  $\nu_b \preceq w_{\max} \mu'$ .

*Proof.* This is [50, Lemma 3.3]. □

**Definition 3.3.** An *extended semi-module*  $(A, \varphi)$  for  $\mu \in X_*(T)_+$  is a normalized semi-module  $A$  for  $m, n$  together with a function  $\varphi: \mathbb{Z} \rightarrow \mathbb{N} \cup \{-\infty\}$  satisfying the following properties:

- (1)  $\varphi(a) = -\infty$  if and only if  $a \notin A$ .
- (2)  $\varphi(a + n) \geq \varphi(a) + 1$  for all  $a \in \mathbb{Z}$ .
- (3)  $\varphi(a) \leq \max\{k \mid a + m - kn \in A\}$  for all  $a \in A$ . If  $b \in A$  for all  $b \geq a$ , then the two sides are equal.
- (4) There is a decomposition of  $A$  into disjoint union of sequences  $a_j^1, \dots, a_j^n$  with  $j \in \mathbb{N}$  and the following properties:
  - (a)  $\varphi(a_{j+1}^l) = \varphi(a_j^l) + 1$ .

- (b) If  $\varphi(a_j^l + n) = \varphi(a_j^l) + 1$ , then  $a_{j+1}^l = a_j^l + n$ . Otherwise  $a_{j+1}^l > a_j^l + n$ .
- (c) The  $n$ -tuple  $(\varphi(a_0^l))$  is a permutation of  $\mu$ .

An extended semi-module such that the equality holds in (3) for all  $a \in A$  is called *cyclic*.

For any  $\lambda \in X_*(T)$ , we denote by  $\lambda_{\text{dom}}$  the dominant conjugate of  $\lambda$ . Let  $\mu'$  be the type of a semi-module for  $m, n$ . Let  $\varphi$  be a function such that (1) and the equation in (3) hold. Then it is easy to check that  $(A, \varphi)$  is a cyclic semi-module for  $\mu'_{\text{dom}}$ . In general, the following lemma holds.

**Lemma 3.4.** Let  $(A, \varphi)$  be an extended semi-module for  $\mu$  and let  $\mu'$  be the type of  $A$ . Then  $\mu'_{\text{dom}} \preceq \mu$  and  $(A, \varphi)$  is cyclic if and only if  $\mu' \in W_0\mu$ . Moreover, if  $\mu$  is minuscule, then all extended semi-modules for  $\mu$  are cyclic.

*Proof.* See [50, Lemma 3.6 & Corollary 3.7]. See also [22, Lemma 5.9].  $\square$

Let  $e_0, \dots, e_{n-1}$  be the standard basis of  $L^n$ . Then the lattice  $\mathcal{O}^n$  is generated by  $e_0, \dots, e_{n-1}$ . For  $i \in \mathbb{Z}$ , we define  $e_i$  by  $e_{i+n} = \varpi e_i$ . Note that we have  $\tau e_i = e_{i+1}$  for any  $i$ . In the sequel, we identify (the  $\overline{\mathbb{F}}_q$ -valued points of)  $\mathcal{G}r$  and  $\{M \subset L^n \text{ lattice}\}$  by  $gK \mapsto g\mathcal{O}^n$ .

Let  $X_\mu(b)^0$  be a  $\overline{\mathbb{F}}_q$ -subscheme of  $X_\mu(b)$  defined as  $X_\mu(b)^0 = \{gK \in X_\mu(b) \mid \kappa(g) = 0\}$ . We associate to  $M \in X_\mu(b)^0$  an extended semi-module for  $\mu$ . Let  $v \in L^n$ . Then we can write  $v = \sum_{i \in \mathbb{Z}} [\alpha_i] e_i$  with  $\alpha_i \in \overline{\mathbb{F}}_q$  and  $\alpha_i = 0$  for sufficiently small  $i$ . Here  $[\alpha_i]$  denotes the Teichmüller lift of  $\alpha_i$  if  $\text{ch } F = 0$  and  $[\alpha_i] = \alpha_i$  if  $\text{ch } F > 0$ . Let

$$\mathcal{I}: L^n \setminus \{0\} \rightarrow \mathbb{Z}, \quad v \mapsto \min\{i \mid \alpha_i \neq 0\}.$$

For  $M \in \mathcal{G}r$ , we define the set

$$A(M) = \{\mathcal{I}(v) \mid v \in M \setminus \{0\}\}.$$

It is easy to check that if  $M \in X_\mu(b)^0$ , then  $A(M)$  is a normalized semi-module for  $m, n$ . We also define  $\varphi(M): \mathbb{Z} \rightarrow \mathbb{N} \cup \{-\infty\}$  by

$$a \mapsto \begin{cases} \max\{k \mid \exists v \in M \setminus \{0\} \text{ with } \mathcal{I}(v) = a, \varpi^{-k} b \sigma(v) \in M\} & (a \in A(M)) \\ -\infty & (a \notin A(M)). \end{cases}$$

**Lemma 3.5.** Let  $M \in X_\mu(b)^0$ . Then  $(A(M), \varphi(M))$  is an extended semi-module for  $\mu$ .

*Proof.* See [50, Lemma 4.1].  $\square$

For an extended semi-module  $(A, \varphi)$  for  $\mu$ , let

$$S_{A,\varphi} = \{M \mid A(M) = A, \varphi(M) = \varphi\} \subset \mathcal{G}r.$$

**Lemma 3.6.** The set  $S_{A,\varphi}$  is a locally closed subscheme of  $X_\mu(b)^0$ .

*Proof.* See [50, Lemma 4.2].  $\square$

Let  $\mathbb{A}_\mu$  be the set of extended semi-modules for  $\mu$ . Set  $\mathbb{A}_\mu^{\text{top}} = \{(A, \varphi) \in \mathbb{A}_\mu \mid \dim S_{A,\varphi} = \dim X_\mu(b)\}$ . By Proposition 3.7 below,  $\mathbb{J} \setminus \text{Irr } X_\mu(b)$  is parametrized by  $\mathbb{A}_\mu^{\text{top}}$ . In the sequel, we also use the symbol  $\mathbb{A}$  to denote the affine space as usual. We hope our notation will not cause confusions.

For an extended semi-module  $(A, \varphi)$  for  $\mu$ , let

$$\mathcal{V}(A, \varphi) = \{(a, c) \in A \times A \mid c > a, \varphi(a) > \varphi(c) > \varphi(a - n)\}.$$

**Proposition 3.7.** Let  $(A, \varphi)$  be an extended semi-module for  $\mu$ . There exists a non-empty open subscheme  $U_{A,\varphi} \subseteq \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$  and a morphism  $U_{A,\varphi} \rightarrow S_{A,\varphi}$  which is bijective on  $\overline{\mathbb{F}}_q$ -valued points. In particular,  $S_{A,\varphi}$  is irreducible and of dimension  $|\mathcal{V}(A, \varphi)|$ . Moreover if  $(A, \varphi)$  is a cyclic extended semi-module, then  $U_{A,\varphi} = \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$ .

*Proof.* See [50, Theorem 4.3].  $\square$

Here we briefly describe  $U_{A,\varphi}$  and the map  $U_{A,\varphi} \rightarrow S_{A,\varphi}$ . For any  $x \in \overline{\mathbb{F}}_q^{|\mathcal{V}(A,\varphi)|} = \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$ , we denote the coordinate of  $x$  by  $x_{a,c}$ . We associate to every  $x$  a set of elements  $\{v(a) \in L^n \mid a \in A\}$  which satisfies the following equations.

If  $a = \max \bar{A}$ , then

$$v(a) = e_a + \sum_{(a,c) \in \mathcal{V}(A,\varphi)} [x_{a,c}]v(c).$$

For any other element  $a \in \bar{A}$ , we want

$$v(a) = v' + \sum_{(a,c) \in \mathcal{V}(A,\varphi)} [x_{a,c}]v(c),$$

where  $v' = \varpi^{-\varphi(a')}b\sigma(v(a'))$  for  $a'$  being minimal satisfying  $a' + m - \varphi(a')n = a$ . For  $a \in n + A$ , we want

$$v(a) = \varpi v(a - n) + \sum_{(a,c) \in \mathcal{V}(A,\varphi)} [x_{a,c}]v(c).$$

The set  $\{v(a) \in L^n \mid a \in A\}$  is uniquely determined by the equations above. Hence the map  $\mathbb{A}^{|\mathcal{V}(A,\varphi)|} \rightarrow \mathcal{G}r, x \mapsto \langle v(a) \rangle_{a \in A}$  is well-defined. By applying  $\sigma$  on the above

equations for  $x$ , we can easily check that this map is compatible with the action of  $\sigma$ , i.e.,  $\sigma(x) := (x_{a,c}^q)$  maps to  $\sigma\langle v(a) \rangle_{a \in A}$ . Let  $U_{A,\varphi}$  be the preimage of  $S_{A,\varphi}$  under this map. Then  $S_{A,\varphi}$  and hence  $U_{A,\varphi}$  are stable under  $\sigma$  (because  $\sigma(b) = b$ ). In particular, we have  $|S_{A,\varphi}^\sigma| = |U_{A,\varphi}^\sigma|$ . So if  $(A, \varphi)$  is cyclic, then  $|S_{A,\varphi}^\sigma| = q^{|\mathcal{V}(A,\varphi)|}$ . Although not needed in this paper, it is also worth mentioning that if  $(A, \varphi)$  is non-cyclic, then  $S_{A,\varphi}$  is never universally homeomorphic to an affine space.

**Proposition 3.8.** If  $(A, \varphi)$  is non-cyclic, then  $|S_{A,\varphi}^\sigma| < q^{|\mathcal{V}(A,\varphi)|}$ . In particular,  $S_{A,\varphi}$  is never universally homeomorphic to an affine space.

*Proof.* Let  $x \in \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$ . Note that if  $x_{a,c} = 0$  for all  $(a, c) \in \mathcal{V}(A, \varphi)$ , then  $v(a) = e_a$  for all  $a \in A$ . Set  $M = \langle e_a \rangle_{a \in A}$ . Then it is easy to check that  $(A(M), \varphi(M))$  is a cyclic semi-module for the dominant conjugate of the type of  $A(M)$ . So if  $(A, \varphi)$  is not cyclic, then  $M \notin S_{A,\varphi}$  and hence  $|S_{A,\varphi}^\sigma| = |U_{A,\varphi}^\sigma| < q^{|\mathcal{V}(A,\varphi)|}$ . The last statement follows from [6, Proposition 4.1.12 & Proposition 8.1.11 (ii)].  $\square$

## 3.2 The Stratification by Extended Semi-Modules

For any  $\lambda \in X_*(T)$ , set  $A^\lambda = \{(i-1) + \lambda(i)n + kn \mid 1 \leq i \leq n, k \in \mathbb{N}\}$ . It is easy to check that for a lattice  $M \in I\varpi^\lambda K/K$ , we have  $A(M) = A^\lambda$ . Thus we have the following lemma, which relates the semi-module stratification to the stratification by extended semi-modules.

**Lemma 3.9.** Let  $\lambda \in X_*(T)$  with  $\lambda(1) + \dots + \lambda(n) = 0$ . Then  $X_\mu^\lambda(b) \neq \emptyset$  if and only if there exists an extended semi-module  $(A^\lambda, \varphi)$  for  $\mu$ . If this is the case, we have

$$X_\mu^\lambda(b) = \bigsqcup_{\varphi} S_{A^\lambda, \varphi},$$

where  $\varphi$  runs over all the functions  $\mathbb{Z} \rightarrow \mathbb{N} \cup \{-\infty\}$  such that the pair of  $A^\lambda$  and the function is an extended semi-module for  $\mu$ .

For  $\lambda \in X_*(T)$  with  $X_\mu^\lambda(b) \neq \emptyset$ , let  $1 \leq i_0 \leq n$  such that  $(i_0 - 1) + \lambda(i_0)n = \min \overline{A^\lambda}$ . Let  $1 \leq m_0 < n$  be the residue of  $m$  modulo  $n$ , and let  $\lambda_{b,\text{dom}}$  be  $((\lfloor \frac{m}{n} \rfloor + 1)^{(m_0)}, \lfloor \frac{m}{n} \rfloor^{(n-m_0)})$ . Then

$$\begin{aligned} & (i_0 - 1) + \lambda(i_0)n + m - (\lambda(i_0) + \lambda_{b,\text{dom}}(c^m(i_0)) - \lambda(c^m(i_0)))n \\ & = c^m(i_0) - 1 + \lambda(c^m(i_0))n \in \overline{A^\lambda}, \end{aligned}$$

where  $c = s_1 \cdots s_{n-1}$ . Repeating the same argument, we can check that the type of  $A^\lambda$  is a conjugate of  $b\lambda - \lambda = c^m\lambda + \lambda_{b,\text{dom}} - \lambda$ . By Lemma 3.4, an extended semi-module  $(A^\lambda, \varphi)$  for  $\mu$  is cyclic if and only if  $b\lambda - \lambda \in W_0\mu$ .

**Corollary 3.10.** Let  $\mu \in X_*(T)_+$ . If there exists a non-cyclic semi-module for  $\mu$ , then the semi-module stratification of  $X_{\leq \mu}(b)$  is not a refinement of the Ekedahl-Oort stratification.

*Proof.* Let  $(A^\lambda, \varphi)$  be a non-cyclic semi-module for  $\mu$ . Then we have  $(b\lambda - \lambda)_{\text{dom}} \prec \mu$  by Lemma 3.4. On the other hand, there always exists a cyclic semi-module  $(A^\lambda, \varphi')$  for  $(b\lambda - \lambda)_{\text{dom}}$ . By Lemma 3.9,  $X_{\leq \mu}^\lambda(b)$  intersects both  $X_\mu(b)$  and  $X_{(b\lambda - \lambda)_{\text{dom}}}(b)$ . This implies that  $X_{\leq \mu}^\lambda(b)$  is not contained in any set of the form  $\pi(X_w(b))$  with  $w \in \tilde{W}$ , which finishes the proof.  $\square$

For  $\mu = (\mu(1), \dots, \mu(n-1), 0) \in X_*(T)_+$ , set  $\mu^* = (\mu(1), \mu(1) - \mu(n-1), \dots, \mu(1) - \mu(2), 0)$  and  $b^* = \tau^{n\mu(1)-m}$ . If  $(A^\lambda, \varphi)$  is an extended semi-module for  $\mu$ , then there exists  $\varphi': \mathbb{Z} \rightarrow \mathbb{N} \cup \{-\infty\}$  such that  $(A^{-w_{\max}\lambda}, \varphi')$  is an extended semi-module for  $\mu^*$  (see §2.3). Clearly,  $b\lambda - \lambda \in W_0\mu$  if and only if  $b^*(-w_{\max}\lambda) + w_{\max}\lambda \in -W_0\mu^*$ . Thus we have the following lemma.

**Lemma 3.11.** There exists a non-cyclic extended semi-module for  $\mu$  if and only if the same is true for  $\mu^*$ .

### 3.3 The Minuscale Case

In this subsection, we treat the minuscale case. Consider  $G^d$  with a Frobenius automorphism  $\sigma_\bullet$  given by

$$(g_1, g_2, \dots, g_d) \mapsto (g_2, \dots, g_d, \sigma(g_1)).$$

For  $\mu_\bullet = (\mu_1, \dots, \mu_d) \in X_*(T)_+^d$  and  $b_\bullet = (1, \dots, 1, b) \in G^d(L)$  with  $b \in G(L)$ , we define  $X_{\mu_\bullet}(b_\bullet) \subset \mathcal{G}r^d = G^d(L)/K^d$  as

$$X_{\mu_\bullet}(b_\bullet) = \{x_\bullet K^d \in \mathcal{G}r^d \mid x_\bullet^{-1} b_\bullet \sigma_\bullet(x_\bullet) \in K^d \varpi^{\mu_\bullet} K^d\}.$$

Let us denote by  $\text{Irr } X_{\mu_\bullet}(b_\bullet)$  the set of irreducible components of  $X_{\mu_\bullet}(b_\bullet)$ . Through the identification  $\mathbb{J} \cong \mathbb{J}_{b_\bullet}$  given by  $g \mapsto (g, \dots, g)$ , this set is equipped with an action of  $\mathbb{J}$ .

For minuscale  $\mu_\bullet \in X_*(T)_+^d$  and  $b_\bullet = (1, \dots, 1, b) \in G^d(L)$ , we define

$$\mathcal{A}_{\mu_\bullet}^{\text{top}} := \{\lambda_\bullet \in X_*(T)^d \mid \dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = \dim X_{\mu_\bullet}(b_\bullet)\}.$$

Here  $X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)$  denotes  $X_{\mu_\bullet}(b_\bullet) \cap I^d \varpi^{\lambda_\bullet} K^d / K^d$ . For  $\lambda_\bullet, \lambda'_\bullet \in \mathcal{A}_{\mu_\bullet}^{\text{top}}$ , we write  $\lambda_\bullet \sim \lambda'_\bullet$  if  $\lambda_\bullet = \tau^k \lambda'_\bullet = (\tau^k \lambda'_1, \dots, \tau^k \lambda'_d)$  for some  $k \in \mathbb{Z}$ . Let  $\mathbb{A}_{\mu_\bullet}^{\text{top}}$  denote the set of equivalence classes with respect to  $\sim$ , and let  $[\lambda_\bullet] \in \mathbb{A}_{\mu_\bullet}^{\text{top}}$  denote the equivalence class represented by  $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet}^{\text{top}}$ . We also define

$$\mathcal{A}_{\mu_\bullet}^j := \{\lambda_\bullet \in X_*(T)^d \mid \dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = j\}$$



for  $1 \leq j \leq \dim X_{\mu_\bullet}(b_\bullet)$ . We can similarly consider the equivalence relation  $\sim$  as above. If  $d = 1$ , then  $\mathbb{A}_\mu^j := \mathcal{A}_\mu^j / \sim$  can be identified with (extended) semi-modules for  $\mu$  whose corresponding stratum has dimension  $j$ ; see Lemma 3.4 and Lemma 3.9.

**Proposition 3.12.** Set  $\mu = \omega_i$ . Then we always have  $|\mathbb{A}_\mu^{\text{top}}| = |\mathbb{A}_\mu^0| = 1$ . If  $i = 2, n - 2$ , then  $|\mathbb{A}_\mu^j| = 1$  for all  $0 \leq j \leq \dim X_\mu(b)$ . If  $i = 3, n - 3$ , then  $|\mathbb{A}_\mu^{\dim X_\mu(b)-1}| = 2$ .

*Proof.* We can easily check the equalities in the proposition using [23, Theorem 4.16] (cf. [5, Remark 6.16]), which gives a combinatorial way of computing  $|\mathbb{A}_\mu^j|$ . In fact, all of the assertions except the last assertion follow from [50, Proposition 5.5].  $\square$

**Example 3.13.** We always have  $\mathbb{A}_{\omega_i}^0 = \{[0]\}$ .

**Remark 3.14.** Set  $R_{\mu_\bullet}(\lambda_\bullet) = \{(l, \chi_{i,j}) \mid 1 \leq l \leq d, \langle \chi_{i,j}, \lambda_l^\natural \rangle = -1, (\lambda_l)_{\chi_{i,j}} \geq 1\}$ . See §5.1 for the notation. By [40, Proposition 2.9],  $X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) \neq \emptyset$  if and only if  $\lambda_\bullet^\natural$  is conjugate to  $\mu_\bullet$ . Moreover, in this case,

$$\dim X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet) = |R_{\mu_\bullet}(\lambda_\bullet)|.$$

Combining this with the dimension formula for  $X_{\mu_\bullet}(b_\bullet)$ , we have

$$\mathcal{A}_{\mu_\bullet}^{\text{top}} = \{\lambda_\bullet \in X_*(T)^d \mid \lambda_\bullet^\natural \in W_0\mu_\bullet, |R_{\mu_\bullet}(\lambda_\bullet)| = \langle \rho, |\mu_\bullet| - \nu_b \rangle - \frac{1}{2} \text{def}(b)\},$$

where  $|\mu_\bullet| = \mu_1 + \cdots + \mu_d$ . Thus we can actually define  $\mathcal{A}_{\mu_\bullet}^{\text{top}}$  without using affine Deligne-Lusztig varieties.

## 4 Crystal Bases

Keep the notation and assumptions above.

### 4.1 Crystals and Young Tableaux

In this subsection, we first recall the definition of  $\widehat{G}$ -crystals from [53, Definition 3.3.1]. After that, we give a realization of  $\mathbb{B}_\mu$  by Young tableaux. This allows us to treat them in a combinatorial way.

**Definition 4.1.** A (normal)  $\widehat{G}$ -crystal is a finite set  $\mathbb{B}$ , equipped with a weight map  $\text{wt}: \mathbb{B} \rightarrow X_*(T)$ , and operators  $\tilde{e}_\alpha, \tilde{f}_\alpha: \mathbb{B} \rightarrow \mathbb{B} \cup \{0\}$  for each  $\alpha \in \Delta$ , such that

- (i) for every  $\mathbf{b} \in \mathbb{B}$ , either  $\tilde{e}_\alpha \mathbf{b} = 0$  or  $\text{wt}(\tilde{e}_\alpha \mathbf{b}) = \text{wt}(\mathbf{b}) + \alpha^\vee$ , and either  $\tilde{f}_\alpha \mathbf{b} = 0$  or  $\text{wt}(\tilde{f}_\alpha \mathbf{b}) = \text{wt}(\mathbf{b}) - \alpha^\vee$ ,

(ii) for all  $\mathbf{b}, \mathbf{b}' \in \mathbb{B}$  one has  $\mathbf{b}' = \tilde{e}_\alpha \mathbf{b}$  if and only if  $\mathbf{b} = \tilde{f}_\alpha \mathbf{b}'$ , and

(iii) if  $\varepsilon_\alpha, \phi_\alpha: \mathbb{B} \rightarrow \mathbb{Z}$ ,  $\alpha \in \Delta$  are the maps defined by

$$\varepsilon_\alpha(\mathbf{b}) = \max\{k \mid \tilde{e}_\alpha^k \mathbf{b} \neq 0\} \quad \text{and} \quad \phi_\alpha(\mathbf{b}) = \max\{k \mid \tilde{f}_\alpha^k \mathbf{b} \neq 0\},$$

then  $\phi_\alpha(\mathbf{b}) - \varepsilon_\alpha(\mathbf{b}) = \langle \alpha, \text{wt}(\mathbf{b}) \rangle$ .

For a  $\widehat{G}$ -crystal  $\mathbb{B}$ , let  $\mathbb{B}^* = \{\mathbf{b}^* \mid \mathbf{b} \in \mathbb{B}\}$  be the dual  $\widehat{G}$ -crystal. Setting  $0^* = 0$ , the maps are given by

$$\text{wt}(\mathbf{b}^*) = -\text{wt}(\mathbf{b}), \quad \tilde{e}_\alpha(\mathbf{b}^*) = (f_\alpha \mathbf{b})^*, \quad \text{and} \quad \tilde{f}_\alpha(\mathbf{b}^*) = (\tilde{e}_\alpha \mathbf{b})^*.$$

For  $\lambda \in X_*(T)$ , we denote by  $\mathbb{B}(\lambda)$  the set of elements with weight  $\lambda$  for  $\widehat{G}$ , called the *weight space* with weight  $\lambda$  for  $\widehat{G}$ . Let  $\mathbb{B}_1$  and  $\mathbb{B}_2$  be two  $\widehat{G}$ -crystals. A morphism  $\mathbb{B}_1 \rightarrow \mathbb{B}_2$  is a map of underlying sets compatible with  $\text{wt}$ ,  $\tilde{e}_\alpha$  and  $\tilde{f}_\alpha$ .

In the sequel, we write  $\tilde{e}_i$  and  $\tilde{f}_i$  (resp.  $\varepsilon_i$  and  $\phi_i$ ) instead of  $\tilde{e}_{\chi_{i,i+1}}$  and  $\tilde{f}_{\chi_{i,i+1}}$  (resp.  $\varepsilon_{\chi_{i,i+1}}$  and  $\phi_{\chi_{i,i+1}}$ ) for simplicity.

**Example 4.2.** Set  $\mathbb{B}_\square = \{\boxed{1}, \boxed{2}, \dots, \boxed{n}\}$ . We define  $\tilde{e}_i, \tilde{f}_i$  and  $\text{wt}$  by

$$\tilde{e}_i \boxed{k} = \begin{cases} \boxed{i} & (k = i + 1) \\ 0 & (k \neq i + 1), \end{cases} \quad \tilde{f}_i \boxed{k} = \begin{cases} \boxed{i + 1} & (k = i) \\ 0 & (k \neq i), \end{cases} \quad \text{wt}(\boxed{k}) = v_k,$$

where  $v_k = (0, \dots, 0, 1, 0, \dots, 0)$  with the nonzero component at position  $k$ . It is easy to check that this defines a  $\widehat{G}$ -crystal structure on  $\mathbb{B}_\square$ .

**Example 4.3.** Let  $\mathbb{B}_\mu$  be the crystal basis of the irreducible  $\widehat{G}$ -module of highest weight  $\mu \in X_*(T)_+$ . Then  $\mathbb{B}_\mu$  is a crystal. We call  $\mathbb{B}_\mu$  a *highest weight crystal* of highest weight  $\mu$  (cf. [53, Definition 3.3.1 (3)]). There exists a unique element  $\mathbf{b}_\mu \in \mathbb{B}_\mu$  satisfying  $\tilde{e}_\alpha \mathbf{b}_\mu = 0$  for all  $\alpha$ ,  $\text{wt}(\mathbf{b}_\mu) = \mu$ , and  $\mathbb{B}_\mu$  is generated from  $\mathbf{b}_\mu$  by the operators  $\tilde{f}_\alpha$ . In particular, for  $\omega_1 = (1, 0, \dots, 0)$ , we can easily check that  $\mathbb{B}_{\omega_1}$  is a crystal isomorphic to  $\mathbb{B}_\square$  and  $\mathbf{b}_{\omega_1}$  corresponds to  $\boxed{1}$ .

Following [53, Definition 3.3.1 (5)], we define the tensor product of  $\widehat{G}$ -crystals.

**Definition 4.4.** Let  $\mathbb{B}_1$  and  $\mathbb{B}_2$  be two  $\widehat{G}$ -crystals. The tensor product  $\mathbb{B}_1 \otimes \mathbb{B}_2$  is the  $\widehat{G}$ -crystal with underlying set  $\mathbb{B}_1 \times \mathbb{B}_2$ , and  $\text{wt}(\mathbf{b}_1 \otimes \mathbf{b}_2) = \text{wt}(\mathbf{b}_1) + \text{wt}(\mathbf{b}_2)$ . The operators  $\tilde{e}_\alpha$  and  $\tilde{f}_\alpha$  are defined by

$$\begin{aligned} \tilde{e}_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \begin{cases} \tilde{e}_\alpha \mathbf{b}_1 \otimes \mathbf{b}_2 & (\phi_\alpha(\mathbf{b}_1) \geq \varepsilon_\alpha(\mathbf{b}_2)) \\ \mathbf{b}_1 \otimes \tilde{e}_\alpha \mathbf{b}_2 & (\phi_\alpha(\mathbf{b}_1) < \varepsilon_\alpha(\mathbf{b}_2)), \end{cases} \\ \tilde{f}_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \begin{cases} \tilde{f}_\alpha \mathbf{b}_1 \otimes \mathbf{b}_2 & (\phi_\alpha(\mathbf{b}_1) > \varepsilon_\alpha(\mathbf{b}_2)) \\ \mathbf{b}_1 \otimes \tilde{f}_\alpha \mathbf{b}_2 & (\phi_\alpha(\mathbf{b}_1) \leq \varepsilon_\alpha(\mathbf{b}_2)). \end{cases} \end{aligned}$$

We have

$$\begin{aligned}\varepsilon_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \max\{\varepsilon_\alpha(\mathbf{b}_1), \varepsilon_\alpha(\mathbf{b}_2) - \langle \alpha, \text{wt}(\mathbf{b}_1) \rangle\}, \\ \phi_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \max\{\phi_\alpha(\mathbf{b}_2), \phi_\alpha(\mathbf{b}_1) + \langle \alpha, \text{wt}(\mathbf{b}_1) \rangle\}.\end{aligned}$$

Taking tensor product of  $\widehat{G}$ -crystals is associative, making the category of  $\widehat{G}$ -crystals a monoidal category. Using this fact, we will endow a  $\widehat{G}$ -crystal structure on the set of Young tableaux. A detailed discussion can be found in [31, chapter 7].

**Definition 4.5.** A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. For a dominant cocharacter  $\mu \in X_*(T)_+$ , we denote by  $Y_\mu$  the Young diagram having  $\mu(i)$  boxes in the  $i$ th row. A *skew Young diagram* is a diagram obtained by removing a smaller Young diagram from a larger one that contains it. For dominant cocharacters  $\mu, \nu \in X_*(T)_+$  with  $\nu(i) \leq \mu(i)$ , we denote by  $Y_{\mu/\nu}$  the skew Young diagram obtained by removing  $Y_\nu$  from  $Y_\mu$ .



**Definition 4.6.** A *tableau* is a (skew) Young diagram filled with numbers, one for each box. A *semi-standard tableau* is a tableau obtained from a (skew) Young diagram by filling the boxes with the numbers  $1, 2, \dots, n$  subject to the conditions

- (i) the entries in each row are weakly increasing from left to right,
- (ii) the entries in each column are strictly increasing from top to bottom.



Let  $K_{\mu/\nu}(\lambda)$  be the number of all semi-standard tableaux  $\mathbf{b}$  of shape  $Y_{\mu/\nu}$  such that the number of  $\boxed{i}$  appearing in  $\mathbf{b}$  is  $\lambda(i)$  for  $1 \leq i \leq n$ . This is sometimes called the *Kostka number*. In §5.4, we need the following well-known result.

**Proposition 4.7.** Let  $\lambda, \lambda' \in X_*(T)_+$ . If  $\lambda \preceq \lambda'$ , then  $K_{\mu/\nu}(\lambda') \leq K_{\mu/\nu}(\lambda)$ . In particular,  $K_{\mu/\nu}(\lambda') \neq 0$  implies  $K_{\mu/\nu}(\lambda) \neq 0$ .

*Proof.* See [8, Proposition 1.2] and the remark right after the proposition.  $\square$

We denote by  $\mathcal{B}(Y)$  the set of all semistandard tableaux of shape  $Y$ .

**Definition 4.8.** Let  $Y$  be a Young diagram and let  $N$  be the number of boxes in  $Y$ . The *Far-Eastern reading* is an embedding  $\mathcal{B}(Y) \rightarrow \mathbb{B}_{\square}^{\otimes N}$  defined by decomposing a semistandard tableau  $\mathbf{b} \in \mathcal{B}(Y)$  into a tensor product of its boxes by proceeding down columns from top to bottom and from right to left.

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & \\ \hline 4 & & & \\ \hline \end{array} = \boxed{4} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{4}$$

**Theorem 4.9.** Let  $Y$  be a Young diagram. Then the image of the Far-Eastern reading  $\mathcal{B}(Y) \rightarrow \mathbb{B}_{\square}^{\otimes N}$  is stable under  $\tilde{e}_i$  and  $\tilde{f}_i$  for any  $i$ . Hence the Far-Eastern reading defines a  $\widehat{G}$ -crystal structure on  $\mathcal{B}(Y)$ .

*Proof.* This follows from [31, Theorem 7.3.6].  $\square$

For a semistandard tableau  $\mathbf{b} \in \mathcal{B}(Y)$ , let  $k_i$  denote the number of  $i$ 's appearing in  $\mathbf{b}$ . Then the weight map  $\text{wt}$  on this  $\widehat{G}$ -crystal structure is given by  $\text{wt}(\mathbf{b}) = (k_1, \dots, k_n)$ . Finally, the following theorem gives a realization of  $\mathbb{B}_{\mu}$ .

**Theorem 4.10.** Let  $\mu = (\mu(1), \dots, \mu(n)) \in X_*(T)_+ \setminus \{0\}$  with  $\mu(n) = 0$ . Let  $Y$  be the Young diagram having  $\mu(i)$  boxes in the  $i$ th row. Then  $\mathbb{B}_{\mu}$  is isomorphic to  $\mathcal{B}(Y)$ .

*Proof.* This is [31, Theorem 7.4.1].  $\square$

In the sequel, we identify  $\mathbb{B}_{\mu}$  and  $\mathcal{B}(Y)$  by this isomorphism. The following result is an explicit description of the actions of  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $\mathbb{B}_{\mu}$ .

**Theorem 4.11.** The actions of  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $\mathbf{b} \in \mathbb{B}_{\mu}$  can be computed by following the steps below:

- (i) In the Far-Eastern reading  $\mathbf{b}_1 \otimes \dots \otimes \mathbf{b}_N$  of  $\mathbf{b}$ , we identify  $\boxed{i}$  (resp.  $\boxed{i+1}$ ) by  $+$  (resp.  $-$ ) and neglect other boxes.
- (ii) Let  $u_i(\mathbf{b}) = u^1 u^2 \dots u^{\ell}$  ( $u^j \in \{\pm\}$ ) be the sequence obtained by (i). If there is “ $+-$ ” in  $u(\mathbf{b})$ , then we neglect such a pair. We continue this procedure as far as we can.
- (iii) Let  $u_i(\mathbf{b})_{\text{red}} = - \dots - + \dots +$  be the sequence obtained by (ii). Then  $\tilde{e}_i$  changes the rightmost  $-$  in  $u_i(\mathbf{b})_{\text{red}}$  to  $+$ , and  $\tilde{f}_i$  changes the leftmost  $+$  in  $u_i(\mathbf{b})_{\text{red}}$  to  $-$ . If there is no such  $-$  (resp.  $+$ ), then  $\tilde{e}_i \mathbf{b} = 0$  (resp.  $\tilde{f}_i \mathbf{b} = 0$ ).

Moreover,  $\varepsilon_i(\mathbf{b})$  (resp.  $\phi_i(\mathbf{b})$ ) is equal to the number of  $-$  (resp.  $+$ ) in  $u_i(\mathbf{b})_{\text{red}}$ .

*Proof.* The first statement is [33, Theorem 3.4.2]. The second statement follows immediately from this.  $\square$

We will see an example of this computation in §5.3. For  $j_1 \leq j_2$ , let  $u^{j_1}u^{j_1+1} \cdots u^{j_2}$  be a part of  $u_i(\mathbf{b})$  above. Then similarly as the notation above, we denote by  $(u^{j_1}u^{j_1+1} \cdots u^{j_2})_{\text{red}}$  the sequence obtained by neglecting “+−” as far as we can. Then Theorem 4.11 tells us that  $\varepsilon_i(\mathbf{b}) = \max\{\text{the number of } - \text{ in } (u^1u^2 \cdots u^j)_{\text{red}} \mid 0 \leq j \leq \ell\}$  (resp.  $\phi_i(\mathbf{b}) = \max\{\text{the number of } + \text{ in } (u^ju^{j+1} \cdots u^\ell)_{\text{red}} \mid 1 \leq j \leq \ell+1\}$ ). If  $\varepsilon_i(\mathbf{b}) > 0$  (resp.  $\phi_i(\mathbf{b}) > 0$ ), then  $\tilde{e}_i$  (resp.  $\tilde{f}_i$ ) changes  $u^j = -$  (resp.  $u^j = +$ ) with  $j$  minimal (resp. maximal) such that the number of  $-$  (resp.  $+$ ) in  $(u^1 \cdots u^j)_{\text{red}}$  (resp.  $(u^j \cdots u^\ell)_{\text{red}}$ ) is  $\varepsilon_i(\mathbf{b})$  (resp.  $\phi_i(\mathbf{b})$ ).

Finally, we recall the Weyl group action on crystals. Let  $\mathbb{B}$  be a  $\widehat{G}$ -crystal. For any  $1 \leq i \leq n-1$  and  $\mathbf{b} \in \mathbb{B}$ , we set

$$s_i \mathbf{b} = \begin{cases} \tilde{f}_i^{\langle \chi_{i,i+1}, \text{wt}(\mathbf{b}) \rangle} \mathbf{b} & \text{if } \langle \chi_{i,i+1}, \text{wt}(\mathbf{b}) \rangle \geq 0 \\ \tilde{e}_i^{-\langle \chi_{i,i+1}, \text{wt}(\mathbf{b}) \rangle} \mathbf{b} & \text{if } \langle \chi_{i,i+1}, \text{wt}(\mathbf{b}) \rangle \leq 0. \end{cases}$$

Then we have the obvious relation

$$\text{wt}(s_i \mathbf{b}) = s_i(\text{wt}(\mathbf{b})).$$

By [32, Theorem 7.2.2], this extends to the action of the Weyl group  $W_0$  on  $\mathbb{B}$ , which is compatible with the action on  $X_*(T)$ . For example,  $w_{\max} \mathbf{b}_\mu \in \mathbb{B}_\mu$  has the *lowest* weight  $w_{\max} \mu$ . It is well-known that the dual of  $\mathbb{B}_\mu$  is isomorphic to  $\mathbb{B}_{-w_{\max} \mu}$  (see for example [31, Lemma 3.5.2]).

**Lemma 4.12.** Let  $w, w' \in W_0$  and  $\mathbf{b} \in \mathbb{B}$ . If  $w(\text{wt}(\mathbf{b})) = w'(\text{wt}(\mathbf{b}))$ , then  $w\mathbf{b} = w'\mathbf{b}$ .

*Proof.* It is enough to show that if  $w(\text{wt}(\mathbf{b})) = \text{wt}(\mathbf{b})$ , then  $w\mathbf{b} = \mathbf{b}$ . By decomposing  $w$  into disjoint cycles and considering the conjugation, we can reduce the general case to the case where  $w = s_i$ . Then the assertion follows immediately from the definition of the Weyl group action on crystals.  $\square$

Let  $\mathbf{b} \in \mathbb{B}(\lambda)$ . If  $\lambda'$  is a conjugate of  $\lambda$ , i.e., there exists  $w \in W_0$  such that  $\lambda' = w\lambda$ , then we call  $w\mathbf{b}$  the conjugate of  $\mathbf{b}$  with weight  $\lambda'$ . By Lemma 4.12, this does not depend on the choice of  $w$ .

## 4.2 The Minuscale Case

If  $\mu \in X_*(T)_+$  is minuscule, then  $\text{wt}: \mathbb{B}_\mu \rightarrow X_*(T)$  gives an identification between  $\mathbb{B}_\mu$  and the set of cocharacters which are conjugate to  $\mu$ . Suppose  $\mu_\bullet = (\mu_1, \dots, \mu_d) \in$

$X_*(T)_+^d$  is minuscule. We can also identify  $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} := \mathbb{B}_{\mu_1} \times \cdots \times \mathbb{B}_{\mu_d}$  with the set of cocharacters in  $X_*(T)^d$  which are conjugate to  $\mu_\bullet$ . Under this identification, set

$$\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda) = \{(\mu'_1, \dots, \mu'_d) \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} \mid \mu'_1 + \cdots + \mu'_d = \lambda\}$$

for any  $\lambda \in X_*(T)$ .

We write  $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$  for the  $\widehat{G}$ -crystal  $\mathbb{B}_{\mu_1} \otimes \cdots \otimes \mathbb{B}_{\mu_d}$ . Note that this is equal to  $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$  as a set. As a  $\widehat{G}$ -crystal, we can decompose  $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$  into simple objects, i.e.,

$$\mathbb{B}_{\mu_\bullet}^{\widehat{G}} = \sqcup_{\mu} \mathbb{B}_{\mu}^{m_{\mu_\bullet}^{\mu}}.$$

Here  $m_{\mu_\bullet}^{\mu}$  denotes the multiplicity with which  $\mathbb{B}_{\mu}$  appears in  $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$ . Using this decomposition, we define a natural map

$$\otimes: \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} \rightarrow \mathbb{B}_{\mu_\bullet}^{\widehat{G}} \rightarrow \sqcup_{\mu} \mathbb{B}_{\mu}$$

as a composition of the map given by taking tensor product and the canonical projection to highest weight  $\widehat{G}$ -crystals.

For  $1 \leq k < n$ , let  $\omega_k$  be the cocharacter of the form  $(1, \dots, 1, 0, \dots, 0)$  in which 1 is repeated  $k$  times. Assume that each  $\mu_i$  is equal to  $\omega_{k_i}$  for some  $1 \leq k_i < n$  and  $i \leq j$  if and only if  $k_i \leq k_j$ . In the rest of paper, we call such  $\mu_\bullet$  *Far-Eastern*. Since  $\mu_\bullet$  is Far-Eastern, then  $|\mu_\bullet| := \mu_1 + \cdots + \mu_d$  is dominant and its last entry is 0. Set  $\mu = |\mu_\bullet|$  for some Far-Eastern  $\mu_\bullet$ . Using Theorem 4.10, we obtain an embedding (i.e., an injective morphism of crystals)

$$\text{FE}: \mathbb{B}_{\mu} \rightarrow \mathbb{B}_{\mu_\bullet}^{\widehat{G}},$$

which decomposes  $\mathbf{b} \in \mathbb{B}_{\mu}$  into the tensor product of its columns from right to left. We also call FE the Far-Eastern reading. By forgetting the  $\widehat{G}$ -crystal structure, we obtain a map  $\mathbb{B}_{\mu} \rightarrow \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$ , which is also denoted by FE.

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 & \\ \hline 4 & & & \\ \hline \end{array} = \begin{array}{|c|} \hline 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array}$$

**Lemma 4.13.** For any  $\mathbf{b} \in \mathbb{B}_{\mu}$ ,  $\text{FE}(\mathbf{b})$  is the unique element in  $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$  such that  $\otimes(\text{FE}(\mathbf{b})) = \mathbf{b}$ .

*Proof.* Let  $\mathbf{b}_{\mu} \in \mathbb{B}_{\mu}$  be the unique element with highest weight  $\mu$ . Then the  $i$ th row of  $\mathbf{b}_{\mu}$  consists of only  $i$ . By the ‘‘Littlewood-Richardson’’ rule (see [31, Theorem 7.4.6]), we can check that  $m_{\mu_\bullet}^{\mu} = 1$  and  $\text{FE}(\mathbf{b}_{\mu}) \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}}$  is the unique maximal vector with weight  $\mu$ . In particular,  $\otimes(\text{FE}(\mathbf{b}_{\mu})) = \mathbf{b}_{\mu}$ . Since FE is a morphism of crystals, we have  $\text{FE}(\tilde{f}_{\alpha} \mathbf{b}) = \tilde{f}_{\alpha} \text{FE}(\mathbf{b})$  for any  $\alpha \in \Delta$ ,  $\mathbf{b} \in \mathbb{B}_{\mu}$ . Therefore  $\otimes(\text{FE}(\mathbf{b})) = \mathbf{b}$ , and such  $\mathbf{b}$  is unique.  $\square$

## 5 Semi-Modules and Crystal Bases

Keep the notation and assumptions above. From now, we set  $c = s_1 s_2 \cdots s_{n-1}$ .

### 5.1 Irreducible Components

Let  $\lambda \in X_*(T)$  and  $\alpha \in \Phi$ . We set  $\lambda_\alpha = \langle \alpha, \lambda \rangle$  if  $\alpha \in \Phi_-$  and  $\lambda_\alpha = \langle \alpha, \lambda \rangle - 1$  if  $\alpha \in \Phi_+$ . Let  $U_\lambda$  be the subgroup of  $G$  generated by  $U_\alpha$  such that  $\lambda_\alpha \geq 0$ . We define  $v_\lambda \in W_0$  to be the unique element such that  $U_\lambda = v_\lambda U v_\lambda^{-1}$ . In particular,  $v_\lambda^{-1} \lambda$  is dominant. Here  $U$  denotes the unipotent radical of  $B$ . It is easy to check  $v_{\tau\lambda} = c v_\lambda$ . For  $\lambda_\bullet = (\lambda_1, \dots, \lambda_d) \in X_*(T)^d$ , set  $v_{\lambda_\bullet} = (v_{\lambda_1}, \dots, v_{\lambda_d})$ .

Let us denote by  $\text{Irr } X_{\mu_\bullet}(b_\bullet)$  the set of irreducible components of  $X_{\mu_\bullet}(b_\bullet)$ . Through the identification  $\mathbb{J} = \mathbb{J}_b \cong \mathbb{J}_{b_\bullet}$  given by  $g \mapsto (g, \dots, g)$ , this set is equipped with an action of  $\mathbb{J}$ . Set  $\mathbb{J}^0 = \mathbb{J} \cap K = \mathbb{J} \cap I$ . Then we have  $\mathbb{J}/\mathbb{J}^0 = \{\tau^k \mathbb{J}^0 \mid k \in \mathbb{Z}\}$  (cf. [4, Lemma 3.3]).

We first consider the case where  $\mu_\bullet$  is minuscule. For  $\lambda_\bullet \in X_*(T)^d$ , set  $\lambda_\bullet^\dagger = b_\bullet \sigma_\bullet(\lambda_\bullet)$ ,  $\lambda_\bullet^\natural = \lambda_\bullet^\dagger - \lambda_\bullet$  and  $\lambda_\bullet^b = v_{\lambda_\bullet^\natural}^{-1}(\lambda_\bullet^\natural)$ . It is easy to check  $(\tau \lambda_\bullet)^b = \lambda_\bullet^b$ . Let  $\lambda_b$  denote the cocharacter whose  $i$ -th entry is  $\lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor$ .

**Theorem 5.1.** Assume that  $\mu_\bullet \in X_*(T)_+^d$  is minuscule. Then  $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet}^{\text{top}}$  if and only if  $\lambda_\bullet^b \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda_b)$ , and  $X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)$  is an affine space for such  $\lambda_\bullet$ . Moreover, the maps  $\lambda_\bullet \mapsto \lambda_\bullet^b$  and  $\lambda_\bullet \mapsto \overline{X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)}$  induce bijections

$$\mathbb{J} \backslash \text{Irr } X_{\mu_\bullet}(b_\bullet) \cong \mathbb{A}_{\mu_\bullet}^{\text{top}} \cong \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda_b).$$

*Proof.* This follows from [40, Proposition 2.9 & Theorem 3.3]. Note that we have  $\text{Stab}_{\mathbb{J}}(X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)) = \mathbb{J}^0$ .  $\square$

We write  $\gamma^{G^d}: \text{Irr } X_{\mu_\bullet}(b_\bullet) \rightarrow \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$  for the map which factors through this bijection. Let  $\mu_\bullet$  be a minuscule cocharacter in  $X_*(T)_+^d$ . Set  $\mu = |\mu_\bullet|$ . By [40, Corollary 1.6], the projection  $\text{pr}: \mathcal{G}r^d \rightarrow \mathcal{G}r$  to the first factor induces a  $\mathbb{J}$ -equivariant map

$$\text{Irr } X_{\mu_\bullet}(b_\bullet) \rightarrow \sqcup_{\mu' \leq \mu} \text{Irr } X_{\mu'}(b), \quad C \mapsto \text{pr}(C),$$

which is also denoted by  $\text{pr}$ . The general case can be characterized by the minuscule case using  $\text{pr}$  and the tensor product of  $\widehat{G}$ -crystals:

**Theorem 5.2.** There exists a map

$$\gamma^G: \text{Irr } X_\mu(b) \rightarrow \mathbb{B}_\mu(\lambda_b)$$

which is characterized by the Cartesian square

$$\begin{array}{ccc} \mathrm{Irr} X_{\mu_\bullet}(b_\bullet) & \xrightarrow{\gamma^{G^d}} & \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} \\ \mathrm{pr} \downarrow & & \downarrow \otimes \\ \sqcup_{\mu' \leq \mu} \mathrm{Irr} X_{\mu'}(b) & \xrightarrow{\gamma^G} & \sqcup_{\mu' \leq \mu} \mathbb{B}_{\mu'}^{\widehat{G}}. \end{array}$$

Moreover,  $\gamma^G$  factors through a bijection

$$\mathbb{J} \setminus \mathrm{Irr} X_\mu(b) \cong \mathbb{B}_\mu(\lambda_b).$$

*Proof.* This follows from [40, Theorem 0.5 & Theorem 0.7].  $\square$

Let us denote by  $\Gamma^{G^d}$  (resp.  $\Gamma^G$ ) the bijection  $\mathbb{A}_{\mu_\bullet}^{\mathrm{top}} \rightarrow \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda_b)$  (resp.  $\mathbb{A}_\mu^{\mathrm{top}} \rightarrow \mathbb{B}_\mu(\lambda_b)$ ) induced by  $\gamma^{G^d}$  (resp.  $\gamma^G$ ). Then by Theorem 5.1 and Theorem 5.2, we have the Cartesian square

$$\begin{array}{ccc} \mathbb{A}_{\mu_\bullet}^{\mathrm{top}} & \xrightarrow{\Gamma^{G^d}} & \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda_b) \\ \mathrm{pr} \downarrow & & \downarrow \otimes \\ \sqcup_{\mu' \leq \mu} \mathbb{A}_{\mu'}^{\mathrm{top}} & \xrightarrow{\Gamma^G} & \sqcup_{\mu' \leq \mu} \mathbb{B}_{\mu'}^{\widehat{G}}(\lambda_b). \end{array}$$

## 5.2 Construction

Let  $\mu \in X_*(T)_+$ . For  $1 \leq k \leq \mu(1)$ , set

$$\mu_k = \begin{cases} \omega_1 & (1 \leq k \leq \mu(1) - \mu(2)), \\ \omega_2 & (\mu(1) - \mu(2) < k \leq \mu(1) - \mu(3)), \\ \vdots & \\ \omega_{n-2} & (\mu(1) - \mu(n-2) < k \leq \mu(1) - \mu(n-1)), \\ \omega_{n-1} & (\mu(1) - \mu(n-1) < k \leq \mu(1)). \end{cases}$$

Set  $d = \mu(1)$ . Obviously  $\mu_\bullet \in X_*(T)_+^d$  is Far-Eastern (§4.2) and  $\mu = |\mu_\bullet|$ .

Let  $w_{\max}$  denote the maximal length element in  $W_0$ . Set  $\lambda_b^{\mathrm{op}} = w_{\max} \lambda_b$ . For any  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ , we denote by  $\mathbf{b}^{\mathrm{op}}$  the conjugate of  $\mathbf{b}$  with weight  $\lambda_b^{\mathrm{op}}$ . Let  $1 \leq m_0 < n$  be the residue of  $m$  modulo  $n$ . Since  $\lfloor \frac{im}{n} \rfloor = i \frac{m-m_0}{n} + \lfloor \frac{im_0}{n} \rfloor$ , we have  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor + \lfloor \frac{im_0}{n} \rfloor - \lfloor \frac{(i-1)m_0}{n} \rfloor$ . So each entry of  $\lambda_b$  is  $\lfloor \frac{m}{n} \rfloor$  or  $\lfloor \frac{m}{n} \rfloor + 1$ , and  $\lambda_b(i) = \lambda_b(n+1-i)$  for any  $2 \leq i \leq n-1$ . For  $0 \leq k \leq m_0$ , let  $1 \leq i_k \leq n$  be the minimal integer such



that  $\lfloor \frac{i_k m_0}{n} \rfloor \geq k$ . In other words, we define  $i_0 = 1 < i_1 < i_2 < \dots < i_{m_0} = n$  as the integers such that  $\lambda_b(i_1) = \lambda_b(i_2) = \dots = \lambda_b(i_{m_0}) = \lfloor \frac{m}{n} \rfloor + 1$ . Then

$$\lambda_b^{\text{op}} = w'_{\max} \lambda_b, \quad \text{where } w'_{\max} = (s_{i_{m_0-1}} \cdots s_{n-1}) \cdots (s_{i_1} \cdots s_{i_2-1})(s_1 \cdots s_{i_1-1}).$$

Here  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$ ) if and only if  $s_{i-1}s_i \leq w'_{\max}$  (resp.  $s_i s_{i+1} \leq w'_{\max}$ ). By Lemma 4.12, it follows that  $\mathbf{b}^{\text{op}}$  can be computed by the action of the Coxeter element  $w'_{\max}$ . In this computation, each  $s_i$  acts as the action of  $\tilde{e}_i$  because  $\lfloor \frac{m}{n} \rfloor - (\lfloor \frac{m}{n} \rfloor + 1) = -1$ . Therefore, if we write

$$\text{FE}(\mathbf{b}) = \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_d,$$

then there exists  $(w_1, \dots, w_d) \in W_0^d$  such that

$$\text{FE}(\mathbf{b}^{\text{op}}) = w_1 \mathbf{b}_1 \otimes \cdots \otimes w_d \mathbf{b}_d$$

and each simple reflection appears exactly once in some  $\text{supp}(w_j)$ .

**Lemma 5.3.** The tuple  $(w_1, \dots, w_d) \in W_0^d$  as above is uniquely determined by  $\mathbf{b}$ . In particular,  $w(\mathbf{b}) := w_1^{-1} \cdots w_d^{-1}$  is a Coxeter element uniquely determined by  $\mathbf{b}$ .

*Proof.* If  $(w'_1, \dots, w'_d) \in W_0^d$  is another tuple such that

$$\text{FE}(\mathbf{b}^{\text{op}}) = w'_1 \mathbf{b}_1 \otimes \cdots \otimes w'_d \mathbf{b}_d$$

and each simple reflection appears exactly once in some  $\text{supp}(w'_j)$ , then each  $s_i$  appearing in this tuple acts as the action of  $\tilde{e}_i$ . This follows from the fact that

$$\lambda_b^{\text{op}} - \lambda_b = (1, 0, \dots, 0, -1) = \chi_{1,2}^\vee + \chi_{2,3}^\vee + \cdots + \chi_{n-1,n}^\vee$$

and  $\chi_{1,2}^\vee, \chi_{2,3}^\vee, \dots, \chi_{n-1,n}^\vee$  are linearly independent. Assume that  $s_i \in \text{supp}(w_j)$ . If  $s_i \notin \text{supp}(w'_j)$ , then the number of 1 appearing at position  $k \leq i$  of  $\text{wt}(w'_j \mathbf{b}_j)$  is different from that of  $\text{wt}(w_j \mathbf{b}_j)$ , which is a contradiction. So  $s_i \in \text{supp}(w'_j)$ . Since this is true for any  $i$ , it follows that  $\text{supp}(w_j) = \text{supp}(w'_j)$  for any  $j$ .

Fix  $j$  and let  $\Sigma$  be a connected component of  $\text{supp}(w_j) = \text{supp}(w'_j)$ . In particular,  $\Sigma = \{\min \Sigma, \min \Sigma + 1, \dots, \max \Sigma - 1, \max \Sigma\}$ . We define  $k_0 = \min \Sigma \leq k_1 < k_2 < \dots < k_l = \max \Sigma$  by

$$\text{wt}(\mathbf{b}_j)(k) = \begin{cases} 1 & (k = k_1 + 1, k_2 + 1, \dots, k_l + 1) \\ 0 & (k \neq k_1 + 1, k_2 + 1, \dots, k_l + 1) \end{cases}$$

for  $k_0 \leq k \leq k_l + 1$ . Since each  $s_i$  with  $i \in \text{supp}(w_j)$  acts as the action of  $\tilde{e}_i$ , we have

$$(s_{k_{l-1}+1} s_{k_{l-1}+2} \cdots s_{k_l}) \cdots (s_{k_1+1} s_{k_1+2} \cdots s_{k_2}) (s_{k_0} s_{k_0+1} \cdots s_{k_1}) \leq w_j.$$

By the above argument, the same is true for  $w'_j$ . Since both  $j$  and  $\Sigma$  are arbitrary, it follows that  $w_j = w'_j$ .  $\square$

We call  $w(\mathbf{b})$  the *Coxeter element associated to  $\mathbf{b}$* . Set  $\Upsilon(\mathbf{b}) = \{v \in W_0 \mid v^{-1}c^m v = w(\mathbf{b})\}$ . Clearly  $|\Upsilon(\mathbf{b})| = n$ .

For any  $\mathbf{b}' \in \mathbb{B}_\mu$ , set

$$\xi(\mathbf{b}') = (\varepsilon_1(\mathbf{b}') + \cdots + \varepsilon_{n-1}(\mathbf{b}'), \varepsilon_2(\mathbf{b}') + \cdots + \varepsilon_{n-1}(\mathbf{b}'), \dots, \varepsilon_{n-1}(\mathbf{b}'), 0).$$

Let  $\lambda_b^-$  be the anti-dominant conjugate of  $\lambda_b$ , and let  $\mathbf{b}^-$  be the conjugate of  $\mathbf{b}$  with weight  $\lambda_b^-$ . For any  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ , we define  $\xi_\bullet(\mathbf{b}, v) \in X_*(T)^d$  by

$$\xi_j(\mathbf{b}, v) = v\xi(v^{-1}\mathbf{b}^-) + \sum_{1 \leq j' < j} v w_1^{-1} \cdots w_{j'-1}^{-1} \text{wt}(\mathbf{b}_{j'}) \quad (1 \leq j \leq d).$$

**Theorem 5.4.** We have  $v\xi_j(\mathbf{b}, v) = v w_1^{-1} \cdots w_{j-1}^{-1}$  and  $\xi_\bullet(\mathbf{b}, v) \in \mathcal{A}_{\mu_\bullet}^{\text{top}}$ . Moreover, if  $v'$  is an element in  $\Upsilon(\mathbf{b})$  different from  $v$ , then  $\xi_\bullet(\mathbf{b}, v) \neq \xi_\bullet(\mathbf{b}, v')$  and  $\xi_\bullet(\mathbf{b}, v) \sim \xi_\bullet(\mathbf{b}, v')$ . Finally, we have  $(\Gamma^{G^d})^{-1}(\text{FE}(\mathbf{b})) = [\xi_\bullet(\mathbf{b}, v)]$ .

This construction itself does not depend on the choice of realization of  $\mathbb{B}_\mu$ .

Let  $C \in \text{Irr } X_\mu(b)^0$ . By Proposition 3.7,  $C = \overline{S_{A, \varphi}}$  for some  $(A, \varphi) \in \mathbb{A}_\mu^{\text{top}}$ . On the other hand, by Theorem 5.1 and [40, Proposition 3.13], there exists a unique  $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet}^{\text{top}}$  with  $\lambda_1(1) + \cdots + \lambda_1(n) = 0$  such that  $C = \text{pr}(\overline{X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)})$ . Therefore Theorem 5.4 can be rephrased as follows:

**Corollary 5.5.** For any  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ , let  $\xi_\bullet^0(\mathbf{b})$  be the unique cocharacter in  $[\xi_\bullet(\mathbf{b}, v)]$  such that  $\xi_1^0(\mathbf{b})(1) + \cdots + \xi_1^0(\mathbf{b})(n) = 0$ . Then for any  $(A, \varphi) \in \mathbb{A}_\mu^{\text{top}}$ , there exists a unique  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  such that  $\overline{S_{A, \varphi}} = \text{pr}(\overline{X_{\mu_\bullet}^{\xi_\bullet^0(\mathbf{b})}(b_\bullet)})$ .

This correspondence between  $\mathbb{A}_\mu^{\text{top}}$  and  $\mathbb{B}_\mu(\lambda_b)$  is compatible with the natural bijection in the Chen-Zhu conjecture constructed by Nie in [40].

**Corollary 5.6.** Let  $(A, \varphi) \in \mathbb{A}_\mu^{\text{top}}$ . Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  such that  $\overline{S_{A, \varphi}} = \text{pr}(\overline{X_{\mu_\bullet}^{\xi_\bullet^0(\mathbf{b})}(b_\bullet)})$ . Then  $(A, \varphi)$  is cyclic if and only if

$$\sum_{1 \leq j \leq d} w_1^{-1} \cdots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j) \in W_0 \mu.$$

*Proof.* By Lemma 3.9, we have  $A = A^{\xi_1^0(\mathbf{b})}$ . Recall that  $(A, \varphi)$  is cyclic if and only if  $b\xi_1^0(\mathbf{b}) - \xi_1^0(\mathbf{b}) \in W_0 \mu$ . Since  $b\xi_1^0(\mathbf{b}) - \xi_1^0(\mathbf{b})$  is a conjugate of  $b\xi_1(\mathbf{b}, v) - \xi_1(\mathbf{b}, v)$ , this is also equivalent to  $v^{-1}b\xi_1(\mathbf{b}, v) - v^{-1}\xi_1(\mathbf{b}, v) \in W_0 \mu$ . By Theorem 5.1 and Theorem 5.4,  $v^{-1}b\xi_1(\mathbf{b}, v) - v^{-1}\xi_d(\mathbf{b}, v) = w_1^{-1} \cdots w_{d-1}^{-1} \text{wt}(\mathbf{b}_d)$ , i.e.,

$$v^{-1}b\xi_1(\mathbf{b}, v) - v^{-1}\xi_1(\mathbf{b}, v) = \sum_{1 \leq j \leq d} w_1^{-1} \cdots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j).$$

This finishes the proof.  $\square$

We say that an element  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  is cyclic if

$$\lambda(\mathbf{b}) := \sum_{1 \leq j \leq d} w_1^{-1} \cdots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j) \in W_0\mu.$$

Now we give another interpretation of Lemma 3.11. Recall that  $\mathbb{B}_\mu^*$  is isomorphic to  $\mathbb{B}_{\mu^*}$ . We denote by  $\mathbf{b}^* \in \mathbb{B}_{\mu^*}$  the dual of  $\mathbf{b} \in \mathbb{B}_\mu$ . Note that we have  $(w\mathbf{b})^* = w\mathbf{b}^*$  for any  $w \in W_0$ . So if  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ , then  $\mathbf{b}^{\text{op}*} = w_{\max}\mathbf{b}^* \in \mathbb{B}_{\mu^*}(\lambda_{b^*})$ .

**Lemma 5.7.** We have  $\lambda(\mathbf{b}^{\text{op}*}) = -w(\mathbf{b})^{-1}\lambda(\mathbf{b}) + (d, \dots, d)$ . In particular,  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  is cyclic if and only if  $\mathbf{b}^{\text{op}*} \in \mathbb{B}_{\mu^*}(\lambda_{b^*})$  is cyclic.

*Proof.* Note that if  $(\mu_1, \dots, \mu_d)$  is Far-Eastern, then  $(\mu_d^*, \dots, \mu_1^*)$  is Far-Eastern. So if we write

$$\text{FE}(\mathbf{b}) = \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_d \quad \text{and} \quad \text{FE}(\mathbf{b}^{\text{op}}) = w_1\mathbf{b}_1 \otimes \cdots \otimes w_d\mathbf{b}_d$$

in  $\mathbb{B}_{\mu_1} \otimes \cdots \otimes \mathbb{B}_{\mu_d}$ , then we have

$$\text{FE}(\mathbf{b}^*) = \mathbf{b}_d^* \otimes \cdots \otimes \mathbf{b}_1^* \quad \text{and} \quad \text{FE}(\mathbf{b}^{\text{op}*}) = w_d\mathbf{b}_d^* \otimes \cdots \otimes w_1\mathbf{b}_1^*$$

in  $\mathbb{B}_{\mu_d^*} \otimes \cdots \otimes \mathbb{B}_{\mu_1^*}$ . Thus  $w(\mathbf{b}^{\text{op}*}) = w_d \cdots w_1 = w(\mathbf{b})^{-1}$ ,  $\Upsilon(\mathbf{b}^{\text{op}*}) = \Upsilon(\mathbf{b})$  and

$$\begin{aligned} \lambda(\mathbf{b}^{\text{op}*}) &= \text{wt}(w_d\mathbf{b}_d^*) + w_d \text{wt}(w_{d-1}\mathbf{b}_{d-1}^*) + \cdots + w_d \cdots w_2 \text{wt}(w_1\mathbf{b}_1^*) \\ &= -w(\mathbf{b})^{-1}\lambda(\mathbf{b}) + (d, \dots, d), \end{aligned}$$

as desired.  $\square$

**Remark 5.8.** For  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ ,  $\xi(\mathbf{b})$  already appeared in [53, Lemma 4.4.3]. In [53, Theorem 4.4.5],  $\xi(\mathbf{b})$  was used to construct the irreducible component corresponding to  $\mathbf{b}$ .

**Remark 5.9.** Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ . In [40, §3.3], Nie defined  $(w'_1, \dots, w'_d) \in W_0^d$  from  $\lambda_\bullet := \xi_\bullet(\mathbf{b}, v)$  as follows. Set  $a_{j,i} = v_{\lambda_j}(i) + n\lambda_j(v_{\lambda_j}(i))$  for  $1 \leq j \leq d$ . By the definition of  $v_{\lambda_j}$ ,  $a_{j,1} > \cdots > a_{j,n}$  is the arrangement of the integers  $i + n\lambda_j(i)$  in the decreasing order. Define  $(w'_1, \dots, w'_d) \in W_0^d$  such that

$$a_{j,i} = \begin{cases} a_{j+1, w'_j(i)} - n\lambda_j^b(i) & (1 \leq j \leq d-1) \\ a_{1, w'_d(i)} - n\lambda_d^b(i) + m & (j = d). \end{cases}$$

Then we have  $(w_1, \dots, w_d) = (w'_1, \dots, w'_d)$ . Indeed, by Theorem 5.4 and [40, Lemma 3.7], we have  $v(w_{j-1} \cdots w_1)^{-1} = v_{\lambda_j} = v(w'_{j-1} \cdots w'_1)^{-1}$  for  $1 \leq j \leq d$ . This implies  $(w_1, \dots, w_{d-1}) = (w'_1, \dots, w'_{d-1})$ . Moreover, by [40, Lemma 3.11],  $\ell(w'_d \cdots w'_1) =$

$\sum_{j=1}^d \ell(w'_j) = n - 1$  and  $w'_d \cdots w'_1$  is a product of distinct simple reflections. So we have  $\text{supp}(w_d) = \text{supp}(w'_d)$ . Let  $\Sigma$  be a connected component of  $\text{supp}(w_d) = \text{supp}(w'_d)$ . We define  $k_0 = \min \Sigma \leq k_1 < k_2 < \cdots < k_l = \max \Sigma$  such that

$$(s_{k_{l-1}+1} s_{k_{l-1}+2} \cdots s_{k_l}) \cdots (s_{k_1+1} s_{k_1+2} \cdots s_{k_2}) (s_{k_0} s_{k_0+1} \cdots s_{k_1}) \leq w'_d.$$

In particular,  $\{\chi_{i,i+1} \in \Delta \mid k_0 \leq i \leq k_l, w'_d \chi_{i,i+1} \in \Phi_-\} = \{\chi_{k_1,k_1+1}, \dots, \chi_{k_l,k_l+1}\}$ . By Theorem 5.1 and [40, Lemma 3.8 (1) & Lemma 3.9],  $w'_d \chi_{i,i+1} \in \Phi_-$  if and only if  $\text{wt}(\mathbf{b}_d)(i+1) - \text{wt}(\mathbf{b}_d)(i) = 1$  for  $i \in \text{supp}(w_d)$ . Thus we have

$$(\text{wt}(\mathbf{b}_d)(k_1), \text{wt}(\mathbf{b}_d)(k_1+1)) = \cdots = (\text{wt}(\mathbf{b}_d)(k_l), \text{wt}(\mathbf{b}_d)(k_l+1)) = (0, 1)$$

and  $\text{wt}(\mathbf{b}_d)(k) \geq \text{wt}(\mathbf{b}_d)(k+1)$  for  $k \in \{k_0, k_0+1, \dots, k_l\} \setminus \{k_1, k_2, \dots, k_l\}$ . Since each  $s_i$  with  $i \in \text{supp}(w_d)$  acts as the action of  $\tilde{e}_i$ , we have

$$\text{wt}(\mathbf{b}_d)(k) = \begin{cases} 1 & (k = k_1+1, k_2+1, \dots, k_l+1) \\ 0 & (k \neq k_1+1, k_2+1, \dots, k_l+1) \end{cases}$$

for  $k_0 \leq k \leq k_l+1$ , and hence

$$(s_{k_{l-1}+1} s_{k_{l-1}+2} \cdots s_{k_l}) \cdots (s_{k_1+1} s_{k_1+2} \cdots s_{k_2}) (s_{k_0} s_{k_0+1} \cdots s_{k_1}) \leq w_d.$$

Since  $\Sigma$  is arbitrary, it follows that  $w_d = w'_d$ .

### 5.3 An Example

In this subsection, we give an example. We consider the case for  $n = 5, m = 12$  and  $\mu = (4, 3, 3, 2, 0)$ . Then  $\mu_1 = (1, 0, 0, 0, 0), \mu_2 = (1, 1, 1, 0, 0), \mu_3 = (1, 1, 1, 1, 0), \mu_4 = (1, 1, 1, 1, 0), \lambda_b = (2, 2, 3, 2, 3)$  and  $\lambda_b^{\text{op}} = (3, 2, 3, 2, 2)$ . Set

$$\mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 5 & 5 & & \\ \hline \end{array} \in \mathbb{B}_\mu(\lambda_b).$$

Then

$$\text{FE}(\mathbf{b}) = \mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4 = \begin{array}{|c|} \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ 4 \\ 5 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ 2 \\ 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ 2 \\ 3 \\ 5 \\ \hline \end{array} \in \mathbb{B}_{\mu_\bullet}^{\hat{G}^d},$$

and

$$\begin{aligned}
u_2(\mathbf{b}) &= \boxed{3} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{3} = - - + + -, \\
u_2(\mathbf{b})_{\text{red}} &= - - +, \varepsilon_2(\mathbf{b}) = 2, \phi_2(\mathbf{b}) = 1, \\
u_4(\mathbf{b}) &= \boxed{4} \otimes \boxed{5} \otimes \boxed{4} \otimes \boxed{5} \otimes \boxed{5} = + - + - -, \\
u_4(\mathbf{b})_{\text{red}} &= -, \varepsilon_4(\mathbf{b}) = 1, \phi_2(\mathbf{b}) = 0.
\end{aligned}$$

So by Theorem 4.11, we have

$$\tilde{e}_2 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 5 & 5 & & \\ \hline \end{array}, \tilde{f}_2 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 3 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 5 & 5 & & \\ \hline \end{array}, \tilde{e}_4 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}, \tilde{f}_4 \mathbf{b} = 0.$$

In a similar way, we compute

$$\begin{aligned}
\tilde{e}_2 \mathbf{b} &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 5 & 5 & & \\ \hline \end{array}, \tilde{e}_2 \tilde{e}_4 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}, \tilde{e}_1 \tilde{e}_2 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 5 & 5 & & \\ \hline \end{array}, \\
\tilde{e}_3 \tilde{e}_2 \tilde{e}_4 \mathbf{b} &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 3 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}, \tilde{e}_1 \tilde{e}_2 \tilde{e}_4 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}, \tilde{e}_3 \tilde{e}_4 \tilde{e}_1 \tilde{e}_2 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 3 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}.
\end{aligned}$$

By Theorem 5.1, we want to find  $\lambda_\bullet$  satisfying

$$\begin{aligned}
[\lambda_\bullet] &= (\Gamma^{G^d})^{-1}(\text{FE}(\mathbf{b})) \\
\Leftrightarrow \lambda_\bullet^b &= \text{FE}(\mathbf{b}) \in \mathbb{B}_{\mu_\bullet}^{\hat{G}^d}(\lambda_b) \\
\Leftrightarrow v_{\lambda_1}^{-1}(\lambda_2 - \lambda_1) &= \text{wt}(\mathbf{b}_1) = (0, 0, 1, 0, 0), \\
v_{\lambda_2}^{-1}(\lambda_3 - \lambda_2) &= \text{wt}(\mathbf{b}_2) = (0, 0, 1, 1, 1), \\
v_{\lambda_3}^{-1}(\lambda_4 - \lambda_3) &= \text{wt}(\mathbf{b}_3) = (1, 1, 0, 1, 1), \\
v_{\lambda_4}^{-1}(b\lambda_1 - \lambda_4) &= \text{wt}(\mathbf{b}_4) = (1, 1, 1, 0, 1).
\end{aligned}$$

In the sequel, we check that for  $v \in \Upsilon(\mathbf{b})$ ,  $\lambda_\bullet = \xi_\bullet(\mathbf{b}, v)$  satisfies these equations. Since

$$\mathbf{b}^{\text{op}} = \tilde{e}_3 \tilde{e}_4 \tilde{e}_1 \tilde{e}_2 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 3 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array} \in \mathbb{B}_\mu(\lambda_b^{\text{op}}),$$

we have

$$\text{FE}(\mathbf{b}^{\text{op}}) = \boxed{3} \otimes s_1 s_2 \begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array} \otimes s_3 \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline \end{array} \otimes s_4 \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline \end{array} \in \mathbb{B}_{\mu_\bullet}^{\hat{G}^d},$$

and

$$w_1 = 1, w_2 = s_1 s_2, w_3 = s_3, w_4 = s_4, w(\mathbf{b}) = w_1^{-1} w_2^{-1} w_3^{-1} w_4^{-1} = s_2 s_1 s_3 s_4.$$

So

$$\begin{aligned} \Upsilon(\mathbf{b}) &= \{v \in W_0 \mid v^{-1} c^{12} v = s_2 s_1 s_3 s_4\} \\ &= \{v \in W_0 \mid (1 \ 3 \ 5 \ 2 \ 4) = (v(1) \ v(3) \ v(4) \ v(5) \ v(2))\} \\ &= \{(1 \ 3 \ 5 \ 4 \ 2), (2 \ 4 \ 5), (1 \ 5)(2 \ 3), (1 \ 2 \ 5 \ 3 \ 4), (1 \ 4 \ 3)\}. \end{aligned}$$

Set  $v_1 = (1 \ 3 \ 5 \ 4 \ 2), v_2 = (2 \ 4 \ 5), v_3 = (1 \ 5)(2 \ 3), v_4 = (1 \ 2 \ 5 \ 3 \ 4), v_5 = (1 \ 4 \ 3)$ . Then

$$\begin{aligned} v_1^{-1} \lambda_b^- &= (2, 2, 3, 2, 3), v_2^{-1} \lambda_b^- = (2, 3, 2, 3, 2), v_3^{-1} \lambda_b^- = (3, 2, 2, 3, 2), \\ v_4^{-1} \lambda_b^- &= (2, 3, 3, 2, 2), v_5^{-1} \lambda_b^- = (3, 2, 2, 2, 3). \end{aligned}$$

The corresponding conjugates of  $\mathbf{b}$  are

$$\begin{aligned} \mathbf{b} &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 5 & 5 & & \\ \hline \end{array}, \quad \tilde{e}_2 \tilde{e}_4 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}, \quad \tilde{e}_1 \tilde{e}_2 \tilde{e}_4 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}, \\ \tilde{e}_3 \tilde{e}_2 \tilde{e}_4 \mathbf{b} &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 3 & 5 & \\ \hline 4 & 5 & & \\ \hline \end{array}, \quad \tilde{e}_1 \tilde{e}_2 \mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & \\ \hline 3 & 4 & 5 & \\ \hline 5 & 5 & & \\ \hline \end{array}, \end{aligned}$$

respectively. From this, we compute

$$\begin{aligned} \xi(v_1^{-1} \mathbf{b}^-) &= (3, 3, 1, 1, 0), \xi(v_2^{-1} \mathbf{b}^-) = (3, 2, 1, 0, 0), \xi(v_3^{-1} \mathbf{b}^-) = (2, 2, 1, 0, 0), \\ \xi(v_4^{-1} \mathbf{b}^-) &= (3, 2, 1, 1, 0), \xi(v_5^{-1} \mathbf{b}^-) = (3, 3, 2, 1, 0), \end{aligned}$$

and

$$\begin{aligned} v_1 \xi(v_1^{-1} \mathbf{b}^-) &= (3, 1, 3, 0, 1), v_2 \xi(v_2^{-1} \mathbf{b}^-) = (3, 0, 1, 2, 0), v_3 \xi(v_3^{-1} \mathbf{b}^-) = (0, 1, 2, 0, 2), \\ v_4 \xi(v_4^{-1} \mathbf{b}^-) &= (1, 3, 0, 1, 2), v_5 \xi(v_5^{-1} \mathbf{b}^-) = (2, 3, 1, 3, 0). \end{aligned}$$

Note that

$$\begin{aligned} v_2\xi(v_2^{-1}\mathbf{b}^-) &= \tau(v_3\xi(v_3^{-1}\mathbf{b}^-)), v_4\xi(v_4^{-1}\mathbf{b}^-) = \tau(v_2\xi(v_2^{-1}\mathbf{b}^-)), \\ v_1\xi(v_1^{-1}\mathbf{b}^-) &= \tau(v_4\xi(v_4^{-1}\mathbf{b}^-)), v_5\xi(v_5^{-1}\mathbf{b}^-) = \tau(v_1\xi(v_1^{-1}\mathbf{b}^-)). \end{aligned}$$

We first consider the case for  $v_3$ . Set  $\xi_\bullet = \xi_\bullet(\mathbf{b}, v_3)$ . Then

$$\begin{aligned} \xi_1 &= (0, 1, 2, 0, 2), \\ \xi_2 &= \xi_1 + v_3 \text{wt}(\mathbf{b}_1) = (0, 2, 2, 0, 2), \\ \xi_3 &= \xi_2 + v_3 \text{wt}(\mathbf{b}_2) = (1, 3, 2, 1, 2), \\ \xi_4 &= \xi_3 + v_3 s_2 s_1 \text{wt}(\mathbf{b}_3) = (2, 4, 2, 2, 3). \end{aligned}$$

We can check that

$$v_{\xi_1} = v_3, v_{\xi_2} = v_3 = v_3 w_1^{-1}, v_{\xi_3} = v_3 s_2 s_1 = v_3 w_1^{-1} w_2^{-1}, v_{\xi_4} = v_3 s_2 s_1 s_3 = v_3 w_1^{-1} w_2^{-1} w_3^{-1},$$

and

$$\begin{aligned} b\xi_1 - \xi_4 &= c^{12}\xi_1 + (3, 3, 2, 2, 2) - \xi_4 \\ &= (0, 2, 0, 1, 2) + (3, 3, 2, 2, 2) - (2, 4, 2, 2, 3) \\ &= (1, 1, 0, 1, 1) = v_{\xi_4} \text{wt}(\mathbf{b}_4). \end{aligned}$$

Thus  $\xi_\bullet^\flat = \text{FE}(\mathbf{b})$ . The same holds for other  $v \in \Upsilon(\mathbf{b})$  because  $v_{\tau\lambda} = cv_\lambda$ .

In the above example, there exists a partial Coxeter element  $w_v$  such that  $v^{-1}\lambda_b^- = w_v\lambda_b$  for any  $v \in \Upsilon(\mathbf{b})$ . In fact the same is true in general; see Lemma 6.7. Here we illustrate this for  $n = 5$  and  $m_0 = 2$ . In this case there are 8 Coxeter elements:

$$\begin{aligned} s_1 s_2 s_3 s_4 &= (1 \ 2 \ 3 \ 4 \ 5), & s_2 s_3 s_4 s_1 &= (1 \ 3 \ 4 \ 5 \ 2), \\ s_3 s_4 s_1 s_2 &= (1 \ 2 \ 4 \ 5 \ 3), & s_4 s_1 s_2 s_3 &= (1 \ 2 \ 3 \ 5 \ 4), \\ s_3 s_4 s_2 s_1 &= (1 \ 4 \ 5 \ 3 \ 2), & s_4 s_2 s_1 s_3 &= (1 \ 3 \ 5 \ 4 \ 2), \\ s_4 s_1 s_3 s_2 &= (1 \ 2 \ 5 \ 4 \ 3), & s_4 s_3 s_2 s_1 &= (1 \ 5 \ 4 \ 3 \ 2). \end{aligned}$$

Note that 1, 2 and 4, 5 are adjacent respectively in these  $n$ -cycles (cf. Lemma 6.1). On the other hand, 4, 5 in  $c^m = (1 \ 3 \ 5 \ 2 \ 4)$  are not adjacent. Since  $(v^{-1}\lambda_b^-)(i) = \lambda_b^-(v(i))$  and  $v^{-1}c^m v$  is one of the Coxeter elements listed above, we have  $v^{-1}\lambda_b^- \neq (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor), (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor + 1)$ . For other conjugate  $\lambda$  of  $\lambda_b$ , there exists a partial Coxeter element  $w$  such that  $\lambda = w\lambda_b$ . Thus our claim is verified in this case.

## 5.4 Non-Cyclic Semi-standard Tableaux

The goal of this subsection is to specify the dominant cocharacters  $\mu$  such that every  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  is cyclic. This will be used in the proof of Theorem 9.2. Set  $d = \mu(1)$ .

**Lemma 5.10.** Assume that  $n \geq 3$ . We have  $d \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$  or  $d \geq 2\lfloor \frac{nd-m}{n} \rfloor + \lfloor \frac{2(n-m_0)}{n} \rfloor + 1$ .

*Proof.* It suffices to show that  $d \leq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor$  is equivalent to  $d \geq 2\lfloor \frac{nd-m}{n} \rfloor + \lfloor \frac{2(n-m_0)}{n} \rfloor + 1$ . Note that  $\lfloor \frac{m}{n} \rfloor = \frac{m-m_0}{n}$ ,  $\lfloor \frac{nd-m}{n} \rfloor = \frac{nd-m-(n-m_0)}{n}$ . So  $d \leq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor$  is equivalent to  $(n-2)d \leq 2(m-d-m_0) + n\lfloor \frac{2m_0}{n} \rfloor$ , and  $d \geq 2\lfloor \frac{nd-m}{n} \rfloor + \lfloor \frac{2(n-m_0)}{n} \rfloor + 1$  is equivalent to  $(n-2)d \leq 2(m-d-m_0) + n(1 - \lfloor \frac{2(n-m_0)}{n} \rfloor)$ . Then the assertion follows from the fact that  $\lfloor \frac{2m_0}{n} \rfloor = 0$  (resp. 1) if and only if  $\lfloor \frac{2(n-m_0)}{n} \rfloor = 1$  (resp. 0).  $\square$

**Lemma 5.11.** Assume that  $n \geq 3$ . Let  $\mu \in X_*(T)_+$  such that  $d \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$ ,  $\mu(2) \geq 2$  and  $\lfloor \frac{m}{n} \rfloor \geq 2$ . Then  $\mathbb{B}_\mu(\lambda_b)$  contains at least one non-cyclic element.

*Proof.* First we consider the case  $n = 3$ . In this case, we have  $2 \leq \mu(2) \leq \lfloor \frac{m}{n} \rfloor$  because  $\mu(3) = 0$ . Let  $\mathbf{b}$  be the unique element in  $\mathbb{B}_\mu(\lambda_b)$  whose second row contains exactly one  $\boxed{3}$ . Then  $w(\mathbf{b}) = s_2 s_1$  and  $s_1 \in \text{supp}(w_{d-\lfloor \frac{m}{n} \rfloor})$ .

1	...	...	1	2	3	...	3
2	...	2	3				

Since  $2 \leq \mu(2) \leq \lfloor \frac{m}{n} \rfloor$ , we have

$$w_1^{-1} \cdots w_{d-\mu(2)}^{-1} \text{wt}(\mathbf{b}_{d-\mu(2)+1}) = (0, 1, 1) \quad \text{and} \quad w_1^{-1} \cdots w_{d-1}^{-1} \text{wt}(\mathbf{b}_d) = (1, 0, 1).$$

Thus  $\lambda(\mathbf{b}) \notin W_0\mu$  because  $\mu(n) = 0$ . This proves the case  $n = 3$ .

In the rest of the proof, we assume that  $n \geq 4$ . Let  $\lambda$  be a conjugate of  $\lambda_b$  such that  $(\lambda(1), \lambda(2), \lambda(3)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$  and  $\lambda(4) \geq \cdots \geq \lambda(n)$ . Set

$$\mu_0 = (3\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1 - \min\{\mu(2), \lfloor \frac{m}{n} \rfloor\}, \min\{\mu(2), \lfloor \frac{m}{n} \rfloor\}, 0, \dots, 0) \in X_*(T)_+$$

and  $\lambda_0 = (\lambda(1), \lambda(2), \lambda(3), 0, \dots, 0) \in X_*(T)$ . Note that we have  $\mu(1) + \mu(2) \geq 3\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$ . Indeed if  $\mu(1) + \mu(2) \leq 3\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor$ , then by  $\mu(1) \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$ , we have  $\mu(2) \leq \lfloor \frac{m}{n} \rfloor - 1$ . This implies  $\mu(3) + \cdots + \mu(n-1) \leq (n-3)(\lfloor \frac{m}{n} \rfloor - 1)$ , or equivalently  $3\lfloor \frac{m}{n} \rfloor + n + m_0 - 3 \leq \mu(1) + \mu(2)$ , which is a contradiction. Thus  $Y_\mu$  contains  $Y_{\mu_0}$ .

Let  $\mathbf{b}_0$  be the unique element in  $\mathbb{B}_{\mu_0}(\lambda_0)$  whose second row contains exactly one  $\boxed{3}$ . We will show that there exists  $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$  that contains  $\mathbf{b}_0$ . It is easy to check





$j$ . So the tableau obtained by replacing  $\mathbf{b}'_1$  by the rightmost one among such  $\boxed{j-1}$  is semi-standard. Repeating the same argument, we may assume  $j = 4$ . Similarly, if  $\lfloor \frac{m}{n} \rfloor \geq 3$ , we may also assume  $j' = 4$ . Indeed if  $j' \geq 6$  and the leftmost column in  $\mathbf{b}'$  contains  $\boxed{j'-1}$  but does not contain  $\boxed{j'}$ , we replace  $\mathbf{b}'_2$  by this  $\boxed{j'-1}$ . In other cases, by  $\lfloor \frac{m}{n} \rfloor \geq 3$ , there exists at least one  $\boxed{j'-1}$  such that there is no box beneath it or the number in the box beneath it is greater than  $j'$ , and we replace  $\mathbf{b}'_2$  by the rightmost  $\boxed{j'-1}$  among such  $\boxed{j'-1}$ . Then the obtained tableau is semi-standard. Thus if  $\lfloor \frac{m}{n} \rfloor \geq 3$ , there exists  $\mathbf{b}'$  containing  $\mathbf{b}_0$  such that  $k(\mathbf{b}') < \lfloor \frac{m}{n} \rfloor$ , which is non-cyclic by the above argument. If  $\lfloor \frac{m}{n} \rfloor = 2$  and  $n = 4$ , then  $\mathbf{b}$  is non-cyclic because  $k(\mathbf{b}') < 2$ . If  $\lfloor \frac{m}{n} \rfloor = 2$  and  $n \geq 5$ , we may also assume  $j' = 4$  and hence  $\mathbf{b}$  is non-cyclic unless the third row of  $\mathbf{b}'$  contains three  $\boxed{5}$ . If  $\lfloor \frac{m}{n} \rfloor = 2, n \geq 5$  and the third row of  $\mathbf{b}'$  contains three  $\boxed{5}$ , then

$$(w_1^{-1} \cdots w_{d-2}^{-1} \text{wt}(\mathbf{b}_{d-1}))(4) = 1 \quad \text{and} \quad (w_1^{-1} \cdots w_{d-1}^{-1} \text{wt}(\mathbf{b}_d))(4) = 0.$$

Thus  $\lambda(\mathbf{b}) \notin W_0\mu$  and hence  $\mathbf{b}$  is non-cyclic.

1	1	2	3	3				
2	3	4	4					
4	5	5	5					

Assume that  $\lfloor \frac{m}{n} \rfloor = 2$  and  $\mu(3) = 1$ . By the same argument as above, we may assume that the leftmost column of  $\mathbf{b}'$  contains  $\boxed{4}$ . So  $\mathbf{b}$  is non-cyclic when  $\lambda(4) = 2$ . If  $\mu(1) > 5 + \lfloor \frac{2m_0}{n} \rfloor$ , we may assume that the first row of  $\mathbf{b}'$  also contains  $\boxed{4}$ . This can be checked easily as above using  $\mu(3) = 1$ . Thus if  $\mu(1) > 5 + \lfloor \frac{2m_0}{n} \rfloor$ , we obtain a non-cyclic  $\mathbf{b}$ .

1	1	2	3	3	4			
2	3	4						
4								

If  $\mu(1) = 5 + \lfloor \frac{2m_0}{n} \rfloor$ , then we have  $n = 4$  or  $5$ . More precisely, we have

$$\mu = (6, 4, 1, 0), (5, 5, 1, 1, 0), (6, 5, 1, 1, 0), (6, 6, 1, 0, 0), \text{ or } (6, 6, 1, 1, 0),$$

and  $\mathbf{b}'$  contains one of the following smaller Young tableaux when  $\lambda(4) = 3$ .

1	1	2	3	3
2	3	4	4	
4				

1	1	2	2	3	3
2	3	4	4		
4					

We can easily check that  $\mathbf{b}$  is non-cyclic in every case.

Putting things together, we have proved the lemma.  $\square$

**Lemma 5.12.** Assume that  $n \geq 4$ . Let  $\mu \in X_*(T)_+$  such that  $d \geq 3 + \lfloor \frac{2m_0}{n} \rfloor$ ,  $\mu(2) \geq 2$  and  $\lfloor \frac{m}{n} \rfloor = 1$ . Then  $\mathbb{B}_\mu(\lambda_b)$  contains at least one non-cyclic element.

*Proof.* Let  $\lambda$  be a conjugate of  $\lambda_b$  such that  $(\lambda(1), \lambda(2), \lambda(3)) = (\lambda_b(1), \lambda_b(2), \lambda_b(3))$  and  $\lambda(4) \geq \dots \geq \lambda(n)$ . Assume that  $(\lambda_b(1), \lambda_b(2), \lambda_b(3)) = (1, 2, 2)$  and  $\mu(2) \geq 3$ . Similarly as the proof of Lemma 5.11, we can easily show that there exists  $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$  containing the following smaller Young tableau.

1	2	3	4
2	3	4	

Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  be the conjugate of  $\mathbf{b}'$ . If  $\mu(3) < 2$ , then  $\mathbf{b}$  is non-cyclic because  $\lambda(\mathbf{b})(2) = 2$ . If  $\mu(3) \geq 2$ , then similarly as the proof of Lemma 5.11, we may assume that the second row of  $\mathbf{b}'$  does not contain  $\boxed{5}$ . In this case, the conjugate  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  of  $\mathbf{b}'$  is non-cyclic because

$$(w_1^{-1} \cdots w_{d-3}^{-1} \text{wt}(\mathbf{b}_{d-2}))(3) = 1 \quad \text{and} \quad (w_1^{-1} \cdots w_{d-1}^{-1} \text{wt}(\mathbf{b}_d))(3) = 0.$$

Assume that  $(\lambda_b(1), \lambda_b(2), \lambda_b(3)) = (1, 2, 2)$  and  $\mu(2) = 2$ . Then there exists  $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$  containing one of the following smaller Young tableaux.

1	2	3	3	4
2	4			

1	2	3	4
2	4		
3			

It is easy to check that the conjugate  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  of  $\mathbf{b}'$  is non-cyclic.

Assume that  $(\lambda_b(1), \lambda_b(2), \lambda_b(3)) \neq (1, 2, 2)$ . Then there exists  $\mathbf{b}' \in \mathbb{B}_\mu(\lambda)$  containing one of the following smaller Young tableaux.

1	2	3	4
2	4		

1	3	3	4
2	4		

1	3	4
2	4	

Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  be the conjugate of  $\mathbf{b}'$ . Since  $\lambda(\mathbf{b})(1) = 1$ ,  $\mathbf{b}$  is non-cyclic if  $\mu(3) = 0$ . If  $\mu(3) \geq 2$ , then similarly as the proof of Lemma 5.11, we may assume that the second row of  $\mathbf{b}'$  does not contain  $\boxed{5}$ . In this case,  $\mathbf{b}$  is non-cyclic because

$$(w_1^{-1} \cdots w_{d-2}^{-1} \text{wt}(\mathbf{b}_{d-1}))(3) = 1 \quad \text{and} \quad (w_1^{-1} \cdots w_{d-1}^{-1} \text{wt}(\mathbf{b}_d))(3) = 0.$$

If  $\mu(3) = 1$  and  $\mu(1) > 3 + \lfloor \frac{2m_0}{n} \rfloor$ , then we may also assume that the second row of  $\mathbf{b}'$  does not contain  $\boxed{5}$  and hence  $\mathbf{b}$  is non-cyclic. If  $\mu(3) = 1$  and  $\mu(1) = 3 + \lfloor \frac{2m_0}{n} \rfloor$ , then we may assume that the leftmost column of  $\mathbf{b}'$  contains  $\boxed{5}$ . We can easily check that  $\mathbf{b}$  is non-cyclic by an easy calculation.

1	2	3	4
2	4	5	
5			

1	3	3	4
2	4	5	
5			

1	3	4
2	4	5
5		

This finishes the proof.  $\square$

**Lemma 5.13.** Assume that  $n \geq 5$ . Let  $\mu \in X_*(T)_+$  such that  $\lfloor \frac{m}{n} \rfloor = 0$ . If (1)  $\mu(2) \geq 2$  or (2)  $d \geq 3, \mu(2) = 1$ , then  $\mathbb{B}_\mu(\lambda_b)$  contains at least one non-cyclic element.

*Proof.* Let  $1 < i_1 < i_2 < \dots < i_{m_0} = n$  be the integers such that  $\lambda_b(i_1) = \lambda_b(i_2) = \dots = \lambda_b(i_{m_0}) = 1$ . Let  $\mathbf{b}$  be the Young tableau in  $\mathbb{B}_\mu(\lambda_b)$  obtained by filling  $Y_\mu$  with  $i_1, \dots, i_{m_0}$  from top to bottom, starting from the leftmost column.

$i_1$	$i_{k+1}$	$\dots$	$i_m$
$i_2$	$i_{k+2}$	$\vdots$	
$\vdots$	$\vdots$		
$i_k$			

If (1) holds, then  $\mathbf{b}$  is non-cyclic because

$$\text{wt}(\mathbf{b}_1)(i_m) = 1 \quad \text{and} \quad (w_1^{-1} \dots w_{d-1}^{-1} \text{wt}(\mathbf{b}_d))(i_m) = 0.$$

Let  $k = \max\{i \mid \mu(i) \neq 0\}$ . If (2) holds, then the Young tableau  $\mathbf{c} \in \mathbb{B}_\mu(\lambda_b)$  obtained by replacing  $\boxed{i_k}$  by  $\boxed{i_{k+1}}$  in  $\mathbf{b}$  is non-cyclic because  $\lambda(\mathbf{c})(i_k) = 2$ .

$i_1$	$i_k$	$i_{k+2}$	$\dots$	$i_m$
$\vdots$				
$i_{k-1}$				
$i_{k+1}$				

This finishes the proof.  $\square$

**Theorem 5.14.** Every  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  is cyclic if and only if  $\mu$  has one of the following forms:

- (i)  $\omega_i$  with  $1 \leq i \leq n-1$  such that  $i$  is coprime to  $n$ .
- (ii)  $\omega_1 + \omega_i$  or  $\omega_{n-1} + \omega_{n-i}$  with  $1 \leq i \leq n-1$  such that  $i+1$  is coprime to  $n$ .
- (iii)  $(nr+i)\omega_1$  or  $(nr+i)\omega_{n-1}$  with  $r \geq 0$  and  $1 \leq i \leq n-1$  such that  $i$  is coprime to  $n$ .
- (iv)  $(nr+i-j)\omega_1 + \omega_j$  or  $(nr+i-j)\omega_{n-1} + \omega_{n-j}$  with  $r \geq 1$ ,  $2 \leq j \leq n-1$  and  $1 \leq i \leq n-1$  such that  $i$  is coprime to  $n$ .

*Proof.* It is easy to check that every  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  is cyclic if  $\mu$  is one of the cocharacters in (i), (ii), (iii) and (iv). It remains to show that if  $\mu$  does not belong to the list above, then  $\mathbb{B}_\mu(\lambda_b)$  contains at least one non-cyclic element. By Lemma 5.7 and Lemma 5.10, we may assume that  $d \geq 2\lfloor \frac{m}{n} \rfloor + \lfloor \frac{2m_0}{n} \rfloor + 1$ . Then this follows from Lemma 5.11, Lemma 5.12 and Lemma 5.13.  $\square$

**Remark 5.15.** Even if every top extended semi-module for  $\mu$  is cyclic, there might be a non-cyclic extended semi-module for  $\mu$ . In fact, such cases exist; see §7.4.

## 6 Proof of Theorem 5.4

Keep the notation and assumptions above. In §6.1, we collect some properties of Coxeter elements and  $c^m$ . We need these facts to study  $v^{-1}\lambda_b^-$  (and hence  $v^{-1}\mathbf{b}^-$ ) in §6.2 because  $v$  is an element defined by measuring the difference between  $w(\mathbf{b})$  and  $c^m$ . In §6.3, we examine the relationship between  $w(\mathbf{b})$  and the computation of  $\varepsilon_i(v^{-1}\mathbf{b}^-)$  from  $\varepsilon_i(\mathbf{b})$ . In §6.4, we will establish some inequalities on  $\varepsilon_i(v^{-1}\mathbf{b}^-)$  from this computation. These inequalities are the key to the proof of  $\xi_l(\mathbf{b}, v)_{vw_1^{-1}w_2^{-1}\dots w_{l-1}^{-1}\chi_{i,j}} \geq 0$  for all  $\chi_{i,j} \in \Phi_+$ ; see §6.5 for details. By definition, this is equivalent to  $v_{\xi_l(\mathbf{b}, v)} = vw_1^{-1}w_2^{-1}\dots w_{l-1}^{-1}$  (cf. §5.1). In §6.6, we finish the proof of Theorem 5.4 using tensor structure of crystals.

### 6.1 Coxeter Elements and $c^m$

Every Coxeter element in the symmetric group  $W_0$  is a cycle of length  $n$ . The next lemma says that the numbers  $1, 2, \dots, j$  (resp.  $n, n-1, \dots, n-j$ ) appearing in the cycle corresponding to a Coxeter element are “successive”.

**Lemma 6.1.** If  $w \in W_0$  is a Coxeter element, then for any  $1 \leq j \leq n$ , there exists  $1 \leq i \leq j$  (resp.  $n-j+1 \leq i \leq n$ ) such that  $\{i, w(i), \dots, w^{j-1}(i)\} = \{1, 2, \dots, j\}$  (resp.  $\{i, w(i), \dots, w^{j-1}(i)\} = \{n, n-1, \dots, n-j+1\}$ ).

*Proof.* It suffices to prove the case for  $\{1, 2, \dots, j\}$ . We argue by induction on  $n$ . If  $n = 2$ , the statement is obvious. Suppose it is true for  $n - 1$ . Then any Coxeter element in  $W_0$  can be written as a product of  $w$  and  $s_{n-1}$  such that  $w$  is a Coxeter element of the symmetric group of degree  $n - 1$ . The case for  $j = n - 1$ ,  $n$  is obvious. If  $1 \leq j < n - 1$ , then by the induction hypothesis, it is easy to check that the statement holds for both  $ws_{n-1}$  and  $s_{n-1}w$ . This completes the proof.  $\square$

Let  $w$  be a Coxeter element in  $W_0$ . Fix a reduced expression  $w = s_{j_1}s_{j_2}\cdots s_{j_{n-1}}$ . Then  $s_{j_h}s_{j_{h-1}} \leq w$  (resp.  $s_{j_h}s_{j_{h+1}} \leq w$ ) if and only if  $j_{h'} = j_h - 1$  (resp.  $j_h + 1$ ) for some  $h' > h$ .

**Corollary 6.2.** Let  $w$  be a Coxeter element in  $W_0$ . Set  $s_0 = s_n = 1$ .

- (i) If  $s_i s_{i-1} \leq w$  and  $s_i s_{i+1} \leq w$ , then

$$\{i, w(i), \dots, w^i(i)\} = \{1, 2, \dots, i + 1\}$$

and  $w^i(i) = i + 1$ .

- (ii) If  $s_{i-1}s_i \leq w$  and  $s_i s_{i+1} \leq w$ , then  $w(i) = i + 1$ .

- (iii) If  $s_i s_{i-1} \leq w$  and  $s_{i+1}s_i \leq w$ , then  $w(i + 1) = i$ .

- (iv) If  $s_{i-1}s_i \leq w$  and  $s_{i+1}s_i \leq w$ , then

$$\{i + 1, w(i + 1), \dots, w^i(i + 1)\} = \{1, 2, \dots, i + 1\}$$

and  $w^i(i + 1) = i$ .

*Proof.* If  $s_i s_{i-1} \leq w$  and  $s_i s_{i+1} \leq w$ , then  $w^{-1}(i) \geq i + 1$ . So by Lemma 6.1, we have  $\{i, w(i), \dots, w^{i-1}(i)\} = \{1, 2, \dots, i\}$ . Moreover if  $i < n - 1$ , then  $w^{-1}(i) > i + 1$ . Again by Lemma 6.1, we have  $w^i(i) = i + 1$ . This proves (i). Note that  $s_j$  with  $j \neq i - 1, i, i + 1$  does not affect  $i, i + 1$ . The assertion of (ii) follows immediately from this. The proof of (iii) and (iv) is similar.  $\square$

The following facts on  $c^m$  are also useful.

**Lemma 6.3.** Let  $1 \leq r < m_0$  be the residue of  $n$  modulo  $m_0$ . Let  $i_k$  be as in §5.2.

- (i) We have  $\{c^{(i_1-1)m}(1), c^{(i_2-1)m}(1), \dots, c^{(i_{m_0-1}-1)m}(1)\} = \{n - m_0 + 1, n - m_0 + 2, \dots, n\}$ .

- (ii) For any  $1 \leq k \leq m_0 - 1$ ,  $c^{(i_k-1)m}(1) - c^{(i_{k+1}-1)m}(1)$  is congruent to  $n$  modulo  $m_0$ . This is also true for  $c^{(i_{m_0-1}-1)m}(1) - c^{(i_1-1)m}(1)$ .

- (iii) For any  $1 \leq k \leq m_0$ ,  $i_k - i_{k-1}$  is equal to  $i_1$  or  $i_1 - 1$  according to whether  $c^{(i_k-1)m}(1) > n - r$  or  $c^{(i_k-1)m}(1) \leq n - r$ .

*Proof.* By the definition of  $i_k$ , we have  $c^{im}(1) = 1 + im_0 - (k-1)n$  for  $i_{k-1} \leq i < i_k$ . The assertion of (ii) follows immediately from this. Note that  $c^{im}(1) > n - m_0$  if and only if  $c^{(i+1)m}(1) < c^{im}(1)$ . This implies (i).

Fix  $k$ . By (i),  $c^{(i_k-1)m}(1) > n - m_0$  and  $1 \leq c^{(i_k-1-j)m}(1) = c^{(i_k-1)m}(1) - jm_0 \leq n - m_0$  for  $1 \leq j < i_k - i_{k-1}$ . So  $i_k - i_{k-1}$  is equal to the minimal integer  $i$  such that  $c^{(i_k-1)m}(1) - im_0 < 0$ . Again by the definition of  $i_1$ ,  $i = i_1$  if  $c^{(i_k-1)m}(1) = n$ . Thus  $i_k - i_{k-1} = i_1$  (resp.  $i_1 - 1$ ) if and only if  $c^{(i_k-1)m}(1) > n - r$  (resp.  $c^{(i_k-1)m}(1) \leq n - r$ ).  $\square$

In below, let  $X_{>a}$  denote the set  $\{x \in X \mid x > a\}$  for a set  $X \subset \mathbb{Z}$  and an integer  $a$ . The following two lemmas will be used in §6.2.

**Lemma 6.4.** Let  $1 \leq r < m_0$  be the residue of  $n$  modulo  $m_0$ . Fix  $2 \leq k \leq m_0$  and let  $z_k \in \{n - m_0 + 1, n - m_0 + 2, \dots, n\}$ . We define  $z_1, \dots, z_{k-1} \in \{n - m_0 + 1, n - m_0 + 2, \dots, n\}$  such that  $z_1 - z_2, \dots, z_{k-1} - z_k$  are congruent to  $n$  modulo  $m_0$ . Then

$$|\{z_1, z_2, \dots, z_k\}_{>n-r}| \geq |\{c^{(i_{m_0-k+1}-1)m}(1), c^{(i_{m_0-k+2}-1)m}(1), \dots, c^{(i_{m_0}-1)m}(1)\}_{>n-r}|.$$

*Proof.* For an integer  $a$ , let  $n - m_0 + 1 \leq [a]_{m_0} \leq n$  denote its residue modulo  $m_0$ . Set  $Z(z_k) = \{z_1, z_2, \dots, z_k\}$ . If  $z_k < n$ , then we have

$$|\{[z_1 + 1]_{m_0}, [z_2 + 1]_{m_0}, \dots, [z_k + 1]_{m_0}\}_{>n-r}| \geq |Z(z_k)_{>n-r}|.$$

This is obvious if  $n \notin Z(z_k)$ . If  $z_l = n$ , then  $l < k$  and  $[z_{l+1} + 1]_{m_0} = z_{l+1} + 1 = n - r + 1$ . Thus the inequality holds. Note that  $c^{(i_{m_0}-1)m}(1) = n - m_0 + 1$ . So by Lemma 6.3 (ii), we have  $Z(n - m_0 + 1) = \{c^{(i_{m_0-k+1}-1)m}(1), c^{(i_{m_0-k+2}-1)m}(1), \dots, c^{(i_{m_0}-1)m}(1)\}$ . Combining this with the above inequality, we obtain the lemma.  $\square$

**Lemma 6.5.** Let  $1 \leq k \leq m_0$  and let  $i_{k-1} < j \leq i_k$ .

- (i) Let  $1 \leq z \leq n$  such that  $c^{(j-1)m}(z) \leq n - m_0$ . Then

$$\begin{aligned} |\{z, c^m(z), \dots, c^{(j-1)m}(z)\}_{>n-m_0}| &= k \Leftrightarrow z > c^{(j-1)m}(z), \\ |\{z, c^m(z), \dots, c^{(j-1)m}(z)\}_{>n-m_0}| &= k - 1 \Leftrightarrow z < c^{(j-1)m}(z). \end{aligned}$$

- (ii) Let  $1 \leq z \leq n$  such that  $c^{(j-1)m}(z) > n - m_0$ . Then

$$\begin{aligned} |\{z, c^m(z), \dots, c^{(j-1)m}(z)\}_{>n-m_0}| &= k + 1 \Leftrightarrow z > c^{(j-1)m}(z), \\ |\{z, c^m(z), \dots, c^{(j-1)m}(z)\}_{>n-m_0}| &= k \Leftrightarrow z < c^{(j-1)m}(z). \end{aligned}$$

*Proof.* By Lemma 6.3 (i),  $c^{im}(1) > n - m_0$  if and only if  $i = i_k - 1$  for some  $1 \leq k \leq m_0$ . So if  $i_{k-1} < j < i_k$  (resp.  $j = i_k$ ), we have

$$|\{1, c^m(1), \dots, c^{(j-1)m}(1)\}_{>n-m_0}| = k - 1 \text{ (resp. } k\text{)}.$$

For  $1 \leq z \leq n - 1$ , set  $Z = \{z, c^m(z), \dots, c^{(j-1)m}(z)\}$ . For an integer  $a$ , let  $1 \leq [a]_n \leq n$  denote its residue modulo  $n$ . If  $c^{(j-1)m}(z) \neq n - m_0$  (resp.  $c^{(j-1)m}(z) = n - m_0$ ), then  $|\{[z+1]_n, [c^m(z)+1]_n, \dots, [c^{(j-1)m}(z)+1]_n\}_{>n-m_0}| = |Z_{>n-m_0}|$  (resp.  $|Z_{>n-m_0}|+1$ ). Note that  $z = [1+(z-1)]_n$ ,  $c^m(z) = [c^m(1)+(z-1)]_n, \dots, c^{(j-1)m}(z) = [c^{(j-1)m}(1)+(z-1)]_n$ . Thus, as in the proof of Lemma 6.4, we can verify the lemma by adding 1 to  $\{1, c^m(1), \dots, c^{(j-1)m}(1)\}$  repeatedly.  $\square$

## 6.2 Allowed Cocharacters

Let  $\lambda$  be a conjugate of  $\lambda_b$ . We say  $\lambda$  is *allowed* if there exists a partial Coxeter element  $w$  such that  $w$  has a reduced expression  $s_{j_1}s_{j_2}\cdots s_{j_h}$  satisfying  $\langle \chi_{j_h, j_h+1}, \lambda_b \rangle = -1$ ,  $\langle \chi_{j_{h-1}, j_{h-1}+1}, s_{j_h}\lambda_b \rangle = -1, \dots, \langle \chi_{j_1, j_1+1}, s_{j_2}\cdots s_{j_h}\lambda_b \rangle = -1$  and  $\lambda = w\lambda_b$ . This means that  $\lambda$  is obtained from  $\lambda_b$  by multiplying each simple reflection at most once and moving  $\lfloor \frac{m}{n} \rfloor + 1$  from right to left. For allowed  $\lambda$ , such  $w$  is unique, and the same holds for any reduced expression of  $w$ . We call this  $w$  the partial Coxeter element associated to  $\lambda$ . In below, let  $c_i$  (resp.  $c'_i$ ) denote the cardinality of the set  $\{j \mid 1 \leq j \leq i, \lambda(j) = \lfloor \frac{m}{n} \rfloor + 1\}$  (resp.  $\{j \mid i \leq j \leq n, \lambda(j) = \lfloor \frac{m}{n} \rfloor + 1\}$ ).

**Lemma 6.6.** Let  $\lambda$  be a conjugate of  $\lambda_b$ . Then  $\lambda$  is allowed if and only if  $c_{i_{k-1}} \leq k$  and  $c'_{i_{m_0-k}+1} \leq k$  for all  $1 \leq k \leq m_0$ . Fix  $1 \leq k \leq m_0$ . If  $\lambda$  is allowed, then for  $i_{k-1} \leq i < i_k$ ,  $i \in \text{supp}(w)$  if and only if  $c_i = k$ , where  $w$  is the partial Coxeter element associated to  $\lambda$ .

*Proof.* Note that if  $\lambda = \lambda_b$ , then  $c_{i_{k-1}} = k - 1$  and  $c'_{i_{m_0-k}+1} = k$  for any  $k$ . If  $\lambda = w\lambda_b$  is allowed, then by  $w$ ,  $c_{i_{k-1}}$  increases at most once, and  $c'_{i_{m_0-k}+1}$  does not increase. So we have  $c_{i_{k-1}} \leq k$  and  $c'_{i_{m_0-k}+1} \leq k$  for all  $k$ . Conversely, if  $c_{i_{k-1}} \leq k$  and  $c'_{i_{m_0-k}+1} \leq k$  for any  $k$ , then in particular, we have  $c_{i_{k-1}} \leq k$  and  $c'_{i_k+1} \leq m_0 - k$ . The latter implies that  $k \leq c_{i_k}$ . So we deduce that  $c_{i_{k-1}} = k - 1$  or  $k$ . If  $c_{i_{k-1}} = k - 1$ , set  $t_k = 1$ . If  $c_{i_{k-1}} = k$ , then  $\{j \mid i_{k-1} \leq j < i_k, \lambda(j) = \lfloor \frac{m}{n} \rfloor + 1\}$  is non-empty, and contains at most two elements. Let  $j_k$  be the greater one among them, and set  $t_k = s_{j_k}s_{j_k+1}\cdots s_{i_{k-1}}$ . It is easy to check that  $\lambda = t_{m_0}\cdots t_2t_1\lambda_b$  and  $\lambda$  is allowed as desired.

Fix  $1 \leq k \leq m_0$  and assume that  $\lambda$  is allowed. For  $i_{k-1} \leq i < i_k$ , we have  $c_{i_{k-1}} \leq c_i \leq c_{i_{k-1}}$ . By the above discussion, we have  $c_i = k - 1$  or  $k$ . Since  $c_i = k - 1$  if  $\lambda = \lambda_b$ , the last assertion follows immediately from the definition of allowed cocharacters.  $\square$



**Lemma 6.7.** Let  $v \in W_0$  such that  $v^{-1}c^mv$  is a Coxeter element. Then  $v^{-1}\lambda_b^-$  is allowed.

The strategy of the proof is the same as the case for  $n = 5$  in §5.3. The key observations are the following: As a  $n$ -cycle, the numbers in a Coxeter element are successive (Lemma 6.1). On the other hand, the numbers greater than  $n - m_0$  in  $c^m$  are apart enough (Lemma 6.3 (iii) and Lemma 6.4).

*Proof.* Set  $\lambda = v^{-1}\lambda_b^-$ , and let  $c_i$  be as above. By Lemma 6.6, we need to show that if  $v^{-1}c^mv$  is a Coxeter element, then  $c_{i_k-1} \leq k$  and  $c'_{i_{m_0-k}+1} \leq k$  for all  $1 \leq k \leq m_0$ . For this, it suffices to show that for any  $k$  and  $1 \leq z \leq n$ , there are at most  $k$  elements greater than  $n - m_0$  among  $z, c^m(z), \dots, c^{(i_k-2)m}(z)$ . Indeed, by Lemma 6.1 and the assumption that  $w := v^{-1}c^mv$  is a Coxeter element, there exists  $j$  (resp.  $j'$ ) such that  $\{j, w(j), \dots, w^{i_k-2}(j)\} = \{1, 2, \dots, i_k - 1\}$  (resp.  $\{j', w(j'), \dots, w^{i_k-2}(j')\} = \{n, n-1, \dots, n - i_k + 2\}$ ) and

$$c^m = v w v^{-1} = (\dots v(j) \ v(w(j)) \ \dots \ v(w^{i_k-2}(j)) \ \dots)$$

Since  $\lambda(i) = \lambda_b^-(v(i))$  and  $n - i_k + 2 = i_{m_0-k} + 1$ , both  $c_{i_k-1}$  and  $c'_{i_{m_0-k}+1}$  are equal to the number of integers greater than  $n - m_0$  appearing in  $z, c^m(z), \dots, c^{(i_k-2)m}(z)$  for some  $z$ .

If  $c^{(i_k-2)m}(z) \leq n - m_0$ , then we have

$$|\{c^{-m}(z), z, \dots, c^{(i_k-3)m}(z)\}_{>n-m_0}| \geq |\{z, c^m(z), \dots, c^{(i_k-2)m}(z)\}_{>n-m_0}|.$$

So we may replace  $z$  by  $c^{-m}(z)$ . Repeating this, we may assume  $c^{(i_k-2)m}(z) > n - m_0$ . It follows from Lemma 6.3 (i) and (iii) that if  $j$  is the minimal positive integer such that  $c^{-jm}(z') > n - m_0$  for some  $n - m_0 < z' \leq n$ , then  $j = i_1$  or  $i_1 - 1$  according to whether  $z' > n - r$  or  $z' \leq n - r$ . So in particular, our claim is true for  $k = 1$ . To show the case for  $2 \leq k \leq m_0$ , we argue by contradiction. Suppose  $|\{z, c^m(z), \dots, c^{(i_k-2)m}(z)\}_{>n-m_0}| > k$ . Set  $z_k = c^{(i_k-2)m}(z)$  and define  $z_1, \dots, z_{k-1}$  as in Lemma 6.4. By Lemma 6.3, we have  $z_{j-1} = c^{-i_1 m}(z_j)$  or  $c^{-(i_1-1)m}(z_j)$  for  $2 \leq j \leq k$ . Set

$$z_0 = \begin{cases} c^{-(i_1-1)m}(z_1) & (\text{if } c^{-(i_1-1)m}(z_1) > n - m_0) \\ c^{-i_1 m}(z_1) & (\text{if } c^{-(i_1-1)m}(z_1) \leq n - m_0). \end{cases}$$

Then  $z_0 > n - m_0$ . By  $|\{z, c^m(z), \dots, c^{(i_k-2)m}(z)\}_{>n-m_0}| > k$ , we have

$$\{z_0, z_1, \dots, z_k\} \subseteq \{z, c^m(z), \dots, c^{(i_k-2)m}(z)\}_{>n-m_0}.$$

Thus

$$\begin{aligned} & |\{z, c^m(z), \dots, c^{(i_k-2)m}(z)\}| \\ & \geq |\{z_0, c^m(z_0), c^{2m}(z_0), \dots, z_1, \dots, c^m(z_{k-1}), c^{2m}(z_{k-1}), \dots, z_k\}|. \end{aligned}$$

As explained above, the latter number is determined by  $|\{z_1, z_2, \dots, z_k\}_{>n-r}|$ . So by Lemma 6.4, we have

$$\begin{aligned} & |\{z_0, c^m(z_0), c^{2m}(z_0), \dots, z_1, \dots, c^m(z_{k-1}), c^{2m}(z_{k-1}), \dots, z_k\}| \\ & \geq |\{c^{(i_{m_0-k}-1)m}(1), c^{i_{m_0-k}m}(1), \dots, c^{(i_{m_0}-1)m}(1)\}| = n - i_{m_0-k} + 1 = i_k. \end{aligned}$$

This combined with the above inequality contradicts to  $|\{z, c^m(z), \dots, c^{(i_k-2)m}(z)\}| = i_k - 1$ . Therefore there are at most  $k$  elements greater than  $n - m_0$  among the sequence  $z, c^m(z), \dots, c^{(i_k-2)m}(z)$  for any  $z$ . This completes the proof.  $\square$

Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ . Then  $v^{-1}\lambda_b^-$  is allowed by Lemma 6.7. We denote by  $w_v$  the partial Coxeter element associated to  $v^{-1}\lambda_b^-$ . By Lemma 4.12, we can compute  $v^{-1}\mathbf{b}^-$  from  $\mathbf{b}$  by  $w_v$ . The following corollary will be used frequently in §6.4.

**Corollary 6.8.** Set  $s_0 = s_n = 1$ . Assume that  $\langle \chi_{i,i+1}, v^{-1}\lambda_b^- \rangle = 0$ , i.e.,  $v(i), v(i+1) \leq n - m_0$  or  $v(i), v(i+1) > n - m_0$ .

- (i) Assume that  $s_i s_{i-1} \leq w(\mathbf{b})$  and  $s_i s_{i+1} \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  if and only if  $i \in \text{supp}(w_v)$ .
- (ii) Assume that  $s_{i-1} s_i \leq w(\mathbf{b})$  and  $s_i s_{i+1} \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  if and only if  $v(i), v(i+1) > n - m_0$ .
- (iii) Assume that  $s_i s_{i-1} \leq w(\mathbf{b})$  and  $s_{i+1} s_i \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  if and only if  $v(i), v(i+1) \leq n - m_0$ .
- (iv) Assume that  $s_{i-1} s_i \leq w(\mathbf{b})$  and  $s_{i+1} s_i \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  if and only if  $i \notin \text{supp}(w_v)$ .

*Proof.* By the definition of  $v$ , we have  $vw(\mathbf{b})v^{-1} = c^m$ . Set  $z = v(i)$ . Then  $v(w(\mathbf{b})(i)) = c^m(z), \dots, v(w(\mathbf{b})^{j-1}(i)) = c^{(j-1)m}(z)$  for any  $j$ . The assertions of (ii) and (iii) follow immediately from this and Corollary 6.2.

It remains to prove (i) and (iv). We only prove (i), and the proof of (iv) is similar. Assume that  $v(i), v(i+1) \leq n - m_0$  (resp.  $v(i), v(i+1) > n - m_0$ ). Let  $k$  such that  $i_{k-1} \leq i < i_k$ . By Corollary 6.2 (i) and Lemma 6.5,  $|\{v(1), v(2), \dots, v(i+1)\}_{>n-m_0}| = |\{v(i), v(w(\mathbf{b})(i)), \dots, v(w(\mathbf{b})^i(i))\}_{>n-m_0}| = k$  (resp.  $k+1$ ) if and only if  $v(i) > v(i+1)$ . By Corollary 6.2 (i) and Lemma 6.6,  $|\{v(1), \dots, v(i+1)\}_{>n-m_0}| = k$  (resp.  $k+1$ ) if and only if  $i \in \text{supp}(w_v)$ . This proves (i).  $\square$

For  $\text{supp}(w_v)$ , we also have the following lemma:

**Lemma 6.9.** Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ . Fix a reduced expression  $s_{j_1}s_{j_2}\cdots s_{j_{n-1}}$  of  $w(\mathbf{b})$ . For  $1 \leq h \leq n-1$ ,  $j_h \in \text{supp}(w_v)$  if and only if  $vs_{j_1}s_{j_2}\cdots s_{j_{h-1}}\chi_{j_h,j_h+1} \in \Phi_-$ .

*Proof.* Set  $\lambda = v^{-1}\lambda_b^-$ , and let  $c_i$  be as above. Assume  $i_{k-1} \leq j_h < i_k$ . By Lemma 6.6,  $j_h \in \text{supp}(w_v)$  if and only if  $c_{j_h} = k$ .

Set  $\underline{j} = s_{j_1}s_{j_2}\cdots s_{j_{h-1}}(j_h)(\leq j_h)$  and  $\bar{j} = s_{j_1}s_{j_2}\cdots s_{j_{h-1}}(j_h+1)(\geq j_h+1)$ . Since  $w(\mathbf{b})^{-1}(\underline{j}) \geq j_h+1$  and  $w(\mathbf{b})^{-1}(\bar{j}) \leq j_h$ , we have

$$\{\underline{j}, w(\mathbf{b})(\underline{j}), \dots, w(\mathbf{b})^{j_h-1}(\underline{j})\} = \{1, 2, \dots, j_h\}$$

and  $w(\mathbf{b})^{j_h}(\underline{j}) = \bar{j}$  by Lemma 6.1. If  $\lambda(\bar{j}) = \lambda_b^-(v(\bar{j})) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lfloor \frac{m}{n} \rfloor + 1$ ), then  $|\{v(1), v(2), \dots, v(j_h)\}_{>n-m_0}| = k$  is equivalent to  $|\{v(1), v(2), \dots, v(j_h), v(\bar{j})\}_{>n-m_0}| = k$  (resp.  $k+1$ ). Thus by  $vw(\mathbf{b})v^{-1} = c^m$  and Lemma 6.5 (i) (resp. (ii)) for  $j = j_h+1$ ,  $c_{j_h} = k$  if and only if  $v(\underline{j}) > v(\bar{j})$ , i.e.,  $vs_{j_1}s_{j_2}\cdots s_{j_{h-1}}\chi_{j_h,j_h+1} \in \Phi_-$ .  $\square$

**Corollary 6.10.** Keep the notation in Lemma 6.9. Set  $s_0 = s_n = 1$ .

- (i) Assume that  $s_{j_h}s_{j_{h-1}} \leq w(\mathbf{b})$  and  $s_{j_h}s_{j_{h+1}} \leq w(\mathbf{b})$  for fixed  $h$ . Assume further that there exists  $h < h'$  such that  $j_{h'} = j_h - 1$  (resp.  $j_h + 1$ ) and  $j_h + 1 \notin \{j_1, j_2, \dots, j_{h'-1}\}$  (resp.  $j_h - 1 \notin \{j_1, j_2, \dots, j_{h'-1}\}$ ). If  $\lambda_b^-(v(j_h)) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lambda_b^-(v(j_h + 1)) = \lfloor \frac{m}{n} \rfloor + 1$ ), then  $vs_{j_1}\cdots s_{j_{h'}}\chi_{j_h,j_h+1} \in \Phi_-$ .
- (ii) Assume that  $s_{j_{h-1}}s_{j_h} \leq w(\mathbf{b})$  and  $s_{j_{h+1}}s_{j_h} \leq w(\mathbf{b})$  for fixed  $h$ . Assume further that there exists  $h' < h$  such that  $j_{h'} = j_h - 1$  (resp.  $j_h + 1$ ) and  $j_h + 1 \notin \{j_1, j_2, \dots, j_{h'-1}\}$  (resp.  $j_h - 1 \notin \{j_1, j_2, \dots, j_{h'-1}\}$ ). If  $\lambda_b^-(v(j_h + 1)) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lambda_b^-(v(j_h)) = \lfloor \frac{m}{n} \rfloor + 1$ ), then  $vs_{j_1}\cdots s_{j_{h'}}\chi_{j_h,j_h+1} \in \Phi_-$ .
- (iii) Assume that  $s_{j_h}s_{j_{h-1}} \leq w(\mathbf{b})$  and  $s_{j_h}s_{j_{h+1}} \leq w(\mathbf{b})$  for fixed  $h$ . If  $\langle \chi_{j_h,j_h+1}, v^{-1}\lambda_b^- \rangle = -1$ , then  $j_h \notin \text{supp}(w_v)$ .
- (iv) Assume that  $s_{j_{h-1}}s_{j_h} \leq w(\mathbf{b})$  and  $s_{j_{h+1}}s_{j_h} \leq w(\mathbf{b})$  for fixed  $h$ . If  $\langle \chi_{j_h,j_h+1}, v^{-1}\lambda_b^- \rangle = 1$ , then  $j_h \in \text{supp}(w_v)$ .

*Proof.* Keep the notation in the proof of Lemma 6.9.

Set  $j = s_{j_{n-1}}\cdots s_{j_{h'+1}}(j_h+1)$  (resp.  $s_{j_{n-1}}\cdots s_{j_{h'+1}}(j_h)$ ). Then, by  $vw(\mathbf{b}) = c^mv$ ,  $vs_{j_1}\cdots s_{j_{h'}}\chi_{j_h,j_h+1} = vw(\mathbf{b})\chi_{j_h,j} = c^mv\chi_{j_h,j}$  (resp.  $c^mv\chi_{j,j_h+1}$ ) and  $w(\mathbf{b})(j) = j_h$  (resp.  $w(\mathbf{b})(j) = j_h+1$ ). Moreover, if  $\lambda_b^-(v(j_h)) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lambda_b^-(v(j_h+1)) = \lfloor \frac{m}{n} \rfloor + 1$ ), i.e.,  $v(j_h) \leq n - m_0$  (resp.  $v(j_h+1) > n - m_0$ ), then again by  $vw(\mathbf{b})v^{-1} = c^m$ ,  $v(j) \leq n - m_0$  (resp.  $v(j) > n - m_0$ ) implies  $v(j) < v(j_h)$  (resp.  $v(j) > v(j_h+1)$ ). Combining these facts, we deduce  $vs_{j_1}\cdots s_{j_{h'}}\chi_{j_h,j_h+1} \in \Phi_-$ . The proof of (i) is finished.

We next prove (ii). Recall that we have  $\{\underline{j}, w(\mathbf{b})(\underline{j}), \dots, w(\mathbf{b})^{j_h-1}(\underline{j})\} = \{1, 2, \dots, j_h\}$  and  $w(\mathbf{b})^{j_h}(\underline{j}) = \bar{j}$ . Combining this with Corollary 6.2 (iv), we can easily check that  $w(\mathbf{b})(j_h+1) = \underline{j}$  and  $w(\mathbf{b})(j_h) = \bar{j}$ . Thus, by  $vw(\mathbf{b})v^{-1} = c^m$ , if  $\lambda_b^-(v(j_h+1)) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lambda_b^-(v(j_h)) = \lfloor \frac{m}{n} \rfloor + 1$ ), i.e.,  $v(j_h+1) \leq n - m_0$  (resp.  $v(j_h) > n - m_0$ ), we have  $v(\underline{j}) > v(j_h+1)$  (resp.  $v(j_h) > v(\bar{j})$ ). By our assumption on  $h'$ , this is equivalent to  $vs_{j_1} \cdots s_{j_{h'}} \chi_{j_h, j_h+1} \in \Phi_-$ . The proof of (ii) is finished.

For (iii), by  $s_{j_h} s_{j_h-1} \leq w(\mathbf{b})$ ,  $s_{j_h} s_{j_h+1} \leq w(\mathbf{b})$  and Lemma 6.9,  $j_h \in \text{supp}(w_v)$  if and only if  $v(j_h) > v(j_h+1)$ . Further,  $\langle \chi_{j_h, j_h+1}, v^{-1} \lambda_b^- \rangle = -1$  implies  $v(j_h) \leq n - m_0 < v(j_h+1)$ . Thus  $j_h \notin \text{supp}(w_v)$ . The proof of (iii) is finished.

For (iv), by  $s_{j_h-1} s_{j_h} \leq w(\mathbf{b})$  and  $s_{j_h+1} s_{j_h} \leq w(\mathbf{b})$ , we have  $\underline{j} = w(\mathbf{b})(j_h+1)$  and  $\bar{j} = w(\mathbf{b})(j_h)$ . So by Lemma 6.9,  $j_h \in \text{supp}(w_v)$  if and only if  $vw(\mathbf{b}) \chi_{j_h+1, j_h} \in \Phi_-$ . By  $vw(\mathbf{b}) = c^m v$ , this is equivalent to saying  $c^m v \chi_{j_h+1, j_h} \in \Phi_-$ . This holds if  $\langle \chi_{j_h, j_h+1}, v^{-1} \lambda_b^- \rangle = 1$ , i.e.,  $v(j_h+1) \leq n - m_0 < v(j_h)$ . Thus  $j_h \in \text{supp}(w_v)$ . The proof of (iv) is finished.  $\square$

### 6.3 Computation of Kashiwara Operators

As explained in §5.2, we can compute  $\mathbf{b}^{\text{op}}$  from  $\mathbf{b}$  using each simple reflection exactly once. Consider  $u_i(\mathbf{b})$  defined in Theorem 4.11. In this computation, the action of  $s_i$  changes some  $-$  to  $+$ , and the action of  $s_{i-1}$  (resp.  $s_{i+1}$ ) deletes  $+$  (resp. adds  $-$ ). Other simple reflections do not affect  $u_i(\mathbf{b})$  (and hence  $\varepsilon_i(\mathbf{b})$ ). Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ . Since  $v^{-1} \lambda_b^-$  is allowed, we can use a part of this computation to obtain  $v^{-1} \mathbf{b}^-$  from  $\mathbf{b}$  by  $w_v$ . Let  $\lambda$  be an allowed conjugate of  $\lambda_b$ , and let  $\mathbf{b}'$  be the conjugate of  $\mathbf{b}$  with weight  $\lambda$ . Let  $w$  be the partial Coxeter element associated to  $\lambda$ . Assume  $\text{supp}(w) \subseteq \text{supp}(w_v)$ . Then  $\lambda$  is a weight appearing in the computation of  $v^{-1} \mathbf{b}^-$  from  $\mathbf{b}$ . If  $i \in \text{supp}(w_v) \setminus \text{supp}(w)$  and  $\langle \chi_{i, i+1}, \lambda \rangle = -1$ , then  $\varepsilon_i(s_i \mathbf{b}') = \varepsilon_i(\tilde{e}_i \mathbf{b}') = \varepsilon_i(\mathbf{b}') - 1$  (and hence  $\phi_i(s_i \mathbf{b}') = \phi_i(\mathbf{b}') + 1$  by Definition 4.1 (iii)). For the action of  $s_{i-1}$  or  $s_{i+1}$ , we have the following lemma.

**Lemma 6.11.** Let  $\lambda, \mathbf{b}'$  and  $w$  be as above. Assume that  $\text{supp}(w_v) \setminus \text{supp}(w)$  contains  $i-1$  (resp.  $i+1$ ) and  $\langle \chi_{i-1, i}, \lambda \rangle = -1$  (resp.  $\langle \chi_{i+1, i+2}, \lambda \rangle = -1$ ). If  $s_i s_{i-1} \leq w(\mathbf{b})$  (resp.  $s_i s_{i+1} \leq w(\mathbf{b})$ ), then  $\varepsilon_i(s_{i-1} \mathbf{b}') = \varepsilon_i(\mathbf{b}')$  (resp.  $\varepsilon_i(s_{i+1} \mathbf{b}') = \varepsilon_i(\mathbf{b}')$ ). Moreover, the converse holds if  $(\lambda_b(i), \lambda_b(i+1)) \neq (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ .

Note that  $\varepsilon_i(s_{i-1} \mathbf{b}'), \varepsilon_i(s_{i+1} \mathbf{b}') \in \{\varepsilon_i(\mathbf{b}'), \varepsilon_i(\mathbf{b}') + 1\}$  (and  $\phi_i(s_{i-1} \mathbf{b}'), \phi_i(s_{i+1} \mathbf{b}') \in \{\phi_i(\mathbf{b}') - 1, \phi_i(\mathbf{b}')\}$ ) in any case. Roughly speaking, this says that  $w(\mathbf{b})$  determines  $\varepsilon_i(s_{i-1} \mathbf{b}')$  or  $\varepsilon_i(s_{i+1} \mathbf{b}')$ . Before beginning the proof, let us illustrate why this lemma holds by an example for the case  $\lambda = \lambda_b$ .

**Example 6.12.** Assume that  $\lfloor \frac{m}{n} \rfloor = 7$  and  $(\lambda_b(i-1), \lambda_b(i), \lambda_b(i+1)) = (7, 8, 8)$ . We can easily find  $\mu$  and  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  such that

$$u_i(\mathbf{b}) = - + + - - + - - / + + - + + + - -, \quad u_i(\mathbf{b})_{\text{red}} = - - + + .$$

Here  $(- + + - - + --)_{\text{red}} = --$  and  $(+ + - + + + --)_{\text{red}} = ++$ . If the action of  $s_{i-1}$  deletes  $+$  on the left (resp. right) of  $/$ , then  $\varepsilon_i(s_{i-1}\mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1 = 3$  (resp.  $\varepsilon_i(s_{i-1}\mathbf{b}) = \varepsilon_i(\mathbf{b}) = 2$ ). Let  $u = -$  be the rightmost  $-$  to  $+$  in  $u_i(\mathbf{b})_{\text{red}}$ , or equivalently, the unique  $-$  in  $u_i(\mathbf{b})$  adjacent to  $/$ . Note that if we apply  $s_i$  on  $s_{i-1}\mathbf{b}$ , then  $u$  changes to  $+$ . So the action of  $s_{i-1}$  deletes  $+$  on the left (resp. right) of  $/$  if and only if  $s_{i-1}s_i \leq w(\mathbf{b})$  (resp.  $s_i s_{i-1} \leq w(\mathbf{b})$ ).

We next consider  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  such that

$$u_i(\mathbf{b}) = - + + - - + - - / + - + + + + - -, \quad u_i(\mathbf{b})_{\text{red}} = - - + + .$$

In this case,  $\varepsilon_i(s_{i-1}\mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$  if and only if  $s_{i-1}$  deletes  $+$  on the left of  $/$ , or the unique  $+$  adjacent to  $/$ . Nevertheless, the equivalence  $\varepsilon_i(s_{i-1}\mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1 \Leftrightarrow s_{i-1}s_i \leq w(\mathbf{b})$  (and hence  $\varepsilon_i(s_{i-1}\mathbf{b}) = \varepsilon_i(\mathbf{b}) \Leftrightarrow s_i s_{i-1} \leq w(\mathbf{b})$ ) still holds. Indeed, if  $s_{i-1}$  deletes the unique  $+$  adjacent to  $/$ , then the action of  $s_i$  on  $s_{i-1}\mathbf{b}$  changes  $-$  next to this  $+$ . So we still have  $s_{i-1}s_i \leq w(\mathbf{b})$ , which implies the equivalence.

Assume that  $\lfloor \frac{m}{n} \rfloor = 7$  and  $(\lambda_b(i-1), \lambda_b(i), \lambda_b(i+1), \lambda_b(i+2)) = (7, 8, 7, 8)$ . Consider  $\mu$  and  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  such that

$$u_i(\mathbf{b}) = - + \hat{+} - + - / + + + - - + + - - .$$

We also assume that the action of  $s_{i-1}$  deletes  $\hat{+}$  and the action of  $s_{i+1}$  adds  $-$  to the place where  $/$  exists. Then  $\varepsilon_i(s_{i-1}\mathbf{b}) = \varepsilon_i(s_{i+1}\mathbf{b}) = \varepsilon_i(\mathbf{b}) = 1$ . On the other hand,  $\varepsilon_i(s_{i-1}s_{i+1}\mathbf{b}) = \varepsilon_i(s_{i+1}s_{i-1}\mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1 = 2$ ,  $s_{i-1}s_i \leq w(\mathbf{b})$  and  $s_{i+1}s_i \leq w(\mathbf{b})$ . The difference from the above example is that we apply both  $s_{i-1}$  and  $s_{i+1}$  on  $\mathbf{b}$  before applying  $s_i$  to compute  $\mathbf{b}^{\text{op}}$ . In other words,  $s_i s_{i-1} \leq w'_{\text{max}}$  and  $s_i s_{i+1} \leq w'_{\text{max}}$ .

We generalize the observation in Example 6.12 as follows:

**Lemma 6.13.** Let  $\lambda$  be a conjugate of  $\lambda_b$ . Let  $\mathbf{b}'$  be the conjugate of  $\mathbf{b}$  with weight  $\lambda$ . Assume that  $\langle \chi_{i-1,i}, \lambda \rangle = -1$  (resp.  $\langle \chi_{i+1,i+2}, \lambda \rangle = -1$ ).

- (i) We write  $u_i(\mathbf{b}') = u^1 \dots u^{\lambda(i)+\lambda(i+1)}$  (resp.  $u_i(s_{i+1}\mathbf{b}') = u^1 \dots u^{\lambda(i)+\lambda(i+1)+1}$ ). Let  $u^{\ell_{i-1}} = +$  (resp.  $u^{\ell_{i+1}} = -$ ) be the box which vanishes in  $u_i(s_{i-1}\mathbf{b}')$  (resp.  $u_i(\mathbf{b}')$ ). If  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$  (resp.  $\varepsilon_i(s_{i+1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$ ), then there exists  $u^\ell = -$  with  $\ell_{i-1} < \ell$  (resp.  $\ell_{i+1} \leq \ell$ ) which remains in  $u_i(s_{i-1}\mathbf{b}')_{\text{red}}$  (resp.  $u_i(s_{i+1}\mathbf{b}')_{\text{red}}$ ).
- (ii) Assume that  $\varepsilon_i(\mathbf{b}') > 0$ . Let  $u$  be the rightmost  $-$  to  $+$  in  $u_i(\mathbf{b}')_{\text{red}}$ , and let  $u'$  be the rightmost  $-$  to  $+$  in  $u_i(s_{i-1}\mathbf{b}')_{\text{red}}$  (resp.  $u_i(s_{i+1}\mathbf{b}')_{\text{red}}$ ). If  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$  (resp.  $\varepsilon_i(s_{i+1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$ ), then  $u = u'$  or  $u'$  is on the right side of  $u$ . If  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}')$  (resp.  $\varepsilon_i(s_{i+1}\mathbf{b}') = \varepsilon_i(\mathbf{b}')$ ), then  $u = u'$ .

*Proof.* We only prove the case for  $i - 1$ . The case for  $i + 1$  follows in a similar way.

If  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$ , then there exists  $u^\ell = -$  which remains in  $u_i(s_{i-1}\mathbf{b}')_{\text{red}}$  but does not remain in  $u_i(\mathbf{b}')_{\text{red}}$ . Note that  $u^\ell = -$  with  $\ell < \ell_{i-1}$  remains in  $u_i(s_{i-1}\mathbf{b}')_{\text{red}}$  if and only if it remains in  $u_i(\mathbf{b}')_{\text{red}}$ . So we must have  $\ell_{i-1} < \ell$ . This proves (i). Note that if  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$  (resp.  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}')$ ),  $u(s_{i-1}\mathbf{b}')_{\text{red}}$  is obtained from  $u_i(\mathbf{b}')_{\text{red}}$  by adding one  $-$  (resp. deleting one  $+$ ). The statement of (ii) follows immediately from this.  $\square$

For  $1 \leq i \leq n - 1$ , let  $l_i$  be a positive integer such that  $i \in \text{supp}(w_{l_i})$ .

*Proof of Lemma 6.11.* We only prove the case for  $i - 1$ . The case for  $i + 1$  follows in a similar way.

First assume that  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$ . Then  $s_i s_{i-1} \leq w(\mathbf{b})$  if and only if  $l_i \leq l_{i-1}$ . Since  $\langle \chi_{i-1,i}, \lambda \rangle = -1$  and  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$ , we have  $i \in \text{supp}(w)$ . We write  $u_i(\mathbf{b}') = u^1 \dots u^{\lambda(i)+\lambda(i+1)}$ . Let  $u^{\ell_i} = +$  be the unique  $+$  which does not exist in  $u_i(\mathbf{b})$  (i.e., the box added by the action of  $s_i$ ), and let  $u^{\ell_{i-1}} = +$  be the box which vanishes in  $u_i(s_{i-1}\mathbf{b})$ . Then  $l_i \leq l_{i-1}$  if and only if  $\ell_i \leq \ell_{i-1}$ . Note that  $(u^1 \dots u^{\ell_{i-1}})_{\text{red}} = - \dots -$  (resp.  $(u^{\ell_i} \dots u^{\lambda(i)+\lambda(i+1)})_{\text{red}} = + \dots +$ ) unless no  $-$  (resp.  $+$ ) remains. So if  $\ell_{i-1} < \ell_i$ , then  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$ . This proves the second statement.

For the first statement, we need to show that if  $l_i \leq l_{i-1}$ , then  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}')$ . To show this, we first check that  $(u^{\ell_{i+1}} \dots u^{\lambda(i)+\lambda(i+1)})_{\text{red}} = + \dots +$  unless no  $+$  remains. This claim is obviously true when  $i + 1 \notin \text{supp}(w)$  or  $s_i s_{i+1} \leq w$ . If  $\lambda(i + 1) = \lfloor \frac{m}{n} \rfloor$  and  $\lambda_b(i + 1) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lfloor \frac{m}{n} \rfloor + 1$ ), then  $s_i s_{i+1} \leq w$  (resp.  $i + 1 \notin \text{supp}(w)$ ), and hence the claim holds. If  $\lambda(i + 1) = \lfloor \frac{m}{n} \rfloor + 1$ , then by  $i \in \text{supp}(w)$ , we must have  $\lambda_b(i + 1) = \lfloor \frac{m}{n} \rfloor + 1$  and  $i + 1 \in \text{supp}(w)$ . By our assumption  $\text{supp}(w) \subset \text{supp}(w_v)$ , we also have  $\lambda_b^-(v(i)) = \lfloor \frac{m}{n} \rfloor$ ,  $\lambda_b^-(v(i+1)) = \lambda(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ . If moreover,  $-$  remains in  $(u^{\ell_{i+1}} \dots u^{\lambda(i)+\lambda(i+1)})_{\text{red}}$ , then we must have  $\ell_i < \ell_{i+1}$ , where  $u^{\ell_{i+1}} = -$  be the box added by the action of  $s_{i+1}$ . Clearly,  $\ell_i < \ell_{i+1}$  implies  $s_i s_{i+1} \leq w(\mathbf{b})$ . By Corollary 6.10 (iii), this and  $\ell_i \leq l_{i-1} (\Leftrightarrow s_i s_{i-1} \leq w(\mathbf{b}))$  imply  $i \notin \text{supp}(w_v)$ . This contradicts to  $i \in \text{supp}(w) \subset \text{supp}(w_v)$ , which shows our claim. If  $\ell_i \leq \ell_{i-1}$ , then by our claim, at most one  $-$  remains after we delete  $u^{\ell_{i-1}}$  and then “ $+$   $-$ ” in  $u^{\ell_{i+1}} \dots u^{\lambda(i)+\lambda(i+1)}$  as far as we can. This  $-$  does not contribute to  $\varepsilon_i(s_{i-1}\mathbf{b}')$  because  $u^{\ell_i} = +$ . Thus we have  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}')$ .

Next assume that  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor + 1$ . Then  $s_i s_{i-1} \leq w(\mathbf{b})$  if and only if  $l_i < l_{i-1}$ . Let  $j = \min\{j' \mid i + 1 \leq j' \leq n, \lambda(j') = \lfloor \frac{m}{n} \rfloor + 1\}$ . Set  $\mathbf{b}'_0 = s_{i+1} s_{i+2} \dots s_{j-1} \mathbf{b}'$ . We write  $u_i(\mathbf{b}'_0) = u^1 \dots u^{2\lfloor \frac{m}{n} \rfloor + 2}$ . Then this is obtained from  $u_i(\mathbf{b}')$  by adding one  $-$ . Let  $u^{\ell_{i-1}} = +$  be the box which vanishes in  $u_i(s_{i-1}\mathbf{b}'_0)$ , and let  $u^{\ell_i} = -$  be the box which vanishes in  $u_i(s_i s_{i-1} \mathbf{b}'_0)$ . Then  $l_i < l_{i-1}$  if and only if  $\ell_i < \ell_{i-1}$ . Note that  $s_{i-1} \mathbf{b}'_0 = s_{i+1} s_{i+2} \dots s_{j-1} (s_{i-1} \mathbf{b}')$ , and  $u^{\ell_{i-1}}$  (regarded as in  $u_i(\mathbf{b}')$ ) also vanishes in  $u_i(s_{i-1} \mathbf{b}')$ . So, by Lemma 6.13 (i), if  $\varepsilon_i(s_{i-1} \mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$ , then there exists  $u^\ell = -$

with  $\ell_{i-1} < \ell$  which remains in  $u_i(s_{i-1}\mathbf{b}'_0)_{\text{red}}$ . Then  $\ell_{i-1} < \ell \leq \ell_i$ , i.e.,  $s_{i-1}s_i \leq w(\mathbf{b})$ . Thus if  $s_i s_{i-1} \leq w(\mathbf{b})$ , then  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}')$ . The first statement is verified.

We further assume that  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor + 1)$ . Then  $\mathbf{b}'_0 = \mathbf{b}'$ . To prove the converse, we argue by contradiction. If  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}')$ , then by Definition 4.1 (iii), we have  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(s_{i-1}\mathbf{b}') > 0$ . By Lemma 6.13 (ii),  $u^{\ell_i}$  is also the rightmost  $-$  to  $+$  in  $u(\mathbf{b}')_{\text{red}}$ . So if moreover  $\ell_{i-1} < \ell_i$ , then  $\varepsilon_i(s_{i-1}\mathbf{b}') = \varepsilon_i(\mathbf{b}') + 1$ , which is a contradiction. This proves the second statement.  $\square$

**Remark 6.14.** In the proof of Lemma 6.11, the assumption  $\text{supp}(w) \subset \text{supp}(w_v)$  is used only in the third paragraph to treat the case where  $\langle \chi_{i,i+1}, \lambda_b \rangle = -1$ . So if  $\langle \chi_{i,i+1}, \lambda_b \rangle \neq -1$ , the lemma is true for any allowed conjugate  $\lambda$  with  $\langle \chi_{i-1,i}, \lambda \rangle = -1$  (resp.  $\langle \chi_{i+1,i+2}, \lambda \rangle = -1$ ) such that  $s_{i-1}\lambda$  (resp.  $s_{i+1}\lambda$ ) is allowed.

We need the following corollary to treat the case  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ . This corollary tells us that the converse of Lemma 6.11 does not hold only if  $s_{i-1}s_i \leq w(\mathbf{b})$  and  $s_{i+1}s_i \leq w(\mathbf{b})$ .

**Corollary 6.15.** Assume that  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ . Let  $j_1 = \min\{j' \mid i+1 < j' \leq n, \lambda(j') = \lfloor \frac{m}{n} \rfloor + 1\}$ , and let  $j_2 = \max\{j' \mid 1 \leq j' < i, \lambda(j') = \lfloor \frac{m}{n} \rfloor\}$ .

(i) Assume that  $s_{i+1}s_i s_{i-1} \leq w(\mathbf{b})$  (resp.  $s_{i-1}s_i s_{i+1} \leq w(\mathbf{b})$ ). Then

$$\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1 \quad (\text{resp. } \varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1).$$

(ii) Assume that  $s_{i-1}s_i \leq w(\mathbf{b})$  and  $s_{i+1}s_i \leq w(\mathbf{b})$ . Then

$$\varepsilon_i(s_{i-1} \cdots s_{j_2} s_{i+1} \cdots s_{j_1-1} \mathbf{b}) \geq \varepsilon_i(\mathbf{b}) + 1.$$

Moreover, if  $\varepsilon_i(s_{i-1} \cdots s_{j_2} s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$  (resp.  $\varepsilon_i(\mathbf{b}) + 2$ ), then  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b})$  (resp.  $\varepsilon_i(\mathbf{b}) + 1$ ).

In particular, if  $s_{i-1}s_i \leq w(\mathbf{b})$  or  $s_{i+1}s_i \leq w(\mathbf{b})$ , then  $\varepsilon_i(s_{i-1} \cdots s_{j_2} s_{i+1} \cdots s_{j_1-1} \mathbf{b}) \geq \varepsilon_i(\mathbf{b}) + 1$ .

*Proof.* If  $\varepsilon_i(\mathbf{b}) = 0$ , then (i) follows from Definition 4.1 (iii) and Lemma 6.11. If  $\varepsilon_i(\mathbf{b}) > 0$ , let  $l_0$  be the minimal integer such that  $\langle \chi_{i+1,i}, \text{wt}(\mathbf{b}_1) + \text{wt}(\mathbf{b}_2) + \cdots + \text{wt}(\mathbf{b}_{l_0}) \rangle = \varepsilon_i(\mathbf{b})$ . If  $s_i s_{i-1} \leq w(\mathbf{b})$  (resp.  $s_i s_{i+1} \leq w(\mathbf{b})$ ) and  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b})$  (resp.  $\varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b})$ ), then by Lemma 6.11 and Lemma 6.13 (ii),  $l_i = l_0$ . However, this and  $s_{i+1}s_i \leq w(\mathbf{b})$  (resp.  $s_{i-1}s_i \leq w(\mathbf{b})$ ) imply  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$  (resp.  $\varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$ ), which is a contradiction. Thus (i) follows.

Set  $\mathbf{b}' = s_{i-1} \cdots s_{j_2} s_{i+1} \cdots s_{j_1-1} \mathbf{b}$ . We show that if  $l_{i+1} < l_i$  (resp.  $l_{i-1} < l_i$ ), then  $\varepsilon_i(\mathbf{b}') \geq \varepsilon_i(\mathbf{b}) + 1$ . This follows from Definition 4.1 (iii) if  $\varepsilon_i(\mathbf{b}) = 0$ . If  $\varepsilon_i(\mathbf{b}) > 0$

and  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b})$ , then  $l_0 = l_i$  by Lemma 6.13 (ii). However,  $l_{i+1} < l_i = l_0$  (resp.  $l_{i-1} < l_i = l_0$ ) implies  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$  (resp.  $\varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$ ), which is a contradiction. This proves the claim. Again by Lemma 6.13 (ii),  $l_i = l_{i-1} = l_{i+1}$  implies  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b}) + 1$ . Putting things together, we have proved the inequality in (ii).

Finally, we prove the “moreover” part in (ii). Assume that  $s_{i-1}s_i \leq w(\mathbf{b})$  and  $s_{i+1}s_i \leq w(\mathbf{b})$ . If  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b}) + 2$ , then the statement is obvious. If  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$  (resp.  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b})$ ) and  $\varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b})$  (resp.  $\varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$ ), then similarly as above, we deduce  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b}) + 2$ , which is a contradiction. So the statement for the case  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b}) + 1$  follows. This finishes the proof.  $\square$

## 6.4 Some Inequalities on $\varepsilon_i(v^{-1}\mathbf{b}^-)$

Keep the notation in §6.3. Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ . In this subsection, we will establish some inequalities on  $\varepsilon_i(v^{-1}\mathbf{b}^-)$  using the results in §6.3. These inequalities are the keys to the proof of  $v\xi_{j(\mathbf{b},v)} = vw_1^{-1} \cdots w_{j-1}^{-1}$ .

Set  $S_l = \langle \chi_{i+1,i}, \text{wt}(\mathbf{b}_1) + \text{wt}(\mathbf{b}_2) + \cdots + \text{wt}(\mathbf{b}_l) \rangle$  for fixed  $1 \leq i \leq n-1$ . We also set  $S_0 = 0$ . Then  $S_l$  is the difference of the number of  $-$  and  $+$  in  $u_i(\mathbf{b})$  which are contained in  $\mathbf{b}_1, \dots, \mathbf{b}_l$ . Thus  $S_l \leq \varepsilon_i(\mathbf{b})$ .

**Lemma 6.16.** We have  $S_l \leq \varepsilon_i(v^{-1}\mathbf{b}^-)$  for  $0 \leq l < l_i$ . If the equality holds for some  $0 \leq l < l_i$ , then  $v\xi_{i,i+1} \in \Phi_-$ .

*Proof.* If  $\varepsilon_i(v^{-1}\mathbf{b}^-) \geq \varepsilon_i(\mathbf{b})$ , then the inequality is obvious. In particular, the inequality holds if  $\varepsilon_i(\mathbf{b}) = 0$ . If  $\varepsilon_i(\mathbf{b}) > 0$ , let  $l_0$  be the minimal integer such that  $S_{l_0} = \varepsilon_i(\mathbf{b})$ . It follows from Lemma 6.13 (ii) that if  $\varepsilon_i(\mathbf{b}) > 0$  and  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b}) - 1$ , then  $i \in \text{supp}(w_v)$  and  $l_0 = l_i$ . This implies the inequality. Note that if the equality holds for some  $0 \leq l < l_i$ , then  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b}) - 1$  or  $\varepsilon_i(\mathbf{b})$ . Set  $\lambda = v^{-1}\lambda_b^-$ .

If  $i \in \text{supp}(w_v)$  and  $(\lambda(i), \lambda(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$ , then  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$  and  $i-1, i+1 \in \text{supp}(w_v)$ . If moreover, the equality holds for some  $0 \leq l < l_i$ , then we must have  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b}) - 1$  because  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$  and hence  $l_0 = l_i$ . By Lemma 6.11, we have  $s_i s_{i-1} \leq w(\mathbf{b})$  and  $s_i s_{i+1} \leq w(\mathbf{b})$ . This contradicts to Corollary 6.10 (iii). If  $i \notin \text{supp}(w_v)$ ,  $(\lambda(i), \lambda(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$  and the equality holds for some  $0 \leq l < l_i$ , then we must have  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b})$ . By Definition 4.1 (iii),  $\varepsilon_i(\mathbf{b}) = \varepsilon_i(v^{-1}\mathbf{b}^-) > 0$ . So by Lemma 6.13 (ii), we have  $l_0 = l_i$  and hence  $S_l \leq \varepsilon_i(\mathbf{b}) - 1$ . This is a contradiction. Thus the equality implies  $\langle \chi_{i,i+1}, \lambda \rangle = 0$  or  $1$ . If  $\langle \chi_{i,i+1}, \lambda \rangle = 1$ , then  $v(i) > n - m_0 \geq v(i+1)$  and hence  $v\xi_{i,i+1} \in \Phi_-$ . It remains to treat the case where  $\lambda(i) = \lambda(i+1)$ .

If  $i \in \text{supp}(w_v)$  and  $\lambda(i) = \lambda(i+1) = \lfloor \frac{m}{n} \rfloor$ , then  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$  or  $(\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor)$ . In the former case, we have  $i-1 \in \text{supp}(w_v)$  and  $i+1 \notin \text{supp}(w_v)$ .



If the equality holds for some  $0 \leq l < l_i$ , then  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b}) - 1$ . It follows from Lemma 6.11 that  $s_i s_{i-1} \leq w(\mathbf{b})$ . In the latter case, we have  $i-1, i+1 \in \text{supp}(w_v)$ . It follows from Lemma 6.11 that if the equality holds for some  $0 \leq l < l_i$  and  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b}) - 1$ , then  $s_i s_{i-1} \leq w(\mathbf{b})$  and  $s_i s_{i+1} \leq w(\mathbf{b})$ . If the equality holds for some  $0 \leq l < l_i$  and  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b})$ , then by Lemma 6.11 and Lemma 6.13 (ii) (or Definition 4.1 (iii) if  $\varepsilon_i(\mathbf{b}) = 0$ ), we must have  $s_{i+1} s_i \leq w(\mathbf{b})$  and hence  $s_i s_{i-1} \leq w(\mathbf{b})$ . Thus the equality for some  $0 \leq l < l_i$  implies  $s_i s_{i-1} \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  follows from Corollary 6.8 (i) and (iii).

If  $i \in \text{supp}(w_v)$  and  $\lambda(i) = \lambda(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ , then  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$  or  $(\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor + 1)$ . In the former case, we have  $i+1 \in \text{supp}(w_v)$  and  $i-1 \notin \text{supp}(w_v)$ . If the equality holds for some  $0 \leq l < l_i$ , then  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b}) - 1$ . It follows from Lemma 6.11 that  $s_i s_{i+1} \leq w(\mathbf{b})$ . In the latter case, we have  $i-1, i+1 \in \text{supp}(w_v)$ . It follows from Lemma 6.11 and Lemma 6.13 (ii) (or Definition 4.1 (iii) if  $\varepsilon_i(\mathbf{b}) = 0$ ) that if the equality holds for some  $0 \leq l < l_i$  and  $\varepsilon_i(v^{-1}\mathbf{b}^-)$  is equal to  $\varepsilon_i(\mathbf{b}) - 1$  (resp.  $\varepsilon_i(\mathbf{b})$ ), then  $s_i s_{i-1} \leq w(\mathbf{b})$  and  $s_i s_{i+1} \leq w(\mathbf{b})$  (resp.  $s_{i-1} s_i \leq w(\mathbf{b})$  and  $s_i s_{i+1} \leq w(\mathbf{b})$ ). Thus the equality for some  $0 \leq l < l_i$  implies  $s_i s_{i+1} \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  follows from Corollary 6.8 (i) and (ii).

If  $i \notin \text{supp}(w_v)$  and  $\lambda(i) = \lambda(i+1) = \lfloor \frac{m}{n} \rfloor$ , then  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor)$  or  $(\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ . Note that if the equality holds for some  $0 \leq l < l_i$ , then  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b})$ . In the former case, let  $j = \min\{j' \mid i+1 < j' \leq n, \lambda(j') = \lfloor \frac{m}{n} \rfloor + 1\}$ . By  $i \notin \text{supp}(w_v)$ ,  $s_i s_{i+1} \cdots s_{j-1} \lambda$  is allowed. Let  $\mathbf{b}'$  be the conjugate of  $\mathbf{b}$  with weight  $s_{i+1} \cdots s_{j-1} \lambda$ . If the equality holds for some  $0 \leq l < l_i$ , then by Lemma 6.13 (ii) (or Definition 4.1 (iii) if  $\varepsilon_i(\mathbf{b}) = 0$ ) and Remark 6.14, we have  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b}) + 1$  and hence  $s_{i+1} s_i \leq w(\mathbf{b})$ . In the latter case, if the equality holds for some  $0 \leq l < l_i$ , then  $s_{i+1} s_i s_{i-1} \leq w(\mathbf{b})$  by Lemma 6.15. Thus the equality for some  $0 \leq l < l_i$  implies  $s_{i+1} s_i \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  follows from Corollary 6.8 (iii) and (iv).

If  $i \notin \text{supp}(w_v)$  and  $\lambda(i) = \lambda(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ , then  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor + 1)$  or  $(\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ . Note that if the equality holds for some  $0 \leq l < l_i$ , then  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b})$ . In the former case, let  $j = \max\{j' \mid 1 \leq j' < i, \lambda(j') = \lfloor \frac{m}{n} \rfloor\}$ . By  $i \notin \text{supp}(w_v)$ ,  $s_i s_{i-1} \cdots s_j \lambda$  is allowed. Let  $\mathbf{b}'$  be the conjugate of  $\mathbf{b}$  with weight  $s_{i-1} \cdots s_j \lambda$ . If the equality holds for some  $0 \leq l < l_i$ , then by Lemma 6.13 (ii) (or Definition 4.1 (iii) if  $\varepsilon_i(\mathbf{b}) = 0$ ) and Remark 6.14, we have  $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b}) + 1$  and hence  $s_{i-1} s_i \leq w(\mathbf{b})$ . In the latter case, if the equality holds for some  $0 \leq l < l_i$ , then  $s_{i-1} s_i s_{i+1} \leq w(\mathbf{b})$  by Lemma 6.15. Thus the equality for some  $0 \leq l < l_i$  implies  $s_{i-1} s_i \leq w(\mathbf{b})$ . Then  $v\chi_{i,i+1} \in \Phi_-$  follows from Corollary 6.8 (ii) and (iv). This finishes the proof.  $\square$

In a similar way, we will prove the following lemma.

**Lemma 6.17.** (i) Assume that  $l_{i+1} < l_i$ . We set  $\delta = 1$  if  $i+1 \in \text{supp}(w_v)$  and  $\delta = 0$  if  $i+1 \notin \text{supp}(w_v)$ . Then we have  $S_l + \delta \leq \varepsilon_i(v^{-1}\mathbf{b}^-)$  for  $l_{i+1} \leq l < l_i$ .

If  $s_i s_{i-1} \leq w(\mathbf{b})$  (resp.  $s_{i-1} s_i \leq w(\mathbf{b})$ ), then the equality for some  $l_{i+1} \leq l < l_i$  implies that  $i \in \text{supp}(w_v)$  (resp.  $\lambda_b^-(v(i)) = \lfloor \frac{m}{n} \rfloor + 1$ ).

- (ii) Assume that  $l_{i-1} < l_i$ . We set  $\delta = 1$  if  $i - 1 \in \text{supp}(w_v)$  and  $\delta = 0$  if  $i - 1 \notin \text{supp}(w_v)$ . Then we have  $S_l + \delta \leq \varepsilon_i(v^{-1}\mathbf{b}^-)$  for  $l_{i-1} \leq l < l_i$ . If  $s_i s_{i+1} \leq w(\mathbf{b})$  (resp.  $s_{i+1} s_i \leq w(\mathbf{b})$ ), then the equality for some  $l_{i+1} \leq l < l_i$  implies that  $i \in \text{supp}(w_v)$  (resp.  $\lambda_b^-(v(i+1)) = \lfloor \frac{m}{n} \rfloor$ ).
- (iii) Assume that  $l_{i+1} < l_i$  and  $l_{i-1} < l_i$ . We set  $\delta = |\{i-1, i+1\} \cap \text{supp}(w_v)|$ . Then we have  $S_l + \delta \leq \varepsilon_i(v^{-1}\mathbf{b}^-)$  for  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$ . The equality for some  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$  implies that  $i \in \text{supp}(w_v)$ .

*Proof.* We first prove (i).

Assume that  $l_{i+1} < l_i$ ,  $s_i s_{i-1} \leq w(\mathbf{b})$  and  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ . Then  $s_i s_{i-1} \leq w(\mathbf{b})$  combined with Lemma 6.11 and Lemma 6.13 (ii) implies  $l_0 = l_i$  and hence  $S_l \leq \varepsilon_i(\mathbf{b}) - 1$  for  $0 \leq l < l_i$ , where  $l_0$  denotes the minimal integer such that  $S_{l_0} = \varepsilon_i(\mathbf{b})$ . By Lemma 6.11 and  $l_{i+1} < l_i$ ,  $i+1 \in \text{supp}(w_v)$  implies  $\varepsilon_i(v^{-1}\mathbf{b}^-) \geq \varepsilon_i(\mathbf{b})$ . Thus the inequality holds. By  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ ,  $i+1 \in \text{supp}(w_v)$  implies  $i \in \text{supp}(w_v)$ . If the equality holds for some  $l_{i+1} \leq l < l_i$  and  $i \notin \text{supp}(w_v)$ , then we must have  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b})$  and hence  $i+1 \in \text{supp}(w_v)$ , which is a contradiction. Thus the equality implies  $i \in \text{supp}(w_v)$ .

Assume that  $l_{i+1} < l_i$ ,  $s_{i-1} s_i \leq w(\mathbf{b})$  and  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$ . Then we have  $S_l \leq \varepsilon_i(\mathbf{b}) - 1$  for  $0 \leq l < l_i$ . The inequality follows from this, Lemma 6.11 and  $l_{i+1} < l_i$ . If the equality holds for some  $l_{i+1} \leq l < l_i$ , then again by Lemma 6.11,  $l_{i+1} < l_i$  and  $s_{i-1} s_i \leq w(\mathbf{b})$ , we have  $i \in \text{supp}(w_v)$  and  $i-1 \notin \text{supp}(w_v)$ . Thus  $\lambda_b^-(v(i)) = \lfloor \frac{m}{n} \rfloor + 1$ .

Assume that  $l_{i+1} < l_i$ ,  $s_{i-1} s_i \leq w(\mathbf{b})$  and  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor + 1)$ . The inequality for the case  $i+1 \notin \text{supp}(w_v)$  follows from Lemma 6.16. If  $i+1 \in \text{supp}(w_v)$ , then we have  $i-1, i \in \text{supp}(w_v)$ . By Lemma 6.11,  $l_{i+1} < l_i$  and  $s_{i-1} s_i \leq w(\mathbf{b})$ , we also have  $\varepsilon_i(v^{-1}\mathbf{b}^-) \geq \varepsilon_i(\mathbf{b}) + 1$ . Hence the inequality holds. Note that if  $i \in \text{supp}(w_v)$ , then  $\lambda_b^-(v(i)) = \lfloor \frac{m}{n} \rfloor + 1$ . So it remains to show that if  $i \notin \text{supp}(w_v)$  and the equality holds for some  $l_{i+1} \leq l < l_i$ , then  $i-1 \notin \text{supp}(w_v)$ . By  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ ,  $i \notin \text{supp}(w_v)$  implies  $i+1 \notin \text{supp}(w_v)$ . So if  $i \notin \text{supp}(w_v)$ , then the equality implies  $\varepsilon_i(v^{-1}\mathbf{b}^-) = \varepsilon_i(\mathbf{b})$ . Hence the assertion follows from Lemma 6.11 and  $s_{i-1} s_i \leq w(\mathbf{b})$ . Thus the equality implies  $\lambda_b^-(v(i)) = \lfloor \frac{m}{n} \rfloor + 1$ .

We next treat the case where  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$ . For this, we need the following claim.

**Claim 1.** Assume that  $l_{i+1} < l_i$  and  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$ . Then  $S_l \leq \varepsilon_i(\mathbf{b}) - 1$  for  $l_{i+1} \leq l < l_i$ .

We follow the notation in Corollary 6.15. To check this claim, it suffices to show that if  $S_l = \varepsilon_i(\mathbf{b})$  for some  $l_{i+1} \leq l$ , then  $l_i \leq l$ . We write  $u_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) =$

$u^1 \dots u^{\lambda_b(i) + \lfloor \frac{m}{n} \rfloor + 1}$ . Let  $u^{\ell_{i+1}} = -$  be the box added by the action of  $s_{i+1} \dots s_{j_1-1}$ . Let  $\ell$  be the maximal integer such that  $u^\ell$  is contained in  $\mathbf{b}_{l'}$  with some  $l' \leq l$ . If  $S_l = \varepsilon_i(\mathbf{b})$  for some  $l_{i+1} \leq l$ , then  $\ell_{i+1} \leq \ell$ ,  $\varepsilon_i(s_{i+1} \dots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$  and the number of  $-$  in  $(u^1 \dots u^\ell)_{\text{red}} = - \dots -$  is  $\varepsilon_i(\mathbf{b}) + 1$ . If  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$ , then  $l_i \leq l' \leq l$  follows immediately from this. If  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor + 1$ , let  $u^{\ell_{i-1}} = +$  be the box deleted by the action of  $s_{i-1} \dots s_{j_2}$  on  $s_{i+1} \dots s_{j_1-1} \mathbf{b}$ . Then the number of  $-$  in  $u^1 \dots u^\ell$  after we delete  $u^{\ell_{i-1}}$  (if  $\ell_{i-1} \leq \ell$ ) and then “ $+ -$ ” is  $\varepsilon_i(s_{i-1} \dots s_{j_2} s_{i+1} \dots s_{j_1-1} \mathbf{b})$ . So we have  $l_{i+1} \leq l' \leq l$ . This finishes the proof of Claim 1.

Assume that  $l_{i+1} < l_i$  and  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$ . Then the inequality follows from Lemma 6.11, Corollary 6.15 and Claim 1. By  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$ ,  $i \in \text{supp}(w_v)$  implies  $i+1 \in \text{supp}(w_v)$ . By  $l_{i+1} < l_i$ , Lemma 6.11 and Corollary 6.15, we have  $\varepsilon_i(v^{-1} \mathbf{b}^-) \geq \varepsilon_i(\mathbf{b})$ . So if the equality holds for some  $l_{i+1} \leq l < l_i$ , then by Claim 1, we must have  $i+1 \in \text{supp}(w_v)$  and  $\varepsilon_i(v^{-1} \mathbf{b}^-) = \varepsilon_i(\mathbf{b})$ . If  $s_i s_{i-1} \leq w(\mathbf{b})$ , then Lemma 6.11 and Corollary 6.15 imply  $\varepsilon_i(s_{i+1} \dots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$ . Thus if  $s_i s_{i-1} \leq w(\mathbf{b})$  and the equality holds for some  $l_{i+1} \leq l < l_i$ , we have  $i \in \text{supp}(w_v)$ . Also, if  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$  and  $s_{i-1} s_i \leq w(\mathbf{b})$ , then by Lemma 6.11, the equality for some  $l_{i+1} \leq l < l_i$  implies  $i \in \text{supp}(w_v)$  and  $i-1 \notin \text{supp}(w_v)$ . Hence  $\lambda_b^-(v(i)) = \lfloor \frac{m}{n} \rfloor + 1$ . Note that if  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor + 1$  and  $i \in \text{supp}(w_v)$ , then  $\lambda_b^-(v(i)) = \lfloor \frac{m}{n} \rfloor + 1$ . Therefore it remains to show that if  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor + 1$ ,  $s_{i-1} s_i \leq w(\mathbf{b})$ ,  $i \notin \text{supp}(w_v)$  and the equality holds for some  $l_{i+1} \leq l < l_i$ , then  $i-1 \notin \text{supp}(w_v)$ . This follows from Corollary 6.15.

Putting things together, we have proved (i). We can similarly prove (ii) using the following claim.

**Claim 2.** Assume that  $l_{i-1} < l_i$  and  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor + 1$ . Then  $S_l \leq \varepsilon_i(\mathbf{b}) - 1$  for  $l_{i-1} \leq l < l_i$ .

The proof of this claim is also similar to that of Claim 1, so we omit the details.

We next prove (iii). For this, we need the following claims.

**Claim 3.** Assume that  $l_{i-1}, l_{i+1} < l_i$ . Then  $S_l \leq \varepsilon_i(\mathbf{b}) - 1$  for  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$ .

This claim is obvious if  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$ . Other cases follow from Claim 1 and Claim 2.

**Claim 4.** Assume that  $l_{i-1}, l_{i+1} < l_i$  and  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ . If  $\varepsilon_i(s_{i+1} \dots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b})$  or  $\varepsilon_i(s_{i-1} \dots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b})$ , then  $S_l \leq \varepsilon_i(\mathbf{b}) - 2$  for  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$ .

It follows from Claim 1 and Claim 3 that  $S_l \leq \varepsilon_i(\mathbf{b}) - 1$  for  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$ . We write  $u_i(\mathbf{b}) = u^1 \dots u^{2\lfloor \frac{m}{n} \rfloor + 1}$ . Let  $\ell$  be the maximal integer such that  $u^\ell$  is contained in  $\mathbf{b}_{l'}$  with some  $l' \leq l$ . If  $S_l = \varepsilon_i(\mathbf{b}) - 1$  for  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$ , then  $(u^1 \dots u^\ell)_{\text{red}} = - \dots -$  or  $- \dots - +$ . Here the number of  $-$  is  $\varepsilon_i(\mathbf{b}) - 1$ .

or  $\varepsilon_i(\mathbf{b})$  respectively. By  $\max\{l_{i-1}, l_{i+1}\} \leq l$  and  $l_{i-1}, l_{i+1} < l_i$ , it follows that  $\varepsilon_i(s_{i-1} \cdots s_{j_2} s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 2$  in both cases. Hence  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$  and  $\varepsilon_i(s_{i-1} \cdots s_{j_2} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$ . This proves the claim.

Assume that  $l_{i-1}, l_{i+1} < l_i$  and  $\delta = 0$ . Then the inequality follows from Lemma 6.16. If the equality holds for some  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$ , then by Claim 3, we have  $i \in \text{supp}(w_v)$ .

Assume that  $l_{i-1}, l_{i+1} < l_i$  and  $\delta = 1$ . Then the inequality follows from (i) and (ii). If  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ , then by Corollary 6.15 and Claim 4, the equality never holds. If the equality holds for some  $\max\{l_{i-1}, l_{i+1}\} \leq l < l_i$  and  $(\lambda_b(i), \lambda_b(i+1)) \neq (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ , then by Lemma 6.11 and Claim 3, we must have  $i \in \text{supp}(w_v)$ .

Assume that  $l_{i-1}, l_{i+1} < l_i$  and  $\delta = 2$ . Then our assertion follows from Claim 3 and Lemma 6.11 (resp. Claim 4) if  $(\lambda_b(i), \lambda_b(i+1)) \neq (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$  (resp.  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ ). This finishes the proof of (iii).  $\square$

For  $S_{l_i}$ , we have the following lemma with the same notation as in Corollary 6.15.

**Lemma 6.18.** If  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$ ,  $s_{i-1}s_i \leq w(\mathbf{b})$ ,  $s_{i+1}s_i \leq w(\mathbf{b})$  and  $\varepsilon_i(s_{i-1} \cdots s_{j_2} s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$ , then  $S_{l_i} = \varepsilon_i(\mathbf{b}) - 1$ . Otherwise, we have  $S_{l_i} = \varepsilon_i(\mathbf{b})$ .

*Proof.* This is obvious if  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$ . If  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor)$  and  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b})$ , then  $S_{l_i} = \varepsilon_i(\mathbf{b})$  follows from Lemma 6.13 (ii). If  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor)$  and  $\varepsilon_i(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1$ , then  $\langle \chi_{i+1,i}, \text{wt}(\mathbf{b}'_1) + \cdots + \text{wt}(\mathbf{b}'_{l_i}) \rangle = \varepsilon_i(\mathbf{b}) + 1$ , where  $\text{FE}(s_{i+1} \cdots s_{j_1-1} \mathbf{b}) = \mathbf{b}'_1 \otimes \cdots \otimes \mathbf{b}'_d$ . By Remark 6.14, the box added by the action of  $s_{i+1} \cdots s_{j_1-1}$  is contained in one of  $\mathbf{b}'_1, \dots, \mathbf{b}'_{l_i}$ . This implies  $S_{l_i} = \varepsilon_i(\mathbf{b})$ . The proof for the case  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor + 1)$  is similar. By Lemma 6.11 and Corollary 6.15, the proof for the case  $(\lambda_b(i), \lambda_b(i+1)) = (\lfloor \frac{m}{n} \rfloor + 1, \lfloor \frac{m}{n} \rfloor)$  is also similar.  $\square$

Set  $T_l = \langle \chi_{i,i+1}, \text{wt}(\mathbf{b}_{l_{i+1}}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle$  for  $l_i < l$ . We also set  $T_{l_i} = 0$ . We will also need the following inequality.

**Lemma 6.19.** For  $l_i \leq l$ , we have  $T_l \geq 0$ . If  $s_i s_{i-1} \leq w(\mathbf{b})$  (resp.  $s_i s_{i+1} \leq w(\mathbf{b})$ ) and the equality holds for some  $l_{i-1} \leq l$  (resp.  $l_{i+1} \leq l$ ), then  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$  (resp.  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ ). Similarly, if  $s_i s_{i-1} \leq w(\mathbf{b})$ ,  $s_i s_{i+1} \leq w(\mathbf{b})$  and  $T_l = 1$  for some  $\max\{l_{i-1}, l_{i+1}\} \leq l$ , then  $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$  or  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ .

*Proof.* Let  $\lambda$  be an allowed cocharacter with  $i \notin \text{supp}(w)$  and  $\langle \chi_{i,i+1}, \lambda \rangle = -1$ , where  $w$  is the partial Coxeter element associated with  $\lambda$ . Let  $\mathbf{b}'$  be the conjugate of  $\mathbf{b}$  with  $\text{wt}(\mathbf{b}') = \lambda$ . Set  $\text{FE}(\mathbf{b}') = \mathbf{b}'_1 \otimes \cdots \otimes \mathbf{b}'_d$ . Then the action of  $s_i$  on  $\mathbf{b}'$

changes a box in  $\mathbf{b}'_{l_i}$ . Since  $\langle \chi_{i,i+1}, \text{wt}(\mathbf{b}'_{l_{i+1}}) + \cdots + \text{wt}(\mathbf{b}'_l) \rangle$  is the difference of the number of  $+$  and  $-$  in  $u_i(\mathbf{b}')$  which are contained in  $\mathbf{b}'_{l_{i+1}}, \dots, \mathbf{b}'_l$ , we have  $T_l \geq \langle \chi_{i,i+1}, \text{wt}(\mathbf{b}'_{l_{i+1}}) + \cdots + \text{wt}(\mathbf{b}'_l) \rangle \geq 0$ .

Assume that  $s_i s_{i+1} \leq w(\mathbf{b})$  and the equality holds for  $l_{i+1} \leq l$ . If  $l = l_i$ , then  $l_i = l_{i+1}$  and hence  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ . Assume moreover that  $l_i < l$ . To show  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor + 1$ , we argue by contradiction. If  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$ , i.e.,  $s_i s_{i+1} \leq w'_{\max}$ , then  $s_i s_{i+1} \leq w(\mathbf{b})$  implies  $l_i < l_{i+1}$ . So if  $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$  and the equality holds for  $l_{i+1} \leq l$ , then  $\langle \chi_{i,i+1}, \text{wt}(\mathbf{b}'_{l_{i+1}}) + \cdots + \text{wt}(\mathbf{b}'_l) \rangle \leq -1$ . This implies  $\langle \chi_{i+1,i}, \text{wt}(\mathbf{b}'_1) + \cdots + \text{wt}(\mathbf{b}'_l) \rangle \geq \varepsilon_i(\mathbf{b}') + 1$ , which is a contradiction. The rest of the statement follows in the same way. The proof is finished.  $\square$

**Remark 6.20.** In §6.3 and §6.4,  $l_i$  denotes an integer such that  $i \in \text{supp}(w_{l_i})$ . However, in the proof of Proposition 6.22 or Proposition 6.24,  $l_h$  denotes an integer such that  $j_h \in \text{supp}(w_{l_h})$ . We hope our notation will not cause confusions.

## 6.5 Proof of $v_{\xi_l(\mathbf{b},v)} = v w_1^{-1} \cdots w_{l-1}^{-1}$

Fix  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ . Set  $v_l = v w_1^{-1} w_2^{-1} \cdots w_{l-1}^{-1}$  for  $v \in \Upsilon(\mathbf{b})$  and  $1 \leq l \leq d$ . We write  $\xi_l$  for  $\xi_l(\mathbf{b}, v)$ . The goal of this section is to prove  $v_{\xi_l} = v_l$ .

Fix a reduced expression  $s_{j_1} s_{j_2} \cdots s_{j_{n-1}}$  of  $w(\mathbf{b})$  such that

$$\begin{aligned} w_1^{-1} &= s_{j_1} s_{j_2} \cdots s_{j_{\ell(w_1)}}, \\ w_2^{-1} &= s_{j_{\ell(w_1)+1}} s_{j_{\ell(w_1)+2}} \cdots s_{j_{\ell(w_1)+\ell(w_2)}}, \\ &\vdots \\ w_d^{-1} &= s_{j_{\ell(w_1)+\cdots+\ell(w_{d-1})+1}} s_{j_{\ell(w_1)+\cdots+\ell(w_{d-1})+2}} \cdots s_{j_{\ell(w_1)+\cdots+\ell(w_d)}}. \end{aligned}$$

Define  $1 \leq l_h \leq d$  by  $j_h \in \text{supp}(w_{l_h})$ . Then  $l_1 \leq l_2 \leq \cdots \leq l_{n-1}$ .

**Lemma 6.21.** For each  $1 \leq h \leq n-1$  such that  $l_h \leq d-1$ , we have

$$\begin{aligned} j_h \in \text{supp}(w_v) &\Leftrightarrow \langle \chi_{j_h, j_h+1}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle = -1, \\ j_h \notin \text{supp}(w_v) &\Leftrightarrow \langle \chi_{j_h, j_h+1}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle = 0. \end{aligned}$$

*Proof.* We first prove the case for  $h = 1$ . Note that  $s_{j_1} s_{j_1-1} \leq w(\mathbf{b})$  and  $s_{j_1} s_{j_1+1} \leq w(\mathbf{b})$ . So by Lemma 6.11,  $j_1 \in \text{supp}(w_v)$  (resp.  $j_1 \notin \text{supp}(w_v)$ ) if and only if  $\varepsilon_{j_1}(v^{-1} \mathbf{b}^-) = \varepsilon_{j_1}(\mathbf{b}) - 1$  (resp.  $\varepsilon_{j_1}(v^{-1} \mathbf{b}^-) = \varepsilon_{j_1}(\mathbf{b})$ ). By Lemma 6.18, we have  $\langle \chi_{j_1+1, j_1}, \text{wt}(\mathbf{b}_1) + \cdots + \text{wt}(\mathbf{b}_{l_1}) \rangle = \varepsilon_{j_1}(\mathbf{b})$ . This and  $v^{-1} \xi_{l_1+1} = \xi(v^{-1} \mathbf{b}^-) + \text{wt}(\mathbf{b}_1) + \cdots + \text{wt}(\mathbf{b}_{l_1})$  imply that we have  $j_1 \in \text{supp}(w_v)$  (resp.  $j_1 \notin \text{supp}(w_v)$ ) if and only if  $\langle \chi_{j_1, j_1+1}, v^{-1} \xi_{l_1+1} \rangle = -1$  (resp.  $\langle \chi_{j_1, j_1+1}, v^{-1} \xi_{l_1+1} \rangle = 0$ ).

Assume that our claim is true for  $1, 2, \dots, h-1$  with  $h \geq 2$ . If  $j_h - 1, j_h + 1 \notin \{j_1, j_2, \dots, j_{h-1}\}$ , then

$$\langle \chi_{j_h, j_h+1}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle = \varepsilon_{j_h}(v^{-1} \mathbf{b}^-) + \langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_1) + \cdots + \text{wt}(\mathbf{b}_{l_h}) \rangle$$

and the statement follows in the same way of the case for  $h = 1$ . If  $j_{h'} = j_h + 1$  for some  $1 \leq h' \leq h-1$  and  $j_h - 1 \notin \{j_1, j_2, \dots, j_{h-1}\}$ , then

$$\begin{aligned} \langle \chi_{j_h, j_h+1}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle &= \varepsilon_{j_h}(v^{-1} \mathbf{b}^-) + \langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_1) + \cdots + \text{wt}(\mathbf{b}_{l_h}) \rangle \\ &\quad + \langle \chi_{j_{h'}, j_{h'}+1}, s_{j_{h'-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_{h'}+1} \rangle. \end{aligned}$$

Then the assertion follows from Lemma 6.11, Corollary 6.15, Lemma 6.18 and the induction hypothesis. The proof for the case  $j_{h'} = j_h - 1$  for some  $1 \leq h' \leq h-1$  and  $j_h + 1 \notin \{j_1, j_2, \dots, j_{h-1}\}$  is similar. If  $\{j_{h'}, j_{h''}\} = \{j_h - 1, j_h + 1\}$  for  $1 \leq h' < h'' \leq h-1$ , then

$$\begin{aligned} \langle \chi_{j_h, j_h+1}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle &= \varepsilon_{j_h}(v^{-1} \mathbf{b}^-) + \langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_1) + \cdots + \text{wt}(\mathbf{b}_{l_h}) \rangle \\ &\quad + \langle \chi_{j_{h'}, j_{h'}+1}, s_{j_{h'-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_{h'}+1} \rangle \\ &\quad + \langle \chi_{j_{h''}, j_{h''}+1}, s_{j_{h''-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_{h''}+1} \rangle. \end{aligned}$$

By Corollary 6.10 (iv), the case where  $s_{j_h-1} s_{j_h} \leq w(\mathbf{b})$ ,  $s_{j_h+1} s_{j_h} \leq w(\mathbf{b})$  and  $j_h - 1, j_h + 1 \notin \text{supp}(w_v)$  does not occur. Then the assertion follows from this, Lemma 6.11, Corollary 6.15, Lemma 6.18 and the induction hypothesis. Thus the statement is true for  $h$ . By induction, this finishes the proof.  $\square$

**Proposition 6.22.** We have  $U_{\xi_l(\mathbf{b}, v)} = v_l U v_l^{-1}$ , i.e.,  $v_{\xi_l} = v_l$  for any  $1 \leq l \leq d$ .

*Proof.* We will prove  $v_{\xi_{l+1}} = v_{l+1}$  for  $0 \leq l \leq d-1$ . For this, we have to check that  $\langle \chi_{i, i+1}, v_{l+1}^{-1} \xi_{l+1} \rangle \geq 0$  for any  $1 \leq i \leq n-1$  and that if  $\langle \chi_{i, i+1}, v_{l+1}^{-1} \xi_{l+1} \rangle = 0$ , then  $v_{l+1} \chi_{i, i+1} \in \Phi_-$ . Set  $\text{supp}_l = \text{supp}(w_1) \cup \cdots \cup \text{supp}(w_l)$  (and  $\text{supp}_0 = \emptyset$ ).

Let  $i \notin \text{supp}_l$ . Note that  $v_{l+1}^{-1} \xi_{l+1} = w_l \cdots w_1 \xi(v^{-1} \mathbf{b}^-) + w_l \cdots w_1 \text{wt}(\mathbf{b}_1) + \cdots + w_l \text{wt}(\mathbf{b}_l)$ . So if  $i-1, i+1 \notin \text{supp}_l$ , then the assertion follows from Lemma 6.16. If  $i+1 \in \text{supp}_l$  and  $i-1 \notin \text{supp}_l$ , then

$$\begin{aligned} \langle \chi_{i, i+1}, v_{l+1}^{-1} \xi_{l+1} \rangle &= \varepsilon_i(v^{-1} \mathbf{b}^-) + \langle \chi_{i, i+1}, \text{wt}(\mathbf{b}_1) + \cdots + \text{wt}(\mathbf{b}_l) \rangle \\ &\quad + \langle \chi_{j_h, j_h+1}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle, \end{aligned}$$

where  $j_h = i+1$ . Thus the assertion follows from Lemma 6.9, Corollary 6.10 (ii), Lemma 6.17 (i) and Lemma 6.21. The proof for the case where  $i-1 \in \text{supp}_l$  and  $i+1 \notin \text{supp}_l$  is similar. If  $i-1, i+1 \in \text{supp}_l$ , then

$$\begin{aligned} \langle \chi_{i, i+1}, v_{l+1}^{-1} \xi_{l+1} \rangle &= \varepsilon_i(v^{-1} \mathbf{b}^-) + \langle \chi_{i, i+1}, \text{wt}(\mathbf{b}_1) + \cdots + \text{wt}(\mathbf{b}_l) \rangle \\ &\quad + \langle \chi_{j_h, j_h+1}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle \\ &\quad + \langle \chi_{j_{h'}, j_{h'}+1}, s_{j_{h'-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_{h'}+1} \rangle, \end{aligned}$$

where  $\{j_h, j_{h'}\} = \{i-1, i+1\}$  with  $h < h'$ . Thus the assertion follows from Lemma 6.9, Lemma 6.17 (iii) and Lemma 6.21. Therefore our assertion is true for  $i \notin \text{supp}_l$ .

Let  $i \in \text{supp}_l$ . Let  $h$  such that  $j_h = i$ . We set  $\text{supp}_{l,h} = \text{supp}_l \setminus \{j_1, \dots, j_h\}$ . If  $j_h - 1, j_h + 1 \notin \text{supp}_{l,h}$ , then

$$\begin{aligned} & \langle \chi_{j_h, j_h+1}, v_{l+1}^{-1} \xi_{l+1} \rangle \\ &= \langle \chi_{j_h+1, j_h}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle + \langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle. \end{aligned}$$

By Lemma 6.19,  $\langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle \geq 0$ . Then the assertion follows from Lemma 6.21. If  $j_h - 1 \notin \text{supp}_{l,h}$  and  $j_h + 1 \in \text{supp}_{l,h}$ , then

$$\begin{aligned} & \langle \chi_{j_h, j_h+1}, v_{l+1}^{-1} \xi_{l+1} \rangle \\ &= \langle \chi_{j_h, j_h+1}, v_{l+1}^{-1} \xi_{l_h'+1} \rangle + \langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h'+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle \\ &= \langle \chi_{j_h+1, j_h}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h'+1} \rangle + \langle \chi_{j_{h'}, j_h'+1}, s_{j_{h'-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h'+1} \rangle \\ & \quad + \langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h'+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle \\ &= \langle \chi_{j_h+1, j_h}, s_{j_{h-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h+1} \rangle + \langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle \\ & \quad + \langle \chi_{j_{h'}, j_h'+1}, s_{j_{h'-1}} \cdots s_{j_2} s_{j_1} v^{-1} \xi_{l_h'+1} \rangle, \end{aligned}$$

where  $j_{h'} = j_h + 1$ . By Lemma 6.19,  $\langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle = 0$  implies  $\lambda_b(j_h + 1) = \lfloor \frac{m}{n} \rfloor + 1$ . Hence  $j_h + 1 \in \text{supp}(w_v)$  implies  $j_h \in \text{supp}(w_v)$ . Thus  $\langle \chi_{j_h, j_h+1}, v_{l_h+1}^{-1} \xi_{l_h+1} \rangle \geq 0$  by Lemma 6.21. The equality holds if and only if one of the following case occurs:

- $j_h \notin \text{supp}(w_v)$ ,  $j_{h'} \in \text{supp}(w_v)$  and  $\langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle = 1$ ,
- $j_h \notin \text{supp}(w_v)$ ,  $j_{h'} \notin \text{supp}(w_v)$  and  $\langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle = 0$ ,
- $j_h \in \text{supp}(w_v)$ ,  $j_{h'} \in \text{supp}(w_v)$  and  $\langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle = 0$ .

In the first case,  $v_{l+1} \chi_{j_h, j_h+1} = v s_{j_1} \cdots s_{j_{h-1}} \chi_{j_h+1, j_h} + v s_{j_1} \cdots s_{j_{h'-1}} \chi_{j_{h'}, j_h'+1} \in \Phi_-$  by Lemma 6.21. In the last two cases, we have  $\lambda_b^-(v(j_h+1)) = \lambda_b(j_h+1) = \lfloor \frac{m}{n} \rfloor + 1$  and hence  $v(j_h+1) > n - m_0$ . If  $j_h - 1 \in \{j_1, \dots, j_{h-1}\}$ ,  $v_{l+1} \chi_{j_h, j_h+1} = v w(\mathbf{b}) \chi_{j_h, j_h+1} = c^m v \chi_{j_h, j_h+1} \in \Phi_-$  by Corollary 6.8 (ii) (if  $v(j_h) > n - m_0$ ). If  $j_h - 1 \notin \{j_1, \dots, j_{h-1}\}$ , then  $v_{l+1} \chi_{j_h, j_h+1} \in \Phi_-$  follows from Corollary 6.10 (i). Thus  $v_{l+1} \chi_{j_h, j_h+1} \in \Phi_-$  holds in every case. The proof for the case where  $j_h + 1 \notin \text{supp}_{l,h}$  and  $j_h - 1 \in \text{supp}_{l,h}$  is similar. If  $j_h - 1, j_h + 1 \in \text{supp}_{l,h}$ , then by Lemma 6.19,  $\langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{l_h+1}) + \cdots + \text{wt}(\mathbf{b}_l) \rangle = 0$  (resp. 1) implies  $(\lambda_b(j_h), \lambda_b(j_h+1)) = (\lfloor \frac{m}{n} \rfloor, \lfloor \frac{m}{n} \rfloor + 1)$  (resp.  $\lambda_b(j_h) = \lfloor \frac{m}{n} \rfloor$  or  $\lambda_b(j_h+1) = \lfloor \frac{m}{n} \rfloor + 1$ ). Thus the inequality follows similarly as above. Using Corollary 6.8 (i), we can also check that the equality implies  $v_{l+1} \chi_{j_h, j_h+1} \in \Phi_-$  in the same way as above. Therefore our assertion is true for  $i \in \text{supp}_l$ . This completes the proof.  $\square$

## 6.6 End of The Proof of Theorem 5.4

In this subsection, we finish the proof of Theorem 5.4.

**Lemma 6.23.** Let  $\mathbf{b}' \in \mathbb{B}_\mu$  and let  $1 \leq i \leq n-1$ . If  $\varepsilon_i(\mathbf{b}') > 0$ , then let  $l$  be the positive integer such that

$$\text{FE}(\tilde{e}_i \mathbf{b}') = \mathbf{b}'_1 \otimes \cdots \otimes \tilde{e}_i \mathbf{b}'_l \otimes \cdots \otimes \mathbf{b}'_d,$$

where  $\text{FE}(\mathbf{b}') = \mathbf{b}'_1 \otimes \cdots \otimes \mathbf{b}'_d$ . If  $\varepsilon_i(\mathbf{b}') = 0$ , set  $l = 0$ . Then the action of  $\tilde{f}_i^{\phi_i(\mathbf{b}')}$  on  $\mathbf{b}'$  does not affect the boxes in  $\mathbf{b}'_1, \dots, \mathbf{b}'_l$  and the following equality holds:

$$\text{wt}(\tilde{f}_i^{\phi_i(\mathbf{b}')} \mathbf{b}') = \sum_{j=1}^l \text{wt}(\mathbf{b}'_j) + s_i \left( \sum_{j=l+1}^d \text{wt}(\mathbf{b}'_j) \right).$$

*Proof.* We naturally identify  $\mathbf{b}'_j$  and  $\text{wt}(\mathbf{b}'_j)$ . We need to check that the  $i, i+1$ -th entries in both sides are equal (because other entries are clearly equal). We write  $u_i(\mathbf{b}') = u^1 \dots u^{\text{wt}(\mathbf{b}')(i) + \text{wt}(\mathbf{b}')(i+1)}$ . Let  $u^\ell = -$  be the box in  $\mathbf{b}'_l$  (which is changed to  $+$  by the action of  $\tilde{e}_i$ ). Then  $(u^{\ell+1} \dots u^{\text{wt}(\mathbf{b}')(i) + \text{wt}(\mathbf{b}')(i+1)})_{\text{red}} = + \dots +$  and the number of  $+$  here is equal to  $\phi_i(\mathbf{b}')$ . Note that  $\tilde{f}_i^{\phi_i(\mathbf{b}')}$  changes all  $+$  in this diagram to  $-$ . Note also that  $\tilde{f}_i^{\phi_i(\mathbf{b}')}$  does not affect the boxes in  $\mathbf{b}'_1, \dots, \mathbf{b}'_l$  and “ $+ -$ ” in  $u_i(\mathbf{b}')$ , which we neglect in  $u_i(\mathbf{b}')_{\text{red}}$ . On the other hand, the action of  $s_i$  on  $\mathbf{b}'_{l+1}, \dots, \mathbf{b}'_d$  changes  $+$  to  $-$  and  $-$  to  $+$ . It is easy to see the total number of  $+(= \boxed{i})$  or  $- (= \boxed{i+1})$  in both sides are equal. Hence the equality holds.  $\square$

**Proposition 6.24.** Let  $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$  and  $v \in \Upsilon(\mathbf{b})$ . Then

$$w(\mathbf{b})(\xi(v^{-1}\mathbf{b}^-) + v^{-1}\lambda_b^-) = \xi(v^{-1}\mathbf{b}^-) + \sum_{1 \leq j \leq d} w_1^{-1} \cdots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j).$$

*Proof.* Fix a reduced expression  $s_{j_1} s_{j_2} \cdots s_{j_{n-1}}$  of  $w(\mathbf{b})$ . We define

$$\Phi_{j_h}(\mathbf{b}, v) := \begin{cases} \phi_{j_h}(v^{-1}\mathbf{b}^-) & (s_{j_h-1} s_{j_h+1} s_{j_h} \leq w(\mathbf{b})) \\ \phi_{j_h}(v^{-1}\mathbf{b}^-) + \Phi_{j_h-1}(\mathbf{b}, v) & (s_{j_h+1} s_{j_h} s_{j_h-1} \leq w(\mathbf{b})) \\ \phi_{j_h}(v^{-1}\mathbf{b}^-) + \Phi_{j_h+1}(\mathbf{b}, v) & (s_{j_h-1} s_{j_h} s_{j_h+1} \leq w(\mathbf{b})) \\ \phi_{j_h}(v^{-1}\mathbf{b}^-) + \Phi_{j_h-1}(\mathbf{b}, v) + \Phi_{j_h+1}(\mathbf{b}, v) & (s_{j_h} s_{j_h-1} s_{j_h+1} \leq w(\mathbf{b})) \end{cases}$$

inductively from  $h = n-1$  to 1, setting  $\Phi_0(\mathbf{b}, v) = \Phi_n(\mathbf{b}, v) = 0$ . In particular, we have  $\Phi_{j_{n-1}}(\mathbf{b}, v) = \phi_{j_{n-1}}(v^{-1}\mathbf{b}^-)$ . Write  $\Phi_{j_h}$  for  $\Phi_{j_h}(\mathbf{b}, v)$ . First, we prove that

$$w(\mathbf{b})(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes v^{-1}\mathbf{b}^-) = \mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes \tilde{f}_{j_1}^{\Phi_{j_1}} \tilde{f}_{j_2}^{\Phi_{j_2}} \cdots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)$$



by induction (see Example 4.3 for  $\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)}$ ). Since

$$\phi_{j_{n-1}}(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)}) = \langle \chi_{j_{n-1}, j_{n-1}+1}, \xi(v^{-1}\mathbf{b}^-) \rangle = \varepsilon_{j_{n-1}}(v^{-1}\mathbf{b}^-)$$

and

$$\begin{aligned} \langle \chi_{j_{n-1}, j_{n-1}+1}, \xi(v^{-1}\mathbf{b}^-) + v^{-1}\lambda_b^- \rangle &= \varepsilon_{j_{n-1}}(v^{-1}\mathbf{b}^-) + \langle \chi_{j_{n-1}, j_{n-1}+1}, v^{-1}\lambda_b^- \rangle \\ &= \phi_{j_{n-1}}(v^{-1}\mathbf{b}^-) \geq 0, \end{aligned}$$

we have

$$\begin{aligned} s_{j_{n-1}}(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes v^{-1}\mathbf{b}^-) &= \mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes \tilde{f}_{j_{n-1}}^{\phi_{j_{n-1}}(v^{-1}\mathbf{b}^-)}(v^{-1}\mathbf{b}^-) \\ &= \mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-), \end{aligned}$$

cf. Definition 4.4. Let  $l_h$  be an integer such that  $j_h \in \text{supp}(w_{l_h})$ . We write  $\text{FE}(v^{-1}\mathbf{b}^-) = \mathbf{b}'_1 \otimes \cdots \otimes \mathbf{b}'_d$ . Let  $u = +$  be the leftmost  $+$  in  $u_{j_{n-1}}(v^{-1}\mathbf{b}^-)_{\text{red}}$  if it exists, i.e.,  $\Phi_{j_{n-1}} \neq 0$ . Note that the action of  $s_{j_{n-1}-1}$  and  $s_{j_{n-1}+1}$  along the way of computing  $v^{-1}\mathbf{b}^-$  from  $\mathbf{b}$  does not increase  $\phi_i$ . So if  $j_{n-1} \in \text{supp}(w_v)$ , then  $u$  is the box in  $\mathbf{b}'_{l_{n-1}}, \dots, \mathbf{b}'_d$ . Equivalently, if  $u$  is the box in  $\mathbf{b}'_1, \dots, \mathbf{b}'_{l_{n-1}-1}$ , then  $j_{n-1} \notin \text{supp}(w_v)$ . Moreover,  $j_{n-1} \in \text{supp}(w_{l_{n-1}})$  implies  $\langle \chi_{j_{n-1}, j_{n-1}+1}, \lambda_b \rangle = \langle \chi_{j_{n-1}, j_{n-1}+1}, v^{-1}\lambda_b^- \rangle = 1$  in this case. By  $s_{j_{n-1}-1}s_{j_{n-1}+1}s_{j_{n-1}} \leq w(\mathbf{b})$ , this contradicts to Corollary 6.10 (iv). Thus  $\tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}$  does not change the boxes in  $\mathbf{b}'_1, \dots, \mathbf{b}'_{l_{n-1}-1}$ . In fact, if  $j_{n-1} \in \text{supp}(w_v)$ , then by  $s_{j_{n-1}-1}s_{j_{n-1}+1}s_{j_{n-1}} \leq w(\mathbf{b})$ ,  $u$  must be the box in  $\mathbf{b}'_{l_{n-1}}$  (which is changed from  $-$  along the computation  $v^{-1}\mathbf{b}^-$  from  $\mathbf{b}$ ). If  $u$  is the box in  $\mathbf{b}'_{l_{n-1}}$  and  $j_{n-1} \notin \text{supp}(w_v)$ , then we must have  $\langle \chi_{j_{n-1}, j_{n-1}+1}, \lambda_b \rangle = \langle \chi_{j_{n-1}, j_{n-1}+1}, v^{-1}\lambda_b^- \rangle = 1$ , which contradicts to Corollary 6.10 (iv). Therefore  $\tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}$  changes the box in  $\mathbf{b}'_{l_{n-1}}$  if and only if  $j_{n-1} \in \text{supp}(w_v)$ .

Assume that

$$s_{j_{h+1}}s_{j_{h+2}} \cdots s_{j_{n-1}}(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes v^{-1}\mathbf{b}^-) = \mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes \tilde{f}_{j_{h+1}}^{\Phi_{j_{h+1}}} \tilde{f}_{j_{h+2}}^{\Phi_{j_{h+2}}} \cdots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)$$

for some  $h < n-1$ . We further assume that for any  $h' > h$ ,  $\tilde{f}_{j_{h'}}^{\Phi_{j_{h'}}}$  does not change the boxes in  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{l_{h'}-1}$ , and  $\tilde{f}_{j_{h'}}^{\Phi_{j_{h'}}}$  changes the box in  $\mathbf{b}_{l_{h'}}$  if and only if  $j_{h'} \in \text{supp}(w_v)$ . It easily follows from Definition 4.1 (i) that

$$\langle \chi_{j_h, j_h+1}, \text{wt}(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes \tilde{f}_{j_{h+1}}^{\Phi_{j_{h+1}}} \tilde{f}_{j_{h+2}}^{\Phi_{j_{h+2}}} \cdots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)) \rangle = \Phi_{j_h}.$$

Moreover, we have

$$\varepsilon_{j_h}(\tilde{f}_{j_{h+1}}^{\Phi_{j_{h+1}}} \tilde{f}_{j_{h+2}}^{\Phi_{j_{h+2}}} \cdots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)) = \varepsilon_{j_h}(v^{-1}\mathbf{b}^-).$$

This is obvious if  $s_{j_h-1}s_{j_h+1}s_{j_h} \leq w(\mathbf{b})$ . If  $j_h \notin \text{supp}(w_v)$  and  $s_{j_h}s_{j_h-1} \leq w(\mathbf{b})$  (resp.  $s_{j_h}s_{j_h+1} \leq w(\mathbf{b})$ ), then by the induction hypothesis,  $\tilde{f}_{j_h-1}^{\Phi_{j_h-1}}$  (resp.  $\tilde{f}_{j_h+1}^{\Phi_{j_h+1}}$ ) does not change the box in  $\mathbf{b}_1, \dots, \mathbf{b}_{l_h}$ . Indeed, if  $j_h-1 \in \text{supp}(w_v)$  (resp.  $j_h+1 \in \text{supp}(w_v)$ ), then  $l_h < l_{h-1}$  (resp.  $l_h < l_{h+1}$ ). Note that the action of  $\tilde{f}_{j_h-1}^{\Phi_{j_h-1}}$  (resp.  $\tilde{f}_{j_h+1}^{\Phi_{j_h+1}}$ ) does not increase  $\varepsilon_{j_h}$  and hence  $\varepsilon_{j_h}(\tilde{f}_{j_h+1}^{\Phi_{j_h+1}} \tilde{f}_{j_h+2}^{\Phi_{j_h+2}} \dots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)) \leq \varepsilon_{j_h}(v^{-1}\mathbf{b}^-)$ . Note also that if  $j_h \notin \text{supp}(w_v)$ , then by Lemma 6.13 (ii), there exists  $0 \leq l \leq l_h$  such that  $\langle \chi_{j_h+1, j_h}, \text{wt}(\mathbf{b}'_1) + \dots + \text{wt}(\mathbf{b}'_l) \rangle = \varepsilon_{j_h}(v^{-1}\mathbf{b}^-)$ . Hence this equality holds if  $j_h \notin \text{supp}(w_v)$ . If  $j_h \in \text{supp}(w_v)$ , then there exists  $0 \leq l < l_h$  such that  $\langle \chi_{j_h+1, j_h}, \text{wt}(\mathbf{b}'_1) + \dots + \text{wt}(\mathbf{b}'_l) \rangle = \varepsilon_{j_h}(v^{-1}\mathbf{b}^-)$  except if  $s_{j_h}s_{j_h-1} \leq w(\mathbf{b})$ ,  $s_{j_h}s_{j_h+1} \leq w(\mathbf{b})$  and  $\langle \chi_{j_h, j_h+1}, \lambda_b \rangle = -1$ . In fact, this exceptional case does not occur by Corollary 6.10 (iii). By the induction hypothesis,  $\tilde{f}_{j_h-1}^{\Phi_{j_h-1}}$  (resp.  $\tilde{f}_{j_h+1}^{\Phi_{j_h+1}}$ ) does not change the box in  $\mathbf{b}_1, \dots, \mathbf{b}_{l_h-1}$ . So this equality also holds in this case.

Thus, by  $\phi_{j_h}(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)}) = \varepsilon_{j_h}(v^{-1}\mathbf{b}^-)$  and the induction hypothesis, we have

$$s_{j_h}s_{j_h+1} \dots s_{j_{n-1}}(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes v^{-1}\mathbf{b}^-) = \mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes \tilde{f}_{j_h}^{\Phi_{j_h}} \tilde{f}_{j_h+1}^{\Phi_{j_h+1}} \dots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-).$$

Moreover  $\tilde{f}_{j_h}^{\Phi_{j_h}}$  does not change the boxes in  $\mathbf{b}'_1, \dots, \mathbf{b}'_{l_h-1}$ , and  $\tilde{f}_{j_h}^{\Phi_{j_h}}$  changes the box in  $\mathbf{b}'_{l_h}$  if and only if  $j_h \in \text{supp}(w_v)$ . Indeed, if  $s_{j_h-1}s_{j_h+1}s_{j_h} \leq w(\mathbf{b})$ , then this follows similarly as above. Assume that  $s_{j_h}s_{j_h-1} \leq w(\mathbf{b})$  or  $s_{j_h}s_{j_h+1} \leq w(\mathbf{b})$ . Note that if  $j_{h'} = j_h - 1$  (resp.  $j_h + 1$ ) for some  $h < h' \leq n-1$ , then  $l_h \leq l_{h'}$  and the action of  $\tilde{f}_{j_{h'}}^{\Phi_{j_{h'}}}$  adds + (resp. deletes -) in  $u_{j_h}(v^{-1}\mathbf{b}^-)$ . Let  $u = +$  be the leftmost + in  $u_{j_h}(\tilde{f}_{j_h}^{\Phi_{j_h}} \tilde{f}_{j_h+1}^{\Phi_{j_h+1}} \dots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-))$ . If  $u$  is the box in  $\mathbf{b}'_1, \dots, \mathbf{b}'_{l_h-1}$ , then  $j_h \notin \text{supp}(w_v)$ . By Lemma 6.11 and our assumption, this contradicts to  $j_h \in \text{supp}(w_{l_h})$ . Thus  $\tilde{f}_{j_h}^{\Phi_{j_h}}$  does not change the boxes in  $\mathbf{b}'_1, \dots, \mathbf{b}'_{l_h-1}$ . If  $j_h \in \text{supp}(w_v)$ , then  $u$  is the box in  $\mathbf{b}'_{l_h}$ . Indeed, if  $l_h = l_{h'}$  and  $j_{h'} \in \text{supp}(w_v)$ , then  $\tilde{f}_{j_{h'}}^{\Phi_{j_{h'}}}$  changes the box in  $\mathbf{b}'_{l_h}$ . If  $u$  is the box in  $\mathbf{b}'_{l_h}$  and  $j_h \notin \text{supp}(w_v)$ , then  $j_h \in \text{supp}(w_{l_h})$  implies  $\langle \chi_{j_h, j_h+1}, \lambda_b \rangle = 1$  and  $j_h - 1, j_h + 1 \in \text{supp}(w_{l_h})$ . This contradicts to our assumption that  $s_{j_h}s_{j_h-1} \leq w(\mathbf{b})$  or  $s_{j_h}s_{j_h+1} \leq w(\mathbf{b})$ . Thus  $\tilde{f}_{j_h}^{\Phi_{j_h}}$  changes the box in  $\mathbf{b}'_{l_h}$  if and only if  $j_h \in \text{supp}(w_v)$ . By induction, this finishes the computation of  $w(\mathbf{b})(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes v^{-1}\mathbf{b}^-)$ .

Since

$$\begin{aligned} w(\mathbf{b})(\xi(v^{-1}\mathbf{b}^-) + v^{-1}\lambda_b^-) &= \text{wt}(w(\mathbf{b})(\mathbf{b}_{\xi(v^{-1}\mathbf{b}^-)} \otimes v^{-1}\mathbf{b}^-)) \\ &= \xi(v^{-1}\mathbf{b}^-) + \text{wt}(\tilde{f}_{j_1}^{\Phi_{j_1}} \tilde{f}_{j_2}^{\Phi_{j_2}} \dots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)), \end{aligned}$$

it remains to show that

$$\text{wt}(\tilde{f}_{j_1}^{\Phi_{j_1}} \tilde{f}_{j_2}^{\Phi_{j_2}} \dots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)) = \sum_{1 \leq j \leq d} w_1^{-1} \dots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j).$$

In the above discussion, we have proved that

$$\phi_{j_h}(\tilde{f}_{j_{h+1}}^{\Phi_{j_{h+1}}} \tilde{f}_{j_{h+2}}^{\Phi_{j_{h+2}}} \cdots \tilde{f}_{j_{n-1}}^{\Phi_{j_{n-1}}}(v^{-1}\mathbf{b}^-)) = \Phi_{j_h}$$

and that  $\tilde{f}_{j_h}^{\Phi_{j_h}}$  changes the box in  $\mathbf{b}_{l_h}$  if and only if  $j_h \in \text{supp}(w_v)$ . Note that  $\Phi_{j_1} \geq \cdots \geq \Phi_{j_{n-1}}$ . Thus we can easily check this equality by applying Lemma 6.23 repeatedly. The proof is finished.  $\square$

*Proof of Theorem 5.4.* We first show

$$b\xi_1(\mathbf{b}, v) = v\xi(v^{-1}\mathbf{b}^-) + \sum_{1 \leq j \leq d} vw_1^{-1} \cdots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j). \quad (*)$$

Note that  $b = c^m \varpi^{\lambda_b^+}$  as an element of  $\tilde{W}$ , where  $\lambda_b^+$  is the dominant conjugate of  $\lambda_b$ . So

$$\begin{aligned} (*) &\Leftrightarrow c^m v \xi(v^{-1}\mathbf{b}^-) + \lambda_b^+ = v \xi(v^{-1}\mathbf{b}^-) + \sum_{1 \leq j \leq d} vw_1^{-1} \cdots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j) \\ &\Leftrightarrow v^{-1}c^m v (\xi(v^{-1}\mathbf{b}^-) + v^{-1}\lambda_b^-) = \xi(v^{-1}\mathbf{b}^-) + \sum_{1 \leq j \leq d} w_1^{-1} \cdots w_{j-1}^{-1} \text{wt}(\mathbf{b}_j). \end{aligned}$$

Since  $v^{-1}c^m v = w(\mathbf{b})$ , the last equality follows from Proposition 6.24. This shows (\*). By (\*) and Proposition 6.22, we have  $\xi_\bullet(\mathbf{b}, v)^b = \text{FE}(\mathbf{b})$ . By Theorem 5.1, this implies  $\xi_\bullet(\mathbf{b}, v) \in \mathcal{A}_{\mu_\bullet}^{\text{top}}, (\Gamma^{G^d})^{-1}(\text{FE}(\mathbf{b})) = [\xi_\bullet(\mathbf{b}, v)]$  and  $\xi_\bullet(\mathbf{b}, v) \sim \xi_\bullet(\mathbf{b}, v')$  for any  $v, v' \in \Upsilon(\mathbf{b})$ . Since  $v_{\xi_1(\mathbf{b}, v)} = v$  and  $v_{\xi_1(\mathbf{b}, v')} = v'$ ,  $v \neq v'$  implies  $\xi_\bullet(\mathbf{b}, v) \neq \xi_\bullet(\mathbf{b}, v')$ . The proof is finished.  $\square$

## 7 The Semi-Module Stratification

Keep the notation and assumptions above.

### 7.1 The Semi-Module Stratification for $\omega_i$

Recall that if  $\mu$  is minuscule, then every extended semi-module is cyclic.

**Lemma 7.1.** For any  $1 \leq j \leq \frac{n-3}{2} (= \dim X_{\omega_2}(\tau^2))$ , we have

$$\mathbb{A}_{\omega_2}^j = \begin{cases} \{[\chi_{2,n-1}^\vee + \chi_{4,n-3}^\vee + \cdots + \chi_{j,n-j+1}^\vee]\} & (j \text{ even}) \\ \{[\chi_{1,n}^\vee + \chi_{3,n-2}^\vee + \cdots + \chi_{j,n-j+1}^\vee]\} & (j \text{ odd}). \end{cases}$$

*Proof.* By (the proof of) [51, Proposition 5.5], each normalized semi-module for  $2, n$  is of the form  $A_j = (2\mathbb{N} - j) \cup (\mathbb{N} + j + 1)$  for some  $1 \leq j \leq \frac{n-3}{2}$ . It is easy to check that

$$A_j = \begin{cases} A^{\chi_{2,n-1}^\vee + \chi_{4,n-3}^\vee + \dots + \chi_{j,n-j+1}^\vee} & (j \text{ even}) \\ A^{\chi_{1,n}^\vee + \chi_{3,n-2}^\vee + \dots + \chi_{j,n-j+1}^\vee} & (j \text{ odd}). \end{cases}$$

Let  $(A_j, \varphi_j)$  be the cyclic semi-module for  $\omega_2$ . Then  $n - 2 - j, n - 1 + j \in \bar{A}_j$  and  $\varphi_j(n - 2 - j) = \varphi_j(n - 1 + j) = 1$ . It is also easy to check that  $|\mathcal{V}(A_j, \varphi_j)| = j$ . This finishes the proof.  $\square$

**Lemma 7.2.** Assume that  $n = 7$ . Then  $\dim X_{\omega_3}(\tau^3) = 3$  and

$$\mathbb{A}_{\omega_3}^1 = \{[\chi_{1,7}^\vee]\}, \quad \mathbb{A}_{\omega_3}^2 = \{[\chi_{1,6}^\vee], [\chi_{2,7}^\vee]\}, \quad \mathbb{A}_{\omega_3}^3 = \{[\chi_{3,5}^\vee]\}.$$

Assume that  $n = 8$ . Then  $\dim X_{\omega_3}(\tau^3) = 4$  and

$$\begin{aligned} \mathbb{A}_{\omega_3}^1 &= \{[\chi_{1,8}^\vee]\}, \quad \mathbb{A}_{\omega_3}^2 = \{[\chi_{1,7}^\vee], [\chi_{2,8}^\vee]\}, \\ \mathbb{A}_{\omega_3}^3 &= \{[\chi_{2,6}^\vee], [\chi_{3,7}^\vee]\}, \quad \mathbb{A}_{\omega_3}^4 = \{[\chi_{1,8}^\vee + \chi_{4,5}^\vee]\}. \end{aligned}$$

*Proof.* Using Lemma 3.2 and Lemma 3.4, we can easily check the lemma by an easy calculation.  $\square$

## 7.2 The Semi-Module Stratification for $\omega_1 + \omega_{n-2}$

Throughout this subsection, we set  $\mu = \omega_1 + \omega_{n-2}$ . Also we assume that  $n \geq 4$ .

**Lemma 7.3.** Every extended semi-module for  $\mu$  is cyclic. For any  $0 \leq j \leq n - 2 (= \dim X_\mu(b))$ , we define  $\mathbb{A}_\mu^j$  similarly as in §3.3. Then we have  $\mathbb{A}_\mu^0 = \emptyset$  and  $|\mathbb{A}_\mu^j| = j$ . More precisely, if  $j$  is odd, then  $\mathbb{A}_\mu^j$  is equal to

$$\begin{aligned} &\{[\chi_{1,n-j+1}^\vee], [\chi_{1,n-j+3}^\vee + \chi_{2,n-j+2}^\vee], \dots, \\ &[\chi_{1,n}^\vee + \chi_{2,n-1}^\vee + \dots + \chi_{\frac{j+1}{2}, n-\frac{j-1}{2}}^\vee], \dots, [\chi_{j-2,n}^\vee + \chi_{j-1,n-1}^\vee], [\chi_{j,n}^\vee]\}, \end{aligned}$$

and if  $j$  is even, then  $\mathbb{A}_\mu^j$  is equal to

$$\begin{aligned} &\{[\chi_{1,n-j+1}^\vee], [\chi_{1,n-j+3}^\vee + \chi_{2,n-j+2}^\vee], \dots, \\ &[\chi_{1,n-1}^\vee + \chi_{2,n-2}^\vee + \dots + \chi_{\frac{j}{2}, n-\frac{j}{2}}^\vee], \dots, [\chi_{j-2,n}^\vee + \chi_{j-1,n-1}^\vee], [\chi_{j,n}^\vee]\}. \end{aligned}$$

*Proof.* Let  $(A, \varphi)$  be an extended semi-module for  $\mu$ . Let  $\mu'$  be the type of  $A$ . If  $(A, \varphi)$  is non-cyclic, then by Lemma 3.4,  $\mu'_{\text{dom}} \prec \mu$ , i.e.,  $\mu'_{\text{dom}} = \omega_{n-1}$ . By Lemma 3.2, we have  $A = \{0, 1, \dots, n - 1, \dots\}$ . By Definition 3.3 (3),  $\varphi(a) = \max\{k \mid$

$a + n - 1 - kn \in A\}$  for all  $a \in A$ . This contradicts to the assumption that  $(A, \varphi)$  is non-cyclic. Thus  $(A, \varphi)$  is cyclic.

Since  $\mu'$  satisfies  $\nu_b \preceq w_{\max}\mu'$ , it is easy to check that

$$w_{\max}\mu' = s_{l+1} \cdots s_{n-3} s_{n-2} s_{k-1} \cdots s_2 s_1 \mu$$

for some  $1 \leq k \leq n-2$  and  $k \leq l \leq n-2$ . Let  $\bar{A} = \{a_0, a_1, \dots, a_{n-1}\}$  with  $a_0 = \min \bar{A}$ . Then we have  $\varphi(a_0) = 0, \varphi(a_{n-l-1}) = 0, \varphi(a_{n-k}) = 2$  and  $\varphi(a_i) = 1$  for  $i \neq 0, n-l-1, n-k$ . Thus

$$\begin{aligned} \mathcal{V}(A, \varphi) = & \{(a_{n-k}, a_{n-l-1} + n), (a_{n-k}, a_{n-l}), (a_{n-k}, a_{n-l+1}), \dots, (a_{n-k}, a_{n-k-1})\} \\ & \sqcup \{(a_{n-k+1}, a_{n-l-1}), (a_{n-k+2}, a_{n-l-1}), \dots, (a_{n-1}, a_{n-l-1})\} \end{aligned}$$

and  $|\mathcal{V}(A, \varphi)| = l$ . Then by Proposition 3.7, the description of  $\mathbb{A}_\mu^l$  for each  $l$  in the lemma follows from direct computation.  $\square$

### 7.3 The Semi-Module Stratification for $\omega_1 + \omega_{n-3}$

Throughout this subsection, we set  $\mu = \omega_1 + \omega_{n-3}$ . Also we assume that  $n \geq 7$ .

**Lemma 7.4.** Every extended semi-module for  $\mu$  is cyclic. For any  $1 \leq j \leq \frac{3n-9}{2}$  ( $= \dim X_\mu(b)$ ), we define  $\mathbb{A}_\mu^j$  similarly as in §3.3. Then  $|\mathbb{A}_\mu^{\frac{3n-9}{2}}| = n-3$  and  $|\mathbb{A}_\mu^{\frac{3n-11}{2}}| \leq 2(n-4)$ .

*Proof.* Using Lemma 7.1, we can show the first assertion similarly as the proof of Lemma 7.3. Indeed, for any semi-module  $A^\lambda$  in Lemma 7.1, there exists a unique  $\varphi$  such that  $(A^\lambda, \varphi)$  is an extended semi-module for some  $\mu \in X_*(T)_+$ . The equality  $|\mathbb{A}_\mu^{\frac{3n-9}{2}}| = n-3$  follows from the Chen-Zhu conjecture.

Let  $(A, \varphi)$  be an extended semi-module for  $\mu$  with type  $\mu' (\in W_0\mu)$ . Let  $0 < k_1 < k_2$  be integers such that  $\mu'(1) = \mu'(k_1+1) = \mu'(k_2+1) = 0$ , and let  $l$  be an integer such that  $\mu'(l+1) = 2$ . Assume that  $\nu_b \preceq w_{\max} s_{k_2+1} \mu'$ . Let  $(B, \psi)$  be an extended semi-module for  $\mu$  with type  $s_{k_2+1} \mu'$ . Let  $a_0 = \min \bar{A}$  (resp.  $b_0 = \min \bar{B}$ ) and let inductively  $a_i = a_{i-1} + n - 2 - \mu'(i)n$  (resp.  $b_i = b_{i-1} + n - 2 - (s_{k_2+1} \mu')(i)n$ ) for  $i = 1, \dots, n$ . Then  $a_0 = a_n$  (resp.  $b_0 = b_n$ ) and  $\{a_0, a_1, \dots, a_{n-1}\} = \bar{A}$  (resp.  $\{b_0, b_1, \dots, b_{n-1}\} = \bar{B}$ ). We will show that if  $l > k_2 + 1$  (resp.  $l = k_2 + 1$ ), then  $|\mathcal{V}(B, \psi)| \leq |\mathcal{V}(A, \varphi)|$  (resp.  $|\mathcal{V}(B, \psi)| < |\mathcal{V}(A, \varphi)| - 1$ ). Moreover, the equality does not hold if  $k_2 - k_1 \leq 3$ .

Note that we have  $\varphi(a_0) = \varphi(a_{k_1}) = \varphi(a_{k_2}) = 0, \varphi(a_l) = 2, \psi(b_0) = \psi(b_{k_1}) = \psi(b_{k_2+1}) = 0, \psi(b_l) = 2$ . Note also that

$$\begin{aligned} \mathcal{V}(A, \varphi) = & \{(a, a') \mid a \in \bar{A} \text{ with } \varphi(a) = 1, a' = a_{k_1} \text{ or } a_{k_2}\} \\ & \sqcup \{(a_l, a') \mid a_l < a', \varphi(a') < 2\} \end{aligned}$$

and

$$\mathcal{V}(B, \psi) = \{(b, b') \mid b \in \bar{B} \text{ with } \psi(b) = 1, b' = b_{k_1} \text{ or } b_{k_2+1}\} \\ \sqcup \{(b_l, b') \mid b_l < b', \psi(b') < 2\}.$$

Let  $\mathcal{V}(A, \varphi)_1$  (resp.  $\mathcal{V}(B, \psi)_1$ ) be the first subset in  $\mathcal{V}(A, \varphi)$  (resp.  $\mathcal{V}(B, \psi)$ ) above, and let  $\mathcal{V}(A, \varphi)_2$  (resp.  $\mathcal{V}(B, \psi)_2$ ) be its complement.

If  $l > k_2 + 1$ , then it follows that

$$b_k = \begin{cases} a_k + 1 & (k \neq k_2 + 1) \\ a_k + 1 - n & (k = k_2 + 1) \end{cases}, \quad \psi(b_k) = \begin{cases} \varphi(a_k) & (k \neq k_2, k_2 + 1) \\ 1 - \varphi(a_k) & (k = k_2, k_2 + 1). \end{cases}$$

In particular,  $b_{k_2+1} - 1 = a_{k_2} - 2$ . So  $|\mathcal{V}(B, \psi)_1| > |\mathcal{V}(A, \varphi)_1|$  implies that  $|\mathcal{V}(B, \psi)_1| = |\mathcal{V}(A, \varphi)_1| + 1$  and  $b_{k_2} < b_{k_1}$ . By the fact  $(a_l, a_{k_2+1}) \in \mathcal{V}(A, \varphi)_2$ , we always have  $|\mathcal{V}(B, \psi)_2| < |\mathcal{V}(A, \varphi)_2|$ . Thus  $|\mathcal{V}(B, \psi)| \leq |\mathcal{V}(A, \varphi)|$ . Moreover, if  $k_2 - k_1 \leq 3$ , then the equality does not hold because  $b_{k_2} \geq b_{k_1}$ .

If  $l = k_2 + 1$ , then it follows that

$$b_k = \begin{cases} a_k + 2 & (k \neq k_2 + 1) \\ a_k + 2 - 2n & (k = k_2 + 1) \end{cases}, \quad \psi(b_k) = \begin{cases} \varphi(a_k) & (k \neq k_2, k_2 + 1) \\ 2 - \varphi(a_k) & (k = k_2, k_2 + 1). \end{cases}$$

In particular,  $b_{k_2+1} - 2 = a_{k_2} - 2 - n$ . By  $\nu_b \preceq w_{\max} s_{k_2+1} \mu'$ , we have  $k_2 \leq \frac{n-3}{2}$ . Using this, we can easily check that  $|\mathcal{V}(B, \psi)| < |\mathcal{V}(A, \varphi)_1|$  and  $\mathcal{V}(A, \varphi)_2 = \{(a_{k_2+1}, a_{k_2} + n)\}$ . Thus  $|\mathcal{V}(B, \psi)| < |\mathcal{V}(A, \varphi)| - 1$ .

Assume that  $\nu_b \preceq w_{\max} s_{k_1+1} \mu'$ . Let  $(C, \chi)$  be an extended semi-module for  $\mu$  with type  $s_{k_1+1} \mu'$ . Similarly as above, we can show that if  $l \geq k_1 + 1$ , then  $|\mathcal{V}(C, \chi)| \leq |\mathcal{V}(A, \varphi)|$ . Therefore,  $|\mathcal{V}(A, \varphi)| \geq \frac{3n-11}{2}$  holds only if  $k_2 = 2$  or  $l > k_2 = 3$ . From this and  $|\mathbb{A}_\mu^{\frac{3n-9}{2}}| = n - 3$ , we obtain  $|\mathbb{A}_\mu^{\frac{3n-11}{2}}| \leq 2(n - 4)$ .  $\square$

## 7.4 The Semi-Module Stratification for $\omega_1 + \omega_2, \omega_4 + \omega_{n-1}$

**Lemma 7.5.** Assume that  $n = 5$ . Set  $\mu = \omega_1 + \omega_2$ . Then every extended semi-module for  $\mu$  is cyclic. For any  $1 \leq j \leq 3 (= \dim X_\mu(b))$ , we define  $\mathbb{A}_\mu^j$  similarly as in §3.3. Then

$$\mathbb{A}_\mu^0 = \emptyset, \mathbb{A}_\mu^1 = \emptyset, \mathbb{A}_\mu^2 = \{\chi_{1,4}^\vee, \chi_{2,5}^\vee\}, \mathbb{A}_\mu^3 = \{\chi_{2,3}^\vee, \chi_{3,4}^\vee\}.$$

*Proof.* The first assertion follows similarly as the proof of Lemma 7.3. The second assertion follows from direct computation.  $\square$

**Lemma 7.6.** Assume that  $n = 7$  or  $8$ . Let  $\mu$  be  $\omega_1 + \omega_2$  or  $\omega_4 + \omega_{n-1}$ . Then there exists a non-cyclic extended semi-module for  $\mu$ .

*Proof.* As described in Lemma 7.2, there exists a unique top cyclic extended semi-module  $(A^\lambda, \varphi)$  for  $\omega_3$ . We define  $\varphi': \mathbb{Z} \rightarrow \mathbb{N} \cup \{-\infty\}$  by setting

$$\varphi'(a) = \begin{cases} \varphi(a) & (a \neq 1) \\ 0 & (a = 1). \end{cases}$$

Then it is straightforward to check that  $(A^\lambda, \varphi')$  is a non-cyclic extended semi-module for  $\omega_1 + \omega_2$ . The proof for  $\omega_4 + \omega_{n-1}$  is similar.  $\square$

## 8 The Ekedahl-Oort Stratification

Keep the notation and assumptions above. For  $\mu \in X_*(T)_+$ , set

$${}^S\text{Adm}(\mu)_{\text{cyc}} = \{w \in {}^S\text{Adm}(\mu) \mid p(w) \text{ is } n\text{-cycle}\}.$$

By Theorem 2.11,  $X_w(b) \neq \emptyset$  if  $w \in {}^S\text{Adm}(\mu)_{\text{cyc}}$ .

### 8.1 The Ekedahl-Oort Stratification for $\omega_i$

Throughout this subsection, we set  $\mu = \omega_i$  and  $c = s_i s_{i+1} \cdots s_{n-1} s_{i-1} \cdots s_2 s_1$ . By [29, Theorem 2.7], we have  $\dim X_{\varpi^\mu c}(b) = \dim X_\mu(b) = \langle \mu, \rho \rangle - \frac{n-1}{2}$ .

Note that  $|W_{\text{supp}_\sigma(w)}|$  is finite if and only if  $\text{supp}_\sigma(w) \neq \tilde{S}$ . Since  $\tau^m$  acts transitively on  $\tilde{S}$ ,  $\text{supp}_\sigma(w) \neq \tilde{S}$  if and only if  $w \in \Omega$ .

**Lemma 8.1.** Assume that  $n \geq 9$  and  $4 \leq i \leq n-4$ . Set  $y = c s_i s_{i+1} s_{i-1} = (1 \ i+1 \ i+3 \ i+4 \ \cdots \ n \ i \ i-2 \ \cdots \ 3 \ 2)(i-1 \ i+2)$ . Then we have  $\varpi^\mu y \in {}^S\text{Adm}(\mu)$  and  $X_{\varpi^\mu y}(b) \neq \emptyset$ .

*Proof.* Under the assumption in the lemma, we have  $\ell(\varpi^\mu y) = \langle \mu, 2\rho \rangle - \ell(y) (> 0)$  and hence  $\varpi^\mu y \in {}^S\text{Adm}(\mu)$  (cf. [39, (2.4.5)]). So, by Lemma 2.10 and Theorem 2.11,  $X_{\varpi^\mu y}(b) \neq \emptyset$  is equivalent to saying  $\text{supp}(ryr^{-1}) \subsetneq S$  for any  $r \in W_0$  such that  $r(\Phi_+ \setminus \Phi_{\varpi^\mu y}) \subset \Phi_+$ . It is easy to check that

$$\Phi_{\varpi^\mu y} = \Phi_{\{\chi_{1,2}, \chi_{2,3}, \dots, \chi_{i,i+1}\}} \cup \Phi_{\{\chi_{i,i+1}, \chi_{i+1,i+2}, \dots, \chi_{n-1,n}\}} \cup \{\chi_{i-2,i+2}, \chi_{i-1,i+2}, \chi_{i-1,i+3}\}.$$

In particular, we have  $\chi_{1,i+2}, \chi_{i-1,n} \in \Phi_+ \setminus \Phi_{\varpi^\mu y}$ . Note that we can decompose  $ryr^{-1}$  into disjoint cycles as

$$(r(1) \ r(i+1) \ r(i+3) \ r(i+4) \ \cdots \ r(n) \ r(i) \ r(i-2) \ \cdots \ r(3) \ r(2))(r(i-1) \ r(i+2))$$

for any  $r \in W_0$ . So if  $ryr^{-1} \in \bigcup_{J \subsetneq S} W_J$ , then  $(r(i-1) \ r(i+2)) = (1 \ 2)$  or  $(n-1 \ n)$ . This implies that  $r\chi_{1,i+2}$  or  $r\chi_{i-1,n}$  is negative and hence that  $r$  does not satisfy  $r(\Phi_+ \setminus \Phi_{\varpi^\mu y}) \subset \Phi_+$ . Thus we have  $X_{\varpi^\mu y}(b) \neq \emptyset$ .  $\square$

**Lemma 8.2.** Assume that  $n \geq 9$  and  $i = 3$  (resp.  $i = n - 3$ ). Set  $y = cs_3s_4s_5s_6s_2$  (resp.  $y = cs_{n-3}s_{n-4}s_{n-5}s_{n-6}s_{n-2}$ ). Then we have  $\varpi^\mu y \in {}^S\text{Adm}(\mu)$  and  $X_{\varpi^\mu y}(b) \neq \emptyset$ .

*Proof.* We only treat the case  $i = 3$ . The proof for the case  $i = n - 3$  is similar.

The first assertion is easy. To show the second assertion, by Lemma 2.10 and Theorem 2.11, it suffices to check that  $ryr^{-1} \notin \bigcup_{J \subsetneq S} W_J$  for any  $r \in W_0$  such that  $r(\Phi_+ \setminus \Phi_{\varpi^\mu y}) \subset \Phi_+$ . By an explicit calculation, it follows that  $\chi_{1,7}, \chi_{2,9} \in \Phi_+ \setminus \Phi_{\varpi^\mu y}$  and

$$ryr^{-1} = (r(1) r(4) r(6) r(8) r(9) \cdots r(n) r(3))(r(2) r(5) r(7)).$$

If  $ryr^{-1} \in \bigcup_{J \subsetneq S} W_J$ , then  $(r(2) r(5) r(7))$  is equal to  $(1 \ 2 \ 3)$  or  $(n-2 \ n-1 \ n)$ . This implies that  $r$  does not satisfy  $r(\Phi_+ \setminus \Phi_{\varpi^\mu y}) \subset \Phi_+$ . Thus we have  $X_{\varpi^\mu y}(b) \neq \emptyset$ .  $\square$

**Lemma 8.3.** Assume that  $n \geq 9$  and  $i = 3$  (resp.  $i = n - 3$ ). Let  $y$  be  $cs_i s_{i-1}$  or  $cs_i s_{i+1}$ . Then we have  $\varpi^\mu y \in {}^S\text{Adm}(\mu)$  and  $X_{\varpi^\mu y}(b) \neq \emptyset$ .

*Proof.* The proof is similar to the proof of Lemma 8.1 and Lemma 8.2. Note that  $y$  is a  $n$ -cycle in this case.  $\square$

**Proposition 8.4.** Assume that  $n \geq 9$  and  $3 \leq i \leq n - 3$ . Then the semi-module stratification of  $X_\mu(b)$  is not a refinement of the Ekedahl-Oort stratification.

*Proof.* First assume that  $n \geq 9$  and  $4 \leq i \leq n - 4$ . Let  $\varpi^\mu y \in {}^S\tilde{W}$  be as in Lemma 8.1. Let  $\mathcal{T}$  be a reduction tree of  $\varpi^\mu y$ . By Proposition 2.8, we have

$$|X_{\varpi^\mu y}(b)^{0,\sigma}| = \sum_{\underline{p}} (q-1)^{\ell_I(\underline{p})} q^{\ell_{II}(\underline{p})},$$

where  $\underline{p}$  runs over all the reduction paths in  $\mathcal{T}$  with  $\text{end}(\underline{p}) = \tau^m$ . Set  $d = \dim X_\mu(b) = \langle \mu, \rho \rangle - \frac{n-1}{2}$ . Suppose that the semi-module stratification of  $X_\mu(b)$  is a refinement of the Ekedahl-Oort stratification. Note that  $Z(\varpi^\mu c) = Z(\varpi^\mu y) = \{1\}$ . By Lemma 2.1, Proposition 2.4 and  $\dim X_{\varpi^\mu c}(b) = d$ , we have  $\ell_I(\underline{p}) + \ell_{II}(\underline{p}) \leq \dim X_{\varpi^\mu y}(b) \leq d - 1$  for any  $\underline{p}$ . On the other hand, we have  $\ell_I(\underline{p}) + 2\ell_{II}(\underline{p}) = \ell(\varpi^\mu y) = 2d - 3$ . Thus we have  $\ell_I(\underline{p}) + \ell_{II}(\underline{p}) = d - 1$  and  $\ell_I(\underline{p}) = 1$  for any  $\underline{p}$ . It follows that

$$|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| = |X_{\varpi^\mu y}(b)^{0,\sigma}| = k(q-1)q^{d-2},$$

where  $k \geq 1$  is the number of irreducible components of  $X_{\varpi^\mu y}(b)^0$ . Again by Lemma 2.1 and the fact that each  $S_{A,\varphi}$  is locally closed, we have  $|\{(A, \varphi) \mid \dim S_{A,\varphi} = d - 1, S_{A,\varphi} \subseteq \pi(X_{\varpi^\mu y}(b)^0)\}| = k$ . By Lemma 3.4, it follows that  $|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| \geq kq^{d-1}$ , which is a contradiction. This implies the proposition in this case.



Next assume that  $n \geq 10$  and  $i = 3, n - 3$ . Let  $\varpi^\mu y \in {}^S\tilde{W}$  be as in Lemma 8.2. Suppose that the semi-module stratification of  $X_\mu(b)$  is a refinement of the Ekedahl-Oort stratification. Similarly as above, we can check that

$$\dim X_{cs_i s_{i-1}}(b) = X_{cs_i s_{i+1}}(b) = d - 1.$$

Note that  $Z(\varpi^\mu c) = Z(\varpi^\mu cs_i s_{i-1}) = Z(\varpi^\mu cs_i s_{i+1}) = Z(\varpi^\mu y) = \{1\}$ . By Lemma 2.1 and Proposition 3.12, we have  $\dim X_{\varpi^\mu y}(b) \leq d - 2$ . Similarly as above, it follows that  $|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| = k(q - 1)q^{d-3}$  and  $|\pi(X_{\varpi^\mu y}(b)^0)^\sigma| \geq kq^{d-2}$ . This is a contradiction, which finishes the proof.  $\square$

The following proposition is the complement of Proposition 8.4.

**Proposition 8.5.** We have

$$\begin{aligned} {}^S\text{Adm}(\omega_1)_{\text{cyc}} &= \{\tau\}, \\ {}^S\text{Adm}(\omega_2)_{\text{cyc}} &= \{\tau^2, s_0 s_{n-1} \tau^2, s_0 s_{n-1} s_{n-2} s_{n-3} \tau^2, \dots, s_0 s_{n-1} \cdots s_5 s_4 \tau^2\} \quad (n \geq 5), \\ {}^S\text{Adm}(\omega_3)_{\text{cyc}} &= \{\tau^3, s_0 s_6 \tau^3, s_0 s_6 s_1 s_0 \tau^3, s_0 s_6 s_5 s_1 \tau^3, s_0 s_6 s_5 s_1 s_0 s_6 \tau^3\} \quad (n = 7), \\ {}^S\text{Adm}(\omega_3)_{\text{cyc}} &= \{\tau^3, s_0 s_1 \tau^3, s_0 s_7 s_6 s_5 \tau^3, s_0 s_7 s_6 s_1 \tau^3, s_0 s_7 s_6 s_5 s_1 s_0 \tau^3, \\ &\quad s_0 s_7 s_6 s_1 s_0 s_7 \tau^3, s_0 s_7 s_6 s_5 s_1 s_0 s_7 s_6 \tau^3\} \quad (n = 8). \end{aligned}$$

Let  $\varpi^\mu y \in {}^S\tilde{W}$  be one of the elements above. Then there exists  $v \in \text{LP}(\varpi^\mu y)$  such that  $v^{-1}yv$  is a Coxeter element. Moreover,  $X_w(b) = \emptyset$  for any  $w \in {}^S\text{Adm}(\mu) \setminus {}^S\text{Adm}(\mu)_{\text{cyc}}$ , and the semi-module stratification of  $X_\mu(b)$  is a refinement of the Ekedahl-Oort stratification.

*Proof.* The equalities in the proposition follow from easy calculations. For other statements, we only prove the case for  $\omega_2$ . Other cases can be checked similarly.

Set  $d = \frac{n-3}{2}$ . For  $0 \leq j \leq d$ , we set  $w_j = s_0 s_{n-1} \cdots s_{n-2j+1} \tau^2$ . Then  $\ell(w_j) = 2j$  and

$$p(w_j) = (1 \ 3 \ 5 \ \cdots \ n - 2j \ n - 2j + 1 \ \cdots \ n \ 2 \ 4 \ \cdots \ n - 2j - 1).$$

Also it is easy to check that

$$\Phi_+ \setminus \Phi_{w_j} = \{\chi_{1, n-2j+1}, \dots, \chi_{1, n-1}, \chi_{1, n}\}.$$

Clearly there exists  $r \in W_0$  with  $r(\Phi_+ \setminus \Phi_{w_j}) \subset \Phi_+$  such that  $rp(w_j)r^{-1}$  is a Coxeter element.

For an integer  $j$ , let  $0 \leq [j] < n$  denote its residue modulo  $n$ . For  $a, b \in \mathbb{N}$  with  $a - b \in 2\mathbb{Z}$ , we define  $t_{a,b} = s_{[b-2]} \cdots s_{[a+2]} s_{[a]}$ . Set

$$\begin{aligned} w_{j,0} &= w_j, w_{j,1} = t_{0, n-2j+1} w_j t_{0, n-2j+1}^{-1}, w_{j,2} = t_{n-1, n-2j+2} w_{j,1} t_{n-1, n-2j+2}^{-1}, \\ &\quad \dots, w_{j,j} = t_{n-j+1, n-j} w_{j, j-1} t_{n-j+1, n-j}^{-1}. \end{aligned}$$

It is easy to check that the simple reflections in  $t_{0,n-2j+1}, t_{n-1,n-2j+2}, \dots, t_{n-j+1,n-j}$  define

$$\begin{aligned} w_j = w_{j,0} \rightarrow_{\sigma} w_{j,1} = s_{n-1}s_{n-2} \cdots s_{n-2j+2}\tau^2 \rightarrow_{\sigma} w_{j,2} = s_{n-2}s_{n-3} \cdots s_{n-2j+3}\tau^2 \\ \rightarrow_{\sigma} \cdots \rightarrow_{\sigma} w_{j,j} = \tau^2. \end{aligned}$$

Let  $\underline{p}_j$  be the reduction path (in a suitable reduction tree) defined by this reduction. Using Lemma 2.1, Proposition 2.7, Proposition 2.8 and Proposition 3.12, we can check that  $X_{w_j}(\tau^2) = X_{\underline{p}_j}$  and  $X_w(\tau^2) = \emptyset$  for any  $w \in {}^S\text{Adm}(\omega_2) \setminus {}^S\text{Adm}(\omega_2)_{\text{cyc}}$  by counting the number of rational points of  $X_{\mu}(\tau^2)^0$  (note that  $X_{\tau^2}(\tau^2)^0 = \{I\}$ ). It is easy to check that

$$\ell(t_{n-j+1,n-j} \cdots t_{n-1,n-2j+2}t_{0,n-2j+1}) = \ell(t_{n-j+1,n-j}) + \cdots + \ell(t_{n-1,n-2j+2}) + \ell(t_{0,n-2j+1}).$$

Thus by Proposition 2.4 (cf. [46, §3.3]), each element  $gI$  in  $X_{w_j}(\tau^2)^0$  is contained in a Schubert cell associated to  $t_{n-j+1,n-j} \cdots t_{n-1,n-2j+2}t_{0,n-2j+1}$ . By Lemma 7.1, it follows that  $\pi(X_{w_j}(b)^0)$  is equal to the unique semi-module stratum of dimension  $j$ . This shows that the semi-module stratification of  $X_{\mu}(b)$  is a refinement of the Ekedahl-Oort stratification.  $\square$

## 8.2 The Ekedahl-Oort Stratification for $\omega_1 + \omega_{n-2}$

Throughout this subsection, we set  $\mu = \omega_1 + \omega_{n-2}$ . Also we assume that  $n \geq 4$ . Note that the unique dominant cocharacter  $\mu'$  with  $\mu' \prec \mu$  is  $\mu' = \omega_{n-1}$ . Clearly we have  ${}^S\text{Adm}(\omega_{n-1})_{\text{cyc}} = \{\tau^{n-1}\}$  and the semi-module stratification of  $X_{\omega_{n-1}}(\tau^{n-1})$  is a refinement of the Ekedahl-Oort stratification.

**Proposition 8.6.** For any  $1 \leq j \leq n-2 (= \dim X_{\mu}(b))$ , there exist exactly  $j$  elements of length  $2j$  in  ${}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ} := {}^S\text{Adm}(\mu)_{\text{cyc}} \setminus \{\tau^{n-1}\}$ . Let  $\varpi^{\mu}y \in {}^S\tilde{W}$  be one of such elements. Then there exists  $v \in \text{LP}(\varpi^{\mu}y)$  such that  $v^{-1}yv$  is a Coxeter element. Moreover,  $X_w(b) = \emptyset$  for any  $w \in {}^S\text{Adm}(\mu) \setminus {}^S\text{Adm}(\mu)_{\text{cyc}}$ , and the semi-module stratification of  $X_{\mu}(b)$  is a refinement of the Ekedahl-Oort stratification.

*Proof.* We first prove by induction on  $n$  that there exist at least  $j$  elements of length  $2j$  in  ${}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$ , each of which has finite part  $y$  such that  $ryr^{-1}$  is a Coxeter element for some  $r \in W_{\{s_2, \dots, s_{n-2}\}}$  satisfying  $r(\Phi_+ \setminus \Phi_{\varpi^{\mu}y}) \subset \Phi_+$  (cf. Lemma 2.10). Note that if  $y \in W_0$  satisfies

$$y^{-1}(2) < y^{-1}(3) < \cdots < y^{-1}(n-2) \text{ and } y^{-1}(n-1) < y^{-1}(n), \quad (*)$$

then by [46, Lemma 4.4], we have  $\varpi^{\mu}y \in {}^S\text{Adm}(\mu)$ . In particular, since  $\ell(\varpi^{\mu}) = 3n-5$ ,  $\varpi^{\mu}y$  is an element of length  $2j$  in  ${}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$  for any  $n$ -cycle  $y$  of length  $3n-2j-5$ .

If  $n = 4$ , then  $s_1s_2s_3, s_2s_3s_1$  and  $s_1s_2s_3s_1s_2$  are 4-cycles satisfying  $(*)$ . Moreover,  $s_2(s_1s_2s_3s_1s_2)s_2 = s_1s_2s_3$  is a Coxeter element and  $s_2(\Phi_+ \setminus \Phi_{\varpi^\mu s_1s_2s_3s_1s_2}) \subset \Phi_+$ . So the claim is true for  $n = 4$ .

Suppose that  $n \geq 5$  and the claim is true for  $n - 1$ . Let  $y$  be a  $(n - 1)$ -cycle in  $W_{\{s_1, s_2, \dots, s_{n-2}\}}$  such that  $y^{-1}(2) < y^{-1}(3) < \dots < y^{-1}(n - 3)$  and  $y^{-1}(n - 2) < y^{-1}(n - 1)$ . Then  $y' := s_1(1 \ 2 \ \dots \ n)y(1 \ 2 \ \dots \ n)^{-1}$  satisfies  $(*)$  and  $\ell(y') = \ell(y) + 1$ . So by the induction hypothesis, there exist at least  $j - 1$  elements in  $W_0$  which are  $n$ -cycles of length  $3n - 2j - 5$  satisfying  $(*)$ . Note that for any  $r \in W_{\{s_2, \dots, s_{n-3}\}}$ , we have  $r'y'r'^{-1} = s_1(1 \ 2 \ \dots \ n)ryr^{-1}(1 \ 2 \ \dots \ n)^{-1}$ , where  $r' = (1 \ 2 \ \dots \ n)r(1 \ 2 \ \dots \ n)^{-1} \in W_{\{s_2, \dots, s_{n-2}\}}$ . So again by the induction hypothesis, it is easy to verify that there exists  $r \in W_{\{s_2, \dots, s_{n-3}\}}$  such that  $r'y'r'^{-1}$  is a Coxeter element and  $r'(\Phi_+ \setminus \Phi_{\varpi^\mu y'}) \subset \Phi_+$ . Set  $c = s_{n-2}s_{n-1}s_{n-3} \dots s_2s_1$ . It is easy to check that if  $n$  is odd (resp. even), then

$$\begin{aligned} & c, \ cs_{n-2}s_{n-3}, \ \dots, \ cs_{n-2}s_{n-3} \dots s_2, \ cs_{n-2}s_{n-3} \dots s_2s_3s_4, \ \dots, \\ & \hspace{15em} cs_{n-2}s_{n-3} \dots s_2s_3s_4 \dots s_{n-2}s_{n-1} \\ (\text{resp. } & c, \ cs_{n-2}s_{n-3}, \ \dots, \ cs_{n-2}s_{n-3} \dots s_3, \ cs_{n-2}s_{n-3} \dots s_3s_2s_3, \ \dots, \\ & \hspace{15em} cs_{n-2}s_{n-3} \dots s_3s_2s_3 \dots s_{n-2}s_{n-1}) \end{aligned}$$

are  $n$ -cycles satisfying  $(*)$ . If  $y'$  is one of the elements above, then  $\Phi_{\{\chi_{2,3}, \dots, \chi_{n-2, n-1}\}} \cap \Phi_+ \subset \Phi_{\varpi^\mu y'}$  and there exists  $r' \in W_{\{s_2, \dots, s_{n-2}\}}$  such that  $r'y'r'^{-1}$  is a Coxeter element. Thus the claim is also true for  $n$ . By induction, our claim is true for any  $n \geq 4$ .

Clearly  $\nu_w = \nu_b$  for any  $w \in {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$ . Since  $b = \tau^{n-1}$  is superbasic, the unique minimal length element in the  $\sigma$ -conjugacy class of  $w$  is  $\tau^{n-1}$  (cf. [28, Proposition 3.5]). By Theorem 2.6, there exist a reduction tree  $\mathcal{T}$  for  $w$  and a reduction path  $\underline{p}$  in  $\mathcal{T}$  such that  $\text{end}(\underline{p}) = \tau^{n-1}$  and  $\ell_I(\underline{p}) = 0$ . Thus by Lemma 2.1 and Proposition 2.8,  $|\pi(X_w(b)^{0,\sigma})| \geq q^{\frac{\ell(w)}{2}}$  for any  $w \in {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$ . By the comparison of  $|\sqcup_{w \in {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ} \pi(X_w(b)^{0,\sigma})|$  and  $|X_\mu(b)^{0,\sigma}|$ , it follows from Lemma 7.3 and the claim we have shown above that there exist exactly  $j$  elements of length  $2j$  in  ${}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$ . Moreover, it follows that  $\pi(X_w(b)^0)$  is irreducible of dimension  $\frac{\ell(w)}{2}$  for any  $w \in {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$  and that  $X_w(b) = \emptyset$  for any  $w \in {}^S\text{Adm}(\mu) \setminus {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$ .

It remains to show that the semi-module stratification of  $X_\mu(b)$  is a refinement of the Ekedahl-Oort stratification. We prove that for any  $w \in {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$ , there exists an extended semi-module  $(A^\lambda, \varphi)$  for  $\mu$  such that  $\pi(X_w(b)^0) = S_{A^\lambda, \varphi} (= X_\mu^\lambda(b))$  by Lemma 3.9 and Lemma 7.3). We argue by induction on  $\ell(w)$ . If  $\ell(w) = 2$ , i.e.,  $w = \varpi^\mu cs_{n-2}s_{n-3} \dots s_2s_3s_4 \dots s_{n-2}s_{n-1} = s_0s_{n-1}\tau^{n-1}$ , then  $w \rightarrow_\sigma s_0ws_0 = \tau^{n-1}$ . It easily follows from Theorem 2.11 that  $X_{\tau^{n-1}s_0}(b) = \emptyset$ . So by Proposition 2.4, we have  $X_w(b)^0 = Is_0I/I$  and hence  $\pi(X_w(b)^0) = X_\mu^{\chi_{1,n}}(b)$ .

Suppose that  $\ell(w) \geq 4$  and the claim is true for any  $w' \in {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$  with  $\ell(w') < \ell(w)$ . Since  $\pi(X_w(b)^0)$  is irreducible of dimension  $\frac{\ell(w)}{2}$ , there exists a unique

extended semi-module  $(A^\lambda, \varphi)$  for  $\mu$  such that  $\dim(\pi(X_w(b)^0) \cap S_{A^\lambda, \varphi}) = \frac{\ell(w)}{2}$ . Also,  $\pi(X_w(b)^0) \cap S_{A^\lambda, \varphi}$  is open in both  $\pi(X_w(b)^0)$  and  $S_{A^\lambda, \varphi}$ . So the closure of  $\pi(X_w(b)^0) \cap S_{A^\lambda, \varphi}$  in  $X_\mu(b)$  is equal to both the closure of  $\pi(X_w(b)^0)$  and  $S_{A^\lambda, \varphi}$  in  $X_\mu(b)$ . By [26, Proposition 2.6] (see also [16, §3.3]), the closure of  $\pi(X_w(b)^0)$  is contained in

$$\bigsqcup_{w' \in {}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}, w' \leq_S w} \pi(X_{w'}(b)).$$

Here we write  $w' \leq_S w$  if there exists  $x \in W_0$  such that  $xw'x^{-1} \leq w$ . By the above description of the finite part of each element in  ${}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$ , it is easily checked that if  $w' \in {}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$  and  $\ell(w) = \ell(w')$ , then there is no  $x \in W_0$  such that  $xwx^{-1} = w'$ . So if  $w' \in {}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}, w' \leq_S w$  and  $\ell(w') = \ell(w)$ , then  $w = w'$ . Thus by the induction hypothesis, we have  $S_{A^\lambda, \varphi} \subseteq \pi(X_w(b)^0)$ . By [4, Proposition 2.11 (5) & Proposition 3.4], the closure of  $S_{A^\lambda, \varphi}$  is contained in a union of semi-module strata  $T_\lambda$  such that  $\dim(T_\lambda \setminus S_{A^\lambda, \varphi}) < \dim S_{A^\lambda, \varphi}$ . Thus by the induction hypothesis and Lemma 7.3, we have  $\pi(X_w(b)^0) \subseteq S_{A^\lambda, \varphi}$ . Therefore it follows that  $\pi(X_w(b)^0) = S_{A^\lambda, \varphi}$ , which completes the proof.  $\square$

### 8.3 The Ekedahl-Oort Stratification for $\omega_1 + \omega_{n-3}$

Throughout this subsection, we set  $\mu = \omega_1 + \omega_{n-3}$ . Also we assume that  $n \geq 7$ . Note that the unique dominant cocharacter  $\mu'$  with  $\mu' \prec \mu$  is  $\mu' = \omega_{n-2}$ .

**Proposition 8.7.** There exist at least  $2(n-4)$  elements of length  $3n-11$  in  ${}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ} := {}^S\text{Adm}(\mu)_{\text{cyc}} \setminus {}^S\text{Adm}(\omega_{n-2})_{\text{cyc}}$ . There also exists an element  $w$  of length  $3n-14$  in  ${}^S\text{Adm}(\mu)$  such that  $p(w)$  is not a  $n$ -cycle and  $X_w(b) \neq \emptyset$ . Moreover, the semi-module stratification of  $X_\mu(b)$  is not a refinement of the Ekedahl-Oort stratification.

*Proof.* For any  $1 \leq j \leq n-4$ , set  $c_j = s_{n-3}s_{n-2}s_{n-1}s_{n-4} \cdots s_{j+2}s_{j+1}s_1 \cdots s_{j-1}s_j$ . For  $j = n-3$ , set  $c_{n-3} = s_1s_2 \cdots s_{n-1}$ . Then we have  $\varpi^\mu c_j \in {}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$  and  $\ell(\varpi^\mu c_j) = 3n-9$  for any  $1 \leq j \leq n-3$ . If  $1 \leq j \leq n-5$ , then  $c_js_{n-3}s_{n-2}$  and  $c_js_{n-3}s_{n-4}$  are  $n$ -cycles of length  $3n-11$  satisfying  $\varpi^\mu c_js_{n-3}s_{n-2}, \varpi^\mu c_js_{n-3}s_{n-4} \in {}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$ . Further  $c_{n-4}s_{n-3}s_{n-2}$  and  $c_{n-3}s_{n-4}s_{n-3}$  are also  $n$ -cycles of length  $3n-11$  satisfying  $\varpi^\mu c_{n-4}s_{n-3}s_{n-2}, \varpi^\mu c_{n-3}s_{n-4}s_{n-3} \in {}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$ . Thus we have found  $2(n-4)$  distinct elements of length  $3n-11$  in  ${}^S\text{Adm}(\mu)_{\text{cyc}}^{\circ}$ .

Set  $y = c_{n-5}s_{n-3}s_{n-2}s_{n-4}s_{n-6}s_{n-5} = (1\ 2 \cdots n-6\ n-2\ n\ n-3)(n-4\ n-5\ n-1)$ . Then  $\varpi^\mu y \in {}^S\text{Adm}(\mu)$  and  $\chi_{1,n-1}, \chi_{n-5,n} \in \Phi_+ \setminus \Phi_{\varpi^\mu y}$ . By Theorem 2.11,  $X_{\varpi^\mu y}(b) \neq \emptyset$ . This shows the second assertion. We can easily check the last assertion using Lemma 7.4, similarly as the proof of Proposition 8.4.  $\square$

## 8.4 The Ekedahl-Oort Stratification for $\omega_1 + \omega_2, \omega_4 + \omega_{n-1}$

Note that the unique dominant cocharacter  $\mu'$  with  $\mu' \prec \omega_1 + \omega_2$  is  $\omega_3$ . By an explicit calculation, it is easy to verify the following statements (cf. Proposition 8.5).

**Proposition 8.8.** Assume that  $n = 5$ . Set  $\mu = \omega_1 + \omega_2$ . For any  $1 \leq j \leq 3 (= \dim X_\mu(b))$ , set  ${}^S\text{Adm}(\mu)_{\text{cyc}}^\circ := {}^S\text{Adm}(\mu)_{\text{cyc}} \setminus {}^S\text{Adm}(\omega_3)_{\text{cyc}}$ . Then we have

$${}^S\text{Adm}(\mu)_{\text{cyc}}^\circ = \{s_0 s_4 s_3 s_2 s_1 s_0 \tau^3, s_0 s_1 s_4 s_3 s_0 s_4 \tau^3, s_0 s_4 s_3 s_2 \tau^3, s_0 s_1 s_4 s_3 \tau^3\}.$$

Let  $\varpi^\mu y \in {}^S\text{Adm}(\mu)_{\text{cyc}}^\circ$ . Then there exists  $v \in \text{LP}(\varpi^\mu y)$  such that  $v^{-1} y v$  is a Coxeter element. Moreover,  $X_w(b) = \emptyset$  for any  $w \in {}^S\text{Adm}(\mu) \setminus {}^S\text{Adm}(\mu)_{\text{cyc}}$ , and the semi-module stratification of  $X_\mu(b)$  is a refinement of the Ekedahl-Oort stratification.

**Lemma 8.9.** Assume that  $n = 7$  or  $8$ . Let  $\mu$  be  $\omega_1 + \omega_2$  (resp.  $\omega_4 + \omega_{n-1}$ ). Set  $c = s_1 s_2 \cdots s_{n-1}$ . Then  $\varpi^\mu c s_1 s_2 s_3 \in {}^S\text{Adm}(\mu)$  and  $X_{\varpi^\mu c s_1 s_2 s_3}(b) \neq \emptyset$  (resp.  $\varpi^\mu c^{-1} s_5 s_4 s_3 \in {}^S\text{Adm}(\mu)$  and  $X_{\varpi^\mu c^{-1} s_5 s_4 s_3}(b) \neq \emptyset$ ). Further  $c s_1 s_2 s_3$  (resp.  $c^{-1} s_5 s_4 s_3$ ) is not  $n$ -cycle.

## 8.5 The Ekedahl-Oort Stratification for $\omega_2 + \omega_{n-3}$

We set  $\mu = \omega_2 + \omega_{n-3}$ . Also we assume that  $n \geq 5$ .

**Lemma 8.10.** If  $n$  is odd (resp. even), set  $y = s_2 s_3 \cdots s_{n-3} s_1 s_2 \cdots s_{n-3}$  (resp.  $y = s_2 s_3 \cdots s_{n-3} s_1 s_2 \cdots s_{n-2}$ ). Then  $\varpi^\mu y \in {}^S\text{Adm}(\mu)$ ,  $X_{\varpi^\mu y}(b) \neq \emptyset$  and  $y$  is not a  $n$ -cycle.

*Proof.* If  $n$  is odd (resp. even), then  $y = (1 \ 3 \ \cdots \ n-2)(2 \ 4 \ \cdots \ n-1 \ n)$  (resp.  $(1 \ 3 \ \cdots \ n-1)(2 \ 4 \ \cdots \ n)$ ) and  $\varpi^\mu y \in {}^S\text{Adm}(\mu)$ . Note that  $\chi_{1,n}, \chi_{2,n-1} \in \Phi_+ \setminus \Phi_{\varpi^\mu y}$ . So by Lemma 2.11,  $X_{\varpi^\mu y}(b) \neq \emptyset$ . The proof is finished.  $\square$

# 9 Comparison of Two Stratifications

Keep the notation and assumptions above.

## 9.1 Known Cases

The following results are known in (the proof of) [46, Corollary 5.5 & Theorem 5.9].

**Proposition 9.1.** Let  $\cong$  denote a universal homeomorphism.

- (i) Assume that  $n \geq 3$ . Set  $\mu = 2\omega_1, w = \varpi^\mu s_1 s_2 \cdots s_{n-1}$  and

$$\lambda = \begin{cases} \chi_{2,n-1}^\vee + \chi_{4,n-3}^\vee + \cdots + \chi_{\frac{n-1}{2}, \frac{n+3}{2}}^\vee & (\frac{n-1}{2} \text{ even}) \\ \chi_{1,n}^\vee + \chi_{3,n-2}^\vee + \cdots + \chi_{\frac{n-1}{2}, \frac{n+3}{2}}^\vee & (\frac{n-1}{2} \text{ odd}). \end{cases}$$

Then we have  $X_\mu(b)^0 = X_\mu^\lambda(b) = \pi(X_w(b)^0) \cong \mathbb{A}^{\frac{n-1}{2}}$ .

- (ii) Assume that  $n \geq 3$ . Set  $\mu = 2\omega_1 + \omega_{n-1}, w_j = \varpi^\mu s_{n-1} s_{n-2} \cdots s_{n-j+1} s_1 s_2 \cdots s_{n-j}$  and

$$\lambda_j = \begin{cases} \chi_{1,2j}^\vee + \chi_{2,2j-1}^\vee + \cdots + \chi_{j,j+1}^\vee & (j \leq \frac{n}{2}) \\ \chi_{2j+1-n,n}^\vee + \chi_{2j+2-n,n-1}^\vee + \cdots + \chi_{j,j+1}^\vee & (j \geq \frac{n}{2}). \end{cases}$$

for  $j = 1, 2, \dots, n-1$ . Then we have  $X_\mu(b)^0 = \bigsqcup_{1 \leq j \leq n-1} X_\mu^{\lambda_j}(b)$  and  $X_\mu^{\lambda_j}(b) = \pi(X_{w_j}(b)^0) \cong \mathbb{A}^{n-1}$  for each  $j$ .

- (iii) Assume that  $n = 5$ . Set  $\mu = 3\omega_1, w = \varpi^\mu s_1 s_2 s_3 s_4$  and  $\lambda = \chi_{1,2}^\vee + \chi_{3,4}^\vee$ . Then we have  $X_\mu(b)^0 = X_\mu^\lambda(b) = \pi(X_w(b)^0) \cong \mathbb{A}^4$ .
- (iv) Assume that  $n = 4$ . Set  $\mu = 3\omega_1, w = \varpi^\mu s_1 s_2 s_3$  and  $\lambda = \chi_{3,2}^\vee$ . Then we have  $X_\mu(b)^0 = X_\mu^\lambda(b) = \pi(X_w(b)^0) \cong \mathbb{A}^3$ .
- (v) Assume that  $n = 3$ . Set  $\mu = 4\omega_1, w = \varpi^\mu s_1 s_2$  and  $\lambda = \chi_{3,1}^\vee$ . Then we have  $X_\mu(b)^0 = X_\mu^\lambda(b) = \pi(X_w(b)^0) \cong \mathbb{A}^3$ .
- (vi) Assume that  $n = 3$ . Set  $\mu = 3\omega_1 + \omega_2, w_1 = \varpi^\mu s_1 s_2, w_2 = \varpi^\mu s_2 s_1, \lambda_1 = \chi_{2,3}^\vee$  and  $\lambda_2 = \chi_{3,2}^\vee$ . Then we have  $X_\mu(b)^0 = X_\mu^{\lambda_1}(b) \sqcup X_\mu^{\lambda_2}(b)$  and  $X_\mu^{\lambda_j}(b) = \pi(X_{w_j}(b)^0) \cong \mathbb{A}^3$  for each  $j$ .
- (vii) Assume that  $n = 2$ . Set  $\mu = m\omega_1$  with  $m \geq 1, w = \varpi^\mu s_1$  and

$$\lambda = \begin{cases} \frac{m-1}{2} \chi_{1,2}^\vee & (\frac{m-1}{2} \text{ odd}) \\ \frac{m-1}{2} \chi_{2,1}^\vee & (\frac{m-1}{2} \text{ even}). \end{cases}$$

Then we have  $X_\mu(b)^0 = X_\mu^\lambda(b) = \pi(X_w(b)^0) \cong \mathbb{A}^{\frac{m-1}{2}}$ .

## 9.2 The Classification in the Superbasic Case

**Theorem 9.2.** Let  $\mu \in X_*(T)_+$ . The following assertions are equivalent.

- (i) The semi-module stratification of  $X_{\leq \mu}(b)$  gives a refinement of the Ekedahl-Oort stratification.

(ii) For any  $w \in {}^S\text{Adm}(\mu)_0$ , there exists  $v \in \text{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element.

(iii) The cocharacter  $\mu$  has one of the following forms:

$$\begin{array}{ll}
\omega_1, & \omega_{n-1}, & (n \geq 1), \\
\omega_2, & 2\omega_1, & \omega_{n-2}, & 2\omega_{n-1}, & (\text{odd } n \geq 3), \\
\omega_2 + \omega_{n-1}, & 2\omega_1 + \omega_{n-1} & \omega_1 + \omega_{n-2}, & \omega_1 + 2\omega_{n-1}, & (n \geq 3), \\
\omega_3, & \omega_{n-3}, & (n = 7, 8), \\
3\omega_1, & 3\omega_{n-1}, & (n = 4, 5), \\
\omega_1 + \omega_2, & \omega_3 + \omega_4, & (n = 5), \\
4\omega_1, & \omega_1 + 3\omega_2, & 4\omega_2, & 3\omega_1 + \omega_2, & (n = 3), \\
m\omega_1 & \text{with } m \text{ odd}, & (n = 2).
\end{array}$$

If one of the above conditions holds, then for any  $w \in {}^S\text{Adm}(\mu)_{\text{cyc}} = {}^S\text{Adm}(\mu)_0$ , there exist  $\mu' \in X_*(T)_+$  with  $\mu' \preceq \mu$  and a cyclic extended semi-module  $(A^\lambda, \varphi)$  for  $\mu'$  such that  $\pi(X_w(b)^0) = X_{\preceq \mu}^\lambda(b) = S_{A^\lambda, \varphi}$ . Moreover  $\pi(X_w(b)^0) \cong \mathbb{A}^{\mathcal{V}(A^\lambda, \varphi)}$ .

*Proof.* For any  $w = \varpi^\mu y \in {}^S\tilde{W}$  with  $\mu$  dominant, set  $w^* = \varpi^{(\mu(1), \dots, \mu(1))} \varsigma(w)$  (cf. §2.5 and §3.2). Then  $w^* \in {}^S\tilde{W}$  and  $p(w^*) = w_{\max} y w_{\max}^{-1}$  (cf. §2.5 and §3.2). Note that the arguments and results in §7 and §8 for  $(\mu, w, b)$  also hold for  $(\mu^*, w^*, b^*)$ . Thus in this proof, it suffices to treat the case for either  $\mu$  or  $\mu^*$ .

First assume that  $n \geq 6$ . Let  $1 \leq m_0 < n$  be the residue of  $m$  modulo  $n$ . If  $4 \leq m_0 \leq n - 4$ , then  $\omega_{m_0} + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$ . So by Lemma 8.1 and Proposition 8.4,  $\mu$  satisfies neither (i) nor (ii). If  $n \geq 10$  and  $m_0 = 3$ , then by Lemma 8.2,  $\mu$  satisfies neither (i) nor (ii). If  $n = 7, 8$  and  $m_0 = 3$ , then by Proposition 8.5,  $\mu = \omega_3$  satisfies (i) and (ii). If moreover,  $\mu \neq \omega_3$ , then  $\omega_1 + \omega_2 + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$  or  $\omega_4 + \omega_{n-1} + (\lfloor \frac{m}{n} \rfloor - 1)\omega_n \preceq \mu$ . So by Lemma 7.6 and Lemma 8.9,  $\mu$  satisfies neither (i) nor (ii). If  $m_0 = n - 2$ , then  $\omega_1 + \omega_{n-3} + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$  unless  $\mu = \omega_{n-2}$  or  $2\omega_{n-1}$ . If  $m_0 = n - 1$ , then  $\omega_2 + \omega_{n-3} + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$  unless  $\mu = \omega_{n-1}, \omega_1 + \omega_{n-2}$  or  $\omega_1 + 2\omega_{n-1}$ . Thus the equivalence of (i), (ii) and (iii) for  $m_0 = n - 2, n - 1$  follows from Theorem 5.14, Proposition 8.5, Proposition 8.7, Proposition 8.10 and Proposition 9.1.

Assume that  $n = 5$ . If  $m_0 = 3$ , then  $\omega_1 + \omega_3 + \omega_4 + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$  unless  $\mu = \omega_3, 2\omega_4, \omega_1 + \omega_2$  or  $3\omega_1$ . If  $m_0 = 4$ , then  $2\omega_2 + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$  unless  $\mu = \omega_4, \omega_1 + \omega_3$  or  $\omega_1 + 2\omega_4$ . Set  $y_5 = (1 \ 5 \ 3)(2 \ 4)$ . Then it is easy to check that  $\varpi^{\omega_1 + \omega_3 + \omega_4} y_5 \in {}^S\text{Adm}(\omega_1 + \omega_3 + \omega_4)$  and  $X_{\varpi^{\omega_1 + \omega_3 + \omega_4} y_5}(\tau^8) \neq \emptyset$ . Assume that  $n = 4$ . If  $m_0 = 3$ , then  $2\omega_2 + \omega_3 + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$  unless  $\mu = \omega_3, \omega_1 + \omega_2, \omega_1 + 2\omega_3$  or  $3\omega_1$ . Set  $y_4 = (1 \ 3)(2 \ 4)$ . Then it is easy to check that  $\varpi^{2\omega_2 + \omega_3} y_4 \in {}^S\text{Adm}(2\omega_2 + \omega_3)$  and  $X_{\varpi^{2\omega_2 + \omega_3} y_4}(\tau^7) \neq \emptyset$ . Assume that  $n = 3$ . If  $m_0 = 2$ , then  $2\omega_1 + 3\omega_2 + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$  unless  $\mu =$

$\omega_2, 2\omega_1, \omega_1 + 2\omega_2, 3\omega_1 + \omega_2$  or  $4\omega_2$ . Set  $y_3 = (1 \ 3)$ . Then it is easy to check that  $\varpi^{2\omega_1+3\omega_2}y_3 \in {}^S\text{Adm}(2\omega_1 + 3\omega_2)$  and  $X_{\varpi^{2\omega_1+3\omega_2}y_3}(\tau^8) \neq \emptyset$ . Thus the equivalence of (i), (ii) and (iii) for  $n = 2, 3, 4, 5$  also follows from Theorem 5.14, Proposition 8.5, Proposition 8.10 and Proposition 9.1. The case for  $n = 1$  is trivially true.

Assume that  $\mu$  satisfies one of the conditions in the theorem, which is equivalent to each other as we have just proved. Except the cases where  $\mu$  or  $\mu^*$  is  $\omega_1 + \omega_{n-2}$  ( $n \geq 4$ ) or  $\omega_1 + \omega_2$  ( $n = 5$ ), it follows from [51, Theorem 5.3] and Proposition 9.1 that each  $X_\mu^\lambda(b) (\neq \emptyset)$  is universally homeomorphic to an affine space. Here we will treat the case  $\mu = \omega_1 + \omega_{n-2}$ . The proof for  $\mu = \omega_1 + \omega_2$  is similar.

Set  $\mu = \omega_1 + \omega_{n-2}$  and  $\mu_\bullet = (\mu_1, \mu_2) = (\omega_1, \omega_{n-2})$ . By [40, Theorem 1.5] and the Cartesian square right after it,  $\text{pr}$  induces a bijection between  $\text{pr}^{-1}(X_\mu(b)) (\subseteq X_{\mu_\bullet}(b_\bullet))$  and  $X_\mu(b)$  (cf. Lemma 4.13). Since  $\text{pr}$  is proper, it induces a universally homeomorphism onto its image. Thus by Theorem 5.1, it suffices to show that for any fixed  $1 \leq j \leq n-2$  and  $[\lambda] \in \mathbb{A}_\mu^j$ , there exists a unique  $\lambda_\bullet = (\lambda_1, \lambda_2) \in \mathcal{A}_{\mu_\bullet}^j$  such that  $\lambda_1 = \lambda$ . If  $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet}^j$ , then by [40, Proposition 2.9], we have

$$\lambda_2 - \lambda_1 \in W_0\omega_1, \quad b\lambda_1 - \lambda_2 \in W_0\omega_{n-2}.$$

By Lemma 7.3, we may assume that  $[\lambda] \in \mathbb{A}_\mu^j$  has one of the following forms:

- (1)  $\lambda = (1, \dots, 1, 0, \dots, 0, -1, \dots, -1, 0, \dots, 0)$ ,
- (2)  $\lambda = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$ ,
- (3)  $\lambda = (1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$ .

Here the numbers of 1 and  $-1$  are equal. In the case (1) (resp. (2)), let  $i = \max\{i' \mid \lambda(i') = -1\}$  (resp.  $\min\{i' \mid \lambda(i') = 1\}$ ). Then  $(\lambda_2 - \lambda)(i) = \lambda_2(i) + 1$  and  $(b\lambda - \lambda_2)(i) = 1 - \lambda_2(i)$  (resp.  $(\lambda_2 - \lambda)(i-1) = \lambda_2(i-1)$  and  $(b\lambda - \lambda_2)(i-1) = 2 - \lambda_2(i-1)$ ). So if  $\lambda_2 - \lambda \in W_0\omega_1$  and  $b\lambda - \lambda_2 \in W_0\omega_{n-2}$ , then  $\lambda_2(i) = 0$  (resp.  $\lambda_2(i-1) = 1$ ). Hence the  $i$ -th (resp.  $(i-1)$ -th) entry of  $\lambda_2 - \lambda$  is equal to 1, and other entries are equal to 0. So  $\lambda_2$  is uniquely determined by  $\lambda$ . In the case (3), we have  $(\lambda_2 - \lambda)(n) = \lambda_2(n) + 1$  and  $(b\lambda - \lambda_2)(n) = 1 - \lambda_2(n)$ . So if  $\lambda_2 - \lambda \in W_0\omega_1$  and  $b\lambda - \lambda_2 \in W_0\omega_{n-2}$ , then  $\lambda_2(n) = 0$ . So  $\lambda_2$  is also uniquely determined by  $\lambda$ .

Other statements follow from the results in §7 and §8. □

## 10 Some Simple Geometric Structure

Keep the notation and assumptions above.



## 10.1 The case of $\omega_2$ when $n$ is odd

In this subsection, we set  $\mu = \omega_2$  and  $b = \tau^2$ . Assume that  $n \geq 5$  is odd, i.e.,  $b$  is superbasic. For any  $1 \leq k \leq \frac{n-3}{2} (= \dim X_\mu(b))$ , set

$$\lambda_k = \begin{cases} \chi_{1,n}^\vee + \chi_{3,n-2}^\vee + \cdots + \chi_{k,n-k+1}^\vee & (k \text{ odd}) \\ \chi_{2,n-1}^\vee + \chi_{4,n-3}^\vee + \cdots + \chi_{k,n-k+1}^\vee & (k \text{ even}). \end{cases}$$

We also set  $\lambda_0 = \omega_0$ . Let  $X_\mu(b)^i = \{gK \in X_\mu(b) \mid \kappa(g) = i\}$ . Then  $X_\mu(b)^i$  is a closed subvariety with  $X_\mu(b) = \bigsqcup_{i \in \mathbb{Z}} X_\mu(b)^i = \bigsqcup_{i \in \mathbb{Z}} \tau^i X_\mu(b)^0$ .

**Proposition 10.1.** We have

$${}^S\text{Adm}(\mu)_0 = \{\tau^2, s_0 s_{n-1} \tau^2, s_0 s_{n-1} s_{n-2} s_{n-3} \tau^2, \dots, s_0 s_{n-1} \cdots s_5 s_4 \tau^2\}.$$

For any  $w \in {}^S\text{Adm}(\mu)_0$ , there exists  $v \in \text{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element. For  $0 \leq k \leq \frac{n-3}{2}$ , let  $w_k$  denote the unique element in  ${}^S\text{Adm}(\mu)_0$  of length  $2k$ . Then there exists an irreducible component  $Y(w_k)$  of  $X_{w_k}(b)$  such that  $X_{w_k}(b) = \mathbb{J}Y(w_k)$ ,  $Y(w_k) \cong \pi(Y(w_k)) = X_\mu^{\lambda_k}(b) \cong \mathbb{A}^k$  and

$$\pi(X_{w_k}(b)) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap I} j X_\mu^{\lambda_k}(b).$$

Moreover, the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$ .

*Proof.* Except the “moreover” part, the proposition follows from Proposition 8.5 and [51, Theorem 5.3]. It is enough to check the closure relation in  $X_\mu(b)^i$  for some  $i$ . Since  $\tau^{\frac{n+3}{2}} \lambda_k \in W_0 \omega_{\frac{n+3}{2}}$  (resp.  $\tau^{\frac{n+5}{2}} \lambda_k \in W_0 \omega_{\frac{n+5}{2}}$ ) for  $1 \leq k \leq \frac{n-3}{2}$  (resp.  $1 \leq k \leq \frac{n-5}{2}$ ), we have

$$X_\mu(b)^{\frac{n+3}{2}} \subset \mathcal{G}r(\omega_{\frac{n+3}{2}}) \quad (\text{resp. } (X_\mu(b)^{\frac{n+5}{2}} \setminus X_\mu^{\tau^{\frac{n+5}{2}} \lambda_{\frac{n-3}{2}}}(b)) \subset \mathcal{G}r(\omega_{\frac{n+5}{2}})),$$

where  $\mathcal{G}r(\lambda) = K\varpi^\lambda K/K$  for  $\lambda \in X_*(T) \cong \mathbb{Z}^n$ .

To show the closure relation, we argue by induction on odd  $n$ . If  $n = 5$ , then this follows from the equidimensionality of  $X_\mu(b)$ . Assume that the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$  for  $5, 7, \dots, n-2$ . Again by the equidimensionality of  $X_\mu(b)$ , the closure of  $X_\mu^{\lambda_{\frac{n-3}{2}}}(b)$  is  $X_\mu(b)^0$ . Let  $K' = \text{GL}_{n-2}(\mathcal{O})$  and let  $I'$  be the standard Iwahori subgroup in it. For  $\lambda' \in \mathbb{Z}^{n-2}$ , let  $\mathcal{G}r'(\lambda')$  denote  $K'\varpi^{\lambda'} K'/K'$ . Let  $\omega'_k = (1, \dots, 1, 0, \dots, 0) \in \mathbb{Z}^{n-2}$  in which 1 is repeated  $k$  times. We define  $\iota: \mathcal{G}r'(\omega'_{\frac{n+1}{2}}) \rightarrow \mathcal{G}r(\omega_{\frac{n+5}{2}})$  by  $g'K' \mapsto \begin{pmatrix} \varpi^{(1,1)} & 0 \\ 0 & g' \end{pmatrix} K$ . Clearly this map is well-defined and (universally) injective. Since  $\mathcal{G}r'(\omega'_{\frac{n+1}{2}})$  and  $\mathcal{G}r(\omega_{\frac{n+5}{2}})$  are projective

over  $\overline{\mathbb{F}}_q$ ,  $\iota$  is a universally homeomorphism onto its image (in fact, it is easy to check that  $\iota$  is a monomorphism and hence a closed immersion). Set  $\mu' = \omega'_2 \in \mathbb{Z}^{n-2}$ ,  $\tau' = \begin{pmatrix} 0 & \varpi \\ 1_{n-3} & 0 \end{pmatrix} \in \mathrm{GL}_{n-2}(F)$  and  $b' = \tau'^2$ . Let  $X_{\mu'}(b')^{\frac{n+1}{2}} \subset \mathcal{G}r'(\omega'_{\frac{n+1}{2}}) \subset \mathcal{G}r_{\mathrm{GL}_{n-2}}$ . For  $0 \leq k \leq \frac{n-5}{2}$ , we define  $\lambda'_k \in \mathbb{Z}^{n-2}$  by  $\tau'^{\frac{n+5}{2}} \lambda_k = (1, 1, \lambda'_k)$ . Then  $X_{\mu'}(b')^{\frac{n+1}{2}} = \bigsqcup_{0 \leq k \leq \frac{n-5}{2}} X_{\mu'}^{\lambda'_k}(b')$ . Note that the first three entries of  $\lambda'_k$  are 1. So for any  $g'K' \in X_{\mu'}^{\lambda'_k}(b')$  ( $gK \in X_{\mu}^{\lambda_k}(b)$ ), there exists

$$h' \in \begin{pmatrix} 1_2 & 0 \\ 0 & \mathrm{GL}_{n-4}(L) \end{pmatrix} \cap I' \quad (\text{resp. } h \in \begin{pmatrix} 1_4 & 0 \\ 0 & \mathrm{GL}_{n-4}(L) \end{pmatrix} \cap I)$$

such that  $g'K' = h'\varpi^{\lambda'_k}K'$  (resp.  $gK = h\varpi^{\lambda_k}K$ ). By  $b = \dot{s}_2\dot{s}_3\dot{s}_1\dot{s}_2 \begin{pmatrix} 1_2 & 0 \\ 0 & b' \end{pmatrix}$ , we have

$$\varpi^{-\lambda_k} \begin{pmatrix} 1_2 & 0 \\ 0 & h'^{-1} \end{pmatrix} b \begin{pmatrix} 1_2 & 0 \\ 0 & \sigma(h') \end{pmatrix} \varpi^{\lambda_k} = \dot{s}_2\dot{s}_3\dot{s}_1\dot{s}_2 \begin{pmatrix} 1_2 & 0 \\ 0 & \varpi^{-\lambda'_k} h'^{-1} b' \sigma(h'\varpi^{\lambda'_k}) \end{pmatrix} \in K\varpi^{\mu}K,$$

Here  $\dot{s}_i$  denotes the permutation matrix corresponding to  $s_i$ . Conversely, for  $h$  above, we define  $h'$  by  $h = \begin{pmatrix} 1_2 & 0 \\ 0 & h' \end{pmatrix}$ . Then we can similarly check that  $h'\varpi^{\lambda'_k}K' \in X_{\mu'}^{\lambda'_k}(b')$ .

Thus  $\iota(X_{\mu'}^{\lambda'_k}(b')) = X_{\mu}^{\tau^{\frac{n+5}{2}} \lambda_k}(b)$  for  $0 \leq k \leq \frac{n-5}{2}$ . It follows from this and the induction hypothesis that the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$  for  $n$ . By induction, this finishes the proof.  $\square$

## 10.2 The case of $\omega_2$ when $n$ is even

In this subsection, we set  $\mu = \omega_2$  and  $b = \tau^2$ . Then the  $F$ -rank of  $\mathbb{J}$  is 1. Assume that  $n \geq 4$  is even.

**Lemma 10.2.** We have

$${}^S\mathrm{Adm}(\mu)_0 = \{\tau^2, s_0\tau^2, s_0s_{n-1}s_{n-2}\tau^2, \dots, s_0s_{n-1} \cdots s_5s_4\tau^2\}.$$

For any  $w \in {}^S\mathrm{Adm}(\mu)_0$ , there exists  $v \in \mathrm{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element.

*Proof.* Set  $u_1 = s_2s_3 \cdots s_{n-1}s_1$ . For  $2 \leq k \leq n-2$ , we also set  $u_k = u_1s_2s_3 \cdots s_k$ . Then  $\varpi^{\mu}u_k \in {}^S\mathrm{Adm}(\mu)$ . Moreover it is easy to check that  $w \in {}^S\mathrm{Adm}(\mu)$  satisfies  $\mathrm{supp}_{\sigma}(w) \neq \tilde{S}$  if and only if  $w$  is  $\varpi^{\mu}u_{n-2} = \tau^2$  or  $\varpi^{\mu}u_{n-3} = s_0\tau^2$ . So it follows from Theorem 2.11 that  ${}^S\mathrm{Adm}(\mu)_0 \subseteq \{\varpi^{\mu}u_1, \dots, \varpi^{\mu}u_{n-2}\}$ .

For even  $k$ , we have

$$u_k = (1 \ 3 \ \cdots \ k+1)(2 \ 4 \ \cdots \ k+2 \ k+3 \ \cdots \ n-1 \ n).$$

For odd  $k$ , we have

$$u_k = (1 \ 3 \ \cdots \ k+2 \ k+3 \ \cdots \ n \ 2 \ 4 \ \cdots \ k+1).$$

It is easy to check that  $\Phi_+ \setminus \Phi_{\varpi^\mu u_k} = \{\chi_{1,k+3}, \dots, \chi_{1,n-1}, \chi_{1,n}\}$ . In particular,  $r(\Phi_+ \setminus \Phi_{\varpi^\mu u_k}) \subset \Phi_+$  for  $r \in W_{\{s_2, \dots, s_{n-1}\}}$ . Again from Theorem 2.11, it follows that  $X_{\varpi^\mu u_k}(b) = \emptyset$  for even  $2 \leq k \leq n-2$ . We can also check that other  $\varpi^\mu u_k$  has positive Coxeter part. This finishes the proof.  $\square$

**Corollary 10.3.** Set  $v_{k,l} = s_{n-l}s_{n-l-1} \cdots s_{n-2k+l+1}\tau^2$  for  $2 \leq k \leq \frac{n-2}{2}$  and  $1 \leq l \leq k-1$ . Then  $X_{v_{k,l}}(b) = \emptyset$ .

*Proof.* We keep the notation in the proof of Lemma 10.2. It is easy to check that  $v_{k,l} \approx_\sigma s_0 s_{n-1} \cdots s_{n-2(k-l)+1}\tau^2 = \varpi^\mu u_{n-2(k-l-1)}$ . Then the statement follows from Proposition 2.4 and Lemma 10.2.  $\square$

For  $w \in {}^S\tilde{W}$ , set  $S_w = \max\{S' \subseteq S \mid \text{Ad}(w)(S') = S'\}$ . Clearly  $S_{\tau^2} = \{s_1, s_3, \dots, s_{n-1}\}$ .

**Corollary 10.4.** Let  $\varpi^\mu y \in {}^S\text{Adm}(\mu) \setminus \{\tau^2\}$  such that  $\ell(y) \geq n-1$ . Then  $S_{\varpi^\mu y} = \emptyset$ .

*Proof.* We keep the notation in the proof of Lemma 10.2. Then  $y = u_k$  for some  $1 \leq k \leq n-3$ , and the lemma follows from explicit computation.  $\square$

There are two  $\tau^2$ -orbit in  $\tilde{S}$ , namely,  $\{s_0, s_2, \dots, s_{n-2}\}$  and  $\{s_1, s_3, \dots, s_{n-1}\}$ . Let  $P_0$  (resp.  $P_1$ ) denote the standard parahoric subgroup of  $G(L)$  corresponding to the former (resp. latter) orbit. Then  $W_a^{\tau^2}$  is the Weyl group with two simple reflections  $s_0 s_2 \cdots s_{n-2}$  and  $s_1 s_3 \cdots s_{n-1}$ . For  $j, j' \in \mathbb{J}^0$ , we have  $\text{inv}(j, j') \in W_a^{\tau^2}$ .

**Lemma 10.5.** For  $1 \leq k \leq \frac{n-2}{2}$ , let  $w_k$  denote the unique element in  ${}^S\text{Adm}(\mu)_0$  of length  $2k-1$  (more precisely,  $w_k = s_0 s_{n-1} \cdots s_{n-2(k-1)}\tau^2$ ). For any  $w_k$ , there exists an irreducible component  $Y(w_k)$  of  $X_{w_k}(b)$  such that

$$X_{w_k}(b) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_{w_k}} jY(w_k),$$

where  $P_{w_k} = P_0$  (resp.  $P_1$ ) if  $k$  is odd (resp. even). Moreover each  $Y(w_k)$  is universally homeomorphic to  $(\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_{q^{\frac{n}{2}}})) \times \mathbb{A}^{k-1}$ , and contained in a  $\mathbb{J}$ -stratum in  $G(L)/I$ .

*Proof.* For an integer  $a$ , let  $0 \leq [a] < n$  denote its residue modulo  $n$ . For  $a, b \in \mathbb{N}$  with  $a - b \in 2\mathbb{Z}$ , we define  $t_{a,b} = s_{[b-2]} \cdots s_{[a+2]} s_{[a]}$ . Set

$$\begin{aligned} w_{k,0} &= w_k, w_{k,1} = t_{0,n-2(k-1)} w_k t_{0,n-2(k-1)}^{-1}, w_{k,2} = t_{n-1,n-2k+3} w_{k,1} t_{n-1,n-2k+3}^{-1}, \\ &\quad \dots, w_{k,k-1} = t_{n-k+2,n-k} w_{k,k-2} t_{n-k+2,n-k}^{-1}. \end{aligned}$$

It is easy to check that the simple reflections in  $t_{0,n-2(k-1)}, t_{n-1,n-2k+3}, \dots, t_{n-k+2,n-k}$  define

$$\begin{aligned} w_k = w_{k,0} &\rightarrow_{\sigma} w_{k,1} = s_{n-1} s_{n-2} \cdots s_{n-2k+3} \tau^2 \rightarrow_{\sigma} w_{k,2} = s_{n-2} s_{n-3} \cdots s_{n-2k+4} \tau^2 \\ &\rightarrow_{\sigma} \cdots \rightarrow_{\sigma} w_{k,k-1} = s_{n-k+1} \tau^2. \end{aligned}$$

Let  $\underline{p}_k$  be the reduction path (in a suitable reduction tree) defined by this reduction. By Proposition 2.7 and Corollary 10.3, we have  $X_{w_k}(b) = X_{\underline{p}_k}$  with  $\text{end}(\underline{p}_k) = s_{n-k+1} \tau^2$ . Let  $f: X_{w_k}(b) \rightarrow X_{s_{n-k+1} \tau^2}(b)$  be the morphism induced by Proposition 2.4. By Proposition 2.3, we have

$$X_{s_{n-k+1} \tau^2}(b) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_{w_k}} jY(s_{n-k+1} \tau^2),$$

where  $Y(s_{n-k+1} \tau^2) = \{gI \in P_{w_k}/I \mid g^{-1} \tau^2 \sigma(g) \tau^{-2} \in Is_{n-k+1}I\}$  is a classical Deligne-Lusztig variety in the finite-dimensional flag variety  $P_{w_k}/I$ . If  $k$  is odd (resp. even), set  $v_k = s_0 s_2 \cdots s_{n-2}$  (resp.  $s_1 s_3 \cdots s_{n-1}$ ). Then by [38, Corollary 2.5] (see also [14, Proposition 1.1]),  $Y(s_{n-k+1} \tau^2)$  is contained in  $Iv_k I/I$ . Moreover, it is easy to check that

$$\begin{aligned} &\ell(v_k t_{n-k+2,n-k} \cdots t_{n-1,n-2k+3} t_{0,n-2(k-1)}) \\ &= \frac{n}{2} + \ell(t_{n-k+2,n-k}) + \cdots + \ell(t_{n-1,n-2k+3}) + \ell(t_{0,n-2(k-1)}). \end{aligned}$$

Indeed  $v_k t_{n-k+2,n-k} \cdots t_{n-k+l+1,n-k-l+1} \alpha_i = v_k v_{k+1} \cdots v_{k+l} \alpha_i$ , where  $\alpha_i$  is the simple affine root corresponding to  $i \in \text{supp}(t_{n-k+l+2,n-k-l})$ . Since  $W_a^{\tau^2}$  is the Weyl group with two simple reflections  $v_k$  and  $v_{k+1}$ ,  $v_k v_{k+1} \cdots v_{k+l} \alpha_i > 0$ . We set  $Y(w_k) = f^{-1}(Y(s_{n-k+1} \tau^2))$ . Then by Proposition 2.4, we have

$$Y(w_k) \subset Iv_k t_{n-k+2,n-k} \cdots t_{n-1,n-2k+3} t_{0,n-2(k-1)} I/I.$$

Note that  $(\tau^2)^{\frac{n}{2}} = \varpi^{(1,\dots,1)}$  belongs to the center of  $G(L)$ . It follows from this fact that  $Y(s_{n-k+1} \tau^2) \cong \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_{q^{\frac{n}{2}}})$ . Thus by Remark 2.5,  $Y(w_k)$  is universally homeomorphic to  $(\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_{q^{\frac{n}{2}}})) \times \mathbb{A}^{k-1}$ .

It remains to show that for all  $j \in \mathbb{J}$ , the value  $\text{inv}(j, -)$  is constant on each  $j'Y(w_k)$ . For this, we argue similarly as [14, §3.3]. Clearly we may assume  $j' = 1$ .

For any  $j \in \mathbb{J}$ , there exists  $\tilde{j} \in \mathbb{J}^0$  such that  $\text{inv}(j, \tilde{j}) = \Omega$ . So we may also assume  $j \in \mathbb{J}^0$ . Fix  $jI$  with  $j \in \mathbb{J}^0$ . Then by [38, Corollary 2.5] and [1, Proposition 5.34] (see also [14, Proposition 1.7]), there exists  $gI$  with  $g \in \mathbb{J}^0 \cap P_{w_k}$  (called the “gate”) such that for any  $y_0I \in Y(s_{n-k+1}\tau^2)$ , we have

$$\text{inv}(j, y_0) = \text{inv}(j, g)v_k(\in W_a) \quad \text{with} \quad \ell(\text{inv}(j, y_0)) = \ell(\text{inv}(j, g)) + \ell(v_k).$$

In particular,  $\text{inv}(j, g)$  has a reduced expression as an element of  $W_a^{\tau^2}$  whose right-most simple reflection is  $v_{k+1}$ . Let  $y \in Y(w_k)$  and set  $y_0 = f(y) \in Y(s_{n-k+1}\tau^2)$ . Note that

$$\begin{aligned} & \ell(\text{inv}(j, y_0)t_{n-k+2, n-k} \cdots t_{n-1, n-2k+3}t_{0, n-2(k-1)}) \\ &= \ell(\text{inv}(j, g)) + \ell(v_k) + \ell(t_{n-k+2, n-k}) + \cdots + \ell(t_{n-1, n-2k+3}) + \ell(t_{0, n-2(k-1)}). \end{aligned}$$

Indeed  $k$  is odd (resp. even) if and only if  $n - k$  is odd (resp. even). Thus

$$\begin{aligned} \text{inv}(j, y) &= \text{inv}(j, y_0)\text{inv}(y_0, y) \\ &= \text{inv}(j, g)v_k t_{n-k+2, n-k} \cdots t_{n-1, n-2k+3}t_{0, n-2(k-1)} \end{aligned}$$

is independent of  $y \in Y(w_k)$ . This finishes the proof.  $\square$

For any  $1 \leq k \leq \frac{n-2}{2} (= \dim X_\mu(b))$ , set

$$\lambda_k = \begin{cases} \chi_{1,n}^\vee + \chi_{3,n-2}^\vee + \cdots + \chi_{k,n-k+1}^\vee & (k \text{ odd}) \\ \chi_{2,n-1}^\vee + \chi_{4,n-3}^\vee + \cdots + \chi_{k,n-k+1}^\vee & (k \text{ even}). \end{cases}$$

We also set  $w_0 = \tau^2$ ,  $P_{w_0} = P_1$ ,  $Y(w_0) = \{pt\}$  and  $\lambda_0 = \omega_0$ .

**Proposition 10.6.** Keep the notation above. Then

$$\pi(X_{w_k}(b)) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_{w_k}} j\pi(Y(w_k))$$

and each  $j\pi(Y(w_k))$  is a  $\mathbb{J}$ -stratum of  $X_\mu(b)$ , which is universally homeomorphic to  $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_{q^{\frac{n}{2}}}) \times \mathbb{A}^{k-1}$  (resp.  $\{pt\}$ ) if  $1 \leq k \leq \frac{n-2}{2}$  (resp.  $k = 0$ ). Moreover, the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$ .

*Proof.* Let  $1 \leq k \leq \frac{n-2}{2}$ . Using Lemma 2.1, we can easily check that the map  $X_{w_k}(b) \rightarrow \pi(X_{w_k}(b))$  induced by  $\pi$  is universally bijective. By Proposition 2.3,  $\pi(X_w(b)) = \pi(X_{\tau^2}(b))$  for any  $w \in W_{\{s_1, s_3, \dots, s_{n-1}\}}\tau^2$ . By [16, Proposition 3.1.1], Proposition 2.4, Lemma 10.2 and Corollary 10.4, we have

$$\pi^{-1}(\pi(X_{w_k}(b))) \cap \left( \bigcup_{w' \leq w_k} X_{w'}(b) \right) = X_w(b)$$

(cf. the proof of [46, Lemma 5.8]). Since  $\pi$  is proper, the map  $X_{w_k}(b) \rightarrow \pi(X_{w_k}(b))$  is also proper. This implies that the map  $X_{w_k}(b) \rightarrow \pi(X_{w_k}(b))$  is a universally homeomorphism. In particular,  $\pi(Y(w_k))$  is universally homeomorphic to  $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_{q^{\frac{n}{2}}}) \times \mathbb{A}^{k-1}$ . Clearly, the same assertions are true for each  $j\pi(Y(w))$ . The case  $k = 0$  is trivial.

We next prove the closure relation. It follows from the proof of Lemma 10.5 that  $\pi(Y(w_k)) \subset X_\mu^{\lambda_k}(b)$  for any  $0 \leq k \leq \frac{n-2}{2}$ . Since  $\tau^{\frac{n+2}{2}}\lambda_k \in W_0\omega_{\frac{n+2}{2}}$ , we have

$$\tau^{\frac{n+2}{2}}\pi(Y(w_k)) \subset X_\mu^{\tau^{\frac{n+2}{2}}\lambda_k}(b) \subset \mathcal{G}r(\omega_{\frac{n+2}{2}}),$$

where  $\mathcal{G}r(\lambda) = K\varpi^\lambda K/K$  for  $\lambda \in X_*(T) \cong \mathbb{Z}^n$ .

Let  $K' = \mathrm{GL}_{n+1}(\mathcal{O})$  and let  $I'$  be the standard Iwahori subgroup in it. For  $\lambda' \in \mathbb{Z}^{n+1}$ , let  $\mathcal{G}r'(\lambda')$  denote  $K'\varpi^{\lambda'}K'/K'$ . Let  $\omega'_k = (1, \dots, 1, 0, \dots, 0) \in \mathbb{Z}^{n+1}$  in which 1 is repeated  $k$  times. We define  $\iota: \mathcal{G}r(\omega_{\frac{n+2}{2}}) \rightarrow \mathcal{G}r'(\omega'_{\frac{n+4}{2}})$  by  $gK \mapsto \begin{pmatrix} \varpi & 0 \\ 0 & g \end{pmatrix} K'$ . Clearly this map is well-defined and (universally) injective. Since  $\mathcal{G}r(\omega_{\frac{n+2}{2}})$  and  $\mathcal{G}r'(\omega'_{\frac{n+4}{2}})$  are projective over  $\overline{\mathbb{F}}_q$ ,  $\iota$  is a universally homeomorphism onto its image (in fact, it is easy to check that  $\iota$  is a monomorphism and hence a closed immersion). For  $0 \leq k \leq \frac{n-2}{2}$ , we define  $\lambda'_k \in \mathbb{Z}^{n+1}$  by  $\lambda'_k = (1, \tau^{\frac{n+2}{2}}\lambda_k)$ . Set  $\mu' = \omega'_2 \in \mathbb{Z}^{n+1}$ ,  $\tau' = \begin{pmatrix} 0 & \varpi \\ 1_n & 0 \end{pmatrix} \in \mathrm{GL}_{n+1}(F)$  and  $b' = \tau'^2$ . Then  $X_{\mu'}(b')^{\frac{n+4}{2}} = \bigsqcup_{0 \leq k \leq \frac{n-2}{2}} X_{\mu'}^{\lambda'_k}(b')$ , where  $X_{\mu'}(b')^i = \{g'K' \in X_{\mu'}(b') \mid \kappa(g') = i\}$ . Similarly as the proof of Proposition 10.1, we can check that  $\iota(X_\mu^{\tau^{\frac{n+2}{2}}\lambda_k}(b)) = X_{\mu'}^{\lambda'_k}(b')$  for  $0 \leq k \leq \frac{n-2}{2}$ . By Proposition 10.1 and  $\dim \tau^{\frac{n+2}{2}}\pi(Y(w_k)) = \dim X_{\mu'}^{\lambda'_k}(b') = k$ , we have

$$\overline{\tau^{\frac{n+2}{2}}\pi(Y(w_k))} = \bigsqcup_{0 \leq k' \leq k} X_\mu^{\tau^{\frac{n+2}{2}}\lambda_{k'}}(b), \quad \text{or equivalently,} \quad \overline{\pi(Y(w_k))} = \bigsqcup_{0 \leq k' \leq k} X_\mu^{\lambda_{k'}}(b).$$

We follow the notation in the proof of Lemma 10.5. Let  $1 \leq k \leq \frac{n-2}{2}$  and  $j \in \mathbb{J}^0$ . Then by the proof of Lemma 10.5, we have

$$jY(w_k) \subset Iv_{k+l} \cdots v_{k+2}v_{k+1}v_k t_{n-k+2, n-k} \cdots t_{n-1, n-2k+3} t_{0, n-2(k-1)} I/I.$$

Indeed, by replacing  $j$  by another representative in  $j(\mathbb{J} \cap P_{w_k})$  if necessary, we may assume  $\mathrm{inv}(1, j) = v_{k+l} \cdots v_{k+2}v_{k+1}$  for some  $l$  (we set  $\mathrm{inv}(1, j) = 1$  if  $l = 0$ ). Thus we can explicitly compute  $\lambda \in X_*(T)$  such that  $j\pi(Y(w_k)) \subset X_\mu^\lambda(b)$ . Combining this with the description of each  $\overline{\pi(Y(w_k))}$  above, we can easily verify the closure relation  $((\mathbb{J} \cap P_{w_k}) \cap j(\mathbb{J} \cap P_{w_{k-1}})) \neq \emptyset$  is equivalent to  $\mathrm{inv}(1, j) = 1$  or  $v_k$ .

It remains to show that each  $j\pi(Y(w_k))$  is a  $\mathbb{J}$ -stratum of  $X_\mu(b)$ . By Lemma 10.5,  $j\pi(Y(w_k))$  is contained in a  $\mathbb{J}$ -stratum. Note that each  $\mathbb{J}$ -stratum of  $X_\mu(b)$  is

contained in  $X_\mu^\lambda(b)$  for some  $\lambda \in X_*(T)$ . So in particular, by an explicit computation as above, the strata  $j\pi(Y(w_k))$  for fixed  $k$  are contained in different  $\mathbb{J}$ -strata from each other. To finish the proof, we need to show that for  $j, j' \in \mathbb{J}^0$  and  $k > k'$  such that  $j\pi(Y(w_k)), j'\pi(Y(w_{k'})) \subset X_\mu^\lambda(b)$  for some  $\lambda \in X_*(T)$ ,  $j\pi(Y(w_k))$  and  $j'\pi(Y(w_{k'}))$  are contained in different  $\mathbb{J}$ -strata. We may assume that  $j = 1$ . Let  $y \in Y(w_k)$  and set  $y_0 = f(y) \in Y(s_{n-k+1}\tau^2)$  as in the proof of Lemma 10.5. Then there exists  $gI$  with  $g \in \mathbb{J}^0 \cap P_{w_k}$  such that

$$\text{inv}(j', y_0) = \text{inv}(j', g)v_k(\in W_a) \quad \text{with} \quad \ell(\text{inv}(j', y_0)) = \ell(\text{inv}(j', g)) + \ell(v_k).$$

Thus  $\text{inv}(j', Y(w_k)) = \text{inv}(j', y) = \text{inv}(j', g)v_k t_{n-k+2, n-k} \cdots t_{n-1, n-2k+3} t_{0, n-2(k-1)}$ . This also implies that  $\text{inv}_K(j', \pi(Y(w_k))) \neq \text{inv}_K(j', j'\pi(Y(w_{k'})))$ . This finishes the proof.  $\square$

### 10.3 The case of $\omega_3$ when $n = 6$

In this subsection, we set  $\mu = \omega_3$  and  $b = \tau^3$ .

There are three  $\tau^3$ -orbit in  $\tilde{S}$ , namely,  $\{s_0, s_3\}$ ,  $\{s_1, s_4\}$  and  $\{s_2, s_5\}$ . Then  $W_a^{\tau^3}$  is the Weyl group with three simple reflections  $s_0 s_3$ ,  $s_1 s_4$  and  $s_2 s_5$ . For  $j, j' \in \mathbb{J}^0$ , we have  $\text{inv}(j, j') \in W_a^{\tau^3}$ .

For  $i, i' \in \{0, 1, \dots, 5\}$ , let  $P_{ii'}$  denote the standard parahoric subgroup of  $G(L)$  corresponding to the union of the orbits of  $s_i$  and  $s_{i'}$ . Let  $\Omega_{\mathbb{F}_{q^2}}^2$  denote the (perfection of) 2-dimensional Drinfeld's upper half-space over  $\mathbb{F}_{q^2}$ . Then the Deligne-Lusztig variety  $\{g \in G(\overline{\mathbb{F}}_q)/B(\overline{\mathbb{F}}_q) \mid \text{inv}(g, \sigma^2(g)) = s_1 s_2\}$  of Coxeter type is isomorphic to  $\Omega_{\mathbb{F}_{q^2}}^2$ .

**Proposition 10.7.** Assume that  $n = 6$ . We have

$${}^S\text{Adm}(\mu)_0 = \{\tau^3, s_0 \tau^3, s_0 s_1 \tau^3, s_0 s_5 \tau^3, s_0 s_1 s_5 s_0 \tau^3\}.$$

For any  $w \in {}^S\text{Adm}(\mu)_0$ , there exists  $v \in \text{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element. Set

$$P_{\tau^2} = P_{12}, \quad P_{s_0 \tau^3} = P_{01} \cap P_{02}, \quad P_{s_0 s_1 \tau^3} = P_{01}, \quad P_{s_0 s_5 \tau^3} = P_{02}, \quad P_{s_0 s_1 s_5 s_0 \tau^3} = P_{12}.$$

For any  $w \in {}^S\text{Adm}(\mu)_0$ , there exists an irreducible component  $Y(w)$  of  $X_w(b)$  such that  $X_w(b) = \mathbb{J}Y(w)$ ,  $Y(w) \cong \pi(Y(w))$  and

$$\pi(X_w(b)) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_w} j\pi(Y(w)).$$

Each  $j\pi(Y(w))$  is a  $\mathbb{J}$ -stratum of  $X_\mu(b)$  with

$$\begin{aligned}\pi(Y(\tau^2)) &\cong \{pt\}, & \pi(Y(s_0\tau^3)) &\cong \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_{q^2}), & \pi(Y(s_0s_1\tau^3)) &\cong \Omega_{\mathbb{F}_{q^2}}^2, \\ \pi(Y(s_0s_5\tau^3)) &\cong \Omega_{\mathbb{F}_{q^2}}^2, & \pi(Y(s_0s_1s_5s_0\tau^3)) &\cong \Omega_{\mathbb{F}_{q^2}}^2 \times \mathbb{A}^1.\end{aligned}$$

Moreover, the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$ .

*Proof.* The assertions for  $w \in {}^S\text{Adm}(\mu)_0 \setminus \{s_0s_1s_5s_0\tau^2\}$  follow from Proposition 2.3, (the proof of) [16, Theorem 7.2.1] and easy computation of finite part.

Set  $w = s_0s_1s_5s_0\tau^3$ . It remains to describe  $\pi(Y(w))$  and  $\overline{\pi(Y(w))}$ . We have

$$w = s_0s_1s_5s_0\tau^3 \xrightarrow{s_0}_\sigma s_3s_1s_5s_0\tau^3 \xrightarrow{s_3}_\sigma s_1s_5\tau^3.$$

By  $p(w) = s_3s_4s_5s_2s_1$  and [29, Theorem 5.1],  $X_{s_1s_5s_0\tau^3}(b) = \emptyset$ . Let  $f: X_w(b) \rightarrow X_{s_1s_5\tau^3}(b)$  be the morphism induced by Proposition 2.4. By Proposition 2.3, we have

$$X_{s_1s_5\tau^3}(b) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_{12}} jY(s_1s_5\tau^3),$$

where  $Y(s_1s_5\tau^3) = \{gI \in P_{12}/I \mid g^{-1}\tau^3\sigma(g)\tau^{-3} \in Is_1s_5I\}$  is a classical Deligne-Lusztig variety in the finite-dimensional flag variety  $P_{12}/I$ . It is easy to check that  $Y(s_1s_5\tau^3) \cong \Omega_{\mathbb{F}_{q^2}}^2$ . Set  $v = s_1s_4s_2s_5s_1s_4 = s_2s_5s_1s_4s_2s_5$ . By [38, Corollary 2.5] (see also [14, Proposition 1.1]),  $Y(s_1s_5\tau^3)$  is contained in  $IvI/I$ . Set  $Y(w) = f^{-1}(Y(s_1s_5\tau^3))$ . Then by Proposition 2.4,  $Y(w) \subset Ivs_3s_0I/I$ . Also by Remark 2.5,  $Y(w)$  is universally homeomorphic to  $\Omega_{\mathbb{F}_{q^2}}^2 \times \mathbb{A}^1$ .

We next show that for all  $j \in \mathbb{J}$ , the value  $\text{inv}(j, -)$  is constant on each  $j'Y(w)$ . Similarly as the proof of Lemma 10.5, we may assume that  $j' = 1$  and  $j \in \mathbb{J}^0$ . Fix  $jI$  with  $j \in \mathbb{J}^0$ . Then by [38, Corollary 2.5] and [1, Proposition 5.34], there exists  $gI$  with  $g \in \mathbb{J}^0 \cap P_{12}$  such that for any  $y_0I \in Y(s_1s_5\tau^3)$ , we have

$$\text{inv}(j, y_0) = \text{inv}(j, g)v \in W_a \quad \text{with} \quad \ell(\text{inv}(j, y_0)) = \ell(\text{inv}(j, g)) + \ell(v).$$

In particular,  $\text{inv}(j, g)$  has a reduced expression as an element of  $W_a^{\tau^3}$  whose right-most simple reflection is  $s_0s_3$  unless  $\text{inv}(j, g) = 1$ . Let  $y \in Y(w)$  and set  $y_0 = f(y) \in Y(s_1s_5\tau^3)$ . Note that  $\ell(\text{inv}(j, y_0)s_3s_0) = \ell(\text{inv}(j, g)) + \ell(v) + \ell(s_3s_0)$ . Thus

$$\text{inv}(j, y) = \text{inv}(j, y_0)\text{inv}(y_0, y) = \text{inv}(j, g)vs_3s_0$$

is independent of  $y \in Y(w)$ . This proves the value  $\text{inv}(j, -)$  (resp.  $\text{inv}_K(j, -)$ ) is constant on each  $Y(w)$  (resp.  $\pi(Y(w))$ ).

We next describe  $\pi(Y(w))$  as a union of other strata. Let  $j \in \mathbb{J}^0$ . We will prove the following two assertions for  $i = 1$  or  $5$ :



(1) If  $(\mathbb{J} \cap P_{12}) \cap j(\mathbb{J} \cap P_{0i}) \neq \emptyset$ , then  $j\pi(Y(s_0 s_i \tau^3)) \subset \overline{\pi(Y(w))}$ .

(2) Otherwise,  $j\pi(Y(s_0 s_i \tau^3)) \cap \overline{\pi(Y(w))} = \emptyset$ .

We only treat the case  $i = 1$ . The proof for the case  $i = 5$  is similar. By replacing  $j$  by another representative in  $j(\mathbb{J} \cap P_{01})$  if necessary, we may assume that  $\text{inv}(1, j)$  is the minimal length representative of its coset in  $W_a/W_{\{s_0, s_3, s_1, s_4\}}$ . By [38, Corollary 2.5],  $Y(s_0 s_1 \tau^3)$  is contained in  $Is_1 s_4 s_0 s_3 s_1 s_4 I/I$ . So there exists  $\mu_j \in X_*(T)_+$  such that  $j\pi(Y(s_0 s_1 \tau^3)) \subset K\varpi^{\mu_j}K/K$ . Note that  $(\mathbb{J} \cap P_{12}) \cap j(\mathbb{J} \cap P_{01}) \neq \emptyset$  is equivalent to  $\text{inv}(1, j) = 1, s_2 s_5$  or  $s_1 s_4 s_2 s_5$ . Moreover, if  $(\mathbb{J} \cap P_{12}) \cap j(\mathbb{J} \cap P_{01}) = \emptyset$ , then  $s_0$  (and hence  $s_3$ ) belongs to  $\text{supp}(\text{inv}(1, j))$ . Combining this with [39, (2.7.11)], we deduce that if  $(\mathbb{J} \cap P_{12}) \cap j(\mathbb{J} \cap P_{01}) = \emptyset$ , then  $(1, 1, 0, 0, -1, -1) \preceq \mu_j$ . This proves (2). Since  $\pi(Y(s_0 s_i \tau^3))$  is irreducible and  $X_\mu(b)$  is equidimensional, there exists  $j_0 \in \mathbb{J}^0$  such that  $j_0 \pi(Y(s_0 s_1 \tau^3)) \subset \overline{\pi(Y(w))}$ . This  $j_0$  must satisfy  $(\mathbb{J} \cap P_{12}) \cap j(\mathbb{J} \cap P_{01}) \neq \emptyset$ . Thus by multiplying  $j_0 \pi(Y(s_0 s_1 \tau^3))$  by elements in  $\mathbb{J} \cap P_{12}$ , we deduce that if  $(\mathbb{J} \cap P_{12}) \cap j(\mathbb{J} \cap P_{01}) \neq \emptyset$ , then  $j\pi(Y(s_0 s_1 \tau^3)) \subset \overline{\pi(Y(w))}$ . This proves (1). We can also verify a similar statement for  $j\pi(Y(s_0 \tau^3))$  and  $j\pi(Y(\tau^2))$ , which proves the closure relation. By computing  $\mathbb{J}$ -invariants explicitly, we can also verify that each  $j\pi(Y(w))$  is a  $\mathbb{J}$ -stratum (cf. the proof of Lemma 10.6). This finishes the proof.  $\square$

## 10.4 The case of $\omega_3$ when $n = 7, 8$

In this subsection, we set  $\mu = \omega_3$  and  $b = \tau^3$ . The following is a well-known fact:

**Lemma 10.8.** If  $X$  is a (separated) variety and  $U \subsetneq X$  is an affine open subscheme, then  $\dim X \setminus U = \dim X - 1$ .

For  $d \leq n$ , every  $d \times n$  matrix  $A$  of rank  $d$  with coefficients in  $\overline{\mathbb{F}}_q$  determines an element  $W_A$  in the Grassmannian  $\text{Gr}_d(\overline{\mathbb{F}}_q^n)$  such that  $W_A$  is generated by the  $d$  rows of  $A$ . Let  $I_{d,n}$  be the set of subsets of  $d$  elements in  $\{1, 2, \dots, n\}$ . For any  $J \in I_{d,n}$ , let  $A_J$  be the  $d \times d$  matrix whose columns are the columns of  $A$  at indices from  $J$ . Let  $L_A = \{J \in I_{d,n} \mid \det A_J \neq 0\}$ . In fact,  $L_A$  only depends on  $W_A$ , and we can associate with every element  $W$  of  $\text{Gr}_d(\overline{\mathbb{F}}_q^n)$  a corresponding set  $L_W$ . This is called the *list* of  $W$ . The locally closed subsets of  $\text{Gr}_d(\overline{\mathbb{F}}_q^n)$  given by fixing  $L_W$  are called thin Schubert cells (cf. [4, §2.1]). For  $W, W' \in \text{Gr}_d(\overline{\mathbb{F}}_q^n)$ , if  $W'$  is in the closure of the thin Schubert cell of  $W$ , then  $L_{W'} \subseteq L_W$ . In general, the converse of this statement (the closure relation) does not hold for this stratification. However, it holds if  $d = 2$  (see [12, §1.10]).

**Proposition 10.9.** Assume that  $n = 7, 8$ . Then  ${}^S\text{Adm}(\omega_3)_0$  is equal to

$$\begin{aligned} & \{\tau^3, s_0 s_6 \tau^3, s_0 s_6 s_1 s_0 \tau^3, s_0 s_6 s_5 s_1 \tau^3, s_0 s_6 s_5 s_1 s_0 s_6 \tau^3\} & (n = 7), \\ & \{\tau^3, s_0 s_1 \tau^3, s_0 s_7 s_6 s_5 \tau^3, s_0 s_7 s_6 s_1 \tau^3, s_0 s_7 s_6 s_5 s_1 s_0 \tau^3, \\ & \quad s_0 s_7 s_6 s_1 s_0 s_7 \tau^3, s_0 s_7 s_6 s_5 s_1 s_0 s_7 s_6 \tau^3\} & (n = 8). \end{aligned}$$

For any  $w \in {}^S\text{Adm}(\mu)_0$ , there exists  $v \in \text{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element, and  $\mathbb{J}$  acts transitively on  $\text{Irr } X_w(b)$ . Each irreducible component of  $X_w(b) \cong \pi(X_w(b))$  is a  $\mathbb{J}$ -stratum universally homeomorphic to an affine space of dimension  $\frac{\ell(w)}{2}$ . Moreover, the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$ .

*Proof.* Except the “moreover” part, the proposition follows from Proposition 8.5 and [51, Theorem 5.3]. Let  $X_\mu(b)^i = \{gK \in X_\mu(b) \mid \kappa(g) = i\}$ .

Assume that  $n = 7$ . Then it follows from Lemma 7.2 that  $X_\mu(b)^4 \subset K\varpi^{\omega_4}K/K$ . Moreover, each  $\mathbb{J}$ -stratum in  $X_\mu(b)^4$  of dimension  $\leq 2$  coincides with a Schubert cell in the Grassmannian  $K\varpi^{\omega_4}K/K$ . So the closure relation follows from this and the equidimensionality of  $X_\mu(b)$ .

Assume that  $n = 8$ . The  $\mathbb{J}$ -strata of dimension  $\leq 2$  in  $X_\mu(b)^2$  coincide with some Schubert cells in the Grassmannian  $K\varpi^{\omega_2}K/K$ . So the closure relation holds for these strata. By the equidimensionality of  $X_\mu(b)$ , it also holds for the  $\mathbb{J}$ -stratum of dimension 4. It remains to show the closure relation for the  $\mathbb{J}$ -strata of dimension 3.

We first treat the case for  $s_0 s_7 s_6 s_1 s_0 s_7 \tau^3$ . Set  $\lambda = (0, 1, 0, 0, 1, 0, 0, 0)$ . It follows from Lemma 7.2 that  $\tau^2 \pi(X_{s_0 s_7 s_6 s_1 s_0 s_7 \tau^3}(b)^0) \subset \tau^2 I\varpi^{\check{\chi}_{3,7}}K/K = I\varpi^\lambda K/K \subset K\varpi^{\omega_2}K/K$ . Fix an isomorphism  $\mathbb{A}^4 \cong I\varpi^\lambda K/K$  which maps  $(s, t, u, v)$  to  $h_{s,t,u,v}\varpi^\lambda K$ , where

$$h_{s,t,u,v} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & u & v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$\varpi^{-\lambda} h_{s,t,u,v}^{-1} b \sigma(h_{s,t,u,v} \varpi^\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \varpi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -s & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varpi \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sigma(s)-v}{\varpi} & 1 & 0 & 0 & 0 & -t & 0 & -u \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \sigma(t) & 0 & \sigma(u) & \sigma(v) & \varpi & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that  $\tau^2 \pi(X_{s_0 s_7 s_6 s_1 s_0 s_7 \tau^3}(b)^0) \cong \mathbb{A}^3$  is the locus  $v = \sigma(s)$  in  $I\varpi^\lambda K/K \cong \mathbb{A}^4$ . The locus  $s = t = v = 0$  and  $u \neq 0$  in  $\tau^2 \pi(X_{s_0 s_7 s_6 s_1 s_0 s_7 \tau^3}(b)^0)$  is a thin Schubert cell whose list is  $\{\{1, 3, 4, 6, 7, 8\}, \{1, 4, 5, 6, 7, 8\}\}$ . Since the closure relation holds for thin Schubert cells in  $K\varpi^{\omega_2} K/K$ , the closure of this Schubert cell intersect  $\tau^2 I\varpi^{\chi_{1,7}^\vee} K/K$ . Indeed, the latter contains the thin Schubert cell with the list  $\{\{1, 4, 5, 6, 7, 8\}\}$ . The locus  $v = \sigma(s) \neq 0$  and  $t = u = 0$  in  $\tau^2 \pi(X_{s_0 s_7 s_6 s_1 s_0 s_7 \tau^3}(b)^0)$  is contained in a thin Schubert cell whose list is

$$L := \{\{1, 3, 4, 6, 7, 8\}, \{1, 3, 5, 6, 7, 8\}, \{2, 3, 4, 6, 7, 8\}, \{2, 3, 5, 6, 7, 8\}\}.$$

Let  $L'$  be the list of a thin Schubert cell in  $\tau^2 I\varpi^{\chi_{1,7}^\vee} K/K$ ,  $\tau^2 I\varpi^{\chi_{1,8}^\vee} K/K$  and  $\{\tau^2 K\}$ . Then it is easy to check that  $L' \not\subseteq L$ . On the other hand, the closure of the locus  $v = \sigma(s) \neq 0$  and  $t = u = 0$  in  $\tau^2 \pi(X_{s_0 s_7 s_6 s_1 s_0 s_7 \tau^3}(b)^0)$  is projective. By [4, Proposition 2.11 (5)], this closure is contained in

$$\tau^2 I\varpi^{\chi_{3,7}^\vee} K/K \sqcup \tau^2 I\varpi^{\chi_{1,7}^\vee} K/K \sqcup \tau^2 I\varpi^{\chi_{2,8}^\vee} K/K \sqcup \tau^2 I\varpi^{\chi_{1,8}^\vee} K/K \sqcup \{\tau^2 K\}.$$

So it must intersect  $\tau^2 I\varpi^{\chi_{2,8}^\vee} K/K$ . Thus both  $\tau^2 I\varpi^{\chi_{1,7}^\vee} K/K$  and  $\tau^2 I\varpi^{\chi_{2,8}^\vee} K/K$  intersect the closure of  $\tau^2 \pi(X_{s_0 s_7 s_6 s_1 s_0 s_7 \tau^3}(b)^0)$ . This combined with Lemma 10.8 imply that they are actually contained in the closure. Note that  $s_0 s_7 s_6 s_1 \tau^3 \leq s_0 s_7 s_6 s_1 s_0 s_7 \tau^3$  and

$$s_7 s_2 (s_0 s_7 s_6 s_5 \tau^3) s_2 s_7 = s_0 s_7 s_0 s_6 \tau^3 \leq s_0 s_7 s_6 s_1 s_0 s_7 \tau^3.$$

Thus the closure relation holds.

We next treat the case for  $s_0 s_7 s_6 s_5 s_1 s_0 \tau^3$ . Set  $\lambda = (1, 1, 1, 0, 1, 1, 0, 1)$ . It follows from Lemma 7.2 that  $\tau^6 \pi(X_{s_0 s_7 s_6 s_5 s_1 s_0 \tau^3}(b)^0) \subset \tau^6 I\varpi^{\chi_{2,6}^\vee} K/K = I\varpi^\lambda K/K \subset K\varpi^{\omega_6} K/K$ . Fix an isomorphism  $\mathbb{A}^4 \cong I\varpi^\lambda K/K$  which maps  $(s, t, u, v)$  to  $h_{s,t,u,v} \varpi^\lambda K$ ,

where

$$h_{s,t,u,v} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u & 0 & 0 & v & 1 \end{pmatrix}.$$

Then we have

$$\varpi^{-\lambda} h_{s,t,u,v}^{-1} b \sigma(h_{s,t,u,v} \varpi^\lambda) = \begin{pmatrix} 0 & 0 & 0 & \sigma(t) & 0 & \varpi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sigma(u) & 0 & 0 & \sigma(v) & \varpi \\ \varpi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -t & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -u & 0 & 0 & \frac{\sigma(s)-v}{\varpi} & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that  $\tau^6 \pi(X_{s_0 s_7 s_6 s_5 s_1 s_0 \tau^3}(b)^0) \cong \mathbb{A}^3$  is the locus  $v = \sigma(s)$  in  $I\varpi^\lambda K/K \cong \mathbb{A}^4$ . The locus  $s = u = v = 0$  and  $t \neq 0$  in  $\tau^6 \pi(X_{s_0 s_7 s_6 s_5 s_1 s_0 \tau^3}(b)^0)$  is a thin Schubert cell whose list is  $\{\{4, 7\}, \{6, 7\}\}$ . Since the closure relation holds for thin Schubert cells in  $K\varpi^{\omega_6} K/K$ , the closure of this Schubert cell intersect  $\tau^6 I\varpi^{\chi_{2,8}} K/K$ . Indeed, the latter contains the thin Schubert cell with the list  $\{\{6, 7\}\}$ . The locus  $v = \sigma(s) \neq 0$  and  $t = u = 0$  in  $\tau^6 \pi(X_{s_0 s_7 s_6 s_5 s_1 s_0 \tau^3}(b)^0)$  is contained in a thin Schubert cell whose list is

$$L := \{\{4, 7\}, \{4, 8\}, \{5, 7\}, \{5, 8\}\}.$$

Let  $L'$  be the list of a thin Schubert cell in  $\tau^6 I\varpi^{\chi_{1,7}} K/K$ ,  $\tau^6 I\varpi^{\chi_{1,8}} K/K$  and  $\{\tau^6 K\}$ . Then it is easy to check that  $L' \not\subseteq L$ . On the other hand, the closure of the locus  $v = \sigma(s) \neq 0$  and  $t = u = 0$  in  $\tau^6 \pi(X_{s_0 s_7 s_6 s_5 s_1 s_0 \tau^3}(b)^0)$  is projective. By [4, Proposition 2.11 (5)], this closure is contained in

$$\tau^6 I\varpi^{\chi_{2,6}} K/K \sqcup \tau^6 I\varpi^{\chi_{1,7}} K/K \sqcup \tau^6 I\varpi^{\chi_{2,8}} K/K \sqcup \tau^6 I\varpi^{\chi_{1,8}} K/K \sqcup \{\tau^6 K\}.$$

So it must intersect  $\tau^6 I\varpi^{\chi_{2,8}} K/K$ . Thus both  $\tau^6 I\varpi^{\chi_{1,7}} K/K$  and  $\tau^6 I\varpi^{\chi_{2,8}} K/K$  intersect the closure of  $\tau^6 \pi(X_{s_0 s_7 s_6 s_5 s_1 s_0 \tau^3}(b)^0)$ . This combined with Lemma 10.8 imply that they are actually contained in the closure. Note that  $s_0 s_7 s_6 s_1 \tau^3 \leq s_0 s_7 s_6 s_5 s_1 s_0 \tau^3$  and  $s_0 s_7 s_6 s_5 \tau^3 \leq s_0 s_7 s_6 s_5 s_1 s_0 \tau^3$ . Thus the closure relation holds. This finishes the proof.  $\square$

## 10.5 Classification

To treat the non-superbasic case, we need the following lemma.

**Lemma 10.10.** Set  $\mu = \omega_2 + \omega_{n-2}$ . Assume that  $n \geq 4$ . There exists  $w \in {}^S\text{Adm}(\mu)^\circ := {}^S\text{Adm}(\mu) \setminus {}^S\text{Adm}(\omega_1 + \omega_{n-1})$  such that  $v^{-1}p(w)v$  is not a Coxeter element for any  $v \in \text{LP}(w)$  and  $X_w(b) \neq \emptyset$ .

*Proof.* First assume that  $n \geq 6$ . Set  $y = s_{n-2}s_{n-1}s_{n-3} \cdots s_2s_1s_3 = (1 \ n-1 \ n \ n-2 \ n-3 \ \cdots \ 2)(3 \ 4) = (1 \ n-1 \ n \ n-2 \ \cdots \ 5 \ 4 \ 2)$ . It is straightforward to check that  $\varpi^\mu y \in {}^S\text{Adm}(\mu)^\circ$  and  $\chi_{1,3}, \chi_{3,n} \in \Phi_+ \setminus \Phi_{\varpi^\mu y}$ . Let  $r \in W_0$ . If  $ryr^{-1} \in \bigcup_{J \subsetneq S} W_J$ , then  $(r(1) \ r(n-1) \ r(n) \ r(n-2) \ \cdots \ r(5) \ r(4) \ r(2)) = (1 \ 2 \ \cdots \ n-1)$  or  $(2 \ 3 \ \cdots \ n)$ . In the former (resp. latter) case, we must have  $r(3) = n$  (resp.  $r(3) = 1$ ) and hence  $r\chi_{3,n}$  (resp.  $r\chi_{1,3}$ ) is negative. Then by Lemma 2.10 and Theorem 2.11, the statement holds for  $w = \varpi^\mu y$ . If  $n = 4$  (resp. 5), set  $y = s_2s_3s_1s_2 = (1 \ 3)(2 \ 4)$  (resp.  $s_3s_4s_2s_1s_3s_2 = (1 \ 4 \ 2 \ 5 \ 3)$ ). Then it is easy to check that the statement holds for  $w = \varpi^\mu y$ .  $\square$

**Theorem 10.11.** The following assertions are equivalent.

- (i) For any  $w \in {}^S\text{Adm}(\mu)_0$ , there exists  $v \in \text{LP}(w)$  such that  $v^{-1}p(w)v$  is a Coxeter element.
- (ii) The cocharacter  $\mu$  has one of the following forms modulo  $\mathbb{Z}\omega_n$ :

$$\begin{array}{ll}
 \omega_1, & \omega_{n-1}, & (n \geq 1), \\
 \omega_1 + \omega_{n-1}, & \omega_2, & 2\omega_1, & \omega_{n-2}, & 2\omega_{n-1}, \\
 \omega_2 + \omega_{n-1}, & 2\omega_1 + \omega_{n-1} & \omega_1 + \omega_{n-2}, & \omega_1 + 2\omega_{n-1}, & (n \geq 3), \\
 \omega_3, & \omega_{n-3}, & (n = 6, 7, 8), \\
 3\omega_1, & 3\omega_{n-1}, & (n = 3, 4, 5), \\
 \omega_1 + \omega_2, & \omega_3 + \omega_4, & (n = 5), \\
 4\omega_1, & \omega_1 + 3\omega_2, & 4\omega_2, & 3\omega_1 + \omega_2, & (n = 3), \\
 m\omega_1 & \text{with } m \in \mathbb{Z}_{>0}, & (n = 2).
 \end{array}$$

Here  $\omega_k$  denotes the cocharacter of the form  $(1, \dots, 1, 0, \dots, 0)$  in which 1 is repeated  $k$  times.

*Proof.* As explained in the last paragraph of §2.5, it is enough to treat one of  $\mu$  or  $-w_{\max}\mu$  for the implication (ii)  $\Rightarrow$  (i). This follows from Theorem 10.11, [46, Corollary 5.5 & Theorem 5.9] and the case-by-case analysis in the previous sections. So we only prove the implication (i)  $\Rightarrow$  (ii) here.

Let  $0 \leq m_0 < n$  be the residue of  $m$  modulo  $n$ . If  $m_0 = 0$  and  $n \geq 4$ , then  $\omega_2 + \omega_{n-2} + (\lfloor \frac{m}{n} \rfloor - 1)\omega_n \preceq \mu$  unless  $\mu = \omega_0$  or  $\omega_1 + \omega_{n-1}$  modulo  $\mathbb{Z}\omega_n$ . So if  $\mu \neq \omega_0, \omega_1 + \omega_{n-1}$  modulo  $\mathbb{Z}\omega_n$ , then  $\mu$  does not satisfy (i) by Lemma 10.10. If  $m_0 = 0$  and  $n = 3$ , then  $2\omega_1 + 2\omega_2 + (\lfloor \frac{m}{n} \rfloor - 2)\omega_n \preceq \mu$  unless  $\mu = \omega_0, \omega_1 + \omega_2, 3\omega_1$  or  $3\omega_2$  modulo  $\mathbb{Z}\omega_n$ . It follows from Lemma 2.11 that  $\varpi^{2\omega_1 + 2\omega_2 + (\lfloor \frac{m}{n} \rfloor - 2)\omega_n}(1 \ 3) \in {}^S\text{Adm}(\mu)$  and  $X_{\varpi^{2\omega_1 + 2\omega_2 + \lfloor \frac{m}{n} \rfloor \omega_n}(1 \ 3)}(b) \neq \emptyset$ . So if  $\mu \neq \omega_0, \omega_1 + \omega_2, 3\omega_1, 3\omega_2$  modulo  $\mathbb{Z}\omega_n$ , then  $\mu$  does not satisfy (i). If  $m_0 = 2$  and  $n = 4$ , then  $3\omega_2 + (\lfloor \frac{m}{n} \rfloor - 1)\omega_n \preceq \mu$  unless  $\mu = \omega_2, 2\omega_1$  or  $2\omega_3$  modulo  $\mathbb{Z}\omega_n$ . It follows from Lemma 2.11 that  $\varpi^{3\omega_2 + (\lfloor \frac{m}{n} \rfloor - 1)\omega_n}(1 \ 3)(2 \ 4) \in {}^S\text{Adm}(\mu)$  and  $X_{\varpi^{3\omega_2 + (\lfloor \frac{m}{n} \rfloor - 1)\omega_n}(1 \ 3)(2 \ 4)}(b) \neq \emptyset$ . So if  $\mu \neq \omega_2, 2\omega_1, 2\omega_3$  modulo  $\mathbb{Z}\omega_n$ , then  $\mu$  does not satisfy (i). If  $m_0 = 2$  (resp.  $m_0 = 4$ ) and  $n = 6$ , then  $\omega_3 + \omega_5 + (\lfloor \frac{m}{n} \rfloor - 1)\omega_n \preceq \mu$  (resp.  $\omega_1 + \omega_3 + \lfloor \frac{m}{n} \rfloor \omega_n \preceq \mu$ ) unless  $\mu = \omega_2$  or  $2\omega_1$  (resp.  $\omega_4$  or  $2\omega_5$ ) modulo  $\mathbb{Z}\omega_n$ . It is easy to check that  $s_0 s_1 s_2 s_5 \tau^8 \in {}^S\text{Adm}(\omega_3 + \omega_5)$ ,  $X_{s_0 s_1 s_2 s_5 \tau^8}(\tau^8) \neq \emptyset$  and  $p(s_0 s_5 s_4 s_1 \tau^4) = (1 \ 6 \ 3)(2 \ 4 \ 5)$  (resp.  $s_0 s_5 s_4 s_1 \tau^4 \in {}^S\text{Adm}(\omega_1 + \omega_3)$ ,  $X_{s_0 s_5 s_4 s_1 \tau^4}(\tau^4) \neq \emptyset$  and  $p(s_0 s_5 s_4 s_1 \tau^4) = (1 \ 4 \ 6)(2 \ 5 \ 3)$ ). So if  $\mu \neq \omega_2, 2\omega_1$  (resp.  $\mu \neq \omega_4, 2\omega_5$ ) modulo  $\mathbb{Z}\omega_n$ , then  $\mu$  does not satisfy (i). Other cases follow from (the proof of) §8 and Theorem 10.11. Note that the proof there literally works even for non-superbasic elements.  $\square$

It is easy to check that the image of  $X_\mu(b)$  under the automorphism of  $\mathcal{G}r$  for  $G = \text{GL}_n$  by  $gK \mapsto w_{\max}^t g^{-1} K$  is  $X_{-w_{\max}\mu}(b^{-1})$ . Also this automorphism maps a  $\mathbb{J}_b$ -stratum of  $\mathcal{G}r$  to a  $\mathbb{J}_{b^{-1}}$ -stratum of  $\mathcal{G}r$ . Thus to study the geometric structure, it is enough to treat one of  $\mu$  or  $-w_{\max}\mu$  modulo  $\mathbb{Z}\omega_n$ . The following theorem follows from this observation and the case-by-case analysis in the previous sections.

**Theorem 10.12.** Let  $\mu \in X_*(T)_+$  be a minuscule cocharacter satisfying the equivalent conditions in Theorem 10.11. Then the following assertions hold:

- For  $w \in {}^S\text{Adm}(\mu)_0$ ,  $\mathbb{J}$  acts transitively on the set of irreducible components of  $X_w(b)$ .
- For  $w \in {}^S\text{Adm}(\mu)_0$ , there exist a parahoric subgroup  $P_w \subset G(L)$  and an irreducible component  $Y(w)$  of  $X_w(b)$  such that  $\pi(X_w(b)) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap P_w} j\pi(Y(w))$ .
- Each  $j\pi(Y(w))$  is a  $\mathbb{J}$ -stratum of  $X_{\preceq \mu}(b)$ .
- $Y(w) \cong \pi(Y(w))$  is universally homeomorphic to the product of a Deligne-Lusztig variety of Coxeter type and a finite-dimensional affine space.

Moreover, the closure relation can be described in terms of  $\mathcal{B}(\mathbb{J}, F)$  (cf. §2.3).

**Remark 10.13.** The first two assertions in Theorem 10.12 follows from [45] (and Proposition 2.1), but we did not use it because we can explicitly describe the reduction tree in our case. The other assertions do not follow from [45].

## References

- [1] P. Abramenko and K. S. Brown, *Buildings*, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008, Theory and applications.
- [2] M. Chen, L. Fargues, and X. Shen, *On the structure of some  $p$ -adic period domains*, Camb. J. Math. **9** (2021), no. 1, 213–267.
- [3] M. Chen and J. Tong, *Weakly admissible locus and Newton stratification in  $p$ -adic Hodge theory*, arXiv:2203.12293 (2022).
- [4] M. Chen and E. Viehmann, *Affine Deligne-Lusztig varieties and the action of  $J$* , J. Algebraic Geom. **27** (2018), no. 2, 273–304.
- [5] A. J. de Jong and F. Oort, *Purity of the stratification by Newton polygons*, J. Amer. Math. Soc. **13** (2000), no. 1, 209–241.
- [6] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 95, Cambridge University Press, Cambridge, 2020.
- [7] G. Faltings, *Algebraic loop groups and moduli spaces of bundles*, J. Eur. Math. Soc. (JEMS) **5** (2003), no. 1, 41–68.
- [8] M. Fayers, *A note on Kostka numbers*, arXiv:1903.12499 (2019).
- [9] M. Fox, B. Howard, and N. Imai, *Rapoport-Zink spaces of type  $\mathrm{GU}(2, n-2)$* , 2308.03816 (2023).
- [10] M. Fox and N. Imai, *The supersingular locus of the Shimura variety of  $\mathrm{GU}(2, n-2)$* , arXiv:2108.03584 (2021).
- [11] Q. R. Gashi, *On a conjecture of Kottwitz and Rapoport*, Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no. 6, 1017–1038.
- [12] I. M. Gel’fand and V. V. Serganova, *Combinatorial geometries and the strata of a torus on homogeneous compact manifolds*, Uspekhi Mat. Nauk **42** (1987), no. 2(254), 107–134, 287.
- [13] U. Görtz, *Affine Springer fibers and affine Deligne-Lusztig varieties*, Affine flag manifolds and principal bundles, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010, pp. 1–50.
- [14] ———, *Stratifications of affine Deligne-Lusztig varieties*, Trans. Amer. Math. Soc. **372** (2019), no. 7, 4675–4699.

- [15] U. Görtz and X. He, *Dimensions of affine Deligne-Lusztig varieties in affine flag varieties*, Doc. Math. **15** (2010), 1009–1028.
- [16] ———, *Basic loci of Coxeter type in Shimura varieties*, Camb. J. Math. **3** (2015), no. 3, 323–353.
- [17] U. Görtz, X. He, and S. Nie,  *$\mathbf{P}$ -alcoves and nonemptiness of affine Deligne-Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 3, 647–665.
- [18] ———, *Fully Hodge-Newton decomposable Shimura varieties*, Peking Math. J. **2** (2019), no. 2, 99–154.
- [19] ———, *Basic loci of Coxeter type with arbitrary parahoric level*, Canadian Journal of Mathematics (2022), First View, 1–47.
- [20] U. Görtz, X. He, and M. Rapoport, *Extremal cases of Rapoport-Zink spaces*, Journal of the Institute of Mathematics of Jussieu (2020), 1–56.
- [21] T. J. Haines, *The combinatorics of Bernstein functions*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 1251–1278.
- [22] P. Hamacher, *The dimension of affine Deligne-Lusztig varieties in the affine Grassmannian*, Int. Math. Res. Not. IMRN (2015), no. 23, 12804–12839.
- [23] P. Hamacher and E. Viehmann, *Irreducible components of minuscule affine Deligne-Lusztig varieties*, Algebra Number Theory **12** (2018), no. 7, 1611–1634.
- [24] U. Hartl and E. Viehmann, *The Newton stratification on deformations of local  $G$ -shtukas*, J. Reine Angew. Math. **656** (2011), 87–129.
- [25] ———, *Foliations in deformation spaces of local  $G$ -shtukas*, Adv. Math. **229** (2012), no. 1, 54–78.
- [26] X. He, *Closure of Steinberg fibers and affine Deligne-Lusztig varieties*, Int. Math. Res. Not. IMRN (2011), no. 14, 3237–3260.
- [27] ———, *Geometric and homological properties of affine Deligne-Lusztig varieties*, Ann. of Math. (2) **179** (2014), no. 1, 367–404.
- [28] X. He and S. Nie, *Minimal length elements of extended affine Weyl groups*, Compos. Math. **150** (2014), no. 11, 1903–1927.
- [29] X. He, S. Nie, and Q. Yu, *Affine Deligne-Lusztig varieties with finite Coxeter parts*, arXiv:2208.14058 (2022).



- [30] X. He and R. Zhou, *On the connected components of affine Deligne-Lusztig varieties*, Duke Math. J. **169** (2020), no. 14, 2697–2765.
- [31] J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, RI, 2002.
- [32] M. Kashiwara, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), no. 2, 383–413.
- [33] M. Kashiwara and T. Nakashima, *Crystal graphs for representations of the  $q$ -analogue of classical Lie algebras*, J. Algebra **165** (1994), no. 2, 295–345.
- [34] R. Kottwitz and M. Rapoport, *Minuscule alcoves for  $GL_n$  and  $GSp_{2n}$* , Manuscripta Math. **102** (2000), no. 4, 403–428.
- [35] R. E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), no. 2, 201–220.
- [36] S. Kudla and M. Rapoport, *Special cycles on unitary Shimura varieties I. Unramified local theory*, Invent. Math. **184** (2011), no. 3, 629–682.
- [37] D. G. Lim, *Nonemptiness of single affine Deligne-Lusztig varieties*, arXiv:2302.04976 (2023).
- [38] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius*, Invent. Math. **38** (1976/77), no. 2, 101–159.
- [39] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, vol. 157, Cambridge University Press, Cambridge, 2003.
- [40] S. Nie, *Irreducible components of affine Deligne-Lusztig varieties*, Cambridge Journal of Mathematics **10** (2022), no. 2, 433–510.
- [41] M. Rapoport and T. Zink, *Period spaces for  $p$ -divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.
- [42] M. Rapoport, *A guide to the reduction modulo  $p$  of Shimura varieties*, Astérisque (2005), no. 298, 271–318.
- [43] F. Schremmer, *Generic Newton points and cordial elements*, arXiv:2205.02039 (2022).
- [44] ———, *Newton strata in Levi subgroups*, arXiv:2305.00683 (2023).

- [45] F. Schremmer, R. Shimada, and Q. Yu, *On affine Weyl group elements of positive Coxeter type*, arXiv:2312.02630 (2023).
- [46] R. Shimada, *On some simple geometric structure of affine Deligne-Lusztig varieties for  $GL_n$* , to appear in Manuscripta Mathematica (2023).
- [47] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, vol. No. 80, American Mathematical Society, Providence, RI, 1968.
- [48] Y. Takaya, *Equidimensionality of the affine Deligne-Lusztig variety in mixed characteristic*, arXiv:2212.13499 (2022).
- [49] S. Trentin, *On the Rapoport-Zink space for  $GU(2, 4)$  over a ramified prime*, 2309.11290 (2023).
- [50] E. Viehmann, *The dimension of some affine Deligne-Lusztig varieties*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 3, 513–526.
- [51] ———, *Moduli spaces of  $p$ -divisible groups*, J. Algebraic Geom. **17** (2008), no. 2, 341–374.
- [52] I. Vollaard and T. Wedhorn, *The supersingular locus of the Shimura variety of  $GU(1, n - 1)$  II*, Invent. Math. **184** (2011), no. 3, 591–627.
- [53] L. Xiao and X. Zhu, *Cycles on Shimura varieties via geometric Satake*, arXiv:1707.05700 (2017).
- [54] W. Zhang, *On arithmetic fundamental lemmas*, Invent. Math. **188** (2012), no. 1, 197–252.
- [55] R. Zhou and Y. Zhu, *Twisted orbital integrals and irreducible components of affine Deligne-Lusztig varieties*, Camb. J. Math. **8** (2020), no. 1, 149–241.