博士論文

論文題目　Studies on F-singularities in equal characteristic zero via ultraproducts
(超積を用いた等標数 0 における F-特異点の研究)

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Preface

This dissertation deals with descent of singularities under pure ring homomorphisms. A ring homomorphism $R \to S$ is said to be *pure* if for any R-module M, the natural map $M \to M \otimes_R S$ is injective. Example of pure morphisms include faithfully flat morphisms and split morphisms. Geometrically, when a linearly reductive group G acts on a ring S, the inclusion $S^G \hookrightarrow S$ from the ring of invariants S^G to S is pure. It is natural to ask what properties descend under pure morphisms $R \to S$. We list some known results below:

(1) Boutot [5] showed that if R and S are essentially of finite type over a field of characteristic zero and if S has rational singularities, then R has rational singularities.

(2) Schoutens [46] showed that if both R and S are $\mathbb{Q}$-Gorenstein normal local domains essentially of finite type over $\mathbb{C}$ and if S has log terminal singularities, then so does R. On the other hand, Braun, Greb, Langlois and Moraga [6] showed that if S is of klt type and a linearly reductive group G acts on S, then S^G is also of klt type. Here singularities of klt type are a natural generalization of log terminal singularities to the non-$\mathbb{Q}$-Gorenstein setting. Recently, Zhuang [67] generalized the above two results: if S is of klt type, then R is of klt type.

(3) Godfrey and Murayama [17] showed that if R and S are essentially of finite type over $\mathbb{C}$, and if S has Du Bois singularities, then R has Du Bois singularities. This is a generalization of a result of Kovács [33], who showed the same result when the morphism $R \to S$ splits.

Zhuang asked two questions in [67]. First, does log canonicity descend under pure morphisms, and second, can his result be generalized to log pairs? Here a log pair is a pair (R, Δ) consisting of a normal domain R and an effective $\mathbb{Q}$-divisor Δ on $\operatorname{Spec} R$. We study these problems using the theory of F-singularities and ultraproducts.

F-singularities are singularities in positive characteristic defined in terms of the Frobenius morphism. F-pure, F-injective and F-regular singularities are major classes of F-singularities. F-pure singularities, introduced by Hochster and Roberts [28], are singularities defined by the purity of the Frobenius morphism. F-injective singularities originate in the study of F-purity and rational singularities by Fedder [13]. F-regular singularities, introduced by Hochster and Huneke [23], came from Hochster-Huneke's tight closure theory. These classes of singularities became increasingly important because it was revealed that F-singularities are closely related to singularities in the minimal model program. Based on this observation, these singularities are extended by [21], [52], [56] and [57] to triples $(R, \Delta, \mathfrak{a}^t)$ of rings R of positive characteristic, effective $\mathbb{Q}$-divisors Δ on $\operatorname{Spec} R$ and nonzero ideals $\mathfrak{a}$ of R with real exponents $t > 0$, which is a setting often appearing in the minimal model program. In this dissertation, we define an analogue of F-singularities in equal characteristic zero using ultraproducts and give a new description of singularities in equal characteristic zero. As an application, we show that some classes of singularities descend under pure morphisms, which is an aim of this dissertation.

Ultraproducts are a fundamental notion in non-standard analysis. Using them, Schoutens [46] gave a characterization of log terminal singularities and show that log terminal singularities descend under pure mophisms if the varieties are $\mathbb{Q}$-Gorenstein. He [45] also gave an explicit construction of a big Cohen-Macaulay algebra $\mathcal{B}(R)$ for a local domain R essentially of finite type over $\mathbb{C}$: $\mathcal{B}(R)$ is described as the ultraproduct of absolute integral closures of Noetherian local domains of positive characteristic. He defined a closure operation associated to $\mathcal{B}(R)$ to introduce the notions of $\mathcal{B}$-rationality and $\mathcal{B}$-regularity, which are closely related to BCM rationality and BCM regularity introduced in [37], [43], and proved that $\mathcal{B}$-rationality is equivalent to having rational singularities. In this dissertation, we generalize his technique and give an affirmative answer to his conjecture in [46] about $\mathcal{B}$-regularity.

As a generalization of the result of Zhuang [67], we study the behavior of adjoint ideals and related classes of singularities under pure morphisms. To do so, we use techniques in the theory of BCM singularities, which is summarized as follows. Pérez and R. G. [43] introduced the notion of the BCM test ideal $\tau_B(R)$ associated to a big Cohen-Macaulay algebra B, which is a generalization and a characteristic-free analogue of the classical test ideal defined by Hochster and Huneke [23]. Also, in order to study singularities in mixed characteristic, Ma and Schwede [37] introduced BCM-regular singularities, a characteristic-free analogue of F-regularity, and extended BCM test ideals to pairs (R, Δ), consisted of complete Noetherian normal local domains R and effective $\mathbb{Q}$-divisors Δ on $\operatorname{Spec} R$ such that $K_R + \Delta$ are $\mathbb{Q}$-Cartier. Furthermore, Ma, Schwede, Tucker, Waldron and Witaszek [39] introduced BCM adjoint ideals, a mixed characteristic analogue of adjoint ideals.

In Chapter 2, we focus on adjoint ideals. Let D be a reduced divisor on a normal variety X and Γ be an effective $\mathbb{Q}$-Weil divisor on X that has no common component with D. When $K_X + D + \Gamma$ is $\mathbb{Q}$-Cartier, the pair $(X, D + \Gamma)$ is said to be *purely log terminal* (plt, for short) along D if its discrepancy at E is greater than -1 for every prime divisor E over X that is not the strict transform of a component of D. When $K_X + D + \Gamma$ is not necessarily $\mathbb{Q}$-Cartier, we say that $(X, D+\Gamma)$ is of *plt type* along D if there exists an effective $\mathbb{Q}$-Weil divisor Δ on X such that Δ has no common component with D, $K_X + D + \Gamma + \Delta$ is $\mathbb{Q}$-Cartier and $(X, D + \Gamma + \Delta)$ is plt along D. Note that when $D = 0$, being of plt type along D is nothing but being of klt type. Our adjoint ideal $\operatorname{adj}_D(X, D+\Gamma)$ is a variant of multiplier ideals and defines the non-plt-type-locus of the pair $(X, D+\Gamma)$ along D. The main theorem of Chapter 2 is the following. Since $\mathrm{adj}_D(X, D+\Gamma)$ is nothing but the multiplier ideal $\mathcal{J}(X,\Gamma)$ when $D=0$, this is a generalization of a previous result of the author of this dissertation [65, Theorem 1.2].

Theorem A (Theorem 2.5.7, Corollary 2.6.8, Theorem 2.6.10)**.** *Let $R \hookrightarrow S$ be a pure ring extension of normal domains of finite type over $\mathbb{C}$ and $f : Y := \operatorname{Spec} S \to X := \operatorname{Spec} R$ denote the corresponding morphism of affine varieties. Let D be a reduced divisor and Γ be an effective $\mathbb{Q}$-Weil divisor on X such that D and Γ have no common components. Suppose that the $\mathcal{O}_X$-algebra $\bigoplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i(K_X+D+\Gamma)\rfloor)$ is finitely generated and the cycle-theoretic pullback $E := f^{\natural}D$ of D is a reduced divisor on Y (see Definition 2.4.1 for the definitions of cycle-theoretic pullback and pullback).*

(1) If f is faithfully flat, then

$$\mathrm{adj}_E(Y, E+f^*\Gamma) \subseteq \mathrm{adj}_D(X, D+\Gamma)\mathcal{O}_Y.$$

(2) Assume that E is a disjoint union of prime divisors and that one of the following conditions holds.

(a) $K_X+D+\Gamma$ is $\mathbb{Q}$-Cartier.

(b) The $\mathcal{O}_Y$-algebra $\bigoplus_{i\geqslant 0}\mathcal{O}_Y(iB)$ is finitely generated for every Weil divisor B on Y (this condition is satisfied, for example, if Y is of klt type).

Then

$$\mathrm{adj}_E(Y, E+f^*\Gamma) \cap \mathcal{O}_X \subseteq \mathrm{adj}_D(X, D+\Gamma).$$

As a corollary of this theorem, we can show the following, which answers Zhuang's question [67, Question 2.13] affirmatively.

Corollary B (Corollary 2.6.11)**.** *Let $f : Y \to X$ be a pure morphism between normal quasi-projective complex varieties, D be a reduced divisor and Γ be an effective $\mathbb{Q}$-Weil divisor on X that has no common component with D. Suppose that the cycle-theoretic pullback $E = f^{\natural}D$ of D under f is a reduced divisor on Y. If $(Y, E+f^*\Gamma)$ is of plt type along E, then $(X, D+\Gamma)$ is of plt type along D. In particular, if $(Y, f^*\Gamma)$ is of klt type, then (X,Γ) is of klt type as well.*

The main tool to show Theorem A is the theory of divisorial test ideals. Takagi [59] introduced divisorial test ideals, a generalization of test ideals, to study adjoint ideals. When $K_X+D+\Gamma$ is $\mathbb{Q}$-Cartier, he showed that the adjoint ideal $\mathrm{adj}_D(X, D+\Gamma)$ of $(X, D+\Gamma)$ coincides, after reduction to sufficiently large p, with the divisorial test ideal $\tau_D(X, D+\Gamma)$. We generalize this coincidence to the case where $K_X+D+\Gamma$ is not necessarily $\mathbb{Q}$-Cartier but the $\mathcal{O}_X$-algebra $\bigoplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i(K_X+D+\Gamma)\rfloor)$ is finitely generated. Then, since flatness is preserved under reduction modulo $p>0$, we can reduce Theorem A (1) to a problem on divisorial test ideals.

The proof of Theorem A (2) is more complicated because purity is not preserved under reduction modulo $p > 0$. Hence, we use ultraproducts rather than reduction modulo p. The author of this dissertation [65] showed that the BCM test ideal $\tau_{\mathcal{B}(R)}(R)$ associated to $\mathcal{B}(R)$ is equal to the multiplier ideal $\mathcal{J}(\operatorname{Spec} R)$ if R is a $\mathbb{Q}$-Gorenstein normal local domain essentially of finite type over $\mathbb{C}$. We introduce a new generalization $\tau_{\mathcal{B},D}(R, D+\Gamma)$ of the ideal $\tau_{\mathcal{B}(R)}(R)$ for a prime divisor D and an effective $\mathbb{Q}$-Weil divisor Γ on $X := \operatorname{Spec} R$ that has no component equal to D, defined in a similar way to BCM adjoint ideals in [39]. We then prove that if $K_X + D + \Gamma$ is $\mathbb{Q}$-Cartier, then this ideal is equal to the adjoint ideal $\operatorname{adj}_D(X, D+\Gamma)$. This characterization of adjoint ideals plays a key role in the proof of Theorem A (2).

As an application of a result of the author of this dissertation [65, Theorem 1.2], which is the prototype of Theorem A (2), we give a partial affirmative answer to Zhuang's question [67, Question 2.11] (cf. [6, Question 8.5]) of whether singularities of lc type descend under pure morphisms.

Theorem C (Theorem 2.6.13)**.** *$f : Y \to X$ be a pure morphism between normal complex affine varieties and suppose that X is $\mathbb{Q}$-Gorenstein. Assume in addition that one of the following conditions holds.*

(i) There exists an effective $\mathbb{Q}$-Weil divisor Δ on Y such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier and no non-klt center of (Y, Δ) dominates X.

(ii) The non-klt-type locus of Y has dimension at most one.

If Y is of lc type, then X has lc singularities.

As another application of Theorem A, we give an affirmative answer to a conjecture proposed by Schoutens [46, Remark 3.10], which says that $\mathcal{B}$-regularity is equivalent to having log terminal singularities (see Theorem 2.7.2).

In Chapter 3, we devote our attention to singularities of dense F-pure type and dense F-injective type. Here, for a given property P defined for schemes of positive characteristic, a scheme X essentially of finite type over a field of characteristic zero is said to be of *dense P-type* if its modulo $p > 0$ reduction X_p satisfies P for infinitely many primes p. Hara and Watanabe [20] showed that if a normal $\mathbb{Q}$-Gorenstein variety over a field of characteristic zero has singularities of dense F-pure type, then it has log canonical singularities. Takagi [60] showed that the converse is also true if the weak ordinarity conjecture, proposed by Mustaţă and Srinivas [41], holds true. As an F-singularity theoretic analogue of Zhuang's question on log canonicity, we discuss whether being of dense F-pure type descends under pure morphisms. As one of the main theorems in Chapter 3, we answer this question affirmatively when the singularity is $\mathbb{Q}$-Gorenstein:

Theorem D (Theorem 3.2.14)**.** *Let $R \to S$ be a pure local $\mathbb{C}$-algebra homomorphism between reduced local rings essentially of finite type over $\mathbb{C}$, $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, where R° denotes the set of elements of R not in any minimal prime of R,* *and t be a positive real number. Suppose that R is $\mathbb{Q}$-Gorenstein normal and $(S, (\mathfrak{a}S)^t)$ is of dense sharply F-pure type. Then $(R, \mathfrak{a}^t)$ is of dense sharply F-pure type.*

F-purity is generalized by Takagi [57] to pairs $(R, \mathfrak{a}^t)$, consisted of rings R of positive characteristic and nonzero ideals $\mathfrak{a}$ of R with real exponents $t > 0$, and sharp F-purity is a variant of Takagi's F-purity introduced by Schwede [49], which behaves better in a geometric setting.

To show Theorem D, we introduce the notion of ultra-F-purity, a variant of F-purity in equal characteristic zero via ultraproducts, defined by the purity of the ultra-Frobenii. We use the ultra-perfect closure R^{upf} (see Definition 1.5.43), an analogue of the big Cohen-Macaulay R-algebra $\mathcal{B}(R)$, to prove the equivalence of ultra-F-purity and being of dense F-pure type when the ring is $\mathbb{Q}$-Gorenstein. Since R^{upf} is not necessarily Cohen-Macaulay, we need to consider p-standard sequences introduced by Kawasaki [31], instead of regular sequences.

In the latter half of Chapter 3, we consider a similar problem for F-injective singularities. We introduce the notion of ultra-F-injectivity, a variant of F-injectivity in equal characteristic zero, defined in a similar way to ultra-F-purity. It follows from a similar argument that ultra-F-injectivity is equivalent to being of dense F-injective type if the residue field is isomorphic to $\mathbb{C}$. This equivalence enables us to show that singularities of dense F-injective type descend under strongly pure morphisms introduced in [8]. Here a ring homomorphism $R \to S$ is said to be *strongly pure* if for any prime ideal $\mathfrak{q}$ of S, the induced morphism $R_{\mathfrak{q} \cap R} \to S_{\mathfrak{q}}$ is pure.

Theorem E (Theorem 3.3.11)**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a strongly pure local $\mathbb{C}$-algebra homomorphism between reduced local rings essentially of finite type over $\mathbb{C}$, $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^{\circ} \neq \emptyset$ and t be a positive real number. Suppose that $R/\mathfrak{m} \cong \mathbb{C}$. If $(S, (\mathfrak{a}S)^t)$ is of dense sharply F-injective type, then $(R, \mathfrak{a}^t)$ is of dense sharply F-injective type.*

Note that strong purity is strictly stronger than purity and F-injectivity does not descend under pure morphisms (see [36, Example 8.6] and [63]). Schwede [50] showed that if a scheme of finite type over a field of characteristic zero is of dense F-injective type, then it has Du Bois singularities. The converse is equivalent to the weak ordinarity conjecture (see [3]). Therefore, compared with the result of Godfrey and Murayama, Theorem E can be regarded as an evidence of the weak ordinarity conjecture, which is wide open.

A part of Chapter 1 is based on [65], Chapter 2 is based on joint work with Shunsuke Takagi [62], and Chapter 3 is based on the preprint [66].

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Notation. Throughout this dissertation, all rings are assumed to be commutative and with unit element.

Chapter 1

Preliminaries

This chapter provides preliminary results needed for the rest of the dissertation.

1.1 Adjoint ideal sheaves

In this section, we define multiplier ideal sheaves and adjoint ideal sheaves in the non-$\mathbb{Q}$-Gorenstein setting. Our main reference is [14], and we use the notation in [32] and [35] freely.

Let X be a normal variety over an algebraically closed field k of characteristic zero, D be a reduced divisor on X and Δ be an effective $\mathbb{Q}$-Weil divisor on X that has no common components with D. Let $t \geqslant 0$ be a real number and $\mathfrak{a} \subseteq \mathcal{O}_X$ be a coherent ideal sheaf such that no components of D are contained in the zero locus of $\mathfrak{a}$.

Definition 1.1.1. (1) Suppose that $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, and take a log resolution $\pi : \widetilde{X} \to X$ of $(X, D + \Delta, \mathfrak{a})$ such that $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F)$ for an effective divisor F on $\widetilde{X}$ and the strict transform $\pi_*^{-1}D$ of D is smooth (but possible disconnected). Then the *adjoint ideal sheaf* $\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t)$ of the triple $(X, D+\Delta, \mathfrak{a}^t)$ along D is defined as

$$\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} - \lfloor \pi^*(K_X + D + \Delta) + tF \rfloor + \pi_*^{-1}D).$$

The definition is independent of the choice of π. When $\mathfrak{a} = \mathcal{O}_X$, the ideal sheaf $\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t)$ is simply denoted by $\mathrm{adj}_D(X, D+\Delta)$.

We say that the pair $(X, D+\Delta)$ is *plt* along D if $\mathrm{adj}_D(X, D+\Delta) = \mathcal{O}_X$.

(2) If $K_X + D + \Delta$ is not $\mathbb{Q}$-Cartier, then the *adjoint ideal sheaf* $\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t)$ of the triple $(X, D+\Delta, \mathfrak{a}^t)$ along D is defined as

$$\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \sum_{\Delta'} \mathrm{adj}_D(X, D+\Delta+\Delta', \mathfrak{a}^t),$$

where Δ' runs through all effective $\mathbb{Q}$-Weil divisors on X such that D and Δ' have no common components and $K_X + D + \Delta + \Delta'$ is $\mathbb{Q}$-Cartier. When $D = 0$, it is denoted by $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ and called the *multiplier ideal sheaf* of the triple $(X, \Delta, \mathfrak{a}^t)$. When $\mathfrak{a} = \mathcal{O}_X$, the ideal sheaf $\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t)$ (resp. $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$) is simply denoted by $\mathrm{adj}_D(X, D+\Delta)$ (resp. $\mathcal{J}(X, \Delta)$).

We say that $(X, D+\Delta)$ is of *plt type* along D if $\mathrm{adj}_D(X, D+\Delta) = \mathcal{O}_X$. We also say that (X, Δ) is of *klt type* if $\mathcal{J}(X, \Delta) = \mathcal{O}_X$.

(3) The pair (X, Δ) is said to be of *lc type* if there exists an effective $\mathbb{Q}$-Weil divisor Δ' on X such that $K_X + \Delta + \Delta'$ is $\mathbb{Q}$-Cartier and the pair $(X, \Delta + \Delta')$ is lc.

Proposition 1.1.2. *There exists an effective $\mathbb{Q}$-Weil divisor Γ on X such that D and Γ have no common components, $K_X + D + \Delta + \Gamma$ is $\mathbb{Q}$-Cartier and*

$$\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \mathrm{adj}_D(X, D+\Delta+\Gamma, \mathfrak{a}^t).$$

Proof. For every integer $m \geqslant 2$ such that $m(K_X+D+\Delta)$ is an integral Weil divisor, take a log resolution $\pi : Y \to X$ of $(X, \mathcal{O}_X(-m(K_X+D+\Delta))\mathfrak{a})$ such that $\mathcal{O}_X(-m(K_X + D+\Delta))\mathcal{O}_Y = \mathcal{O}_Y(-G_m)$ and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ for effective divisors F and G_m on Y. Then we define an ideal sheaf $\mathrm{adj}_D^{(m)}(X, D+\Delta, \mathfrak{a}^t)$ as

$$\mathrm{adj}_D^{(m)}(X, D+\Delta, \mathfrak{a}^t) = \mathcal{O}_Y(K_Y - \lfloor \frac{G_m}{m} + tF \rfloor + \pi_*^{-1}D).$$

Note that for every effective $\mathbb{Q}$-Weil divisor Δ' on X such that D and Δ' have no common components and $m(K_X+D+\Delta+\Delta')$ is Cartier, $G_m \leqslant m\pi^*(K_X+D+\Delta+\Delta')$ and consequently,

$$\mathrm{adj}_D(X, D+\Delta+\Delta', \mathfrak{a}^t) \subseteq \mathrm{adj}_D^{(m)}(X, D+\Delta, \mathfrak{a}^t).$$

The family $\{\mathrm{adj}_D^{(m)}(X, D+\Delta, \mathfrak{a}^t)\}_m$ of ideal sheaves has a unique maximal element, which is denoted by $\mathrm{adj}'_D(X, D+\Delta, \mathfrak{a}^t)$. By the above observation, we have an inclusion $\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) \subseteq \mathrm{adj}'_D(X, D+\Delta, \mathfrak{a}^t)$. On the other hand, it follows from a similar argument to the proof of [14, Proposition 5.4] that there exists an effective $\mathbb{Q}$-Weil divisor Γ on X such that D and Γ have no common components, $K_X + D + \Delta + \Gamma$ is $\mathbb{Q}$-Cartier and

$$\mathrm{adj}'_D(X, D+\Delta, \mathfrak{a}^t) = \mathrm{adj}_D(X, D+\Delta+\Gamma, \mathfrak{a}^t) \subseteq \mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t).$$

Thus, we have $\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \mathrm{adj}_D(X, D+\Delta+\Gamma, \mathfrak{a}^t)$. □

Remark 1.1.3. When Δ is an $\mathbb{R}$-Weil divisor, we can still define the adjoint ideal sheaf $\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t)$ as follows:

$$\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \sum_{\Delta'} \mathrm{adj}_D(X, D+\Delta+\Delta', \mathfrak{a}^t),$$

where Δ' runs through all effective $\mathbb{R}$-Weil divisors on X such that $K_X + D + \Delta + \Delta'$ is $\mathbb{R}$-Cartier. Then, by using essentially the same argument as the proof of Proposition 1.1.2, there exists an effective $\mathbb{R}$-Weil divisor Γ on X such that $K_X + D + \Delta + \Gamma$ is $\mathbb{Q}$-Cartier and

$$\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \mathrm{adj}_D(X, D+\Delta+\Gamma, \mathfrak{a}^t).$$

1.2 Test ideals along divisors

In this section, we recall the definition of test ideals along divisors.[1] The reader is referred to [59] and [60] for details. We will freely use the notation in [61].

Suppose that R is a normal domain of characteristic $p > 0$ and D is a reduced divisor on $X := \operatorname{Spec} R$. Then $R^{\circ,D}$ denotes the set of elements of R not in any minimal prime of $I_D := R(-D)$. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X that has no common components with D, let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^{\circ,D} \neq \emptyset$ and $t \geqslant 0$ be a real number. We assume that R is *F-finite*, that is, the Frobenius map $F : R \to R$ is finite. This is equivalent to saying that the Frobenius pushforward $F_* R$ is a finitely generated R-module.

Remark 1.2.1. Every F-finite Noetherian ring is excellent (see [34]) and has a dualizing complex (see [16, Remark 13.6], [36, Theorem 10.9]).

Definition 1.2.2 (cf. [60, Proposition 1.1])**.** The test ideal $\tau_D(R, D + \Delta, \mathfrak{a}^t)$ of the triple $(R, D + \Delta)$ along D is defined as the unique smallest ideal J of R satisfying the following conditions:

(a) $J \cap R^{\circ,D} \neq \emptyset$.

(b) For every integer $e \geqslant 0$ and every $\varphi \in \operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)(D + \Delta)\rceil), R) \subseteq \operatorname{Hom}_R(F^e_* R, R)$, one has $\varphi(F^e_*(\mathfrak{a}^{\lceil t(p^e-1)\rceil} J)) \subseteq J$.

It is simply denoted by $\tau_D(R, D + \Delta)$ when $\mathfrak{a} = R$.

Definition 1.2.3. Suppose that $(R, \mathfrak{m})$ is local of dimension d.

(1) For an R-module M, $0_M^{*_D(D+\Delta,\mathfrak{a}^t)}$ is the submodule of M consisting of all elements $z \in M$ for which there exists an element $c \in R^{\circ,D}$ such that

$$F^e_*(c\mathfrak{a}^{\lceil tp^e\rceil}) \otimes z = 0 \in F^e_* R((p^e - 1)D + \lceil p^e \Delta\rceil) \otimes_R M$$

for all large e.

(2) The following ideals are equal to each other (cf. [23, Proposition 8.23]), and are collectively denoted by $\tau_D(R, D + \Delta, \mathfrak{a}^t)$.

(a) $\bigcap_M \operatorname{Ann}_R 0_M^{*_D(D+\Delta,\mathfrak{a}^t)}$, where M runs through all R-modules.

(b) $\operatorname{Ann}_R 0_E^{*_D(D+\Delta,\mathfrak{a}^t)}$, where $E = E_R(R/\mathfrak{m})$ is an injective hull of the residue field $R/\mathfrak{m}$.

It is simply denoted by $\tau_D(R, D)$ when $\Delta = 0$ and $\mathfrak{a} = R$.

Remark 1.2.4. The formation of test ideals along divisors commutes with localization and completion (cf. [59, Corollary 3.6]). Therefore, by gluing, we can define test ideals along divisors for any F-finite normal schemes.

[1] Test ideals along divisors are referred to as divisorial test ideals in [59].

1.3 F-pure and F-injective singularities

This section includes the definitions of notions concerning F-pure and F-injective singularities.

Definition 1.3.1 ([28], [49], [57]). Let R be a Noetherian ring of characteristic $p > 0$, $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, where R° denotes the set of elements of R not in any minimal prime of R, and t be a positive real number.

(1) R is said to be *F-pure* if the Frobenius morphism $F : R \to F_*R$ is pure.

(2) $(R, \mathfrak{a}^t)$ is said to be *sharply F-pure* if for infinitely many $e \in N$, there exists $f \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$ such that $\cdot F^e_* f : R \to F^e_* R$ is pure.

Remark 1.3.2. Schwede gave a refined definition of sharp F-purity in [51], which is equivalent to the above definition if the ring R is local.

Definition 1.3.3. Let R be an F-finite reduced ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and t be a positive real number. We define $\sigma(R, \mathfrak{a}^t)$ as follows:

$$\sigma(R, \mathfrak{a}^t) = \sum_{e \geqslant 1} \sum_{\varphi} \varphi(F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil}),$$

where φ runs through all elements of $\mathrm{Hom}_R(F^e_* R, R)$.

Remark 1.3.4. This definition is different from more complicated one in [15]. σ in *loc. cit.* was shown to be contained in a non-lc ideal for sufficiently large $p > 0$ after reduction modulo $p > 0$.

Proposition 1.3.5. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism between F-finite reduced local rings of characteristic $p > 0$ such that the induced morphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ is a separable field extension. Suppose that $\mathfrak{a} \subseteq R$ is an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and t is a positive real number. Then we have the following:*

(1) $(R, \mathfrak{a}^t)$ *is sharply F-pure if and only if* $\sigma(R, \mathfrak{a}^t) = R$.

(2) For any $\mathfrak{p} \in \operatorname{Spec} R$, *we have* $\sigma(R_\mathfrak{p}, (\mathfrak{a}R_\mathfrak{p})^t) = \sigma(R, \mathfrak{a}^t)R_\mathfrak{p}$.

(3) $\sigma(\widehat{R}, (\mathfrak{a}\widehat{R})^t) = \sigma(R, \mathfrak{a}^t)\widehat{R}$.

(4) $\sigma(S, (\mathfrak{a}S)^t) = \sigma(R, \mathfrak{a}^t)S$.

Proof. The conclusion follows from an argument similar to [15, Proposition 14.10] and [55, Lemma 1.5]. $\square$

Definition 1.3.6 ([13], [52, Definition 2.8]). Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic $p > 0$, $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and t be a positive real number.

(1) R is said to be *F-injective* if for any $i \in \mathbb{Z}$, $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F_*R)$ is injective.

(2) $(R, \mathfrak{a}^t)$ is said to be *sharply F-injective* if for any $i \in \mathbb{Z}$ and a nonzero element $\eta \in H^i_{\mathfrak{m}}(R)$, for infinitely many $e \in \mathbb{N}$, there exists $f \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$ such that the image of η under $\cdot F^e_* f : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F^e_*R)$ is nonzero.

Definition 1.3.7. With notation as in Definition 1.3.6, suppose that R is F-finite and $\omega_R^\bullet$ is the normalized dualizing complex of R. For $i \in \mathbb{Z}$, $\sigma^{(i)}_{F\text{-inj}}(\omega_R^\bullet, \mathfrak{a}^t) \subseteq h^{-i}\omega_R^\bullet$ is defined to be

$$\sum_{e \geqslant 1} \sum_{f \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}} \operatorname{Im}\left(h^{-i}\operatorname{RHom}_R(F^e_*R, \omega_R^\bullet) \to h^{-i}\operatorname{RHom}_R(R, \omega_R^\bullet)\right),$$

where the above morphisms are induced by $\cdot F^e_* f : R \to F^e_* R$.

Proposition 1.3.8. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism between F-finite reduced local rings of characteristic $p > 0$ such that the induced morphism $R/\mathfrak{m} \to S/\mathfrak{m}S$ is a separable field extension. Suppose that $\mathfrak{a} \subseteq R$ is an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and t is a positive real number. Then we have the following:*

(1) $(R, \mathfrak{a}^t)$ is sharply F-injective if and only if for all i, $\sigma^{(i)}_{F\text{-inj}}(\omega_R^\bullet, \mathfrak{a}^t) = h^{-i}\omega_R^\bullet$.

(2) For any $i \in \mathbb{Z}$ and $\mathfrak{p} \in \operatorname{Spec} R$, we have $\sigma^{(i)}_{F\text{-inj}}(\omega_{R_\mathfrak{p}}^\bullet, (\mathfrak{a}R_\mathfrak{p})^t) = \sigma^{(i)}_{F\text{-inj}}(\omega_R^\bullet, \mathfrak{a}^t)_\mathfrak{p}$.

(3) For any $i \in \mathbb{Z}$, $\sigma^{(i)}_{F\text{-inj}}(\omega_{\widehat{R}}^\bullet, (\mathfrak{a}\widehat{R})^t) = \sigma^{(i)}_{F\text{-inj}}(\omega_R^\bullet, \mathfrak{a}^t) \otimes_R \widehat{R}$.

(4) For any $i \in \mathbb{Z}$, $\sigma^{(i)}_{F\text{-inj}}(\omega_S^\bullet, (\mathfrak{a}S)^t) = \sigma^{(i)}_{F\text{-inj}}(\omega_R^\bullet, \mathfrak{a}^t) \otimes_R S \subseteq h^{-i}\omega_S^\bullet$.

Proof. The conclusion follows from an argument similar to Proposition 1.3.5. Note that since the morphism $R \to S$ is local flat and $\mathfrak{m}S = \mathfrak{n}$, $\omega_S^\bullet = \omega_R^\bullet \otimes_R S$. We also refer the reader to [50, Proposition 4.3] for details. $\square$

We explain the definition of models and reductions modulo $p > 0$.

Definition 1.3.9. Let R be a ring of finite type over $\mathbb{C}$, $\mathfrak{a}$ be an ideal of R and $\mathfrak{p}$ be a prime ideal of R.

(1) A quadruple $(A, R_A, \mathfrak{a}_A, \mathfrak{p}_A)$ is said to be a *model* of the triple $(R, \mathfrak{a}, \mathfrak{p})$ if the following conditions hold:

   (a) A is a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$.

   (b) R_A is a finitely generated A-algebra such that $R_A \otimes_A \mathbb{C} \cong R$.

   (c) $\mathfrak{a}_A$ and $\mathfrak{p}_A$ are ideals of R_A such that $\mathfrak{a} = \mathfrak{a}_A R$ and $\mathfrak{p} = \mathfrak{p}_A R$.

(2) Let $(A, R_A, \mathfrak{a}_A, \mathfrak{p}_A)$ be a model of the triple $(R, \mathfrak{a}, \mathfrak{p})$. For a maximal ideal μ of A, a quadruple $(\kappa(\mu), R_\mu, \mathfrak{a}_\mu, \mathfrak{p}_\mu)$ is said to be a *reduction modulo $p > 0$* if the following conditions hold:

(a) $\kappa(\mu) = A/\mu$.

(b) $R_\mu = R_A \otimes_A \kappa(\mu)$.

(c) $\mathfrak{a}_\mu = \mathfrak{a}_A R_\mu$, $\mathfrak{p}_\mu = \mathfrak{p}_A R_\mu$.

Definition 1.3.10. Let R be a ring of finite type over $\mathbb{C}$, $\mathfrak{a}$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, $\mathfrak{p}$ be a prime ideal of R and t be a positive real number. A pair $(R_\mathfrak{p}, (\mathfrak{a}R_\mathfrak{p})^t)$ is said to be of *dense sharply F-pure (resp. dense sharply F-injective) type* if there exists a subset D of $\operatorname{Spm} A$, the set of all maximal ideals of A, such that D is a dense subset of $\operatorname{Spec} A$ and, for any $\mu \in D$, $\mathfrak{p}_\mu \in \operatorname{Spec} R_\mu$ and the pair $((R_\mu)_{\mathfrak{p}_\mu}, (\mathfrak{a}_\mu (R_\mu)_{\mathfrak{p}_\mu})^t)$ is sharply F-pure (resp. sharply F-injective). When $\mathfrak{a} = R$, we simply say that $R_\mathfrak{p}$ is of *dense F-pure (resp. dense F-injective) type* if $(R_\mathfrak{p}, R_\mathfrak{p}^t)$ is of sharply F-pure (resp. sharply F-injective) type.

Remark 1.3.11. (1) This definition depends only on $R_\mathfrak{p}$, $\mathfrak{a}R_\mathfrak{p}$ and t and is independent of the choice of models. We refer the reader to [41, Remark 2.5] for details.

(2) Hara and Watanabe [20] showed that singularities of dense F-pure type are log canonical if the ring is $\mathbb{Q}$-Gorenstein. Schwede [50] showed that if a ring of finite type over a field of characteristic zero has dense F-injective type, then it has Du Bois singularities. In [38], the result is generalized to the case of rings essentially of finite type over a field of characteristic zero.

1.4 BCM singularities

In this section, we will briefly review the theory of BCM singularities. Throughout this section, we assume that local rings $(R, \mathfrak{m})$ are Noetherian.

Definition 1.4.1. Let $(R, \mathfrak{m})$ be a local ring, and let $\boldsymbol{x} = x_1, \dots, x_n$ be a system of parameters. R-algebra B is said to be *big Cohen-Macaulay* with respect to $\boldsymbol{x}$ if $\boldsymbol{x}$ is a regular sequence on B. B is called a *(balanced) big Cohen-Macaulay algebra* if it is big Cohen-Macaulay with respect to $\boldsymbol{x}$ for every system of parameters $\boldsymbol{x}$.

Remark 1.4.2 ([7, Corollary 8.5.3]). If B is big Cohen-Macaulay with respect to $\boldsymbol{x}$, then the $\mathfrak{m}$-adic completion $\widehat{B}$ is (balanced) big Cohen-Macaulay.

Let R be a domain with fractional field K. We fix an algebraic closure $\overline{K}$ of K. The integral closure of R in $\overline{K}$, denoted by R^+, is called an *absolute integral closure* of R. Note that R^+ is independent, up to isomorphism, of the choice of $\overline{K}$. We refer the reader to [29] for an overview of the theory of absolute integral closure.

About the relation between absolute integral closures and big Cohen-Macaulay algebras, the following result is known (see [2], [24]).

Theorem 1.4.3. *If $(R, \mathfrak{m})$ is an excellent local domain of residue characteristic $p > 0$, then the p-adic completion of an absolute integral closure R^+ is a (balanced) big Cohen-Macaulay R-algebra.*

Using big Cohen-Macaulay algebras, we can define a class of singularities.

Definition 1.4.4. If R is an excellent local ring of dimension d and let B be a big Cohen-Macaulay R-algebra. We say that R is *big Cohen-Macaulay-rational* with respect to B (or simply BCM_B-rational) if R is Cohen-Macaulay and if $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(B)$ is injective. We say that R is *BCM-rational* if R is BCM_B-rational for any big Cohen-Macaulay algebra B.

We explain BCM test ideals introduced in [37], [43].

Setting 1.4.5. Let $(R, \mathfrak{m})$ be a normal local domain of dimension d.

(1) $\Delta \geqslant 0$ is a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier.

(2) Fixing Δ, we also fix an embedding $R \subseteq \omega_R \subseteq \operatorname{Frac} R$, where ω_R is the canonical module.

(3) Since $K_R + \Delta$ is effective and $\mathbb{Q}$-Cartier, there exist an integer $n > 0$ and $f \in R$ such that $n(K_R + \Delta) = \operatorname{div}(f)$.

Definition 1.4.6. With notation as in Setting 1.4.5, if B is a big Cohen-Macaulay $R[f^{1/n}]$-algebra, then we define $0^{B, K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)}$ to be $\ker \psi$, where ψ is the homomorphism determined by the below commutative diagram:

$$\begin{tikzcd}
H^d_{\mathfrak{m}}(R) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(B) \arrow[r, "\cdot f^{1/n}"] \arrow[d] & H^d_{\mathfrak{m}}(B) . \\
H^d_{\mathfrak{m}}(\omega_R) \arrow[r] \arrow[urr, bend right, "\psi"'] & H^d_{\mathfrak{m}}(B \otimes_R \omega_R) \arrow[ur] &
\end{tikzcd}$$

If R is $\mathfrak{m}$-adically complete, then we define

$$\tau_B(R, \Delta) = \operatorname{Ann}_R 0^{B, K_R+\Delta}_{H^d_{\mathfrak{m}}(\omega_R)}.$$

We call $\tau_B(R, \Delta)$ *the BCM test ideal of* (R, Δ) *with respect to* B. We say that (R, Δ) *is big Cohen-Macaulay regular* with respect to B (or simply BCM_B regular) if $\tau_B(R, \Delta) = R$.

Proposition 1.4.7 ([37])**.** *Let* $(R, \mathfrak{m})$ *be a complete normal local domain of characteristic* $p > 0$, $\Delta \geqslant 0$ *an effective* $\mathbb{Q}$*-Weil divisor on* $\operatorname{Spec} R$ *and* B *a big Cohen-Macaulay* R^+*-algebra. Fix an effective canonical divisor* $K_R \geqslant 0$. *Suppose that* $K_R + \Delta$ *is* $\mathbb{Q}$*-Cartier. Then*

$$\tau_B(R, \Delta) = \tau(R, \Delta).$$

Now suppose that $(R,\mathfrak{m})$ is a complete Noetherian local domain and D is a prime divisor on $\operatorname{Spec} R$ with defining ideal $I_D = R(-D)$, and fix a $\mathbb{Q}$-divisor $\Delta \geqslant 0$ such that $K_R + D + \Delta$ is $\mathbb{Q}$-Cartier and no component of Δ is equal to D. Fix a canonical divisor $K_R = -D + G$ of $\operatorname{Spec} R$ such that $G \geqslant 0$ and G has no component equal to D. Let $f \in R$ be an element such that $\operatorname{div}(f) = r(K_R + D + \Delta)$ for some $r \in \mathbb{Z}_{>0}$. Moreover, we fix an algebraic closure $\overline{K}$ of K and an absolute integral closure R^+ of R. $(R/I_D)^+$ is defined in a similar way. We fix a prime ideal I_D^+ of R^+ lying over I_D such that $R^+/I_D^+ \cong (R/I_D)^+$. This is an abuse of notation since I_D^+ is not uniquely determined by D. Assume that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I_D \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/I_D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_D^+ \arrow[r] \arrow[d] & R^+ \arrow[r] \arrow[d] & (R/I_D)^+ \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_{B\to C} \arrow[r] & B \arrow[r] & C \arrow[r] & 0,
\end{tikzcd}$$

where B and C are big Cohen-Macaulay R^+ (respectively $(R/I_D)^+$) algebras.

Remark 1.4.8. In [39], $I_{B\to C}$ is defined in the derived category $D^b(R)$ when the morphism $B \to C$ is not surjective. As remarked there, if the residue characteristic of R is $p > 0$, then we may simply let B be the p-adic completion of R^+ and C be the p-adic completion of $(R/I_D)^+$. In this case, we do not need to work in $D^b(R)$.

Definition 1.4.9 ([39])**.** With notation as above, we define the *BCM adjoint ideal* with respect to B, C, denoted $\operatorname{adj}^D_{B\to C}(R, D+\Delta)$, to be

$$\operatorname{Ann}_R \ker\left(H^d_{\mathfrak{m}}(\omega_R) \xrightarrow{\cdot f^{\frac{1}{n}}} H^d_{\mathfrak{m}}(I_{B\to C}) \right).$$

If R is of positive characteristic, we use $\tau_{+,D}(R, D+\Delta)$ to denote $\operatorname{adj}^D_{R^+\to(R/I_D)^+}(R, D+\Delta)$.

Proposition 1.4.10 ([39, Theorem 6.6])**.** *With notation as above, suppose that R is of characteristic $p > 0$ and F-finite. Then we have*

$$\tau_D(R, D+\Delta) = \operatorname{adj}^D_{B\to C}(R, D+\Delta).$$

Let S be a module-finite extension of a normal domain R with S normal and $\pi : \operatorname{Spec} S \to \operatorname{Spec} R$ denote a finite surjective morphism induced by the inclusion $R \hookrightarrow S$. The trace map Tr_π of π is the map

$$\omega_S \cong \operatorname{Hom}_R(S, \omega_R) \to \operatorname{Hom}_R(R, \omega_R) = R$$

induced by the inclusion $R \hookrightarrow S$.

We use the following proposition in Section 2.1.

Proposition 1.4.11 (cf. [39, Proposition 6.4]). *Suppose that $(R, \mathfrak{m})$ be an F-finite complete normal local ring of characteristic $p > 0$. Let D be a prime divisor and Δ be an effective $\mathbb{Q}$-Weil divisor on $X := \operatorname{Spec} R$ such that $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier and no component of Δ is equal to D. We fix a choice of I_D^+. For every module-finite extension S of R contained in R^+ with S normal, the trace map Tr_π induces a map*

$$\pi_* \omega_S(D_S - \lfloor \pi^*(K_X + D + \Delta) \rfloor) \to R$$

and the ideal $\tau_D(R, D + \Delta)$ is contained in its image, where $\pi : \operatorname{Spec} S \to \operatorname{Spec} R$ is a finite surjective morphism induced by the inclusion $R \hookrightarrow S$ and D_S is the prime divisor on $\operatorname{Spec} S$ such that $I_D^+ \cap S = S(-D_S)$.

Proof. By assumption, $r(K_X + D + \Delta) = \operatorname{div} f$ for some integer $r \geqslant 1$ and a nonzero element $f \in R$. Let $S \hookrightarrow T$ be a module-finite extension of S contained in R^+ such that T is a normal domain and $f^{1/r} \in T$, and let $\psi : \operatorname{Spec} T \to \operatorname{Spec} R$ denote the morphism corresponding to the inclusion $R \hookrightarrow T$. Since the image of

$$\operatorname{Tr}_\psi : \psi_* \omega_T(D_T - \lfloor \psi^*(K_X + D + \Delta) \rfloor) \to R,$$

where D_T is the prime divisor on $\operatorname{Spec} T$ such that $I_D^+ \cap T = T(-D_T)$, is contained in

$$\mathcal{I}_{S,D}(R, D + \Delta) := \operatorname{Im} \left(\operatorname{Tr}_\pi : \pi_* \omega_S(D_S - \lfloor \pi^*(K_X + D + \Delta) \rfloor) \to R \right),$$

we may assume that $f^{1/r} \in S$. On the other hand, It follows from [39, Proposition 6.4] that $\tau_D(R, D + \Delta)$ is contained in the BCM adjoint ideal $\tau_{+,D}(R, D + \Delta)$. Therefore, it suffices to show that $\tau_{+,D}(R, D + \Delta) \subseteq \mathcal{I}_{S,D}(R, D + \Delta)$. However, this is immediate because

$$\mathcal{I}_{S,D}(R, D + \Delta) = \operatorname{Ann}_R \ker \left(H^d_{\mathfrak{m}}(\omega_R) \xrightarrow{\cdot f^{1/r}} H^d_{\mathfrak{m}}(S(-D_S)) \right)$$

by Matlis duality. □

1.5 Ultraproducts

1.5.1 Basic notions

In this subsection, we quickly review basic notions from the theory of ultraproduct. The reader is referred to [44], [48] for details. We fix an infinite set W.

Definition 1.5.1. A non-empty subset $\mathcal{F}$ of the power set of W is said to be a *non-principal ultrafilter* if the following four conditions hold.

(1) If $A, B \in \mathcal{F}$, then we have $A \cap B \in \mathcal{F}$.

(2) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq W$, then we have $B \in \mathcal{F}$.

(3) For any $A \subseteq W$, we have $A \in \mathcal{F}$ or $W \setminus A \in \mathcal{F}$.

(4) For any finite subset $A \subseteq W$, we have $A \notin \mathcal{F}$.

Proposition 1.5.2. *For any infinite subset A of W, there exists a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{P}$ such that $A \in \mathcal{F}$.*

Proof. The conclusion follows from Zorn's lemma. □

Definition 1.5.3. Let A_w be a family of non-empty sets indexed by W and $\mathcal{F}$ be an ultrafilter on W. Suppose that $a_w \in A_w$ for all $w \in W$ and φ is a predicate. We say $\varphi(a_w)$ holds *for almost all* w if $\{w \in W | \varphi(a_w) \text{ holds}\} \in \mathcal{F}$.

Definition 1.5.4. Let A_w be a family of non-empty sets indexed by W and $\mathcal{F}$ be a non-principal ultrafilter on W. The *ultraproduct* of A_w is defined by

$$\operatorname*{ulim}_w A_w = A_\infty := \prod_w A_w / \sim,$$

where $(a_w) \sim (b_w)$ if and only if $\{w \in W | a_w = b_w\} \in \mathcal{F}$. We denote the equivalence class of (a_w) by $\operatorname{ulim}_w a_w$.

Example 1.5.5. We use ${}^*\mathbb{N}$ and ${}^*\mathbb{R}$ to denote the ultraproduct of $|W|$ copies of $\mathbb{N}$ and $\mathbb{R}$ respectively. ${}^*\mathbb{N}$ is a semiring and ${}^*\mathbb{R}$ is a field, see Definition-Proposition 1.5.6, Theorem 1.5.15. ${}^*\mathbb{N}$ is a non-standard model of Peano arithmetic. ${}^*\mathbb{R}$ is a system of hyperreal numbers used in non-standard analysis.

Definiton-Proposition 1.5.6. *Let $A_{1w}, \ldots, A_{nw}$, B_w be families of nonempty sets indexed by W and $\mathcal{F}$ be a non-principal ultrafilter. Suppose that $f_w : A_{1w} \times \cdots \times A_{nw} \to B_w$ is a family of maps. Then we define the ultraproduct $f_\infty = \operatorname{ulim}_w f_w : A_{1\infty} \times \cdots \times A_{n\infty} \to B_\infty$ of f_w by*

$$f_\infty(\operatorname*{ulim}_w a_{1w}, \ldots, \operatorname*{ulim}_w a_{nw}) := \operatorname*{ulim}_w f_w(a_{1w}, \ldots, a_{nw}).$$

This is well-defined.

Corollary 1.5.7. *Let A_w be a family of rings. Suppose that B_w is an A_w-algebra and M_w is an A_w-module for almost all w. Then the following hold:*

(1) A_∞ is a ring.

(2) B_∞ is an A_∞-algebra.

(3) M_∞ is an A_∞-module.

Proof. Let $0 := \operatorname{ulim}_w 0$, $1 := \operatorname{ulim}_w 1$ in A_∞, B_∞ and $0 := \operatorname{ulim}_w 0$ in M_∞. By the above Definition-Proposition, A_∞, B_∞ have natural additions, subtractions and multiplications and we have a natural ring homomorphism $A_\infty \to B_\infty$. Similarly, M_∞ has a natural addition and a scalar multiplication between elements of M_∞ and A_∞. □

Proposition 1.5.8. *Suppose that, for almost all w, we have an exact sequence*

$$0 \to L_w \to M_w \to N_w \to 0$$

of abelian groups. Then

$$0 \to \operatorname*{ulim}_w L_w \to \operatorname*{ulim}_w M_w \to \operatorname*{ulim}_w N_w \to 0$$

is an exact sequence of abelian groups. In particular, $\mathrm{ulim}_w : \prod_w \mathrm{Ab} \to \mathrm{Ab}$ *is an exact functor.*

Proof. Let $f_w : L_w \to M_w$ and $g_w : M_w \to N_w$ be the morphisms in the given exact sequence. Here we only prove the injectivity of $\mathrm{ulim}_w f_w$ and the surjectivity of $\mathrm{ulim}_w g_w$. Suppose that $\mathrm{ulim}_w f_w(a_w) = 0$ for $\mathrm{ulim}_w a_w \in \mathrm{ulim}_w L_w$. Then $f_w(a_w) = 0$ for almost all w. Since f_w is injective for almost all w, we have $a_w = 0$ for almost all w. Therefore, $\mathrm{ulim}_w a_w = 0$ in $\mathrm{ulim}_w L_w$. Hence, $\mathrm{ulim}_w f_w$ is injective. Next, let $\mathrm{ulim}_w c_w$ be any element in $\mathrm{ulim}_w N_w$. Since g_w is surjective for almost all w, there exists $b_w \in M_w$ such that $g_w(b_w) = c_w$ for almost all w. Let $b = \mathrm{ulim}_w b_w$. Then we have $(\mathrm{ulim}_w g_w)(b) = \mathrm{ulim}_w g_w(b_w) = \mathrm{ulim}_w c_w$. Hence, $\mathrm{ulim}_w g_w$ is surjective. The rest of the proof is similar. □

Łoś's theorem is a fundamental theorem in the theory of ultraproducts. We will prepare some notions needed to state the theorem.

Definition 1.5.9. The *language $\mathcal{L}$ of rings* is the set defined by

$$\mathcal{L} := \{0, 1, +, -, \cdot\}.$$

Definition 1.5.10. *Terms of $\mathcal{L}$* are defined as follows:

(1) 0,1 are terms.

(2) Variables are terms.

(3) If s, t are terms, then $-(s), (s) + (t), (s) \cdot (t)$ are terms.

(4) A string of symbols is a term only if it can be shown to be a term by finitely many applications of the above three rules.

We omit parentheses and "$\cdot$" if there is no ambiguity.

Example 1.5.11. $1 + 1$, $x_1(x_2 + 1)$,$-(-x)$ are terms.

Definition 1.5.12. *Formulas of $\mathcal{L}$* are defined as follows:

(1) If s, t are terms, then $(s = t)$ is a formula.

(2) If φ, ψ are formulas, then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi), (\neg\varphi)$ are formulas.

(3) If φ is a formula and x is a variable, then $\forall x\varphi, \exists x\varphi$ are formulas.

(4) A string of symbols is a formula only if it can be shown to be a formula by finitely many applications of the above three rules.

We omit parentheses if there is no ambiguity and use $\neq$, $\nexists$ in the usual way.

Remark 1.5.13. $\varphi \wedge \psi$ means "φ and ψ," $\varphi \vee \psi$ means "φ or ψ," $\varphi \to \psi$ means "φ implies ψ" and $\neg\varphi$ means "φ does not hold."

Remark 1.5.14. Variables in a formula φ which is not bounded by $\forall$ or $\exists$ are called free variables of φ. If $x_1, \dots, x_n$ are free variables of φ, we denote $\varphi(x_1, \dots, x_n)$ and we can substitute elements of a ring for $x_1, \dots, x_n$.

Theorem 1.5.15 (Łoś's theorem in the case of rings). *Suppose that $\varphi(x_1, \dots, x_n)$ is a formula of $\mathcal{L}$ and A_w is a family of rings indexed by a set W endowed with a non-principal ultrafilter. Let $a_{iw} \in A_w$. Then $\varphi(\operatorname{ulim}_w a_{1w}, \dots, \operatorname{ulim}_w a_{nw})$ holds in A_∞ if and only if $\varphi(a_{1w}, \dots, a_{nw})$ holds in A_w for almost all w.*

Remark 1.5.16. Even if A_w are not rings, replacing $\mathcal{L}$ properly, we can get the same theorem as above. We use one in the case of modules.

Example 1.5.17. Let A be a ring. If a property of rings is written by some formula, we can apply Łoś's theorem.

(1) A is a field if and only if $\forall x(x = 0 \vee \exists y(xy = 1))$ holds.

(2) A is a domain if and only if $\forall x\forall y(xy = 0 \to (x = 0 \vee y = 0))$ holds.

(3) A is a local ring if and only if

$$\forall x\forall y(\nexists z(xz = 1) \wedge \nexists w(yw = 1) \to \nexists u((x + y)u = 1))$$

holds.

(4) The condition that A is an algebraically closed field is written by countably many formulas, i.e., the formula in (1) and for all $n \in \mathbb{N}$,

$$\forall a_0 \dots a_{n-1} \exists x(x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0).$$

(5) The condition that A is Noetherian cannot be written by formulas. Indeed, if $W = \mathbb{N}$ with some non-principal ultrafilter and $A_w = \mathbb{C}[\![x]\!]$, then $\operatorname{ulim}_n x^n \neq 0$ is in $\cap_n \mathfrak{m}_\infty^n$, where $\mathfrak{m}_\infty$ is the maximal ideal of A_∞. Hence, A_∞ is not Noetherian.

Proposition 1.5.18 ([44, 2.8.2], see Example 1.5.17). *If almost all K_w are algebraically closed field, then K_∞ is an algebraically closed field.*

Theorem 1.5.19 (Lefschetz principle, [44, Theorem 2.4])**.** *Let W be the set of prime numbers endowed with some non-principal ultrafilter. Then*

$$\operatorname*{ulim}_{p \in \mathcal{W}} \overline{\mathbb{F}_p} \cong \mathbb{C}.$$

Proof. Let $C = \operatorname{ulim}_p \overline{\mathbb{F}_p}$. By the above theorem, C is an algebraically closed field. For any prime number q, we have $q \neq 0$ in $\overline{\mathbb{F}_p}$ for almost all p. Hence, $q \neq 0$ in C, i.e., C is of characteristic zero. We can check that C has the same cardinality as $\mathbb{C}$. If two algebraically closed uncountable field of characteristic zero have the equal cardinality, then they are isomorphic. Hence, $C \cong \mathbb{C}$ (Note that this isomorphism is not canonical). □

1.5.2 Non-standard hulls

In this subsection, we will introduce the notion of non-standard hulls along [44], [48]. Throughout this subsection, let $\mathcal{P}$ be the set of prime numbers and we fix a non-principal ultrafilter on $\mathcal{P}$ and an isomorphism $\operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$.

Let $\mathbb{C}[X_1, \dots, X_n]_\infty := \operatorname{ulim}_p \overline{\mathbb{F}_p}[X_1, \dots, X_n]$. Then we have the following proposition.

Proposition 1.5.20 ([44, Theorem 2.6])**.** *We have a natural map $\mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}[X_1, \dots, X_n]_\infty$, which is faithfully flat.*

Definition 1.5.21. The ring $\mathbb{C}[X_1, \dots, X_n]_\infty$ is said to be the *non-standard hull of* $\mathbb{C}[X_1, \dots, X_n]$.

Definition 1.5.22. Suppose that R is a finitely generated $\mathbb{C}$-algebra. Let

$$R \cong \mathbb{C}[X_1, \dots, X_n]/I$$

be a presentation of R. The *non-standard hull* R_∞ of R is defined by

$$R_\infty := \mathbb{C}[X_1, \dots, X_n]_\infty / I\mathbb{C}[X_1, \dots, X_n]_\infty.$$

Remark 1.5.23. Let R be as above.

(1) The non-standard hull is independent of a presentation of R. If

$$R \cong \mathbb{C}[X_1, \dots, X_n]/I \cong \mathbb{C}[Y_1, \dots, Y_m]/J,$$

then $\overline{\mathbb{F}_p}[X_1, \dots, X_n]/I_p \cong \overline{\mathbb{F}_p}[Y_1, \dots, Y_m]/J_p$ for almost all p, see Definition 1.5.26, Definition 1.5.27.

(2) The natural map $R \to R_\infty$ is faithfully flat since this is a base change of the homomorphism $\mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}[X_1, \dots, X_n]_\infty$.

Definition 1.5.24. Let $a \in \mathbb{C}$. Since $\operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$, we have a family $(a_p)_p$ of elements of $\overline{\mathbb{F}_p}$ such that $\operatorname{ulim} a_p = a$. Then we call $(a_p)_p$ an *approximation* of a.

Proposition 1.5.25. *Let $I = (f_1, \dots, f_s)$ be an ideal of $\mathbb{C}[X_1, \dots, X_n]$ and $f_i = \sum a_{i\nu} X^\nu$. Let $I_p = (f_{1p}, \dots, f_{sp})\overline{\mathbb{F}_p}[X_1, \dots, X_n]$, where $f_{ip} = \sum a_{i\nu p} X^\nu$ and each $(a_{i\nu p})_p$ is an approximation of $a_{i\nu}$. Then we have*

$$I\mathbb{C}[X_1, \dots, X_n]_\infty = \operatorname*{ulim}_p I_p$$

and

$$R_\infty \cong \operatorname*{ulim}_p (\overline{\mathbb{F}_p}[X_1, \dots, X_n]/I_p).$$

Definition 1.5.26. Let R be a finitely generated $\mathbb{C}$-algebra.

(1) In the setting of Proposition 1.5.25, a family (R_p) is said to be an *approximation* of R if R_p is an $\overline{\mathbb{F}_p}$-algebra and $R_p \cong \overline{\mathbb{F}_p}[X_1, \dots, X_n]/I_p$ for almost all p. Then we have $R_\infty \cong \operatorname{ulim}_p R_p$.

(2) For an element $f \in R$, a family (f_p) is said to be an *approximation* of f if $f_p \in R_p$ and $f = \operatorname{ulim}_p f_p$ in R_∞. For $f \in R_\infty$, we define an *approximation* of f in the same way.

(3) For an ideal $I = (f_1, \dots, f_s) \subseteq R$, a family (I_p) is said to be an *approximation* of I if I_p is an ideal of R_p and $I_p = (f_{1p}, \dots, f_{sp})$ for almost all p. For finitely generated ideal $I \subseteq R_\infty$, we define *an approximation* of I in the same way.

Definition 1.5.27. Let $\varphi : R \to S$ be a $\mathbb{C}$-algebra homomorphism between finitely generated $\mathbb{C}$-algebras. Suppose that $R \cong \mathbb{C}[X_1, \dots, X_n]/I$ and $S \cong \mathbb{C}[Y_1, \dots, Y_m]/J$. Let $f_i \in \mathbb{C}[Y_1, \dots, Y_m]$ be a lifting of the image of $X_i \bmod I$ under φ. Then we define an *approximation* $\varphi_p : R_p \to S_p$ of φ as the morphism induced by $X_i \longmapsto f_{ip}$. Let $\varphi_\infty := \operatorname{ulim}_p \varphi_p$, then the following diagram commutes.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ R_\infty \arrow[r, "\varphi_\infty"] & S_\infty \end{tikzcd}$$

Proposition 1.5.28 ([44, Corollary 4.2],[48, Theorem 4.3.4])**.** *Let R be a finitely generated $\mathbb{C}$-algebra. An ideal $I \subseteq R$ is prime if and only if I_p is prime for almost all p if and only if IR_∞ is prime.*

Definition 1.5.29. Let R be a local ring essentially of finite type over $\mathbb{C}$. Suppose that $R \cong S_\mathfrak{p}$, where S is a finitely generated $\mathbb{C}$-algebra and $\mathfrak{p}$ is a prime ideal of S. Then we define the *non-standard hull* R_∞ of R by

$$R_\infty := (S_\infty)_{\mathfrak{p}S_\infty}.$$

Remark 1.5.30. Since $S \to S_\infty$ is faithfully flat, $R \to R_\infty$ is faithfully flat.

Definition 1.5.31. Let S be a finitely generated $\mathbb{C}$-algebra, $\mathfrak{p}$ a prime ideal of S and $R \cong S_\mathfrak{p}$.

(1) A family R_p is said to be an *approximation* of R if R_p is an $\overline{\mathbb{F}_p}$-algebra and $R_p \cong (S_p)_{\mathfrak{p}_p}$ for almost all p. Then we have $R_\infty \cong \operatorname{ulim}_p R_p$.

(2) For an element $f \in R$, a family f_p is said to be an *approximation* of f if $f_p \in R_p$ for almost all p and $f = \operatorname{ulim}_p f_p$ in R_∞. For $f \in R_\infty$, we define an *approximation* of f in the same way.

(3) For an ideal $I = (f_1, \ldots, f_s) \subseteq R$, a family I_p is said to be an *approximation* of I if I_p is an ideal of R_p and $I_p = (f_{1p}, \ldots, f_{sp})$ for almost all p. For finitely generated ideal $I \subseteq R_\infty$, we define *an approximation* of I in the same way.

Definition 1.5.32. Let S_1, S_2 be finitely generated $\mathbb{C}$-algebras and $\mathfrak{p}_1, \mathfrak{p}_2$ prime ideals of S_1, S_2 respectively. Suppose that $R_i \cong (S_i)_{\mathfrak{p}_i}$ and $\varphi : R_1 \to R_2$ is a local $\mathbb{C}$-algebra homomorphism. Let $S_1 \cong \mathbb{C}[X_1, \ldots, X_n]/I$ and f_j/g_j be the image of X_j under φ, where $f_j \in S_2$, $g_j \in S_2 \setminus \mathfrak{p}_2$. Then we say that a homomorphism $R_{1p} \to R_{2p}$ induced by $X_j \longmapsto f_{jp}/g_{jp}$ is an *approximation* of φ. Let $\varphi_\infty := \operatorname{ulim}_p \varphi_p$. Then the following commutative diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ R_\infty \arrow[r, "\varphi_\infty"] & S_\infty \end{tikzcd}.$$

Proposition 1.5.33 ([44, Theorem 4.7])**.** *Let $\iota : R \hookrightarrow S$ be an injective local $\mathbb{C}$-algebra homomorphism between local domains essentially of finite type over $\mathbb{C}$. If $(\iota_p : R_p \to S_p)$ is an approximation of ι, then ι_p is injective for almost all p.*

Definition 1.5.34. Let R be a finitely generated $\mathbb{C}$-algebra or a local ring essentially of finite type over $\mathbb{C}$ and let M be a finitely generated R-module. Write M as the cokernel of a matrix A, i.e., given by an exact sequence

$$R^m \xrightarrow{A} R^n \to M \to 0,$$

where m, n are positive integers. Let A_p be an approximation of A defined by entrywise approximations. Then the cokernel M_p of the matrix A_p is called an *approximation* of M and the ultraproduct $M_\infty := \operatorname{ulim}_p M_p$ is called the *non-standard hull* of M. M_∞ is a finitely generated R_∞-module and independent of the choice of matrix A.

Remark 1.5.35. Tensoring the above exact sequence with R_∞, we have an exact sequence

$$R_\infty^m \xrightarrow{A} R_\infty^n \to M \otimes_R R_\infty \to 0.$$

Taking the ultraproduct of exact sequences

$$R_p^m \xrightarrow{A_p} R_p^n \to M_p \to 0,$$

we have an exact sequence

$$R_\infty^m \xrightarrow{A} R_\infty^n \to M_\infty \to 0.$$

Therefore, $M_\infty \cong M \otimes_R R_\infty$. Note that if m, n is not integers but infinite cardinals, then the naive definition of an approximation of A does not work and the ultraproduct of $R_p^{\oplus n}$ is not necessarily equal to $R_\infty^{\oplus n}$.

Here we state basic properties about non-standard hulls and approximations.

Proposition 1.5.36 ([44, 2.9.5, 2.9.7, Theorem 4.5, Theorem 4.6],[48, Section 4.3], cf. [1, 5.1])**.** *Let R be a local ring esseentially of finite type over $\mathbb{C}$, then the following hold:*

(1) R has dimension d if and only if R_p has dimension d for almost all p.

(2) $\boldsymbol{x} = x_1, \dots, x_i$ is an R-regular sequence if and only if $\boldsymbol{x}_p = x_{1p}, \dots, x_{ip}$ is an R_p-regular sequence for almost all p if and only if $\boldsymbol{x}$ is an R_∞-regular sequence.

(3) $\boldsymbol{x} = x_1, \dots, x_d$ is a system of parameters of R if and only if $\boldsymbol{x}_p$ is a system of parameters of R_p for almost all p.

(4) R is regular if and only if R_p is regular for almost all p.

(5) R is Gorenstein if and only if R_p is Gorenstein for almost all p.

(6) R is Cohen-Macaulay if and only if R_p is Cohen-Macaulay for almost all p.

Proposition 1.5.37 ([64, Proposition 3.9])**.** *Let R be a local ring essentially of finite type over $\mathbb{C}$. The following conditions are equivalent to each other.*

(1) R is normal.

(2) R_p is normal for almost all p.

(3) R_∞ is normal.

Definition 1.5.38. Let R be a normal local domain essentially of finite type over $\mathbb{C}$ and $\Delta = \sum_i a_i \Delta_i$ a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$. Assume that Δ_i are prime divisors and $\mathfrak{p}_i$ is a prime ideal associated to Δ_i for each i. Suppose that $\mathfrak{p}_{ip}$ is an approximation of $\mathfrak{p}_i$ and Δ_{ip} is a divisor associated to $\mathfrak{p}_{ip}$. We say $\Delta_p := \sum_i a_i \Delta_{ip}$ is an *approximation* of Δ.

Remark 1.5.39. If Δ is an effective integral divisor, then this definition is compatible with Definition 1.5.26 by [44, Theorem 4.4]. Hence, if Δ is $\mathbb{Q}$-Cartier, then Δ_p is $\mathbb{Q}$-Cartier for almost all p.

We give the definition of ultra-Frobenii and ideals with a non-standard integer exponent.

Definition 1.5.40 ([46, 3.2])**.** Let R be a local ring essentially of finite type over $\mathbb{C}$ and $\varepsilon = \operatorname{ulim}_p e_p$ be a non-standard integer (i.e. an element of ${}^*\mathbb{N}$). Then an *ultra-Frobenius* $F^\varepsilon : R \to R_\infty$ associated to ε is defined to be the morphism determined by $x \longmapsto \operatorname{ulim}_p x_p^{p^{e_p}}$. We use $F_*^\varepsilon R_\infty$ to denote the R-module such that $F_*^\varepsilon R_\infty$ is isomorphic to R_∞ as an abelian group but, for any $a \in R$ and $b \in R_\infty$, the scalar multiplication on $F_*^\varepsilon R_\infty$ is defined by $a \cdot F_*^\varepsilon b = F_*^\varepsilon(F^\varepsilon(a)b)$.

Definition 1.5.41 ([64, Notation 5.1])**.** With notation as above, for any $\varepsilon = \operatorname{ulim}_p e_p \in {}^*\mathbb{N}$ and an ideal $\mathfrak{a}$ of R, $\mathfrak{a}^\varepsilon$ is defined to be

$$\operatorname*{ulim}_p \mathfrak{a}_p^{e_p}.$$

Remark 1.5.42. For any $n \in \mathbb{N}$, if ν is the image of n under the diagonal embedding $\mathbb{N} \hookrightarrow {}^*\mathbb{N}$, then we have $\mathfrak{a}^\nu = \mathfrak{a}^n R_\infty$.

Here we introduce a new notion, the ultra-perfect closure.

Definition 1.5.43. Let R be a reduced local ring essentially of finite type over $\mathbb{C}$. The *ultra-perfect closure* of R^{upf} is defined to be $\operatorname{ulim}_p R_p^{1/p^\infty}$.

Proposition 1.5.44. *Let R be a reduced local ring essentially of finite type over $\mathbb{C}$. Then we have*

$$R^{\mathrm{upf}} \cong \varinjlim_{\varepsilon \in {}^*\mathbb{N}} F_*^\varepsilon R_\infty.$$

Proof. Take $\mu \leqslant \nu \in {}^*\mathbb{N}$ and let $\nu = \operatorname{ulim}_p n_p$, $\mu = \operatorname{ulim}_p m_p$. Since $m_p \leqslant n_p$ for almost all p, we have $F^{m_p} R_p \hookrightarrow F^{n_p} R_p \hookrightarrow R_p^{1/p^\infty}$ for almost all p. Hence, we have

$$F_*^\mu R_\infty \hookrightarrow F_*^\nu R_\infty \hookrightarrow R^{\mathrm{upf}}.$$

Therefore, we can define $\varinjlim_{\varepsilon \in {}^*\mathbb{N}} F_*^\varepsilon R_\infty$ and we have $\varinjlim_{\varepsilon \in {}^*\mathbb{N}} F_*^\varepsilon R_\infty \hookrightarrow R^{\mathrm{upf}}$. In order to prove the surjectivity, take any $x = \operatorname{ulim} x_p \in R^{\mathrm{upf}}$. For any p, there exists $e_p \in \mathbb{N}$ such that $x_p \in F_*^{e_p} R_p$. Let $\varepsilon = \operatorname{ulim} e_p \in {}^*\mathbb{N}$. Then we have $x \in \operatorname{ulim}_p F_*^{e_p} R_p \cong F_*^\varepsilon R_\infty$. $\square$

Lastly, we will explain the relation between approximations and reductions modulo $p > 0$.

Proposition 1.5.45 ([45, Lemma 4.10])**.** *Let A be a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$. There exists a family $(\gamma_p)_p$ which satisfies the following two conditions:*

(1) $\gamma_p : A \to \overline{\mathbb{F}_p}$ is a ring homomorphism for almost all p.

(2) For any $x \in A$, $x = \operatorname{ulim}_p \gamma_p(x)$.

Proposition 1.5.46 (cf. [45, Corollary 4.10]). *Let R be a finitely generated $\mathbb{C}$-algebra and let $\mathbf{a} = a_1, \ldots, a_l$ be finitely many elements of R. Let R_p be an approximation of R. Then there exists a model (A, R_A) which satisfies the following conditions:*

(1) There exists a family (γ_p) as in Proposition 1.5.45.

(2) $\mathbf{a} \subseteq R_A$.

(3) $R_A \otimes_A \overline{\mathbb{F}_p} \cong R_p$ for almost all p.

(4) For any $x \in R_A$, the ultraproduct of the image of x under $\mathrm{id}_{R_A} \otimes_A \gamma_p$ is x.

Proof. Let $X = X_1, \ldots, X_n$ and $R \cong \mathbb{C}[X]/I$ for some ideal $I \subseteq \mathbb{C}[X]$. Take any model (A, R_A) which contains $\mathbf{a}$. Enlarging this model, we may assume that there exits an ideal $I_A \subseteq A[X]$ such that $R_A \cong A[X]/I_A$ and $I_A \otimes_A \mathbb{C} = I$ in $\mathbb{C}[X]$. Take (γ_p) as in Proposition 1.5.45. Let $I = (f_1, \ldots, f_m)$. For $f = \sum_\nu c_\nu X^\nu \in A[X] \subseteq \mathbb{C}[X]$, by the definition of approximations, $f_p := \sum_\nu \gamma_p(c_\nu) X^\nu \in \overline{\mathbb{F}_p}[X]$ is an approximation of f. Hence, by the definition of approximations of finitely generated $\mathbb{C}$-algebras, $R_A \otimes_A \overline{\mathbb{F}_p} \cong \overline{\mathbb{F}_p}[X]/(f_{1p}, \ldots, f_{mp})\overline{\mathbb{F}_p}[X]$ is an approximation of R. Since two approximations are isomorphic for almost all p, $R_A \otimes_A \overline{\mathbb{F}_p} \cong R_p$ for almost all p. The condition (4) is clear by the above argument. □

Remark 1.5.47. Let $\mathfrak{p} = (x_1, \ldots, x_n) \subseteq R$ be a prime ideal. Enlarging the model (A, R_A), we may assume that $x_1, \ldots, x_n \in R_A$. Let μ_p be the kernel of $\gamma_p : A \to \overline{\mathbb{F}_p}$. Then this is a maximal ideal of A and A/μ_p is a finite field. $\mathfrak{p}_{\mu_p} = (x_1, \ldots, x_n) R_A/\mu_p R_A$ is prime for almost all p since this is a reduction to $p \gg 0$. On the other hand, $\mathfrak{p}_p := (x_1, \ldots, x_n) R_A \otimes_A \overline{\mathbb{F}_p} \subseteq R_p$ is an approximation of $\mathfrak{p}$. Hence, $\mathfrak{p}_p$ is prime for almost all p. Here, $(R_p)_{\mathfrak{p}_p}$ is an approximation of $R_\mathfrak{p}$. Thus we have a flat local homomorphism $(R_A/\mu_p R_A)_{\mathfrak{p}_{\mu_p}} \to R_p$ with $\mathfrak{p}_{\mu_p} R_p = \mathfrak{p}_p$. Moreover, if $\mathfrak{p}$ is maximal, then $\mathfrak{p}_{\mu_p}, \mathfrak{p}_p$ are maximal for almost all p. Then, the map $R_A/\mathfrak{p}_{\mu_p} \to R_p/\mathfrak{p}_p \cong \overline{\mathbb{F}_p}$ is a separable field extension since $R_A/\mathfrak{p}_{\mu_p}$ is a finite field.

The next result is a generalization of [62, Theorem 4.6] from ideal pairs to triples.

Proposition 1.5.48 ([65, Proposition 5.5]). *Let R be a normal local domain essentially of finite type over $\mathbb{C}$, $\Delta \geqslant 0$ an effective $\mathbb{Q}$-Weil divisor such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier, $\mathfrak{a}$ a nonzero ideal and $t > 0$ a real number. Suppose that R_p, Δ_p, $\mathfrak{a}_p$ are approximations. Then $\tau(R_p, \Delta_p, \mathfrak{a}_p^t)$ is an approximation of $\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$.*

Proof. Let $R = S_\mathfrak{p}$, where S is a normal domain of finite type over $\mathbb{C}$ and $\mathfrak{p}$ is a prime ideal. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then there exists a model (A, S_A) of S such that the properties in Proposition 1.5.46 hold and S_A containing a system of generators of $\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$ and Δ_A, $\mathfrak{a}_A$ can be defined properly. Let μ_p be maximal ideals of S_A as in Remark 1.5.47 and let $\mathfrak{m}_{\mu_p}, \mathfrak{p}_{\mu_p}$ be reductions to $p \gg 0$.

Since, for almost all p, $(S_A/\mu_p)_{\mathfrak{m}_{\mu_p}} \to (S_\mathfrak{m})_p$ is a flat local homomorphism such that $S_A/\mathfrak{m}_{\mu_p} \to (S/\mathfrak{m})_p \cong \overline{\mathbb{F}_p}$ is a separable field extension, we have

$$\tau((S_A/\mu_p)_{\mathfrak{m}_{\mu_p}}, \Delta_{(S_A/\mu_p)_{\mathfrak{m}_{\mu_p}}}, \mathfrak{a}^t_{(S_A/\mu_p)_{\mathfrak{m}_{\mu_p}}})(S_\mathfrak{m})_p = \tau((S_\mathfrak{m})_p, \Delta_{\mathfrak{m}_p}, \mathfrak{a}^t_{\mathfrak{m}_p}),$$

by a generalization of [55, Lemma 1.5]. Since the localization commutes with test ideals ([19, Proposition 3.1]), we have

$$\tau((S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}, \Delta_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}}, \mathfrak{a}^t_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}})R_p = \tau(R_p, \Delta_p, \mathfrak{a}^t_p)$$

for almost all p. Since the reduction of multiplier ideals modulo $p \gg 0$ is the test ideal ([56, Theorem 3.2]), $\tau((S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}, \Delta_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}}, \mathfrak{a}^t_{(S_A/\mu_p)_{\mathfrak{p}_{\mu_p}}})$ is a reduction of

$$\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$$

to characteristic $p \gg 0$. Hence, $\tau(R_p, \Delta_p, \mathfrak{a}^t_p)$ is an approximation of $\mathcal{J}(\operatorname{Spec} R, \Delta, \mathfrak{a}^t)$. □

1.5.3 Relative hulls

In this subsection we introduce the concept of relative hulls and approximations of schemes, cohomologies, etc. We refer the reader to [44], [46], [47].

Definition 1.5.49 (cf. [47])**.** Let R be a local ring essentially of finite type over $\mathbb{C}$. Suppose that X is a finite tuple of indeterminates and $f \in R[X]$ is a polynomial such that $f = \sum_\nu a_\nu X^\nu$, where ν is a multi-index. If $a_{\nu p}$ is an approximation of a_ν for each ν, then the sequence of polynomials $f_p := \sum_\nu a_{\nu p} X^\nu$ is said to be an *R-approximation* of f. If $I := (f_1, \ldots, f_s)$ is an ideal in $R[X]$, then we call $I_p := (f_{1p}, \ldots, f_{sp})R_p[X]$ an *R-approximation* of I, and if $S = R[X]/I$, then we call $S_p := R_p[X]/I_p$ an *R-approximation* of S.

Remark 1.5.50. Any two R-approximations of a polynomial f are almost equal. Similarly, any two R-approximations of an ideal I are almost equal.

Definition 1.5.51 (cf. [47])**.** Let S be a finitely generated R-algebra and S_p an R-approximation of S, then we call $S_\infty = \operatorname{ulim}_p S_p$ the *(relative) R-hull* of S.

Definition 1.5.52 (cf. [46])**.** If X is an affine scheme $\operatorname{Spec} S$ of finite type over $\operatorname{Spec} R$, then we call $X_p := \operatorname{Spec} S_p$ is an *R-approximation* of X.

Definition 1.5.53 (cf. [46])**.** Suppose that $f : Y \to X$ is a morphism of affine schemes of finite type over $\operatorname{Spec} R$. If $X = \operatorname{Spec} S, Y = \operatorname{Spec} T$ and $\varphi : S \to T$ is the morphism corresponding to f, then we call $f_p : Y_p \to X_p$ is an *R-approximation* of f, where f_p is a morphism of R_p-schemes induced by an R-approximation $\varphi_p : S_p \to T_p$.

Definition 1.5.54 (cf. [46])**.** Let S be a finitely generated R-algebra and M a finitely generated S-module. Write M as the cokernel of a matrix A, i.e., given by an exact sequence

$$S^m \xrightarrow{A} S^n \to M \to 0,$$

where m, n are positive integers. Let A_p be an R-approximation of A defined by entrywise R-approximations. Then the cokernel M_p of the matrix A_p is called an *R-approximation* of M and the ultraproduct $M_\infty := \operatorname{ulim}_p M_p$ is called the *R-hull* of M. M_∞ is independent of the choice of the matrix A and $M_\infty \cong M \otimes_S S_\infty$.

Remark 1.5.55. If M is not finitely generated, then we cannot define an R-approximation of M in this way. It is crucial that any two R-approximations of A is equal for almost all p.

Lemma 1.5.56. *Suppose that S is a module-finite extension of R contained in R^+. Let (S_p) be an R-approximation of S, (M_p) be a family of S_p-modules indexed by $\mathcal{P}$ and N be a finite S-module. Then*

$$(\operatorname*{ulim}_p M_p) \otimes_S N \cong \operatorname*{ulim}_p (M_p \otimes_{S_p} N_p).$$

Proof. Take a finite presentation

$$S^m \xrightarrow{A} S^n \to N \to 0$$

of the S-module N, where m, n are positive integers and A is an $n \times m$ matrix with entries in the maximal ideal $\mathfrak{m}$. Then we have an exact sequence

$$S_p^m \xrightarrow{A_p} S_p^n \to N_p \to 0$$

for almost all p, where N_p and A_p are approximations of N and A, respectively. Tensoring with M_p yields the exact sequence

$$M_p^m \xrightarrow{A_p} M_p^n \to M_p \otimes_{S_p} N_p \to 0$$

for almost all p. Taking its ultraproduct, we have an exact sequence

$$(\operatorname*{ulim}_p M_p)^m \xrightarrow{A} (\operatorname*{ulim}_p M_p)^n \to \operatorname*{ulim}_p (M_p \otimes_{S_p} N_p) \to 0,$$

which induces the isomorphism

$$(\operatorname*{ulim}_p M_p) \otimes_S N \cong \operatorname*{ulim}_p (M_p \otimes_{S_p} N_p).$$

$\square$

Definition 1.5.57 ([46])**.** Let X be a scheme of finite type over $\operatorname{Spec} R$. Let $\mathfrak{U} = \{U_i\}$ is a finite affine open covering of X and U_{ip} be an R-approximation of U_i. Gluing $\{U_{ip}\}$ together, we obtain a scheme X_p of finite type over $\operatorname{Spec} R_p$. We call X_p an *R-approximation* of X.

Remark 1.5.58. Suppose that $\{U_{ijk}\}_k$ is a finite affine open covering of $U_i \cap U_j$ and $\varphi_{ijk} : \mathcal{O}_{U_i}|_{U_{ijk}} \cong \mathcal{O}_{U_j}|_{U_{ijk}}$ are isomorphisms. Then R-approximations $\varphi_p : \mathcal{O}_{U_{ip}}|_{U_{ijkp}} \to \mathcal{O}_{U_{jp}}|_{U_{ijkp}}$ are isomorphisms for almost all p (note that indices ijk are finitely many). Hence, we can glue these together. For any other choice of finite affine open covering $\mathfrak{U}'$ of X , the resulting R-approximation X'_p is isomorphic to X_p for almost all p.

Definition 1.5.59 (cf. [46])**.** Suppose that $f : Y \to X$ is a morphism between schemes of finite type over $\operatorname{Spec} R$. Let $\mathfrak{U}$, $\mathfrak{V}$ be finite affine open coverings of X and Y respectively such that for any $V \in \mathfrak{V}$, there exists some $U \in \mathfrak{U}$ such that $f(V) \subseteq U$. Let $\mathfrak{U}_p$, $\mathfrak{V}_p$ be R-approximations of $\mathfrak{U}$, $\mathfrak{V}$ and $(f|_V)_p$ an *R-approximation* of $f|_V$. We define an R-approximation f_p of f by the morphism determined by $(f|_V)_p$.

Remark 1.5.60. In the same way as above Remark, $(f|_V)_p$ and $(f|_{V'})_p$ agree on $V \cap V'$ for any two open subsets $V, V' \in \mathfrak{V}$ for almost all p.

Definition 1.5.61 (cf. [46])**.** Let X be a scheme of finite type over $\operatorname{Spec} R$ and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Let $\mathfrak{U}$ be a finite affine open covering of X. For any $U \in \mathfrak{U}$, we have an R-approximation M_{Up} of M_U such that M_U is a finitely generated $\mathcal{O}_U$-module and $\widetilde{M_U} \cong \mathcal{F}|_U$. We define an *$R$ -approximation* $\mathcal{F}_p$ of $\mathcal{F}$ by the coherent $\mathcal{O}_{X_p}$-module determined by $\widetilde{M_{Up}}$.

Definition 1.5.62 (cf. [46])**.** Let X be a separated scheme of finite type over $\operatorname{Spec} R$ and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then the *ultra-cohomology* of $\mathcal{F}$ is defined by

$$H^i_\infty(X, \mathcal{F}) := \operatorname*{ulim}_p H^i(X_p, \mathcal{F}_p).$$

Remark 1.5.63. In the above setting, let $\mathfrak{U} = \{U_i\}_{i=1,\dots,n}$ be a finite affine open covering of X, let

$$C^j(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_j} \mathcal{F}(U_{i_0 \dots i_j}),$$

where $U_{i_0 \dots i_j} := U_{i_0} \cap \cdots \cap U_{i_j}$, and let

$$(C^j(\mathfrak{U}, \mathcal{F}))_p := \prod_{i_0 < \cdots < i_j} (\mathcal{F}(U_{i_0 \dots i_j}))_p,$$

where $\mathcal{F}(U_{i_0 \dots i_j})_p$ is an R-approximation considered as an $\mathcal{O}(U_{i_0 \dots i_j})$-module. Then

$$(C^j(\mathfrak{U}, \mathcal{F}))_p$$

coincides with the j-th term of the Čech complex associated to $\mathcal{F}_p$ and $\mathfrak{U}_p$. We have a commutative diagram

$$\begin{tikzcd}
C^{j-1}(\mathfrak{U},\mathcal{F}) \arrow[r] \arrow[d] & C^{j}(\mathfrak{U},\mathcal{F}) \arrow[r] \arrow[d] & C^{j+1}(\mathfrak{U},\mathcal{F}) \arrow[d] \\
\operatorname*{ulim}_p (C^{j-1}(\mathfrak{U},\mathcal{F}))_p \arrow[r] & \operatorname*{ulim}_p (C^{j}(\mathfrak{U},\mathcal{F}))_p \arrow[r] & \operatorname*{ulim}_p (C^{j+1}(\mathfrak{U},\mathcal{F}))_p.
\end{tikzcd}$$

Since ulim_p(-) is an exact functor, we have the induced morphism

$$\check{H}^j(\mathfrak{U},\mathcal{F}) \to \operatorname*{ulim}_p \check{H}^j(\mathfrak{U}_p,\mathcal{F}_p).$$

If X is separated, then X_p is separated for almost all p. This can be checked by taking a finite affine open covering and observing that if the diagonal morphism $\Delta_{X/\operatorname{Spec} R}$ is a closed immersion, then $\Delta_{X_p/\operatorname{Spec} R_p}$ is also a closed immersion for almost all p. Hence, we have the map

$$H^j(\mathfrak{U},\mathcal{F}) \to \operatorname*{ulim}_p H^j(\mathfrak{U}_p,\mathcal{F}_p).$$

Note that this map may not be injective.

Similarly, we discuss ultraproduts of local cohomologies following Schoutens [47, Section 5]. Let R be a local ring essentially of finite type over $\mathbb{C}$ of dimension d and $x_1,\ldots,x_d$ be a system of parameters for R. Suppose that M_p is an R_p-module for almost all p and $M_\infty = \operatorname{ulim}_p M_p$. For $n \in \mathbb{N}$ and $1 \leqslant i_1 < \cdots < i_n \leqslant d$, there exists a natural morphism

$$(M_\infty)_{x_{i_1}\cdots x_{i_d}} \to \operatorname*{ulim}_p (M_p)_{x_{i_1,p}\cdots x_{i_d,p}}.$$

Considering the Čech complexes associated to M_∞ and M_p for any p, we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{1\leqslant i_1<\cdots<i_n\leqslant d} (M_\infty)_{x_{i_1}\ldots x_{i_n}} \arrow[r] \arrow[d] & \bigoplus_{1\leqslant j_1<\cdots<j_{n+1}\leqslant d} (M_\infty)_{x_{j_1}\ldots x_{j_{n+1}}} \arrow[d] \\
\bigoplus_{1\leqslant i_1<\cdots<i_n\leqslant d} \operatorname*{ulim}_p (M_p)_{x_{i_1,p}\ldots x_{i_n,p}} \arrow[r] & \bigoplus_{1\leqslant j_1<\cdots<j_{n+1}\leqslant d} \operatorname*{ulim}_p (M_p)_{x_{j_1,p}\ldots x_{j_{n+1},p}}
\end{tikzcd}.$$

Hence, we have a natural morphism

$$\mathrm{H}^n_{\mathfrak{m}}(M_\infty) \to \operatorname*{ulim}_p \mathrm{H}^n_{\mathfrak{m}_p}(M_p)$$

for any $n \in \mathbb{N}$. For an element η of $\mathrm{H}^n_{\mathfrak{m}}(M_\infty)$, a family (η_p) of elements of $H^n_{\mathfrak{m}_p}(M_p)$ is said to be an *approximation* of η if $\operatorname{ulim}_p \eta_p$ is equal to the image of η under the above natural morphism. In later chapters, we will show the injectivity of this map in some situations, which plays an important role there.

1.5.4 Singularities introduced by Schoutens

In this subsection, we provide a quick review on the definition of classes of singularities introduced by Schoutens. Here we suppose that R is a $\mathbb{Q}$-Gorenstein normal local domain essentially of finite type over $\mathbb{C}$ and fix a non-principal ultrafilter on the set $\mathcal{P}$ of prime numbers and an isomorphism $\operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$.

Definition 1.5.64 ([44, Definition 5.2],[47, Definition 3.1])**.** Let $I \subseteq R$ be an ideal. The *generic tight closure* $I^{*\,\mathrm{gen}}$ of I is defined by

$$I^{*\,\mathrm{gen}} = (\operatorname*{ulim}_p I_p)^* \cap R.$$

Remark 1.5.65. The generic tight closure $I^{*\,\mathrm{gen}}$ of I does not depend on the choice of approximation of I since any two approximations are almost equal.

Definition 1.5.66 ([47, Definition 4.1, Remark 4.7],[45, Definition 4.3])**.** Let R be as in the above.

(1) R is said to be *weakly generically F-regular* if $I^{*\,\mathrm{gen}} = I$ for any ideal $I \subseteq R$.

(2) R is said to be *generically F-regular* if $R_\mathfrak{p}$ is weakly generically F-regular for any prime ideal $\mathfrak{p} \in \operatorname{Spec} R$.

Definition 1.5.67 ([46, Definition 3.3])**.** Let R be as above. R is said to be *ultra-F-regular* if, for each $c \in R^\circ$, there exists $\varepsilon \in {}^*\mathbb{N}$ such that

$$R \xrightarrow{cF^\varepsilon} R_\infty$$

is pure.

Proposition 1.5.68 ([46, Theorem A])**.** *R is ultra-F-regular if and only if R has log terminal singularities.*

Definition 1.5.69 ([45, Definition 4.3])**.** Let R be as above.

(1) R is said to be *weakly $\mathcal{B}$-regular* if $R \to \mathcal{B}(R)$ is cyclically pure, i.e., for any ideal I of R, we have $I\mathcal{B}(R) \cap R = I$.

(2) R is said to be *$\mathcal{B}$-regular* if every localization of R at a prime ideal is weakly $\mathcal{B}$-regular.

1.5.5 Big Cohen-Macaulay algebras in equal characteristic zero

Here we provide a brief overview of the canonical big Cohen-Macaulay algebra in equal characteristic zero, constructed by Schoutens [45]. Suppose that $(R, \mathfrak{m})$ is a local domain essentially of finite type over $\mathbb{C}$ and R_p is an approximation of R.

Definition 1.5.70 ([45, Section 2])**.** Suppose that R is a local domain essentially of finite type over $\mathbb{C}$. Then we define the *canonical big Cohen-Macaulay algebra* $\mathcal{B}(R)$ *of* R by

$$\mathcal{B}(R) := \operatorname*{ulim}_p R_p^+.$$

Remark 1.5.71. (1) $\mathcal{B}(R)$ is an R^+-algebra by [45, Proposition 3.2].

(2) R is $\mathrm{BCM}_{\mathcal{B}(R)}$-rational if and only if R has rational singularities. This follows from [45, Theorem 4.2].

The following is a useful lemma to compare local cohomologies of $\mathcal{B}(R)$ and R_p^+.

Lemma 1.5.72. *The natural homomorphism* $H^d_{\mathfrak{m}}(\mathcal{B}(R)) \to \operatorname{ulim}_p H^d_{\mathfrak{m}_p}(R_p^+)$ *is injective.*

Proof. Let $x = x_1 \cdots x_d$ be the product of a system of parameters and $[\frac{z}{x^t}]$ be an element of $H^d_{\mathfrak{m}}(\mathcal{B}(R))$ such that the image in $\operatorname{ulim}_p H^d_{\mathfrak{m}_p}(R_p^+)$ is zero. Then there exists $s_p \in \mathbb{N}$ such that $x_p^{s_p} z \in (x_{1p}^{s_p+t}, \ldots, x_{dp}^{s_p+t})R_p^+$ for almost all p. Since R_p^+ is a big Cohen-Macaulay R_p-algebra for almost all p, $z \in (x_{1p}^t, \ldots, x_{dp}^t)R_p^+$ for almost all p. Hence, $z \in (x_1^t, \ldots, x_d^t)\mathcal{B}(R)$ and $[\frac{z}{x^t}] = 0$ in $H^d_{\mathfrak{m}}(\mathcal{B}(R))$. $\square$

Chapter 2

On the behavior of adjoint ideals under pure morphisms

2.1 Test submodules along divisors

In this section, we develop the theory of test submodules along divisors. Throughout this section, we work with the following setting.

Setting 2.1.1. Let R be an F-finite normal domain of characteristic $p > 0$, D be a reduced divisor on $X := \operatorname{Spec} R$ and Γ be an effective $\mathbb{Q}$-Weil divisor on X that has no common components with D. We assume that $F^! \omega_X^\bullet \cong \omega_X^\bullet$, where $F : X \to X$ is the Frobenius morphism and $\omega_X^\bullet$ is a normalized dualizing complex for X. This condition is satisfied, for example, when R is essentially of finite type over an F-finite local ring (see [4, Example 2.15]).

Definition 2.1.2. Let the notation be as in Setting 2.1.1. The *parameter test submodule* $\tau_D(\omega_R, \Gamma)$ of the pair (R, Γ) along D is defined as the unique smallest submodule M of $\omega_R(D)$ satisfying the following conditions:

(a) M coincides with $\omega_R(D)$ at every generic point of D.

(b) For every integer $e \geqslant 0$ and every $\varphi \in \operatorname{Hom}_R(F^e_* \omega_R(p^e D + \lceil (p^e - 1)\Gamma \rceil), \omega_R(D)) \subseteq \operatorname{Hom}_R(F^e_* \omega_R(D), \omega_R(D))$, one has $\varphi(F^e_* M) \subseteq M$.

Remark 2.1.3. Given an ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^{\circ, D} \neq \emptyset$ and a real number $t \geqslant 0$, we can define the parameter test submodule $\tau_D(\omega, \mathfrak{a}^t)$ of the pair $(R, \mathfrak{a}^t)$ along D similarly. This is the unique smallest submodule M of $\omega_R(D)$ satisfying the following conditions:

(a) M coincides with $\omega_R(D)$ at every generic point of D.

(b') For every integer $e \geqslant 0$ and every $\varphi \in \operatorname{Hom}_R(F^e_* \omega_R(p^e D), \omega_R(D))$, which is viewed as an element of $\operatorname{Hom}_R(F^e_* \omega_R(D), \omega_R(D))$, one has $\varphi(F^e_* \mathfrak{a}^{\lceil t(p^e - 1) \rceil} M) \subseteq M$.

Lemma 2.1.4. *With notation as in Setting 2.1.1, choose a canonical divisor K_X of $X := \operatorname{Spec} R$ such that $-(K_X + D)$ is an effective Weil divisor G with no common components with D, and fix $\omega_R = R(K_X)$ to be the corresponding fractional ideal of R. Then*
$$\tau_D(\omega_R, \Gamma) = \tau_D(R, D + \Gamma + G)$$
as fractional ideals of R. In particular, $\tau_D(\omega_R, \Gamma)$ exists.

Proof. By the definition of test ideals along D, we see that $\tau_D(R, D + \Gamma + G)$ is a submodule of $\omega_R(D) = R(-G) \subseteq R$. Since
$$\operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)(D + \Gamma + G)\rceil), R) \cong \operatorname{Hom}_R(F^e_* \omega_R(p^e D + \lceil (p^e - 1)\Gamma\rceil), \omega_R(D)),$$
$\tau_D(R, D+\Gamma+G)$ is the smallest submodule M of $\omega_R(D)$ with $M \cap R^{\circ,D} \neq \emptyset$ satisfying the condition (b) in Definition 2.1.2. On the other hand, a submodule N of $\omega_R(D)$ satisfies that $N \cap R^{\circ,D} \neq \emptyset$ if and only if there exists an element $c \in R^{\circ,D}$ such that $c\omega_R(D) \subseteq N$, which is equivalent to the condition (a) in Definition 2.1.2. Therefore, $\tau_D(R, D + \Gamma + G)$ coincides with $\tau_D(\omega_R, \Gamma)$. □

Remark 2.1.5. Thanks to Lemma 2.1.4, several basic properties of $\tau_D(\omega_R, \Gamma)$ can be deduced from the corresponding properties of $\tau_D(R, \Delta)$. For example,

(1) the formation of $\tau_D(\omega_R, \Gamma)$ commutes with localization,

(2) if $(R, \mathfrak{m})$ is local, then the formation of $\tau_D(\omega_R, \Gamma)$ commutes with $\mathfrak{m}$-adic completion, and

(3) if B is an effective Cartier divisor on $\operatorname{Spec} R$ such that B has no common component with D, then $\tau_D(\omega_R, \Gamma + B) = \tau_D(\omega_R, \Gamma) \otimes_R R(-B)$.

For each integer $e \geqslant 1$, let
$$R \to F^e_* R \hookrightarrow F^e_* R(\lceil (p^e - 1)(D + \Gamma)\rceil) \tag{$\star$}$$
be the composite of the e-times iterated Frobenius map $R \to F^e_* R$ and the pushforward of the natural inclusion $R \hookrightarrow R(\lceil (p^e - 1)(D + \Gamma)\rceil)$ by F^e.

Definition 2.1.6. With notation as in Setting 2.1.1, suppose that $(R, \mathfrak{m})$ is local of dimensional d. Tensoring ($\star$) with $I_D := R(-D)$ and taking local cohomology, one has a map
$$F^e_{D,\Gamma} : H^d_{\mathfrak{m}}(I_D) \to H^d_{\mathfrak{m}}(I_D(\lceil (p^e - 1)\Gamma\rceil)),$$
where $I_D(\lceil (p^e - 1)\Gamma\rceil) := R(-D + \lceil (p^e - 1)\Gamma\rceil)$. The submodule $0^{*_D\Gamma}_{H^d_{\mathfrak{m}}(I_D)}$ of $H^d_{\mathfrak{m}}(I_D)$ consists of all elements $z \in H^d_{\mathfrak{m}}(I_D)$ for which there exists an element $c \in R^{\circ,D}$ such that
$$cF^e_{D,\Gamma}(z) = 0 \in H^d_{\mathfrak{m}}(I_D(\lceil (p^e - 1)\Gamma\rceil))$$
for all large e.

Lemma 2.1.7. *With notation as in Setting 2.1.6, let $E_R(R/\mathfrak{m})$ be an injective hull of the residue field $R/\mathfrak{m}$. Then*

$$\tau_D(\omega_R, \Gamma) = \operatorname{Ann}_{\omega_R(D)} 0^{*_D\Gamma}_{H^d_{\mathfrak{m}}(I_D)},$$

the annihilator of $0^{*_D\Gamma}_{H^d_{\mathfrak{m}}(I_D)}$ in $\omega_R(D)$ with respect to the duality pairing

$$\omega_R(D) \times H^d_{\mathfrak{m}}(I_D) \to E_R(R/\mathfrak{m}).$$

Proof. This follows from an argument analogous to [51, Theorem 6.3]. □

In Proposition 2.1.8 and Lemma 2.1.9, we assume that D is a prime divisor for simplicity. We then fix a choice of I_D^+ and use the following notation. Given a module-finite extension S of R contained in R^+ with S normal, we define the submodule $\mathcal{I}_S(\omega_R(D), \Gamma)$ of $\omega_R(D)$ as

$$\mathcal{I}_S(\omega_R(D), \Gamma) = \operatorname{Im}(\operatorname{Tr}_\pi : \pi_*\omega_S(D_S - \lfloor \pi^*\Gamma \rfloor) \to \omega_R(D)),$$

where $\pi : \operatorname{Spec} S \to \operatorname{Spec} R$ is the finite surjective morphism induced by the inclusion $R \hookrightarrow S$ and D_S is the prime divisor on $\operatorname{Spec} S$ such that $I_D^+ \cap S = S(-D_S)$. When $\Gamma = 0$, this submodule is simply denoted by $\mathcal{I}_S(\omega_R(D))$.

Proposition 2.1.8. *With notation as above and as in Setting 2.1.1, suppose that D is a prime divisor and Γ is $\mathbb{Q}$-Cartier.*

(1) For every module-finite extension S of R contained in R^+ with S normal, one has

$$\tau_D(\omega_R, \Gamma) \subseteq \mathcal{I}_S(\omega_R(D), \Gamma).$$

(2) There exists a module-finite extension S of R contained in R^+ such that S is normal, $\pi^*\Gamma$ is Cartier, and the equality holds in (1), that is,

$$\tau_D(\omega_R, \Gamma) = \mathcal{I}_S(\omega_R(D), \Gamma).$$

Proof. (1) First note that the formation of $\mathcal{I}_S(\omega_R(D), \Gamma)$ commutes with localization. Therefore, by Remark 2.1.5 (1), we may assume that $(R, \mathfrak{m})$ is local. By the minimality of $\tau_D(\omega_R, \Gamma)$, it suffices to show that the submodule $\mathcal{I}_S(\omega_R(D), \Gamma)$ of $\omega_R(D)$ satisfies conditions (a) and (b) in Definition 2.1.2.

To verify the condition (a), by localizing at the generic point of D, we may assume that R is an F-finite DVR, S is a Dedekind domain and $\Gamma = 0$. Let $\widehat{R}$ denote the completion of R, $\widehat{D}$ denote the flat pullback of D via the canonical morphism $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$, and set $\widehat{S} := S \otimes_R \widehat{R}$. The $\widehat{R}$-algebra $\widehat{S}$ is isomorphic to a finite product $S_1 \times \cdots \times S_r$ of complete DVRs $(S_i, \mathfrak{n}_i)$, and $S(-D_S)\widehat{S}$ is a maximal ideal of $\widehat{S}$. After reindexing, we may assume that $S(-D_S)\widehat{S} \cong \mathfrak{n}_1 \times S_2 \times \cdots \times S_r$. Then one has

$$\mathcal{I}_S(\omega_R(D)) \otimes_R \widehat{R} = \operatorname{Im}\left(\pi_{1*}\omega_{S_1}(D_{S_1}) \to \omega_{\widehat{R}}(\widehat{D})\right) + \sum_{i=2}^{r} \operatorname{Im}\left(\pi_{i*}\omega_{S_i} \to \omega_{\widehat{R}}(\widehat{D})\right),$$

where $\pi_i : \operatorname{Spec} S_i \to \operatorname{Spec} \widehat{R}$ is the finite surjective morphism induced by $\widehat{R} \to \widehat{S} \to S_i$ and D_{S_1} is the prime divisor on $\operatorname{Spec} S_1$ corresponding to $\mathfrak{n}_1$. To verify that $\mathcal{I}_S(\omega_R(D)) = \omega_R(D)$, it suffices to show that $\operatorname{Im}(\pi_{1*}\omega_{S_1}(D_{S_1}) \to \omega_{\widehat{R}}(\widehat{D})) = \omega_{\widehat{R}}(\widehat{D})$. Therefore, we can restrict our attention to the case where R and S are both complete DVRs. It follows from Proposition 1.4.11 and Lemma 2.1.4 that $\tau_D(\omega_R) \subseteq \mathcal{I}_S(\omega_R(D))$. Conversely, since R is an F-finite DVR and D is the divisor corresponding to the maximal ideal $\mathfrak{m}$, it is straightforward to check that $\tau_D(\omega_R) = \omega_R(D)$. Consequently, we conclude that $\mathcal{I}_S(\omega_R(D)) = \omega_R(D)$.

It remains to verify that $\mathcal{I}_S(\omega_R(D), \Gamma)$ satisfies the condition (b) in Definition 2.1.2. For any nonzero element $F^e_* c \in F^e_* R$, we have the following commutative diagram:

$$\begin{tikzcd}
F^e_*\pi_*\omega_S(D_S - \lfloor \pi^*\Gamma \rfloor) \arrow[r, hook] \arrow[d, "F^e_*\mathrm{Tr}_\pi"'] & F^e_*\pi_*\omega_S(p^e D_S - \lfloor \pi^*\Gamma \rfloor) \arrow[r, "\pi_*\mathrm{Tr}_{F^e}(F^e_*c\cdot_)"] \arrow[d, "F^e_*\mathrm{Tr}_\pi"'] & \pi_*\omega_S(D_S - \lfloor \pi^*\Gamma \rfloor) \arrow[d, "\mathrm{Tr}_\pi"'] \\
F^e_*\omega_R(D) \arrow[r, hook] & F^e_*\omega_R(p^e D) \arrow[r, "\mathrm{Tr}_{F^e}(F^e_*c\cdot_)"] & \omega_R(D).
\end{tikzcd}$$

Since

$$\operatorname{Hom}_R(F^e_*\omega_R(p^e D + \lceil (p^e-1)\Gamma \rceil), \omega_R(D)) \subseteq \operatorname{Hom}_R(F^e_*\omega_R(p^e D), \omega_R(D))$$

and $\operatorname{Hom}_R(F^e_*\omega_R(p^e D), \omega_R(D))$ is generated by $\mathrm{Tr}_{F^e} : F^e_*\omega_R(D) \to \omega_R(D)$ as an F^e_*R-module, the commutativity of the above diagram ensures that $\mathcal{I}_S(\omega_R(D), \Gamma)$ satisfies the condition (b).

(2) By [4, Lemma 4.15], there exists a finite separable extension R' of R contained in R^+ such that R' is normal and $\nu^*\Gamma$ is Cartier, where $\nu : \operatorname{Spec} R' \to \operatorname{Spec} R$ is the finite surjective morphism induced by the inclusion $R \hookrightarrow R'$. Since Γ has no component equal to D, the morphism ν is étale over the generic point of D by its construction (see the first paragraph of the proof of [39, Theorem 6.6]). Let D' be the prime divisor on $\operatorname{Spec} R'$ such that $I^+_D \cap R' = R(-D')$. It then follows from [39, Proposition 6.5] and Lemma 2.1.4 that

$$\tau_D(\omega_R, \Gamma) = \mathrm{Tr}_\nu(\nu_*\tau_{D'}(\omega_{R'}, \nu^*\Gamma)).$$

On the other hand, for every finite surjective morphism $\rho : \operatorname{Spec} S \to \operatorname{Spec} R'$ with S normal, one has

$$\mathrm{Tr}_{\nu\circ\rho}((\nu\circ\rho)_*\omega_S(D_S - \lfloor (\nu\circ\rho)^*\Gamma \rfloor)) = \mathrm{Tr}_\nu(\nu_*\mathrm{Tr}_\rho(\rho_*\omega_S(D_S - \lfloor \rho^*\nu^*\Gamma \rfloor))).$$

Therefore, replacing R with R' and Γ with Γ', we may assume that Γ is a Cartier divisor. Furthermore, by Remark 2.1.5 (3) and the projection formula, we can reduce the problem to the case where $\Gamma = 0$.

Finally, we will prove that there exists a module-finite extension S of R contained in R^+ such that S is normal and $\tau_D(\omega_R) = \mathcal{I}_S(\omega_R(D))$. It follows from repeated applications of Lemma 2.1.9. $\square$

Lemma 2.1.9. *With notation as in Proposition 2.1.8, let S be a module-finite extension of R contained in R^+ with S normal. Note that $\tau_D(\omega_R) \subseteq \mathcal{I}_S(\omega_R(D))$ by Proposition 2.1.8 (1). If $\tau_D(\omega_R) \neq \mathcal{I}_S(\omega_R(D))$, then there exists a module-finite extension T of S contained in R^+ such that T is normal and*

$$\operatorname{Supp} \mathcal{I}_T(\omega_R(D))/\tau_D(\omega_R) \subsetneq \operatorname{Supp} \mathcal{I}_S(\omega_R(D))/\tau_D(\omega_R).$$

Proof. Let η be a minimal prime of $\operatorname{Supp} \mathcal{I}_S(\omega_R(D))/\tau_D(\omega_R)$, and R_η, S_η and D_η denote the localization of R, S and D at η, respectively, and let $d = \dim R_\eta$. Note that taking absolute integral closure commutes with localization, that is, $(R^+)_\eta \cong (R_\eta)^+$ and $(I_D^+)_\eta \cong I_{D_\eta}^+$. Since the formation of $\mathcal{I}_T(\omega_R(D))$ and $\tau_D(\omega_R)$ commutes with localization (see Remark 2.1.5 (1)), we have the following sequence:

$$\omega_{S_\eta}(D_{S_\eta}) \twoheadrightarrow \mathcal{I}_{S_\eta}(\omega_{R_\eta}(D_\eta))/\tau_{D_\eta}(\omega_{R_\eta}) \hookrightarrow \omega_{R_\eta}(D_\eta)/\tau_{D_\eta}(\omega_{R_\eta}).$$

By Lemma 2.1.7, applying the Matlis dual functor $(-)^\vee := \operatorname{Hom}_{R_\eta}(-, E_{R_\eta}(R_\eta/\eta R_\eta))$ yields the sequence

$$H^d_{\eta R_\eta}(S_\eta(-D_{S_\eta})) \hookleftarrow (\mathcal{I}_{S_\eta}(\omega_{R_\eta}(D_\eta))/\tau_{D_\eta}(\omega_{R_\eta}))^\vee \twoheadleftarrow 0^{*_{D_\eta} D_\eta}_{H^d_{\eta R_\eta}(I_{D_\eta})}.$$

Here, to obtain the isomorphism $(\omega_{R_\eta}(D_\eta)/\tau_{D_\eta}(\omega_{R_\eta}))^\vee \cong 0^{*_{D_\eta} D_\eta}_{H^d_{\eta R_\eta}(I_{D_\eta})}$, we utilized the fact that the formation of $\tau_{D_\eta}(\omega_{R_\eta})$ commutes with completion (see Remark 2.1.5 (2)). We will show below that there exists a module-finite extension T of S contained in R^+ such that T is normal and the image of $(\mathcal{I}_{S_\eta}(\omega_{R_\eta}(D_\eta))/\tau_{D_\eta}(\omega_{R_\eta}))^\vee$ vanishes in $H^d_{\eta R_\eta}(T_\eta(-D_{T_\eta}))$. By the commutativity of the diagram

$$\begin{tikzcd}
H^d_{\eta R_\eta}(S_\eta(-D_{S_\eta})) \arrow[d] & (\mathcal{I}_{S_\eta}(\omega_{R_\eta}(D_\eta))/\tau_{D_\eta}(\omega_{R_\eta}))^\vee \arrow[l, hook'] \arrow[d, two heads] \\
H^d_{\eta R_\eta}(T_\eta(-D_{T_\eta})) & (\mathcal{I}_{T_\eta}(\omega_{R_\eta}(D_\eta))/\tau_{D_\eta}(\omega_{R_\eta}))^\vee, \arrow[l, hook']
\end{tikzcd}$$

this vanishing ensures that $(\mathcal{I}_{T_\eta}(\omega_{R_\eta}(D_\eta))/\tau_{D_\eta}(\omega_{R_\eta}))^\vee = 0$. Consequently, η does not lie in the support of $\mathcal{I}_T(\omega_R(D))/\tau_D(\omega_R)$, which implies the assertion of Lemma 2.1.9.

Set $N_S := (\mathcal{I}_{S_\eta}(\omega_{R_\eta}(D_\eta))/\tau_{D_\eta}(\omega_{R_\eta}))^\vee$. The R_η-module N_S has finite length by the choice of η, and we have the following commutative diagram:

$$\begin{tikzcd}
0^{*_{D_\eta} D_\eta}_{H^d_{\eta R_\eta}(I_{D_\eta})} \arrow[r, hook] \arrow[d, two heads] & H^d_{\eta R_\eta}(I_{D_\eta}) \arrow[r, two heads] \arrow[d] & H^d_{\eta R_\eta}(R_\eta) \arrow[d] \\
N_S \arrow[r, hook] & H^d_{\eta R_\eta}(S_\eta(-D_{S_\eta})) \arrow[r, two heads] & H^d_{\eta R_\eta}(S_\eta).
\end{tikzcd}$$

Since the image of $0^{*_{D_\eta} D_\eta}_{H^d_{\eta R_\eta}(I_{D_\eta})}$ in $H^d_{\eta R_\eta}(R_\eta)$ is stable under Frobenius action, the image of N_S in $H^d_{\eta R_\eta}(S_\eta)$ is also stable under Frobenius action. It then follows from the equational lemma [30, Lemma 2.2] (see also [4]) that the image of N_S vanishes in $H^d_{\eta R_\eta}(R_\eta^+)$.

Noting that R_η^+ is a big Cohen-Macaulay R_η-algebra, and therefore $H^{d-1}_{\eta R_\eta}(R_\eta^+) = 0$, we can consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & H^{d-1}_{\eta R_\eta}(S_\eta/S_\eta(-D_{S_\eta})) \arrow[r] \arrow[d] & H^d_{\eta R_\eta}(S_\eta(-D_{S_\eta})) \arrow[r] \arrow[d, "\alpha"] & H^d_{\eta R_\eta}(S_\eta) \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & H^{d-1}_{\eta R_\eta}((R_\eta/I_{D_\eta})^+) \arrow[r, "f"] & H^d_{\eta R_\eta}(I^+_{D_\eta}) \arrow[r, "g"] & H^d_{\eta R_\eta}(R^+_\eta) \arrow[r] & 0.
\end{tikzcd}$$

Simple diagram chasing shows the existence of a finitely generated R_η-submodule M_+ of $H^{d-1}_{\eta R_\eta}((R_\eta/I_{D_\eta})^+)$ such that $f(M_+) = \alpha(N_S)$. Since $0^{*_{D_\eta} D_\eta}_{H^d_{\eta R_\eta}(I_{D_\eta})}$ is stable under the map $F_{D_\eta} : H^d_{\eta R_\eta}(I_{D_\eta}) \to H^d_{\eta R_\eta}(I_{D_\eta})$, its image $\alpha(N_S)$ is also stable under the induced map $H^d_{\eta R_\eta}(I^+_{D_\eta}) \to H^d_{\eta R_\eta}(I^+_{D_\eta})$. The injectivity of f consequently ensures that M_+ is stable under Frobenius action. Applying the equational lemma again, we deduce that $\alpha(N_S) \cong M_+ = 0$. Thus, by the finite generation of N_S, there exists a module-finite extension T of S contained in R^+ such that T is normal and the image of N_S vanishes in $H^d_{\eta R_\eta}(T_\eta(-D_{T_\eta}))$. $\square$

Corollary 2.1.10. *With notation as in Setting 2.1.1, suppose that D is a prime divisor and $K_X + D + \Gamma$ is $\mathbb{Q}$-Cartier. Fix a choice of I_D^+.*

(1) For every module-finite extension S of R contained in R^+ with S normal, one has

$$\tau_D(R, D+\Gamma) \subseteq \mathrm{Im}(\mathrm{Tr}_\pi : \pi_*\mathcal{O}_Y(K_Y - \lfloor \pi^*(K_X + D + \Gamma) \rfloor + D_S) \to K(X)),$$

where $Y := \operatorname{Spec} S \xrightarrow{\pi} X$ is the finite morphism induced by the inclusion $R \hookrightarrow S$ and D_S is the prime divisor such that $S(-D_S) = I_D^+ \cap S$.

(2) There exists a module-finite extension S of R contained in R^+ such that S is a normal domain and the equality holds in the inclusion in (1).

Proof. The assertion follows directly from combining Lemma 2.1.4 with Proposition 2.1.8. $\square$

2.2 A generalization of plus closure

In this section, we introduce a generalization of plus closure to give another description of test ideals along divisors. For the theory of classical plus closure, the reader is referred to [24] and [53].

We work with the following setting.

Setting 2.2.1. Let R be a d-dimensional F-finite normal local domain, Δ be an effective $\mathbb{Q}$-Weil divisor and D be a prime divisor on $X := \operatorname{Spec} R$ such that no component of Δ is equal to D. We fix a choice of I_D^+, and let Λ denote the set of module-finite extensions R_λ of R, contained in R^+, such that each R_λ is a normal domain. When R_λ belongs to Λ, we write the morphism corresponding to the inclusion $R \hookrightarrow R_\lambda$ by $\pi_\lambda : X_\lambda := \operatorname{Spec} R_\lambda \to X$.

Definition 2.2.2. With notation as in Setting 2.2.1, for each $R_\lambda \in \Lambda$, let D_λ denotes the prime divisor on $\operatorname{Spec} R_\lambda$ such that $R_\lambda(-D_\lambda) = I_D^+ \cap R_\lambda$.

(1) The R^+-module $I_D^+(D+\Delta)$ is defined as

$$I_D^+(D+\Delta) = \varinjlim_{R_\lambda} R_\lambda(\lfloor \pi_\lambda^*(D+\Delta) - D_\lambda \rfloor).$$

(2) Given an ideal J of R, the $(D+\Delta)$-*plus closure* $J^{+_D(D+\Delta)}$ of J along D is defined to be the ideal $J(I_D^+(D+\Delta)) \cap R$.

(3) Given an R-module M, the $(D+\Delta)$-*plus closure* $0_M^{+_D(D+\Delta)}$ of the zero submodule along D is defined to be the kernel of the natural map $M \to M \otimes_R I_D^+(D+\Delta)$.

Remark 2.2.3. An element $x \in R$ belongs to $J^{+_D(D+\Delta)}$ if and only if $\overline{x}$ does to $0_{R/J}^{+_D(D+\Delta)}$, where $\overline{x}$ is the image of x under the canonical surjection $R \to R/J$.

Proposition 2.2.4. *With notation as in Setting 2.2.1, suppose that $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, that is, $r(K_X + D + \Delta) = \operatorname{div} f$ for some integer $r \geqslant 1$ and some nonzero element $f \in R$. Then*

$$0_{H_\mathfrak{m}^d(\omega_R)}^{*_D(D+\Delta)} = \ker\left(H_\mathfrak{m}^d(\omega_R) \xrightarrow{\cdot f^{\frac{1}{r}}} H_\mathfrak{m}^d(I_D^+)\right),$$

where $H_\mathfrak{m}^d(\omega_R) \xrightarrow{\cdot f^{\frac{1}{r}}} H_\mathfrak{m}^d(I_D^+)$ is a map induced by the multiplication by $f^{\frac{1}{r}}$.

Proof. By Corollary 2.1.10, there exists a module-finite extension $R_\lambda \in \Lambda$ such that $f^{1/r} \in R_\lambda$ and

$$\tau_D(R, D+\Delta) = \operatorname{Im}(\operatorname{Tr}_{\pi_\mu} : \pi_{\mu *}\mathcal{O}_{X_\mu}(K_{X_\mu} - \pi_\mu^*(K_X + D + \Delta) + D_\mu) \to \mathcal{O}_X)$$

holds for all $R_\mu \in \Lambda$ containing R_λ. Taking its Matlis dual, one has

$$0_{H_\mathfrak{m}^d(\omega_R)}^{*_D(D+\Delta)} = \ker\left(H_\mathfrak{m}^d(\omega_R) \xrightarrow{\cdot f^{\frac{1}{r}}} H_\mathfrak{m}^d(I_{D_\lambda})\right).$$

Then taking its direct limit yields the desired equality

$$0_{H_\mathfrak{m}^d(\omega_R)}^{*_D(D+\Delta)} = \ker\left(H_\mathfrak{m}^d(\omega_R) \xrightarrow{\cdot f^{\frac{1}{r}}} H_\mathfrak{m}^d(I_D^+)\right).$$

$\square$

Lemma 2.2.5. *In the setting of Proposition 2.2.4,*

$$H^d_{\mathfrak{m}}(\omega_R \otimes_R I_D^+(D+\Delta)) \xrightarrow{\cdot f^{\frac{1}{r}}} H^d_{\mathfrak{m}}(I_D^+)$$

is an isomorphism.

Proof. We consider only $R_\lambda \in \Lambda$ such that $f^{\frac{1}{r}} \in R_\lambda$, that is, $\pi_\lambda^*(K_X + D + \Delta)$ is a Cartier divisor on $\operatorname{Spec} R_\lambda$. A natural injection

$$\omega_R \otimes_R R_\lambda(\pi_\lambda^*(D+\Delta) - D_\lambda) \hookrightarrow R_\lambda(\pi_\lambda^*(K_X + D + \Delta) - D_\lambda).$$

is an isomorphism on the regular locus of X, that is, an isomorphism in codimension one, which yields an isomorphism

$$H^d_{\mathfrak{m}}(\omega_R \otimes_R R_\lambda(\pi_\lambda^*(D+\Delta) - D_\lambda)) \cong H^d_{\mathfrak{m}}(R_\lambda(\pi_\lambda^*(K_X + D + \Delta) - D_\lambda)).$$

Therefore, we have

$$\begin{aligned} H^d_{\mathfrak{m}}(\omega_R \otimes_R I_D^+(D+\Delta)) &\cong \varinjlim_{R_\lambda} H^d_{\mathfrak{m}}(\omega_R \otimes_R R_\lambda(\pi_\lambda^*(D+\Delta) - D_\lambda)) \\ &\cong \varinjlim_{R_\lambda} H^d_{\mathfrak{m}}(R_\lambda(\pi_\lambda^*(K_X + D + \Delta) - D_\lambda)) \\ &\cong H^d_{\mathfrak{m}}(\varinjlim_{R_\lambda}(R_\lambda(\pi_\lambda^*(K_X + D + \Delta) - D_\lambda))) \\ &\cong H^d_{\mathfrak{m}}(I_D^+ \otimes_{R_\lambda} R_\lambda(\operatorname{div} f^{\frac{1}{r}})) \\ &\cong H^d_{\mathfrak{m}}(I_D^+), \end{aligned}$$

where the last isomorphism is induced by the multiplication by $f^{\frac{1}{r}}$. □

Proposition 2.2.6. *With notation as in Proposition 2.2.4, we have*

$$\tau_D(R, D+\Delta) = \bigcap_J (J : J^{+_D(D+\Delta)}),$$

where J runs through all ideals of R.

Proof. By Proposition 2.2.4 and Lemma 2.2.5,

$$\begin{aligned} 0^{*_D(D+\Delta)}_{H^d_{\mathfrak{m}}(\omega_R)} &= \ker(H^d_{\mathfrak{m}}(\omega_R) \to H^d_{\mathfrak{m}}(\omega_R \otimes_R I_D^+(D+\Delta))) \\ &= \ker(H^d_{\mathfrak{m}}(\omega_R) \to H^d_{\mathfrak{m}}(\omega_R) \otimes_R I_D^+(D+\Delta)) \\ &= 0^{+_D(D+\Delta)}_{H^d_{\mathfrak{m}}(\omega_R)}. \end{aligned}$$

Since R is approximately Gorenstein, the assertion follows from an argument similar to [12, Proposition 3.3.1 (4)] and [23, Proposition 8.23]. □

2.3 A characterization of adjoint ideals via ultraproducts

In this section, we give a characterization of the adjoint ideal $\mathrm{adj}_D(X, D+\Delta)$ via ultraproducts when $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier. We work with the following setting.

Setting 2.3.1. Let $(R, \mathfrak{m})$ be a d-dimensional normal local domain essentially of finite type over $\mathbb{C}$, Δ be an effective $\mathbb{Q}$-Weil divisor and D be a prime divisor on $X := \operatorname{Spec} R$ such that no component of Δ is equal to D. Let $(R_p)_{p\in\mathcal{P}}$, $(D_p)_{p\in\mathcal{P}}$ and $(\Delta_p)_{p\in\mathcal{P}}$ be approximations of R, D and Δ, respectively. Fix choices of $I^+_{D_p}$, which is equivalent to fixing local ring homomorphisms $R_p^+ \to (R_p/I_{D_p})^+$, for almost all p.

First we generalize Schoutens' "canonical" big Cohen-Macaulay algebras $\mathcal{B}(R)$ to the pair setting.

Definition 2.3.2. With notation as in Setting 2.3.1, the R^+-algebra $\mathcal{B}(R)$ is defined as

$$\mathcal{B}(R) = \operatorname*{ulim}_p R_p^+.$$

The $\mathcal{B}(R)$-modules $\mathcal{B}(I_D)$ and $\mathcal{B}(I_D, D+\Delta)$ are defined as

$$\mathcal{B}(I_D) = \operatorname*{ulim}_p(I^+_{D_p}), \quad \mathcal{B}(I_D, D+\Delta) = \operatorname*{ulim}_p(I^+_{D_p}(D_p+\Delta_p)),$$

respectively.

Remark 2.3.3. Definition 2.3.2 is an abuse of notation since $\mathcal{B}(I_D)$ and $\mathcal{B}(I_D, D+\Delta)$ depend on the choices of $(I^+_{D_p})_{p\in\mathcal{P}}$ and are not uniquely determined by I_D and $D+\Delta$. If $\sigma : \mathcal{B}(R) \to \mathcal{B}(R/I_D)$ is the homomorphism induced by the fixed local ring homomorphisms $(R_p)^+ \to (R_p/I_{D_p})^+$, then

$$0 \to \mathcal{B}(I_D) \to \mathcal{B}(R) \xrightarrow{\sigma} \mathcal{B}(R/I_D) \to 0$$

is an exact sequence.

We define a closure operation in equal characteristic zero, using $\mathcal{B}(I_D, D+\Delta)$.

Definition 2.3.4. (1) Given an ideal $J \subseteq R$, the ideal $J^{\mathcal{B}_D(D+\Delta)} \subseteq R$ is defined to be $J\mathcal{B}(I_D, D+\Delta) \cap R$.

(2) Given an R-module M, the submodule $0_M^{\mathcal{B}_D(D+\Delta)}$ is defined to be the kernel of the natural map $M \to M \otimes_R \mathcal{B}(I_D, D+\Delta)$.

(3) The following ideals are equal to each other (cf. [23, Proposition 8.23] and [12, Proposition 3.3.1]), and are collectively denoted by $\tau_{\mathcal{B},D}(R, D+\Delta)$.

   (a) $\bigcap_M \operatorname{Ann}_R 0_M^{\mathcal{B}_D(D+\Delta)}$, where M runs through all R-modules.

(b) $\operatorname{Ann}_R 0_E^{\mathcal{B}_D(D+\Delta)}$, where $E = E_R(R/\mathfrak{m})$ is an injective hull of the residue field $R/\mathfrak{m}$.

(c) $\bigcap_J (J : J^{\mathcal{B}_D(D+\Delta)})$, where J runs through all ideals of R.

In order to prove the main theorem in this section, we need the following two lemmas.

Lemma 2.3.5. *If $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, then $(\tau_{D_p}(R_p, D_p + \Delta_p))_{p \in \mathcal{P}}$ is an approximation of the adjoint ideal* $\operatorname{adj}_D(X, D + \Delta)$.

Proof. When $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, modulo p reductions of the adjoint ideal $\operatorname{adj}_D(X, D + \Delta)$ coincide with the test ideals $\tau_{D_\mu}(R_\mu, D_\mu + \Delta_\mu)$, where $(R_\mu, D_\mu, \Delta_\mu)$ are modulo p reductions of (R, D, Δ), by essentially the same argument as the proof of [59, Theorem 5.3]. The assertion then follows from an argument similar to Proposition 1.5.48. ☐

Lemma 2.3.6. *With notation as in Setting 2.3.1, the natural map*

$$\beta_D : H^d_{\mathfrak{m}}(\mathcal{B}(I_D)) \to \operatorname*{ulim}_p H^d_{\mathfrak{m}_p}(I^+_{D_p})$$

is injective.

Proof. As mentioned in Remark 2.3.3, the exact sequences

$$0 \to I^+_{D_p} \to R^+_p \to (R_p/I_{D_p})^+ \to 0$$

for almost all p induce the exact sequence

$$0 \to \mathcal{B}(I_D) \to \mathcal{B}(R) \to \mathcal{B}(R/I_D) \to 0.$$

Note that R^+_p and $(R_p/I_{D_p})^+$ are big Cohen-Macaulay algebras for almost all p by [24] and that $\mathcal{B}(R)$ and $\mathcal{B}(R/I_D)$ are big Cohen-Macaulay R^+-algebras by [45]. Thus, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^{d-1}_{\mathfrak{m}}(\mathcal{B}(R/I_D)) \arrow[r] \arrow[d, "\alpha"] & H^d_{\mathfrak{m}}(\mathcal{B}(I_D)) \arrow[r] \arrow[d, "\beta_D"] & H^d_{\mathfrak{m}}(\mathcal{B}(R)) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \operatorname*{ulim}_p H^{d-1}_{\mathfrak{m}_p}((R_p/I_{D_p})^+) \arrow[r] & \operatorname*{ulim}_p H^d_{\mathfrak{m}_p}(I^+_{D_p}) \arrow[r] & \operatorname*{ulim}_p H^d_{\mathfrak{m}_p}(R^+_p) \arrow[r] & 0.
\end{tikzcd}$$

Since α and γ are injective by Lemma 1.5.72, so is the homomorphism β_D. ☐

The main result in this section is now stated as follows.

Theorem 2.3.7. *If $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, then*

$$0^{\mathcal{B}_D(D+\Delta)}_{H^d_{\mathfrak{m}}(\omega_R)} = \operatorname{Ann}_{H^d_{\mathfrak{m}}(\omega_R)} \operatorname{adj}_D(X, D + \Delta).$$

Taking the annihilator of both sides in R yields the equality

$$\tau_{\mathcal{B},D}(R, D + \Delta) = \operatorname{adj}_D(X, D + \Delta).$$

Proof. First we will prove that $0^{\mathcal{B}_D(D+\Delta)}_{H^d_{\mathfrak{m}}(\omega_R)} \subseteq \operatorname{Ann}_{H^d_{\mathfrak{m}}(\omega_R)} \operatorname{adj}_D(X, D+\Delta)$. It suffices to show that $\tau_{\mathcal{B},D}(R, D+\Delta) \supseteq \operatorname{adj}_D(X, D+\Delta)$, that is, $J : J^{\mathcal{B}_D(D+\Delta)} \supseteq \operatorname{adj}_D(R, D+\Delta)$ for every ideal $J \subseteq R$. Fix $x \in J^{\mathcal{B}_D(D+\Delta)}$ and $a \in \operatorname{adj}_D(R, D+\Delta)$, and let $(x_p)_{p\in\mathcal{P}}$ and $(a_p)_{p\in\mathcal{P}}$ be approximations of x and a, respectively. By the definition of $J^{\mathcal{B}_D(D+\Delta)}$, x_p is contained in $J_p I^+_{D_p}(D_p+\Delta_p)$, that is, $x_p \in J_p^{+_{D_p}(D_p+\Delta_p)}$ for almost all p. On the other hand, by Lemma 2.3.5, a_p is contained in $\tau_{D_p}(R_p, D_p+\Delta_p)$ for almost all p. It follows from Proposition 2.2.6 that $a_p x_p \in J_p$ for almost all p, which implies that $ax \in J$. Thus, we have the desired containment.

Next we will prove the reverse containment. We may assume that $d \geqslant 1$. Take a log resolution $\mu : Y \to X$ of the pair $(X, D+\Delta)$ and let $Z = \mu^{-1}(\mathfrak{m})$ denote the closed fiber of μ. Set

$$\mathcal{L} = \mathcal{O}_Y(\mu^*(K_X + D + \Delta) - \mu_*^{-1} D)$$

and let $\delta : H^d_{\mathfrak{m}}(\omega_R) \to H^d_Z(Y, \mathcal{L})$ be the map induced by the edge maps of the spectral sequence $H^p_{\mathfrak{m}}(R^q\mu_*\mathcal{L}) \Longrightarrow H^{p+q}_Z(\mathcal{L})$. Let $(Y_p)_{p\in\mathcal{P}}$, $(Z_p)_{p\in\mathcal{P}}$ and $(\mathcal{L}_p)_{p\in\mathcal{P}}$ be approximations of Y, Z and $\mathcal{L}$, respectively. Note that one has log resolutions $\mu_p : Y_p \to \operatorname{Spec} R_p$ and $Z_p = \mu_p^{-1}(\mathfrak{m}_p)$ for almost all p. Then we have a commutative diagram

$$\begin{tikzcd}
 & H^{d-1}(X \setminus \{\mathfrak{m}\}, \omega_R) \arrow[r, two heads] \arrow[d, "\gamma"] & H^d_{\mathfrak{m}}(\omega_R) \arrow[d, "\delta"] \\
H^{d-1}(Y, \mathcal{L}) \arrow[r] \arrow[d] & H^{d-1}(Y \setminus Z, \mathcal{L}) \arrow[r] \arrow[d, "u^{d-1}"] & H^d_Z(Y, \mathcal{L}) \\
H^{d-1}_\infty(Y, \mathcal{L}) \arrow[r, "\rho^{d-1}_\infty"] & H^{d-1}_\infty(Y \setminus Z, \mathcal{L}), &
\end{tikzcd}$$

where the top horizontal map is surjective and the middle row is exact (see Definition 1.5.62 for the definition of H_∞). Similarly, we have the following commutative diagram where the top horizontal map is surjective and the bottom row is exact for almost all p:

$$\begin{tikzcd}
 & H^{d-1}(X_p \setminus \{\mathfrak{m}_p\}, \omega_{R_p}) \arrow[r, two heads] \arrow[d, "\gamma_p"] & H^d_{\mathfrak{m}_p}(\omega_{R_p}) \arrow[d, "\delta_p"] \\
H^{d-1}(Y_p, \mathcal{L}_p) \arrow[r, "\rho^{d-1}_p"] & H^{d-1}(Y_p \setminus Z_p, \mathcal{L}_p) \arrow[r] & H^d_{Z_p}(\mathcal{L}_p).
\end{tikzcd}$$

It is enough to show that $\ker \delta \subseteq 0^{\mathcal{B}_D(D+\Delta)}_{H^d_{\mathfrak{m}}(\omega_R)}$, because δ is the Matlis dual of the inclusion $\operatorname{adj}_D(X, D+\Delta) \hookrightarrow R$ and $\ker \delta = \operatorname{Ann}_{H^d_{\mathfrak{m}}(\omega_R)} \operatorname{adj}_D(X, D+\Delta)$. Suppose $\eta \in \ker \delta$ and take an element $\zeta \in H^{d-1}(X \setminus \{\mathfrak{m}\}, \omega_R)$ that maps to η. Let $(\eta_p)_{p\in\mathcal{P}}$ and $(\zeta_p)_{p\in\mathcal{P}}$ be approximations of η and ζ, respectively. By the commutativity of the first diagram, $u^{d-1}(\gamma(\zeta)) \in \operatorname{Im} \rho^{d-1}_\infty$, which implies that $\gamma_p(\zeta_p) \in \operatorname{Im} \rho^{d-1}_p$ for almost all p. Then by the commutativity of the second diagram, $\eta_p \in \ker \delta_p$ for almost all p. It follows from the dual form of Lemma 2.3.5, Proposition 2.2.4 and Lemma 2.2.5 that

$$\ker \delta_p = 0^{*_{D_p}(D_p+\Delta_p)}_{H^d_{\mathfrak{m}_p}(\omega_{R_p})} = \ker(H^d_{\mathfrak{m}_p}(\omega_{R_p}) \to H^d_{\mathfrak{m}_p}(\omega_{R_p} \otimes_{R_p} I^+_{D_p}(D_p + \Delta_p)))$$

for almost all p. Therefore, the image of η vanishes in $\operatorname{ulim}_p H^d_{\mathfrak{m}_p}(\omega_{R_p} \otimes_{R_p} I^+_{D_p}(D_p + \Delta_p))$. Since $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, $r(K_X + D + \Delta) = \operatorname{div} f$ for some integer $r \geqslant 1$ and some nonzero element $f \in R$. Fix a module-finite extension S of R contained in R^+ such that S is normal and $f^{1/r} \in S$, and let $\pi : \operatorname{Spec} S \to \operatorname{Spec} R = X$ denote the morphism corresponding to the inclusion $R \hookrightarrow S$. We now consider a commutative diagram

$$\begin{tikzcd}
H^d_{\mathfrak{m}}(\omega_R) \arrow[rr] \arrow[d, "\cong"'] & & \operatorname*{ulim}_p H^d_{\mathfrak{m}_p}(\omega_{R_p}) \arrow[dd, "\cdot f^{1/r}"] \\
H^d_{\mathfrak{m}}(\omega_R \otimes_R R(D) \otimes_R R(-D)) \arrow[d] & & \\
H^d_{\mathfrak{m}}(\omega_R \otimes_R S(\pi^*(D+\Delta)) \otimes_S \mathcal{B}(I_D)) \arrow[r, "\cdot f^{1/r}", "\cong"'] \arrow[d] & H^d_{\mathfrak{m}}(\mathcal{B}(I_D)) \arrow[r, "\beta_D"] & \operatorname*{ulim}_p H^d_{\mathfrak{m}_p}(I^+_{D_p}) \\
H^d_{\mathfrak{m}}(\omega_R \otimes_R \mathcal{B}(I_D, D+\Delta)) \arrow[d, "\cong"'] & & \\
H^d_{\mathfrak{m}}(\operatorname*{ulim}_p \omega_{R_p} \otimes_{R_p} I^+_{D_p}(D_p + \Delta_p)) \arrow[rr] & & \operatorname*{ulim}_p H^d_{\mathfrak{m}_p}(\omega_{R_p} \otimes_{R_p} I^+_{D_p}(D_p + \Delta_p)), \arrow[uu, "\cong"', "\cdot f^{1/r}"]
\end{tikzcd}$$

where β_D is injective by Lemma 2.3.6 and the isomorphisms in the lower left and lower right are consequences of Lemma 1.5.56 and Lemma 2.2.5, respectively. By the commutativity of this diagram, the image of η has to be zero in $H^d_{\mathfrak{m}}(\omega_R \otimes_R S(\pi^*(D + \Delta)) \otimes_S \mathcal{B}(I_D))$. Thus,

$$\begin{aligned}
\eta &\in \ker(H^d_{\mathfrak{m}}(\omega_R) \to H^d_{\mathfrak{m}}(\omega_R \otimes_R \mathcal{B}(I_D, D+\Delta))) \\
&= \ker(H^d_{\mathfrak{m}}(\omega_R) \to H^d_{\mathfrak{m}}(\omega_R) \otimes_R \mathcal{B}(I_D, D+\Delta)) \\
&= 0^{\mathcal{B}_D(D+\Delta)}_{H^d_{\mathfrak{m}}(\omega_R)}.
\end{aligned}$$

□

2.4 Pullback of divisors

In this section, we discuss how to pullback Weil divisors. Our main reference is [14, Section 2]. Although morphisms are assumed to be birational in *loc. cit.*, essentially the same arguments work in our setting.

Definition 2.4.1 (cf. [14, Section 2])**.** Let $R \hookrightarrow S$ be an injective homomorphism between Noetherian normal domains and $\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$ denote the corresponding morphism.

(1) Suppose that D is a Weil divisor D on $\operatorname{Spec} R$. The *cycle-theoretic pullback* $\varphi^\natural D$ of D under φ is the Weil divisor

$$\varphi^\natural D = \sum_E v_E(R(-D))E,$$

where E runs through all prime divisors on $\operatorname{Spec} S$ and v_E is the discrete valuation associated to E.

(2) Suppose that Γ is a $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$. The *pullback* $\varphi^*\Gamma$ of Γ under φ is the $\mathbb{R}$-Weil divisor

$$\varphi^*\Gamma = \sum_E \left(\inf_m \frac{v_E(R(-m\Gamma))}{m} \right) E,$$

where E runs through all prime divisors on $\operatorname{Spec} S$ and the infimum is taken over all integers $m \geqslant 1$ such that $m\Gamma$ is an integral Weil divisor. If Γ is $\mathbb{Q}$-Cartier, then this definition coincides with the classical definition of pullback.

Remark 2.4.2. (1) $S(-\varphi^\natural D) = (R(-D)S)^{**}$, where $(-)^{**}$ denotes the reflexive hull as an S-module.

(2) If D_1 and D_2 are Weil divisors on $\operatorname{Spec} R$, then $\varphi^\natural(D_1+D_2) \leqslant \varphi^\natural D_1 + \varphi^\natural D_2$ holds and the inequality is strict in general.

(3) One generally has the inequality $\varphi^* D \leqslant \varphi^\natural D$. If φ is flat, then $\varphi^\natural D = \varphi^* D$, which also coincides with the flat pullback of D under φ.

(4) Definition 2.4.1 can be generalized to the case of dominant morphisms $\varphi : Y \to X$ between normal (not necessarily affine) varieties. If φ is a small birational morphism, then $\varphi^* D$ is nothing but the strict transform of D on Y (see [9, Remark 2.12]).

Throughout this section, we work with the following setting.

Setting 2.4.3. Let k be an algebraically closed field. Suppose that $R \hookrightarrow S$ is an injective k-algebra homomorphism between normal domains essentially of finite type over k and $\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$ is the corresponding morphism. Let Λ (resp. M) be the set of module-finite extensions R_λ (resp. S_μ) of R (resp. S), contained in R^+ (resp. S^+), such that each R_λ (resp. S_μ) is a normal domain. When R_λ (resp. S_μ) belongs to Λ (resp. M), we write the morphism corresponding to the inclusion $R \hookrightarrow R_\lambda$ (resp. $S \hookrightarrow S_\mu$) by $\pi_\lambda : \operatorname{Spec} R_\lambda \to \operatorname{Spec} R$ (resp. $\rho_\mu : \operatorname{Spec} S_\mu \to \operatorname{Spec} S$).

Proposition 2.4.4. *With notation as in Setting 2.4.3, take $R_\lambda \in \Lambda$ and $S_\mu \in M$ such that R_λ is contained in S_μ and let $\varphi_{\lambda\mu} : \operatorname{Spec} S_\mu \to \operatorname{Spec} R_\lambda$ denote the corresponding morphism. For a Weil divisor D on* $\operatorname{Spec} R$*, one has an inequality*

$$\rho_\mu^* \varphi^\natural D \geqslant \varphi_{\lambda\mu}^\natural \pi_\lambda^* D$$

of Weil divisors on $\operatorname{Spec} S_\mu$.

Proof. Let F_μ be a prime divisor on $\operatorname{Spec} S_\mu$ and $F = \rho_\mu(F_\mu)$ denote the image of F_μ under ρ_μ. Then

$$\begin{aligned} \operatorname{ord}_{F_\mu}(\varphi^\natural_{\lambda\mu}\pi^*_\lambda D) = v_{F_\mu}((I_D R_\lambda)^{**}) &\leqslant v_{F_\mu}(I_D) \\ &= v_{F_\mu}(I_F) v_F(I_D) \\ &= \operatorname{ord}_{F_\mu}(F) \operatorname{ord}_F(\varphi^\natural D) \\ &= \operatorname{ord}_{F_\mu}(\rho^*_\mu \varphi^\natural D), \end{aligned}$$

where $(I_D R_\lambda)^{**}$ is the reflexive hull of $I_D R_\lambda$ as an R_λ-module. □

Remark 2.4.5. Cycle-theoretic pullback does not commute with finite pullback, that is, the inequality in Proposition 2.4.4 is strict in general. For example, let $S = \mathbb{C}[x, y, z]$ be the 3-dimensional polynomial ring over $\mathbb{C}$ and $R = \mathbb{C}[xy^2, xyz, xz^2]$ be a subring of S. Consider the module-finite extension $R_\lambda = \mathbb{C}[\sqrt{x}y, \sqrt{x}z]$ of R and the module-finite extension $S_\mu = \mathbb{C}[\sqrt{x}, y, z]$ of S.

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C}[\sqrt{x}, y, z] & \xrightarrow{\varphi_{\lambda\mu}} & \operatorname{Spec}\mathbb{C}[\sqrt{x}y, \sqrt{x}z] \\ {\scriptstyle \rho_\mu}\downarrow & & \downarrow{\scriptstyle \pi_\lambda} \\ \operatorname{Spec}\mathbb{C}[x, y, z] & \xrightarrow{\varphi} & \operatorname{Spec}\mathbb{C}[xy^2, xyz, xz^2] \end{array}$$

Let D be a prime divisor on $\operatorname{Spec}\mathbb{C}[xy^2, xyz, xz^2]$ defined by the prime ideal (xy^2, xyz) of height one. Then $\pi^*_\lambda D = \operatorname{div}\sqrt{x}y$ and $\varphi^\natural_{\lambda\mu}\pi^*_\lambda D = \operatorname{div}\sqrt{x} + \operatorname{div} y$. On the other hand, $\varphi^\natural D = \operatorname{div} x + \operatorname{div} y$ and $\rho^*_\mu \varphi^\natural D = 2\operatorname{div}\sqrt{x} + \operatorname{div} y$.

Proposition 2.4.6. *With notation as in Setting 2.4.3, suppose in addition that $R \hookrightarrow S$ is a pure local homomorphism. For a prime divisor D on $\operatorname{Spec} R$, one has*

$$S(-\varphi^\natural D) \cap R = R(-D).$$

Proof. First note that φ is surjective since $R \hookrightarrow S$ is pure. Pick a prime ideal $\mathfrak{r}$ of S lying over $R(-D)$. Let A and B be normal domains of finite type over k and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A and B, respectively such that $R \cong A_\mathfrak{p}$, $S \cong B_\mathfrak{q}$ and the inclusion $R \hookrightarrow S$ is induced by a k-algebra homomorphism $A \to B$. Take a minimal prime divisor $\mathfrak{s}$ of $(I_D \cap A)B$ contained in $\mathfrak{r} \cap B$. It is easy to see from [40, Theorem 15.1] that $\operatorname{ht}\mathfrak{s} = 1$. Then $\mathfrak{s}S$ is a height one prime of S and we have containments $I_D S \subseteq \mathfrak{s}S \subseteq \mathfrak{r}$, which implies that $I_{\varphi^\natural D} \subseteq \mathfrak{s}S$ and consequently $I_{\varphi^\natural D} \cap R = I_D$. □

The following proposition is one of the key ingredients in the study of the behavior of adjoint ideals under pure morphisms.

Proposition 2.4.7. *With notation as in Setting 2.4.3, let D be a prime divisor on $\operatorname{Spec} R$, and suppose that the cycle-theoretic pullback $E := \varphi^\natural D$ of D under φ is a prime divisor and dominates D. Let Γ (resp. Δ) be an effective $\mathbb{Q}$-Weil divisor on*

$\operatorname{Spec} R$ (resp. $\operatorname{Spec} S$) that has no component equal to D (resp. E), and suppose that $\Delta \geqslant \varphi^*\Gamma$. Fix choices of I_D^+ and I_E^+ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & I_D^+ \arrow[r] \arrow[d] & R^+ \arrow[r] \arrow[d] & (R/I_D)^+ \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_E^+ \arrow[r] & S^+ \arrow[r] & (S/I_E)^+ \arrow[r] & 0
\end{tikzcd}$$

Then there exists a natural inclusion

$$I_D^+(D+\Gamma) \hookrightarrow I_E^+(E+\Delta).$$

Proof. Take $\lambda \in \Lambda$ and $\mu \in M$ such that R_λ is contained in S_μ, and let $\varphi_{\lambda\mu} : \operatorname{Spec} S_\mu \to \operatorname{Spec} R_\lambda$ denote the corresponding morphism and D_λ (resp. E_μ) denote the prime divisor on $\operatorname{Spec} R_\lambda$ (resp. $\operatorname{Spec} S_\mu$) such that $R_\lambda(-D_\lambda) = I_D^+ \cap R_\lambda$ (resp. $S_\mu(-E_\mu) = I_E^+ \cap S_\mu$). It suffices to show the inclusion

$$R_\lambda(\lfloor \pi_\lambda^*(D+\Gamma) - D_\lambda \rfloor) \hookrightarrow S_\mu(\lfloor \rho_\mu^*(E+\Delta) - E_\mu \rfloor).$$

For any nonzero element $f \in R_\lambda(\lfloor \pi_\lambda^*(D+\Gamma) - D_\lambda \rfloor)$ and any prime divisor F_μ on $\operatorname{Spec} S_\mu$, we will show that $\operatorname{ord}_{F_\mu}(\operatorname{div}_{S_\mu} f + \rho_\mu^*(E+\Delta) - E_\mu) \geqslant 0$. First consider the case where $F_\mu \neq E_\mu$. By assumption,

$$\inf_m \frac{v_{F_\mu}(R(-m\Gamma))}{m} = \operatorname{ord}_{F_\mu}(\rho_\mu^*\varphi^*\Gamma) \leqslant \operatorname{ord}_{F_\mu}(\rho_\mu^*\Delta),$$

where the infimum is taken over all integers $m \geqslant 1$ such that mD is an integral Weil divisor. Since

$$f^m R_\lambda(-\pi_\lambda^* D)^m R(-m\Gamma) \subseteq R_\lambda(-mD_\lambda) \subseteq S_\mu$$

for such m, one has

$$\begin{aligned}
\operatorname{ord}_{F_\mu}(\operatorname{div}_{S_\mu} f + \rho_\mu^*(E+\Delta) - E_\mu) &= \operatorname{ord}_{F_\mu}(\operatorname{div}_{S_\mu} f + \rho_\mu^* E + \rho_\mu^*\Delta) \\
&\geqslant \operatorname{ord}_{F_\mu}(\operatorname{div}_{S_\mu} f + \varphi_{\lambda\mu}^\natural \pi_\lambda^* D + \rho_\mu^*\Delta) \\
&\geqslant 0,
\end{aligned}$$

where the middle inequality follows from Proposition 2.4.4.

Next we treat the case where $F_\mu = E_\mu$. Since E dominates D, the prime divisor E_μ dominates D_λ. Also, $\operatorname{ord}_{D_\lambda} \pi_\lambda^*\Gamma = \operatorname{ord}_{E_\mu} \rho_\mu^*\Delta = 0$ by assumption. Therefore, by Proposition 2.4.4,

$$\begin{aligned}
\operatorname{ord}_{E_\mu}(\operatorname{div}_{S_\mu} f + \rho_\mu^*(E+\Delta) - E_\mu) &\geqslant \operatorname{ord}_{E_\mu}(\operatorname{div}_{S_\mu} f + \varphi_{\lambda\mu}^\natural \pi_\lambda^* D) - 1 \\
&\geqslant \operatorname{ord}_{D_\lambda}(\operatorname{div}_{R_\lambda} f + \pi_\lambda^* D) - 1 \\
&= \operatorname{ord}_{D_\lambda}(\operatorname{div}_{R_\lambda} f + \pi_\lambda^*(D+\Gamma) - D_\lambda) \\
&\geqslant 0.
\end{aligned}$$

$\square$

Cycle-theoretic pullback commutes with taking approximations.

Proposition 2.4.8. *Suppose that $R \hookrightarrow S$ is an injective local $\mathbb{C}$-algebra homomorphism between normal local rings essentially of finite type over $\mathbb{C}$ and $\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$ is the corresponding morphism. Let D be a Weil divisor on $\operatorname{Spec} R$ and $E := \varphi^{\natural} D$ be the cycle-theoretic pullback of D under φ. If $(\varphi_p : \operatorname{Spec} S_p \to \operatorname{Spec} R_p)_{p \in \mathcal{P}}, (D_p)_{p \in \mathcal{P}}, (E_p)_{p \in \mathcal{P}}$ are approximations of φ, D, E, respectively, then E_p is the cycle-theoretic pullback of D_p under φ_p for almost all p.*

Proof. Suppose that $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are all the minimal prime ideals of $I_D S$. Let $(\mathfrak{p}_{ip})_{p \in \mathcal{P}}$ be an approximation of $\mathfrak{p}_i$ for each $i = 1, \dots, n$, and then by [44, Theorem 4.4], $\mathfrak{p}_{1p}, \dots, \mathfrak{p}_{np}$ are all the minimal prime ideals of $I_{D_p} S_p$ for almost all p. By reindexing, if necessary, we may assume that $\operatorname{ht} \mathfrak{p}_i = 1$ for $i = 1, \dots, m$ and $\operatorname{ht} \mathfrak{p}_i \geqslant 2$ for $i = m+1, \dots, n$. Let E_i denote the prime divisor on $\operatorname{Spec} S$ defined by $\mathfrak{p}_i$ and l_i denote the positive integer such that $I_D S_{\mathfrak{p}_i} = t_i^{l_i} S_{\mathfrak{p}_i}$, where t_i is a uniformizer of the DVR $S_{\mathfrak{p}_i}$, for $i = 1, \dots, m$. It then follows from Remark 2.4.2 (1) that $E = \sum_{1 \leqslant i \leqslant m} l_i E_i$.

On the other hand, let E_{ip} be the prime divisor on $\operatorname{Spec} S_p$ defined by $\mathfrak{p}_{ip}$ for $i = 1, \dots, m$ and for almost all p. Since $I_{D_p} (S_p)_{\mathfrak{p}_{ip}} = (I_D S_{\mathfrak{p}_i})_p$ and $\operatorname{ht} \mathfrak{p}_i = \operatorname{ht} \mathfrak{p}_{ip}$ for almost all p (see [44, Theorem 4.5] for the second equality), the cycle-theoretic pullback of D_p under φ_p is $\sum_{1 \leqslant i \leqslant m} l_i E_{ip}$ for almost all p, which completes the proof. $\square$

The pullback of an approximation of a Weil divisor can be estimated from above by using an approximation of the pullback of this divisor.

Proposition 2.4.9. *Suppose that $R \hookrightarrow S$ is an injective local $\mathbb{C}$-algebra homomorphism between normal local domains essentially of finite type over $\mathbb{C}$ and $\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$ is the corresponding morphism. Let Γ be an effective $\mathbb{Q}$-Weil divisor on $\operatorname{Spec} R$, and fix an integer $m \geqslant 1$ such that $m\Gamma$ is an integral Weil divisor. Considering approximations of $\Gamma, \varphi^* \Gamma$ and $\varphi^{\natural} m\Gamma$, for every real number $\varepsilon > 0$, one has*

$$(\varphi^* \Gamma)_p + \varepsilon (\varphi^{\natural} m\Gamma)_p \geqslant \varphi_p^* \Gamma_p$$

for almost all p.

Proof. Fix a real number $\varepsilon > 0$ and take a sufficiently large integer n so that

$$\frac{v_E(R(-mn\Gamma))}{mn} \leqslant \operatorname{ord}_E(\varphi^* \Gamma + \varepsilon \varphi^{\natural} m\Gamma)$$

for all prime divisors E on $\operatorname{Spec} S$. Since $v_{E_p}(R_p(-mn\Gamma_p)) = v_E(R(-mn\Gamma))$ for almost all p, where $(E_p)_{p \in \mathcal{P}}$ is an approximation of E, this inequality implies that

$$\operatorname{ord}_{E_p} \varphi_p^* \Gamma_p \leqslant \frac{v_{E_p}(R_p(-mn\Gamma_p))}{mn} \leqslant \operatorname{ord}_{E_p}((\varphi^* \Gamma)_p + \varepsilon (\varphi^{\natural} m\Gamma)_p)$$

for almost all p. $\square$

2.5 Faithfully flat descent of adjoint ideals

In this section, we prove the faithfully flat descent property of adjoint ideals. First, following an idea from [9], we extend the correspondence between adjoint ideals and test ideals along divisors to rings with finitely generated anti-canonical algebras.

Lemma 2.5.1. *Suppose that X is an F-finite normal integral scheme, D is a reduced divisor and Δ is an effective $\mathbb{Q}$-Weil divisor on X such that D and Δ have no common components and the $\mathcal{O}_X$-algebra $\bigoplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i(K_X+D+\Delta)\rfloor)$ is finitely generated. Let $\mathfrak{a}\subseteq\mathcal{O}_X$ be an coherent ideal sheaf whose zero locus contains no components of D and $t>0$ be a rational number. Choose an integer $m\geqslant 1$ such that mt is an integer, $\mathfrak{m}\Delta$ is an integral Weil divisor, and the m-th Veronese subring of $\bigoplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i(K_X+D+\Delta)\rfloor)$ is generated in degree one. Then*

$$\tau_D(X, D+\Delta, \mathfrak{a}^t) = \tau_D(\omega_X, (\mathcal{O}(-m(K_X+D+\Delta))\mathfrak{a}^{tm})^{\frac{1}{m}}).$$

For the definition of the right hand side, see Remark 2.1.3.

Proof. This follows from an argument similar to [9, Lemma 5.2]. □

Lemma 2.5.2. *Suppose that X is a normal complex variety, D is a reduced divisor and Δ is an effective $\mathbb{Q}$-Weil divisor on X such that D and Δ have no common components and the $\mathcal{O}_X$-algebra $\oplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i(K_X+D+\Delta)\rfloor)$ is finitely generated. Let $\mathfrak{a}\subseteq\mathcal{O}_X$ be a coherent ideal sheaf whose zero locus contains no components of D and $t>0$ be a real number. Let $\rho : X' = \mathbf{Proj}\bigoplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i(K_X+D+\Delta)\rfloor)\to X$ be the $\mathbb{Q}$-Cartierization of $-(K_X+D+\Delta)$. Then*

$$\mathrm{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \rho_*\mathrm{adj}_{\rho^*D}(X', \rho^*(D+\Delta), (\mathfrak{a}\mathcal{O}_{X'})^t).$$

Proof. An argument analogous to that in [9, Corollary 2.25] is applicable here, by utilizing $\mathrm{adj}_D^{(m)}(X, D+\Delta, \mathfrak{a}^t)$ in the proof of Proposition 1.1.2 instead of $\mathcal{J}_m(X,\Delta,\mathfrak{a}^t)$. □

We now briefly explain the method for reducing triples $(X, D, \mathfrak{a})$, consisting of varieties, divisors and ideal sheaves, from characteristic zero to positive characteristic. Our main reference is [26, Chapter 2]. For the case of local rings, see Definition 1.3.9.

Let X be a normal variety over a field k of characteristic zero, $D=\sum_i d_iD_i$ be an $\mathbb{Q}$-Weil divisor on X and $\mathfrak{a}\subseteq\mathcal{O}_X$ be a nonzero coherent ideal sheaf. Choosing a suitable finitely generated $\mathbb{Z}$-subalgebra A of k, we can construct a scheme X_A of finite type over A and closed subschemes $D_{i,A}\subseteq X_A$ such that there exist isomorphisms

$$\begin{array}{ccc} X & \xrightarrow{\cong} & X_A \times_{\operatorname{Spec} A} \operatorname{Spec} k \\ \uparrow & & \uparrow \\ D_i & \xrightarrow{\cong} & D_{i,A} \times_{\operatorname{Spec} A} \operatorname{Spec} k, \end{array}$$

and set $\mathfrak{a}_A := \rho_*\mathfrak{a} \cap \mathcal{O}_{X_A}$, where $\rho : X \to X_A$ is the projection. We can enlarge A by localizing at a single nonzero element and subsequently replace X_A and $D_{i,A}$ with their corresponding open subschemes. This enables us to assume that $X_A, D_{i,A}$ and $\mathfrak{a}_A$ are flat over $\operatorname{Spec} A$ due to generic freeness. Further enlarging A if necessary, we can then assume that X_A is a normal integral scheme, $D_{i,A}$ is a prime divisor on X_A and $\mathfrak{a}_A\mathcal{O}_X = \mathfrak{a}$. We refer to the triple $(X_A, D_A := \sum_i d_i D_{i,A}, \mathfrak{a}_A)$ as a *model* of $(X, D, \mathfrak{a})$ over A.

Given an closed point $\mu \in \operatorname{Spec} A$, let X_μ (resp. $D_{i,\mu}$) denote the fiber of $X_A \to \operatorname{Spec} A$ (resp. $D_{i,A} \to \operatorname{Spec} A$) over μ, and set $D_\mu = \sum_i D_{i,\mu}$ and $\mathfrak{a}_\mu = \mathfrak{a}_A\mathcal{O}_{X_\mu}$. Then X_μ is a scheme of finite type over the finite field A/μ. Furthermore, X_μ is a normal variety over A/μ and D_μ is a $\mathbb{Q}$-Weil divisor on X_μ for general closed points $\mu \in \operatorname{Spec} A$.

Theorem 2.5.3. *With notation as in Lemma 2.5.2, suppose that t is a rational number. Given a model over a finitely generated $\mathbb{Z}$-subalgebra A of $\mathbb{C}$, we have*

$$\operatorname{adj}_D(X, D + \Delta, \mathfrak{a}^t)_\mu = \tau_{D_\mu}(X_\mu, D_\mu + \Delta_\mu, \mathfrak{a}_\mu^t)$$

for general closed points $\mu \in \operatorname{Spec} A$.

Proof. By virtue of Lemmas 2.5.2 and 2.5.1, this follows from an argument similar to [9, Theorem 6.4]. $\square$

We remark that flatness is preserved under reduction modulo p.

Proposition 2.5.4. *Let $g : B \to C$ be a $\mathbb{C}$-algebra homomorphism between rings of finite type over $\mathbb{C}$, and $\mathfrak{q}$ be a prime ideal of C. Moreover, set $\mathfrak{p} = \mathfrak{q} \cap B$, $R = B_\mathfrak{p}$ and $S = C_\mathfrak{q}$, and suppose that the induced local ring homomorphism $g_\mathfrak{p} : R \to S$ is flat. Given a model over a finitely generated $\mathbb{Z}$-subalgebra A of $\mathbb{C}$, the local ring homomorphism $g_{\mathfrak{p},\mu} : R_\mu \to S_\mu$ is flat for general closed points $\mu \in \operatorname{Spec} A$.*

Proof. Since the flat locus of $g : B \to C$ is open, by localizing C at a nonzero element if necessary, we may assume that g is flat. Then, enlarging A if necessary, we may assume that $\operatorname{Tor}_1^{B_A}(B_A/\mathfrak{p}_A, C_A) = 0$. It follows from [26, Theorem 2.3.5 (e)] that

$$\begin{aligned}\operatorname{Tor}_1^{R_\mu}(\kappa(\mathfrak{p}_\mu), S_\mu) &= \operatorname{Tor}_1^{B_\mu}(B_\mu/\mathfrak{p}_\mu, C_\mu) \otimes_{C_\mu} S_\mu \\ &= \operatorname{Tor}_1^{B_A}(B_A/\mathfrak{p}_A, C_A) \otimes_{C_A} S_\mu \\ &= 0,\end{aligned}$$

which implies that $g_{\mathfrak{p},\mu} : R_\mu \to S_\mu$ is flat, for general closed points $\mu \in \operatorname{Spec} A$. $\square$

The following lemma is seemingly well-known to experts. However, we include it here due to the lack of a direct reference.

Lemma 2.5.5. *$R \to S$ be a flat local ring homomorphism between Noetherian local rings and M be an R-module. If R is complete, then we have*

$$(\operatorname{Ann}_R M)S = \operatorname{Ann}_S(M \otimes_R S).$$

Proof. For each element x of M, tensoring the exact sequence

$$0 \to \operatorname{Ann}_R x \to R \xrightarrow{\cdot x} M$$

with S yield an exact sequence

$$0 \to (\operatorname{Ann}_R x)S \to S \xrightarrow{\cdot(x\otimes 1)} M \otimes_R S,$$

which implies that $(\operatorname{Ann}_R x)S = \operatorname{Ann}_S(x \otimes 1)$. Therefore, we have

$$\begin{aligned}(\operatorname{Ann}_R M)S = \left(\bigcap_{x\in M} \operatorname{Ann}_R x\right) S &= \bigcap_{x\in M} ((\operatorname{Ann}_R x)S) \\ &= \bigcap_{x\in M} \operatorname{Ann}_S(x \otimes 1) \\ &= \operatorname{Ann}_S(M \otimes_R S),\end{aligned}$$

where the second equality follows from the fact that the homomorphism $R \to S$ is intersection flat by [27, Proposition 5.7 (e)]. $\square$

Proposition 2.5.6. *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism between F-finite normal local rings of characteristic $p > 0$, and let $\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$ denote the corresponding morphism. Let D be a reduced divisor and Γ be an effective $\mathbb{Q}$-Weil divisor on $X := \operatorname{Spec} R$ such that D and Γ have no common components. Suppose that the flat pullback $E := \varphi^* D$ of D under φ is a reduced divisor on $Y := \operatorname{Spec} S$. For any ideal $\mathfrak{a} \subseteq R$ whose zero locus contains no components of D and for any real number $t > 0$, one has*

$$\tau_E(S, E + \varphi^*\Gamma, (\mathfrak{a}S)^t) \subseteq \tau_D(R, D + \Gamma, \mathfrak{a}^t)S.$$

Proof. The inclusion $R \hookrightarrow S$ induces the flat injective local ring homomorphism $\widehat{R} \hookrightarrow \widehat{S}$ (see, for example, [54, 0C4G]), and let $\widehat{\varphi} : \operatorname{Spec} \widehat{S} \to \operatorname{Spec} \widehat{R}$ denote the corresponding morphism. Then we have the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \widehat{S} \arrow[r, "\widehat{\varphi}"] \arrow[d, "\iota_S"'] & \operatorname{Spec} \widehat{R} \arrow[d, "\iota_R"] \\ \operatorname{Spec} S \arrow[r, "\varphi"] & \operatorname{Spec} R, \end{tikzcd}$$

where $\iota_R : \operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$ and $\iota_S : \operatorname{Spec} \widehat{S} \to \operatorname{Spec} S$ are the canonical morphisms, and therefore, $\widehat{\varphi}^* \iota_R^* D = \iota_S^* E$. Note that both $\iota_R^* D$ and $\iota_S^* E$ are reduced divisors. Since the formation of test ideals along divisors commutes with completion, we can reduce the problem to the case where both R and S are complete.

For each integer $e \geqslant 1$, the inclusion $R \hookrightarrow S$ induces an inclusion

$$R((p^e - 1)D + \lceil p^e \Gamma \rceil) \hookrightarrow S((p^e - 1)E + \lceil p^e f^* \Gamma \rceil).$$

Additionally, it induces a containment $R^{\circ,D} \subseteq S^{\circ,E}$. This follows from the fact that every irreducible component of E dominates an irreducible component of D by the flatness of φ. It is then easy to see that $0_M^{*_D(D+\Gamma,\mathfrak{a}^t)} \otimes_R S \subseteq 0_{M\otimes_R S}^{*_E(E+f^*\Gamma,(\mathfrak{a}S)^t)}$ for all R-modules M, which implies that

$$\begin{aligned}\bigcap_M \operatorname{Ann}_S(0_M^{*_D(D+\Gamma,\mathfrak{a}^t)} \otimes_R S) &\supseteq \bigcap_M \operatorname{Ann}_S(0_{M\otimes_R S}^{*_E(E+f^*\Gamma,(\mathfrak{a}S)^t)}) \\ &\supseteq \bigcap_N \operatorname{Ann}_S 0_N^{*_E(E+f^*\Gamma,(\mathfrak{a}S)^t)} \\ &= \tau_E(S, E + f^*\Gamma, (\mathfrak{a}S)^t),\end{aligned}$$

where M (resp. N) runs through all R-modules (resp. S-modules). On the other hand, $R \hookrightarrow S$ is intersection flat by the completeness of R and [27, Proposition 5.7 (e)], and consequently,

$$\begin{aligned}\bigcap_M \operatorname{Ann}_S(0_M^{*_D(D+\Gamma,\mathfrak{a}^t)} \otimes_R S) &= \bigcap_M \Big((\operatorname{Ann}_R 0_M^{*_D(D+\Gamma,\mathfrak{a}^t)})S\Big) \\ &= \left(\bigcap_M \operatorname{Ann}_R 0_M^{*_D(D+\Gamma,\mathfrak{a}^t)}\right) S \\ &= \tau_D(R, D + \Gamma, \mathfrak{a}^t)S,\end{aligned}$$

where the first equality follows from Lemma 2.5.5. Thus, we obtain the desired containment. $\square$

Here is the main result of this section.

Theorem 2.5.7. *Let $f : Y \to X$ be a faithfully flat morphism between normal complex varieties. Let D be a reduced divisor and Γ be an effective $\mathbb{Q}$-Weil divisor on X such that D and Γ have no common components. Suppose that the flat pullback $E := f^*D$ of D under f is a reduced divisor on Y and the $\mathcal{O}_X$-algebra $\bigoplus_{i\geqslant 0} \mathcal{O}_X(\lfloor -i(K_X + D + \Gamma)\rfloor)$ is finitely generated. For any coherent ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ whose zero locus contains no components of D and for any real number $t > 0$, one has*

$$\operatorname{adj}_E(Y, E + f^*\Gamma, (\mathfrak{a}\mathcal{O}_Y)^t) \subseteq \operatorname{adj}_D(X, D + \Gamma, \mathfrak{a}^t)\mathcal{O}_Y.$$

Proof. Take an effective $\mathbb{Q}$-Weil divisor Δ on Y such that Δ and E have no common components, $K_Y + E + f^*\Gamma + \Delta$ is $\mathbb{Q}$-Cartier and

$$\operatorname{adj}_E(Y, E + f^*\Gamma, (\mathfrak{a}\mathcal{O}_Y)^t) = \operatorname{adj}_E(Y, E + f^*\Gamma + \Delta, (\mathfrak{a}\mathcal{O}_Y)^t).$$

After a small perturbation, we may assume that t is a rational number. Given that the question is local, we can further assume that $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$, where

R and S are normal local rings essentially of finite type over $\mathbb{C}$. Given a model over a finitely generated $\mathbb{Z}$-subalgebra A of $\mathbb{C}$, it follows from Theorem 2.5.3 that

$$\begin{aligned}\mathrm{adj}_D(X, D+\Gamma, \mathfrak{a}^t)_\mu &= \tau_{D_\mu}(X_\mu, D_\mu+\Gamma_\mu, \mathfrak{a}_\mu^t),\\ \mathrm{adj}_E(Y, E+f^*\Gamma+\Delta, (\mathfrak{a}S)^t)_\mu &= \tau_{E_\mu}(Y_\mu, E_\mu+f_\mu^*\Gamma_\mu+\Delta_\mu, (\mathfrak{a}_\mu S_\mu)^t)\end{aligned}$$

for general closed points $\mu \in \operatorname{Spec} A$. Therefore, it is enough to show that

$$\tau_{E_\mu}(Y_\mu, E_\mu+f_\mu^*\Gamma_\mu+\Delta_\mu, (\mathfrak{a}_\mu S_\mu)^t) \subseteq \tau_{D_\mu}(X_\mu, D_\mu+\Gamma_\mu, \mathfrak{a}_\mu^t)S_\mu$$

for general closed points $\mu \in \operatorname{Spec} A$. By observing that all assumptions are preserved after reduction to characteristic $p \gg 0$, such as f_μ being flat for general closed points $\mu \in \operatorname{Spec} A$ by Proposition 2.5.4, this is a direct consequence of Proposition 2.5.6. $\square$

The assertion of Theorem 2.5.7 does not hold for pure morphisms.

Example 2.5.8. Let $S = \mathbb{C}[x,y,z]_{(x,y,z)}$ be a localization of the three-dimensional polynomial ring over the field $\mathbb{C}$ of complex numbers, equipped with an action of the multiplicative group $G = \mathbb{C}^\times$ defined by

$$t : \begin{cases} x \longmapsto t^2 x \\ y \longmapsto t^{-1} y \\ z \longmapsto t^{-1} z \end{cases},$$

where $t \in G$. Then the subring $R := S^G$ of invariants under the action of G is described as

$$\mathbb{C}[xy^2, xz^2, xyz]_{(xy^2, xz^2, xyz)} \cong (\mathbb{C}[u,v,w]/(uv-w^2))_{(u,v,w)}.$$

Note that the inclusion $R \hookrightarrow S$ is pure and not flat. Writing $X := \operatorname{Spec} R$, $Y := \operatorname{Spec} S$, and $\mathfrak{m}$ for the maximal ideal of R, we observe that $\mathcal{J}(X, \mathfrak{m}) = \mathfrak{m}$ and $\mathcal{J}(Y, \mathfrak{m}\mathcal{O}_Y) = (xy, xz)\mathcal{O}_Y$, which implies that

$$\mathcal{J}(Y, \mathfrak{m}\mathcal{O}_Y) \cap \mathcal{O}_X \subseteq \mathcal{J}(X, \mathfrak{m}), \quad \mathcal{J}(Y, \mathfrak{m}\mathcal{O}_Y) \nsubseteq \mathcal{J}(X, \mathfrak{m})\mathcal{O}_Y.$$

2.6 The behavior of adjoint ideals under pure extensions

In this section, we study the behavior of adjoint ideals under pure ring extensions. This gives a generalization of [65, Theorem 1.2].

In the first half of this section, we work with the following setting.

Setting 2.6.1. Let $(R, \mathfrak{m})$ be a d-dimensional normal local domain essentially of finite type over $\mathbb{C}$, Δ be an effective $\mathbb{Q}$-Weil divisor and D be a prime divisor on $X := \operatorname{Spec} R$ such that no component of Δ is equal to D. Let $\mathfrak{a}$ be an ideal of R not contained in $R(-D)$ and $t > 0$ be a real number.

First we generalize the definition of $\tau_{\mathcal{B},D}(R, D+\Delta)$ to the case of triples.

Definition 2.6.2. (1) Given an R-module M, the submodule $0_M^{\mathcal{B}_D(D+\Delta,\mathfrak{a}^t)}$ of M is defined as

$$0_M^{\mathcal{B}_D(D+\Delta,\mathfrak{a}^t)} = \bigcap_{n \geqslant 1} \bigcap_{f \in \mathfrak{a}^{\lceil tn \rceil} \cap R^{\circ,D}} 0_M^{\mathcal{B}_D(D+\Delta+\frac{1}{n}\operatorname{div} f)},$$

where the first intersection is taken over all positive integers n and the second intersection is taken over all nonzero elements $f \in \mathfrak{a}^{\lceil tn \rceil} \cap R^{\circ,D}$.

(2) The following ideals are equal to each other (cf. [23, Proposition 8.23]), and are collectively denoted by $\tau_{\mathcal{B},D}(R, D+\Delta, \mathfrak{a}^t)$.

(a) $\bigcap_M \operatorname{Ann}_R 0_M^{\mathcal{B}_D(D+\Delta,\mathfrak{a}^t)}$, where M runs through all R-modules.

(b) $\operatorname{Ann}_R 0_E^{\mathcal{B}_D(D+\Delta,\mathfrak{a}^t)}$, where $E = E_R(R/\mathfrak{m})$ is an injective hull of the residue field $R/\mathfrak{m}$.

Lemma 2.6.3. *If $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, then*

$$\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \sum_{n \geqslant 1} \sum_{f \in \mathfrak{a}^{\lceil tn \rceil} \cap R^{\circ,D}} \operatorname{adj}_D(X, D+\Delta+\frac{1}{n}\operatorname{div} f),$$

where the first summation is taken over all positive integers n and the second summation is taken over all nonzero elements $f \in \mathfrak{a}^{\lceil tn \rceil} \cap R^{\circ,D}$.

Proof. It is clear that the right hand side is contained in the left hand side. We will show the reverse containment. First note that the filtration of adjoint ideals $\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t)$ is right continuous in t, that is, $\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^{t+\varepsilon})$ for all $0 \leqslant \varepsilon \ll 1$. Therefore, we may assume that t is a rational number. Let $f_1, \dots, f_l$ be a system of generators for $\mathfrak{a}$ such that $f_i \notin R(-D)$ for each $i = 1, \dots, l$. Since the adjoint ideal $\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t)$ coincides after reduction to characteristic $p \gg 0$ with the test ideal $\tau_D(R, D+\Delta, \mathfrak{a}^t)$ along D by [59], it follows from an argument similar to the proof of [58, Theorem 3.2] that

$$\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t) = \sum_{\lambda_1+\cdots+\lambda_l = t} \operatorname{adj}_D(X, D+\Delta+\lambda_1 \operatorname{div} f_1 + \cdots + \lambda_l \operatorname{div} f_l),$$

where the summation is taken over all nonnegative rational numbers $\lambda_1, \dots, \lambda_l$ with $\lambda_1 + \cdots + \lambda_l = t$. Fix such nonnegative rational numbers $\lambda_1, \dots, \lambda_l$ and choose an integer $m \geqslant 1$ so that $m\lambda_i$ is an integer for each $i = 1, \dots, l$. Then $f := f_1^{m\lambda_1} \dots f_l^{m\lambda_l}$ is an element of $\mathfrak{a}^{mt} \cap R^{\circ,D}$ and $\frac{1}{m}\operatorname{div} f = \lambda_1 \operatorname{div} f_1 + \cdots + \lambda_l \operatorname{div} f_l$. Thus,

$$\operatorname{adj}_D(X, D+\Delta+\lambda_1 \operatorname{div} f_1 + \cdots + \lambda_l \operatorname{div} f_l) \subseteq \sum_{n \geqslant 1} \sum_{f \in \mathfrak{a}^{\lceil tn \rceil} \cap R^{\circ,D}} \operatorname{adj}_D(X, D+\Delta+\frac{1}{n}\operatorname{div} f),$$

which completes the proof. $\square$

We can now generalize Theorem 2.3.7 to the case of triples.

Proposition 2.6.4. *If $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, then*

$$\tau_{\mathcal{B},D}(R, D+\Delta, \mathfrak{a}^t) = \operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t).$$

Proof. It follows from Theorem 2.3.7, Lemma 2.6.3 and Matlis duality that

$$\begin{aligned}
\tau_{\mathcal{B},D}(R, D+\Delta, \mathfrak{a}^t) &= \operatorname{Ann}_R 0^{\mathcal{B}_D(D+\Delta,\mathfrak{a}^t)}_{H^d_{\mathfrak{m}}(\omega_R)} \\
&= \operatorname{Ann}_R \left(\bigcap_{n \geqslant 1} \bigcap_{f \in \mathfrak{a}^{\lceil tn \rceil} \cap R^{\circ,D}} \operatorname{Ann}_{H^d_{\mathfrak{m}}(\omega_R)} \operatorname{adj}_D(X, D+\Delta + \frac{1}{n} \operatorname{div} f) \right) \\
&= \operatorname{Ann}_R \operatorname{Ann}_{H^d_{\mathfrak{m}}(\omega_R)} \left(\sum_{n \geqslant 1} \sum_{f \in \mathfrak{a}^{\lceil tn \rceil} \cap R^{\circ,D}} \operatorname{adj}_D(X, D+\Delta + \frac{1}{n} \operatorname{div} f) \right) \\
&= \operatorname{Ann}_R \operatorname{Ann}_{H^d_{\mathfrak{m}}(\omega_R)} \operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t) \\
&= \operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t).
\end{aligned}$$

□

Theorem 2.6.5. *With notation as in Setting 2.6.1, let $R \hookrightarrow S$ be a pure local $\mathbb{C}$-algebra homomorphism between normal local domains essentially of finite type over $\mathbb{C}$, and $\varphi : Y := \operatorname{Spec} S \to \operatorname{Spec} R = X$ denote the corresponding morphism. Suppose that $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier and the cycle-theoretic pullback $E := \varphi^\natural D$ of D under φ is a prime divisor. Then*

$$\operatorname{adj}_E(Y, E + \varphi^*\Delta, \mathfrak{a}S^t) \cap R \subseteq \operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t).$$

Proof. Choose an integer $m \geqslant 1$ such that $m\Delta$ is a Weil divisor. Note by Proposition 2.4.6 that $\varphi^\natural m\Delta$ has no component equal to E. We take an effective Cartier divisor G on Y whose support contains that of $\varphi^\natural m\Delta$ and which has no component equal to E. We also take an effective $\mathbb{Q}$-Weil divisor Γ on Y such that no component of Γ equal to E, $K_Y + E + \varphi^*\Delta + \Gamma$ is $\mathbb{Q}$-Cartier and

$$\operatorname{adj}_E(Y, E + \varphi^*\Delta, \mathfrak{a}S^t) = \operatorname{adj}_E(Y, E + \varphi^*\Delta + \Gamma, \mathfrak{a}S^t).$$

Let $(R_p)_{p \in \mathcal{P}}$, $(D_p)_{p \in \mathcal{P}}$ and $(E_p)_{p \in \mathcal{P}}$ be approximations of R, D and E, respectively. Fix choices of $I^+_{D_p}$ and $I^+_{E_p}$ so that the diagram

$$\begin{tikzcd}
0 \arrow[r] & I^+_{D_p} \arrow[r] \arrow[d] & R^+_p \arrow[r] \arrow[d] & (R/I_D)^+_p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^+_{E_p} \arrow[r] & S^+_p \arrow[r] & (S/I_E)^+_p \arrow[r] & 0
\end{tikzcd}$$

commutes for almost all p. Considering approximations of Δ, Γ and G, and applying Proposition 1.5.33, Proposition 2.4.7 and Proposition 2.4.9, we find that the following inclusion holds for any rational number $\varepsilon > 0$:

$$I^+_{D_p}(D_p + \Delta_p) \hookrightarrow I^+_{E_p}(E_p + (\varphi^*\Delta)_p + \varepsilon G_p + \Gamma_p)$$

for almost all p. Furthermore, for every nonzero element $f \in R^{\circ,D}$ and every rational number $s > 0$, this inclusion induces an inclusion

$$I^+_{D_p}(D_p + \Delta_p + s\operatorname{div}_{R_p} f_p) \hookrightarrow I^+_{E_p}(E_p + (\varphi^*\Delta)_p + \varepsilon G_p + \Gamma_p + s\operatorname{div}_{S_p} f_p)$$

for almost all p, and consequently, an inclusion

$$\mathcal{B}(I_D, D + \Delta + s\operatorname{div}_R f) \hookrightarrow \mathcal{B}(I_E, E + \varphi^*\Delta + \varepsilon G + \Gamma + s\operatorname{div}_S f).$$

Given an R-module M, we now have the following commutative diagram:

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d] & M \otimes_R S \arrow[d] \\
M \otimes_R \mathcal{B}(I_D, D + \Delta + s\operatorname{div}_R f) \arrow[r] & M \otimes_R \mathcal{B}(I_E, E + \varphi^*\Delta + \varepsilon G + \Gamma + s\operatorname{div}_S f),
\end{tikzcd}$$

where the upper horizontal map is injective due to the purity of the inclusion $R \hookrightarrow S$. Therefore, $0_M^{\mathcal{B}_D(D+\Delta+s\operatorname{div}_R f)}$ can be viewed as a submodule of $0_{M\otimes_R S}^{\mathcal{B}_E(E+\varphi^*\Delta+\varepsilon G+\Gamma+s\operatorname{div}_S f)}$. When combined with Proposition 2.6.4, this yields

$$\begin{aligned}
\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t) &= \bigcap_M \operatorname{Ann}_R \left(\bigcap_{n\geqslant 1} \bigcap_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} 0_M^{\mathcal{B}_D(D+\Delta+\frac{1}{n}\operatorname{div}_R f)} \right) \\
&\supseteq \bigcap_M \operatorname{Ann}_R \left(\bigcap_{n\geqslant 1} \bigcap_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} 0_{M\otimes_R S}^{\mathcal{B}_E(E+\varphi^*\Delta+\varepsilon G+\Gamma+\frac{1}{n}\operatorname{div}_S f)} \right) \\
&\supseteq \bigcap_N \operatorname{Ann}_S \left(\bigcap_{n\geqslant 1} \bigcap_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} 0_N^{\mathcal{B}_E(E+\varphi^*\Delta+\varepsilon G+\Gamma+\frac{1}{n}\operatorname{div}_S f)} \right) \cap R,
\end{aligned}$$

where M and N run through all R-modules and all S-modules, respectively. On the other hand, by an argument similar to the proof of [43, Proposition 3.9] (cf. [23, Proposition 8.23]), we have

$$\begin{aligned}
&\bigcap_N \operatorname{Ann}_S \left(\bigcap_{n\geqslant 1} \bigcap_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} 0_N^{\mathcal{B}_E(E+\varphi^*\Delta+\varepsilon G+\Gamma+\frac{1}{n}\operatorname{div}_S f)} \right) \\
&= \operatorname{Ann}_S \left(\bigcap_{n\geqslant 1} \bigcap_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} 0_{H^e_{\mathfrak{n}}(\omega_S)}^{\mathcal{B}_E(E+\varphi^*\Delta+\varepsilon G+\Gamma+\frac{1}{n}\operatorname{div}_S f)} \right),
\end{aligned}$$

where $e = \dim S$, $\mathfrak{n}$ is the maximal ideal of S and N runs through all S-modules. It follows from Theorem 2.3.7 that

$$\begin{aligned}
&\bigcap_{n\geqslant 1}\ \bigcap_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} 0^{\mathcal{B}_E(E+\varphi^*\Gamma+\varepsilon G+\Delta+\frac{1}{n}\operatorname{div}_S f)}_{H^e_{\mathfrak{n}}(\omega_S)}\\
&=\bigcap_{n\geqslant 1}\ \bigcap_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} \operatorname{Ann}_{H^e_{\mathfrak{n}}(\omega_S)} \operatorname{adj}_E(Y, E+\varphi^*\Delta+\varepsilon G+\Gamma+\frac{1}{n}\operatorname{div}_S f)\\
&=\operatorname{Ann}_{H^e_{\mathfrak{n}}(\omega_S)}\left(\sum_{n\geqslant 1}\ \sum_{f\in\mathfrak{a}^{\lceil tn\rceil}\cap R^{\circ,D}} \operatorname{adj}_E(Y, E+\varphi^*\Delta+\varepsilon G+\Gamma+\frac{1}{n}\operatorname{div}_S f)\right)\\
&=\operatorname{Ann}_{H^e_{\mathfrak{n}}(\omega_S)} \operatorname{adj}_E(Y, E+\varphi^*\Delta+\varepsilon G+\Gamma, (\mathfrak{a}S)^t),
\end{aligned}$$

with the last equality deduced from essentially the same argument as the proof of Lemma 2.6.3 by noting that $R^{\circ,D} \subseteq S^{\circ,E}$. Summing up the above containments and applying Matlis duality (see, for example, [18, Lemma 3.3]), we obtain

$$\begin{aligned}
\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t) &\supseteq \big(\operatorname{Ann}_S \operatorname{Ann}_{H^e_{\mathfrak{n}}(\omega_S)} \operatorname{adj}_E(Y, E+\varphi^*\Delta+\varepsilon G+\Gamma, (\mathfrak{a}S)^t)\big)\cap R\\
&= \operatorname{adj}_E(Y, E+\varphi^*\Delta+\varepsilon G+\Gamma, (\mathfrak{a}S)^t)\cap R.
\end{aligned}$$

As ε approaches zero, the limit results in the desired inclusion

$$\begin{aligned}
\operatorname{adj}_D(X, D+\Delta, \mathfrak{a}^t) &\supseteq \operatorname{adj}_E(Y, E+\varphi^*\Delta+\Gamma, (\mathfrak{a}S)^t)\cap R\\
&= \operatorname{adj}_E(Y, E+\varphi^*\Delta, (\mathfrak{a}S)^t)\cap R.
\end{aligned}$$

□

We now shift our focus to a global setting. First, we recall the definition of purity in the non-affine context.

Definition 2.6.6 ([67, Appendix])**.** A morphism $f : Y \to X$ between Noetherian schemes is said to be *pure* if for all $x \in X$, there exists $y \in Y$ such that $f(y) = x$ and the local ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is pure.

Remark 2.6.7. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine schemes, then $f : Y \to X$ is pure if and only if the induced ring homomorphism $A \to B$ is pure by [25, Lemma 2.2].

In the global setting, Theorem 2.6.5 can be reformulated as follows.

Corollary 2.6.8. *Let $f : Y \to X$ be a pure morphism between normal complex varieties, D be a reduced divisor and Γ be an effective $\mathbb{Q}$-Weil divisor on X that has no common components with D. Suppose that $K_X + D + \Gamma$ is $\mathbb{Q}$-Cartier and the cycle-theoretic pullback $E := f^\natural D$ of D under f is a disjoint union of prime divisors on Y. For any coherent ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ whose zero locus contains no components of D and for any real number $t > 0$, one has*

$$f_*\operatorname{adj}_E(Y, E+f^*\Gamma, (\mathfrak{a}\mathcal{O}_Y)^t)\cap\mathcal{O}_X \subseteq \operatorname{adj}_D(X, D+\Gamma, \mathfrak{a}^t).$$

Proof. Since E is a disjoint union of prime divisors, the same holds true for D. Considering that the question is local, we may assume that both X and Y are spectra of local rings and that both D and E are prime divisors. The assertion is then simply Theorem 2.6.5. $\square$

Next we consider the case where $K_X + D + \Gamma$ is not necessarily $\mathbb{Q}$-Cartier but the log anti-canonical ring $\bigoplus_{i \geqslant 0} \mathcal{O}_X(\lfloor -i(K_X + D + \Gamma) \rfloor)$ is finitely generated.

Lemma 2.6.9. *Let Y be a normal affine variety such that the $\mathcal{O}_Y$-algebra $\bigoplus_{i \geqslant 0} \mathcal{O}(iB)$ is finitely generated for every Weil divisor B on Y and let $V \subseteq Y$ be an open subset.*

(1) Let $V_1 = \operatorname{Spec} H^0(\mathcal{O}_V)$. Then the natural morphism $V \to V_1$ is an open immersion whose complement has codimension greater than or equal to 2.

(2) Suppose that E is a prime divisor and F is an effective $\mathbb{R}$-Weil divisor on Y such that no component of F is equal to E, with their strict transforms on V_1 denoted by E_1 and F_1, respectively. Let $\mathfrak{b} \subseteq \mathcal{O}_Y$ be an ideal not contained in $\mathcal{O}_Y(-E)$ and $t > 0$ be a real number. For any element

$$c \in \operatorname{adj}_E(Y, E + F, \mathfrak{b}^t),$$

there exists an effective $\mathbb{R}$-Weil divisor Δ_1 on V_1 such that $K_{V_1} + E_1 + F_1 + \Delta_1$ is $\mathbb{Q}$-Cartier, no component of Δ_1 is equal to E_1, and

$$c \in \operatorname{adj}_{E_1}(V_1, E_1 + F_1 + \Delta_1, (\mathfrak{b}\mathcal{O}_{V_1})^t).$$

Proof. The assertion follows from an argument similar to that in [67, Lemma 2.6], but we include the proof here for the reader's convenience.

Let D be the divisorial part of $Y \setminus V$, considered as a reduced divisor on Y, and then take the $\mathbb{Q}$-Cartierization $Y' := \operatorname{Proj} \bigoplus_{i \geqslant 0} \mathcal{O}_X(iD) \xrightarrow{\rho} Y$ of D. Since the strict transform $D' := \rho_*^{-1} D$ of D is ρ-ample, the complement $Y' \setminus D'$ is affine. By the choice of D, we note that $V \cong \rho^{-1}(V)$ and $\rho^{-1}(V)$ is an open subset of $Y' \setminus D'$ whose complement has codimension greater than or equal to 2. Consequently, we obtain

$$V_1 = \operatorname{Spec} H^0(\mathcal{O}_V) = \operatorname{Spec} H^0(\mathcal{O}_{Y' \setminus D'}) = Y' \setminus D'.$$

Therefore, the complement of the inclusion $V \cong \rho^{-1}(V) \hookrightarrow Y' \setminus D' \cong V_1$ also has codimension greater than or equal to 2.

Choose an effective $\mathbb{R}$-Weil divisor Δ on Y such that Δ has no component equal to E, $K_Y + E + F + \Delta$ is $\mathbb{Q}$-Cartier and $\operatorname{adj}_E(Y, E + F, \mathfrak{b}^t) = \operatorname{adj}_E(Y, E + F + \Delta, \mathfrak{b}^t)$. Let E', F', and Δ' denote the strict transforms of E, F, and Δ on Y', respectively, and take a log resolution $\pi : \widetilde{Y} \to Y'$ of $(Y', E' + F' + \Delta', \mathfrak{b}\mathcal{O}_{Y'})$ such that the strict transform $\pi_*^{-1} E'$ of E' is smooth and $\mathfrak{b}\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_{\widetilde{Y}}(-\widetilde{B})$ is invertible. Then the condition $c \in \operatorname{adj}_E(Y, E + F, \mathfrak{b}^t)$ is equivalent to the inequality

$$K_{\widetilde{Y}} - \lfloor \pi^*(K_{Y'} + E' + F' + \Delta') + t\widetilde{B} \rfloor + \pi_*^{-1} E' + \operatorname{div}_{\widetilde{Y}} c \geqslant 0.$$

Viewing V_1 as a open subset of Y', we set $\widetilde{V} := \pi^{-1}(V_1)$ and $\pi_{\widetilde{V}} := \pi|_{\widetilde{V}} : \widetilde{V} \to V_1$. Restricting the above inequality to $\widetilde{V}$ yields the inequality

$$K_{\widetilde{V}} - \lfloor \pi_{\widetilde{V}}^*(K_{V_1} + E_1 + F_1 + \Delta_1) + t\widetilde{B}|_{\widetilde{V}} \rfloor + {\pi_{\widetilde{V}}}_*^{-1} E_1 + \mathrm{div}_{\widetilde{V}}\, c \geqslant 0,$$

where Δ_1 is the strict transform of Δ on V_1. This implies that

$$c \in \mathrm{adj}_{E_1}(V_1, E_1 + F_1 + \Delta_1, (\mathfrak{b}\mathcal{O}_{V_1})^t) \subseteq \mathrm{adj}_{E_1}(V_1, E_1 + F_1, (\mathfrak{b}\mathcal{O}_{V_1})^t).$$

□

Theorem 2.6.10. *Let $f : Y \to X$ be a pure morphism between normal complex varieties, D be a reduced divisor and Γ be an effective $\mathbb{Q}$-Weil divisor on X that has no common components with D. Suppose that the $\mathcal{O}_X$-algebra $\bigoplus_{i \geqslant 0} \mathcal{O}_X(\lfloor -i(K_X + D + \Gamma) \rfloor)$ is finitely generated, the $\mathcal{O}_Y$-algebra $\bigoplus_{i \geqslant 0} \mathcal{O}_Y(iB)$ is finitely generated for every Weil divisor B on Y, and the cycle-theoretic pullback $E := f^{\natural} D$ of D is a disjoint union of prime divisors on Y. For any coherent ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ whose zero locus contains no components of D and for any real number $t > 0$, one has*

$$f_* \mathrm{adj}_E(Y, E + f^*\Gamma, (\mathfrak{a}\mathcal{O}_Y)^t) \cap \mathcal{O}_X \subseteq \mathrm{adj}_D(X, D + \Gamma, \mathfrak{a}^t).$$

Proof. Since the question is local, we may assume that X and Y are both affine and D and E are both prime divisors. We use a similar strategy to the proof of [67, Lemma 2.8]. Take any nonzero element

$$c \in \mathrm{adj}_E(Y, E + f^*\Gamma, (\mathfrak{a}\mathcal{O}_Y)^t) \cap \mathcal{O}_X.$$

Let $\pi : X' := \mathrm{Proj} \bigoplus_{i \geqslant 0} \mathcal{O}_X(\lfloor -i(K_X + D + \Gamma) \rfloor) \to X$ be the $\mathbb{Q}$-Cartierization of $-(K_X + D + \Gamma)$, that is, a projective birational morphism such that the strict transform of $-(K_X + D + \Gamma)$ is $\mathbb{Q}$-Cartier and ample. By Lemma 2.5.2, we have

$$\mathrm{adj}_D(X, D + \Gamma, \mathfrak{a}^t) = \pi_* \mathrm{adj}_{D'}(X', D' + \Gamma', (\mathfrak{a}\mathcal{O}_{X'})^t),$$

where D' and Γ' are the strict transforms on X' of D and Γ, respectively. Since $-(K_X' + D' + \Gamma')$ is ample, take an effective Cartier divisor G' on X' that is linearly equivalent to $-m(K_{X'} + D' + \Gamma')$ for sufficiently divisible integer $m \gg 0$. Consequently, $U_1 = X' \setminus G'$ is an affine open subset of X'. As G' varies, the corresponding U_1 cover X'. Therefore, it suffices to show that $c \in \mathrm{adj}_{D'|_{U_1}}(U_1, (D' + \Gamma')|_{U_1}, (\mathfrak{a}\mathcal{O}_{U_1})^t)$.

Set $G := \pi_* G'$, $U := X \setminus (X_{\mathrm{sing}} \cup G)$ and $V := f^{-1}(U) \subseteq Y$, where X_{sing} is the singular locus of X. By [67, Lemma 2.2], $H^0(\mathcal{O}_{U_1}) = H^0(\mathcal{O}_U)$ is a pure subring of $H^0(\mathcal{O}_V)$. By setting $V_1 := \mathrm{Spec}\, H^0(\mathcal{O}_V)$, this inclusion induces the morphism of affine varieties $g : V_1 \to U_1$. Note that $g^*(D'|_{U_1})$ is a prime divisor on V_1 and $g^*(\Gamma'|_{U_1})$ has no component equal to $g^*(D'|_{U_1})$. Applying Lemma 2.6.9, one can find an effective $\mathbb{R}$-divisor Δ_1 on V_1 such that Δ_1 has no component equal to $g^*(D'|_{U_1})$, $K_{V_1} + g^*((D' + \Gamma')|_{U_1}) + \Delta_1$ is $\mathbb{Q}$-Cartier and

$$c \in \mathrm{adj}_{g^*(D'|_{U_1})}(V_1, g^*((D' + \Gamma')|_{U_1}) + \Delta_1, (\mathfrak{a}\mathcal{O}_{V_1})^t).$$

It follows from the definition of U_1 that $K_{U_1} + (D' + \Gamma')|_{U_1}$ is $\mathbb{Q}$-Cartier. Thus, by applying Corollary 2.6.8, we have $c \in \mathrm{adj}_{D'|_{U_1}}(U_1, (D' + \Gamma')|_{U_1}, (\mathfrak{a}\mathcal{O}_{U_1})^t)$. □

The following corollary generalizes Zhuang's result [67, Theorem 2.10] and provides an affirmative answer to his question [67, Question 2.13] in the klt case.

Corollary 2.6.11 (cf. [67, Theorem 2.10])**.** *Let $f : Y \to X$ be a pure morphism between normal complex varieties, D be a reduced divisor and Γ be an effective $\mathbb{Q}$-Weil divisor on X that has no common components with D. Suppose that the cycle-theoretic pullback $E = f^{\natural}D$ of D under f is a reduced divisor on Y. If $(Y, E + f^*\Gamma)$ is of plt type along E, then $(X, D+\Gamma)$ is of plt type along D. In particular, if $(Y, f^*\Gamma)$ is of klt type, then (X, Γ) is of klt type as well.*

Proof. Given that Y is of klt type, the $\mathcal{O}_Y$-algebra $\bigoplus_{i\geqslant 0}\mathcal{O}_Y(iB)$ is finitely generated for every Weil divisor B on Y. The $\mathcal{O}_X$-algebras $\bigoplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i\Gamma\rfloor)$ and $\bigoplus_{i\geqslant 0}\mathcal{O}_X(\lfloor -i(K_X+\Gamma)\rfloor)$ are also finitely generated, as shown in [67, Lemma 2.7]. Note that $f^*\Gamma$ is a $\mathbb{Q}$-Weil divisor due to the finite generation of the former graded ring. Since $(Y, E + f^*\Gamma)$ is of plt type along E, which forces E to be supported on a disjoint union of prime divisors, it follows from Theorem 2.6.10 that

$$\mathrm{adj}_D(X, D+\Gamma) \supseteq f_*\mathrm{adj}_E(Y, E+f^*\Gamma) \cap \mathcal{O}_X = f_*\mathcal{O}_Y \cap \mathcal{O}_X = \mathcal{O}_X.$$

Thus, the pair $(X, D+\Gamma)$ is of plt type along D. $\square$

We conclude this section by focusing on the lc case. The following lemma gives a characterization of lc singularities in terms of multiplier ideals.

Lemma 2.6.12 (cf. [57, Lemma 1.3])**.** *Let X be a normal affine variety over an algebraically closed field of characteristic zero. Take a nonzero element f of the multiplier ideal $\mathcal{J}(X)$. If X is of lc type, then $f \in \mathcal{J}(X, (1-\varepsilon)\operatorname{div} f)$ for every $0 < \varepsilon < 1$. When X is $\mathbb{Q}$-Gorenstein, then converse also holds.*

Proof. The latter assertion follows immediately from [57, Lemma 1.3], and therefore, we suppose that X is of lc type. By taking an m-compatible boundary of the pair $(X, \operatorname{div} f)$ for sufficiently divisible m (see Definition 5.1 and Theorem 5.4 in [14] for the definition and the existence of m-compatible boundaries), we can find an effective $\mathbb{Q}$-Weil divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier, (X,Δ) is lc and $\mathcal{J}(X) = \mathcal{J}(X,\Delta)$. Let $\mu : Y \to X$ be a log resolution of the pair $(X, \Delta + \operatorname{div} f)$ with exceptional divisor $E = \bigcup_i E_i$. The containment $f \in \mathcal{J}(X) = \mathcal{J}(X,\Delta)$ implies that

$$\lceil K_Y - \mu^*(K_X+\Delta)\rceil + \operatorname{div} f \geqslant 0.$$

On the other hand, since (X,Δ) is lc, $\mathrm{ord}_{E_i}(K_Y - \mu^*(K_X+\Delta)) \geqslant -1$ for all i. If $\mathrm{ord}_{E_i}(K_Y - \mu^*(K_X+\Delta)) = -1$, then by the above inequality, $\mathrm{ord}_{E_i}\operatorname{div} f$ must be positive. Therefore, for all $1 > \varepsilon > 0$, one has

$$\lceil K_Y - \mu^*(K_X+\Delta) - (1-\varepsilon)\operatorname{div} f\rceil + \operatorname{div} f = \lceil K_Y - \mu^*(K_X+\Delta) + \varepsilon \operatorname{div} f\rceil \geqslant 0,$$

which is equivalent to saying that $f \in J(X, \Delta + (1-\varepsilon)\operatorname{div} f) \subseteq \mathcal{J}(X, (1-\varepsilon)\operatorname{div} f)$. $\square$

As an application of Corollary 2.6.8, which has its roots in [65, Theorem 1.2], we can provide a partial affirmative answer to another question posed by Zhuang [67, Question 2.11].

Theorem 2.6.13. *Let $f : Y \to X$ be a pure morphism between normal complex varieties, and suppose that X is $\mathbb{Q}$-Gorenstein. Assume in addition that one of the following conditions holds.*

(i) There exists an effective $\mathbb{Q}$-Weil divisor Δ on Y such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier and no non-klt center of (Y, Δ) dominates X.

(ii) The non-klt-type locus of Y has dimension at most one.

If Y is of lc type, then X has lc singularities.

Proof. Since the question is local, we may assume that X and Y are both affine. First, we consider case (i). This condition is equivalent to stating that $\mathcal{J}(Y) \cap \mathcal{O}_X \neq 0$, and therefore, we take a nonzero element $f \in \mathcal{J}(Y) \cap \mathcal{O}_X$. Since Y is of lc type, by Lemma 2.6.12, f lies in $\mathcal{J}(Y, (1-\varepsilon) \operatorname{div} f)$ for all $1 > \varepsilon > 0$. Applying Corollary 2.6.8, we have

$$f \in \mathcal{J}(Y, (1-\varepsilon) \operatorname{div}_Y f) \cap \mathcal{O}_X \subseteq \mathcal{J}(X, (1-\varepsilon) \operatorname{div}_X f)$$

for all $1 > \varepsilon > 0$. Then, by Lemma 2.6.12 once again, we can conclude that X has only lc singularities.

Next, we turn our attention to case (ii). If $\mathcal{J}(Y) \cap \mathcal{O}_X \neq 0$, then we can reduce this to case (i). Thus, we may assume that $\mathcal{J}(Y) \cap \mathcal{O}_X = 0$. Given that $\mathcal{J}(Y)$ defines the non-klt-type locus Z of Y, this implies that Z dominates X. However, by assumption, Z has dimension at most one, which forces X to be the same. Therefore, X is smooth, and in particular, has lc singularities. $\square$

2.7 $\mathcal{B}$-regularity

In this section, as an application of Theorem 2.3.7, we prove the equivalence of some classes singularities introduced by Schoutens (see Subsection 1.5.4 for their definitions).

Proposition 2.7.1. *With notation as in Subsection 1.5.4, then we have*

$$0_E^{\widehat{\mathcal{B}(R)}, K_R} = 0_E^{\mathrm{cl}_{\widehat{\mathcal{B}(R)}}},$$

where E is an injective hull of the residue field of R and the right hand side is defined by

$$0_E^{\mathrm{cl}_{\widehat{\mathcal{B}(R)}}} = \ker(E \to E \otimes_{\widehat{R}} \widehat{\mathcal{B}(R)}).$$

Proof. Let $f \in R$ be a nonzero element such that $rK_R = \operatorname{div}(f)$ for some $r \in \mathbb{N}$ and S be a normal domain such that S is a finite extension of R in $\mathcal{B}(R)$ and contains $f^{1/r}$. Since the reflexive hull $(S \otimes_R \omega_R)^{**}$ is equal to $S(\operatorname{div}(f^{\frac{1}{r}}))$, we have $H^d_{\mathfrak{m}}(S \otimes_R \omega_R) \cong H^d_{\mathfrak{m}}(S(\operatorname{div}(f^{\frac{1}{r}})))$. Hence, we have

$$\begin{aligned} \widehat{\mathcal{B}(R)} \otimes_{\widehat{R}} E &\cong \mathcal{B}(R) \otimes_S H^d_{\mathfrak{m}}(S \otimes_R \omega_R) \\ &\cong \mathcal{B}(R) \otimes_S H^d_{\mathfrak{m}}(S(\operatorname{div}(f^{\frac{1}{r}}))). \end{aligned}$$

Then there exists a commutative diagram

$$\begin{tikzcd}
E \cong H^d_{\mathfrak{m}}(\omega_R) \arrow[r] \arrow[ddd] & \widehat{\mathcal{B}(R)} \otimes_{\widehat{R}} E \arrow[d, "\cong"] \\
& \mathcal{B}(R) \otimes_S H^d_{\mathfrak{m}}(S(\operatorname{div}(f^{\frac{1}{r}}))) \arrow[d, "\mathrm{id} \otimes (\cdot f^{1/r})"] \\
& \mathcal{B}(R) \otimes_S H^d_{\mathfrak{m}}(S) \arrow[d, "\cong"] \\
H^d_{\mathfrak{m}}(\mathcal{B}(R) \otimes_R \omega_R) \arrow[r, "\psi"'] & H^d_{\mathfrak{m}}(\mathcal{B}(R))
\end{tikzcd},$$

where ψ is the second map of

$$\cdot f^{\frac{1}{r}} : H^d_{\mathfrak{m}}(\mathcal{B}(R)) \to H^d_{\mathfrak{m}}(\mathcal{B}(R) \otimes_R \omega_R) \to H^d_{\mathfrak{m}}(\mathcal{B}(R)).$$

The conclusion follows from the above commutative diagram. $\square$

Theorem 2.7.2. *With notation as above. Then the following are equivalent:*

(1) R has log-terminal singularities.

(2) R is ultra-F-regular.

(3) R is weakly generically F-regular.

(4) R is generically F-regular.

(5) R is weakly $\mathcal{B}$-regular.

(6) R is $\mathcal{B}$-regular.

(7) $\widehat{R}$ is $BCM_{\widehat{\mathcal{B}(R)}}$-regular.

Proof. The equivalence of (1) and (2) follows from Proposition 1.5.68 and the equivalence of (1) and (7) follows from Theorem 2.3.7 (let $D = \Delta = 0$). Since if R has log terminal singularities, then every localization of R at a prime ideal is log terminal, it is enough to show the equivalence of (1), (3) and (5). (1) is equivalent to (3) by [62, Theorem 5.24, Proof of Theorem 5.25]. Lastly, we will show the equivalence of (5) and (7). Let E be the injective hull of the residue field of R. By Proposition 2.7.1, we have $0_E^{\mathrm{cl}_{\widehat{\mathcal{B}(R)}}} = 0_E^{\widehat{\mathcal{B}(R)},K_R}$. Hence, $E \to \mathcal{B}(R) \otimes_R E$ is injective if and only if $\widehat{R}$ is $\mathrm{BCM}_{\widehat{\mathcal{B}(R)}}$-regular. $R \to \mathcal{B}(R)$ is pure if and only if $E \to \mathcal{B}(R) \otimes_R E$ is injective by [25, Lemma 2.1 (e)]. $R \to \mathcal{B}(R)$ is pure if and only if $R \to \mathcal{B}(R)$ is cyclically pure by [22, Theorem 1.7]. Therefore, (5) is equivalent to (7). $\square$

Remark 2.7.3. For the equivalence of (5) and (7), see [37, Proposition 6.14].

Chapter 3

F-pure and F-injective singularities in equal characteristic zero

3.1 p-standard sequences and ultraproducts

In this section, we define p-standard sequences following [31] and apply them to the non-standard setting.

Definition 3.1.1 ([31, Definition 2.2])**.** Let R be a Noetherian ring, M be an R-module and d be a positive integer. A sequence $x_1, \ldots, x_d$ in R is said to be a p-*standard sequence* on M if

$$(x_\lambda^{n_\lambda} | \lambda \in \Lambda)M : x_i^{n_i} x_j^{n_j} = (x_\lambda^{n_\lambda} | \lambda \in \Lambda)M : x_j^{n_j}$$

for any positive integers $n_1, \ldots, n_d$, any subset $\Lambda \subsetneq \{1, ..., d\}$ and $i, j \in \{1, \ldots, d\} \setminus \Lambda$ such that $i \leqslant j$.

Given a Noetherian local ring $(R, \mathfrak{m})$, for a finitely generated R-module M, the ideal $\mathfrak{a}(M)$ is defined to be

$$\mathfrak{a}(M) = \prod_{0 \leqslant i < \dim M} \operatorname{Ann}_R H^i_{\mathfrak{m}}(R).$$

Definition 3.1.2 ([31, Definition 3.1])**.** Let R be a Noetherian local ring with a dualizing complex, M be a finitely generated R-module and $d = \dim M$. A system of parameters $x_1, \ldots, x_d$ for M is said to be a p-*standard system of parameters* for M if

$$x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M)$$

for $1 \leqslant i \leqslant d$.

The following are important properties of p-standard systems of parameters.

Proposition 3.1.3. *Suppose that R is a Noetherian local ring with a dualizing complex and M is a finitely generated R-module.*

(1) ([10, p. 482]) There exists a p*-standard system of parameters for M.*

(2) ([31, Theorem 3.3]) A p*-standard system of parameters for M is a* p*-standard sequence on M.*

Lemma 3.1.4. *Let R be a local ring essentially of finite type over $\mathbb{C}$, and M and N be finitely generated R-modules. Then $(\operatorname{Ext}_{R_p}(M_p, N_p))_p$ is an approximation of $\operatorname{Ext}_R(M, N)$.*

Proof. Comparing approximations with reductions modulo p, this follows from [26, Theorem 2.3.5 (e)]. □

Proposition 3.1.5. *Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over $\mathbb{C}$ and M be a finitely generated R-module of* $\dim s$*. If $x_1, \ldots, x_s$ is a* p*-standard system of parameters for M, then $x_{1,p}, \ldots, x_{s,p}$ is a* p*-standard system of parameters for M_p for almost all p.*

Proof. Since $x_1, \ldots, x_s$ is a system of parameters for M, $x_{1,p} \ldots, x_{s,p}$ is a system of parameters for M_p for almost all p. Let $(S, \mathfrak{n})$ be a regular local ring essentially of finite type over $\mathbb{C}$ such that R is isomorphic to a homomorphic image of S and $t = \dim S$. By the local duality, we have

$$H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{n}}(M) \cong \operatorname{Hom}_S(\operatorname{Ext}^{t-i}_S(M, S), E_S)$$

for $1 \leqslant i \leqslant t$, where E_S is the injective hull of $S/\mathfrak{n}$ as an S-module. Hence, we have

$$\operatorname{Ann}_S H^i_{\mathfrak{m}}(M) = \operatorname{Ann}_S \operatorname{Ext}^{t-i}_S(M, S).$$

Similarly, we have $\operatorname{Ann}_{S_p} H^i_{\mathfrak{m}_p}(M_p) = \operatorname{Ann}_{S_p} \operatorname{Ext}^{t-i}_{S_p}(M_p, S_p)$ for almost all p. By Lemma 3.1.4, $(\operatorname{Ext}^{t-i}_{S_p}(M_p, S_p))$ is an approximation of $\operatorname{Ext}^{t-i}_S(M, S)$. Given an element x of $\operatorname{Ann}_S \operatorname{Ext}^{t-i}_S(M, S)$, we have $x_p \in \operatorname{Ann}_{S_p} \operatorname{Ext}^{t-i}_{S_p}(M_p, S_p)$ for almost all p. Therefore, $x_{i,p} \in \mathfrak{a}(M_p/(x_{i+1,p}, \ldots, x_{s,p})M_p)$ for any $1 \leqslant i \leqslant s$ for almost all p, which completes the proof. □

Proposition 3.1.6. *Let R be a reduced local ring essentially of finite type over $\mathbb{C}$ and $\varepsilon \in {}^*\mathbb{N}$. A* p*-standard system of parameters $x_1, \ldots, x_d$ for R is a* p*-standard sequence on $F^{\varepsilon}_* R_\infty$ and R^{upf}.*

Proof. Take any $\varepsilon = \operatorname{ulim}_p e_p \in {}^*\mathbb{N}$. For any $n_1, \ldots, n_d \in \mathbb{N}$, any subset $\Lambda \subsetneq \{1, \ldots, d\}$ and any $i, j \in \{1, \ldots, d\} \setminus \Lambda$ such that $i \leqslant j$, take $y \in (x_\lambda^{n_\lambda} | \lambda \in \Lambda) F^{\varepsilon}_* R_\infty : x_i^{n_i} x_j^{n_j}$. Suppose that $y = \operatorname{ulim}_p y_p = \operatorname{ulim}_p F^{e_p}_* z_p$. Then we have $x_{i,p}^{n_i p^{e_p}} x_{j,p}^{n_j p^{e_p}} z_p \in (x_{\lambda,p}^{n_\lambda p^{e_p}} | \lambda \in \Lambda) R_p$ for almost all p. Since $x_{1,p}, \ldots, x_{d,p}$ is a p-standard sequence on R_p for almost all p by Proposition 3.1.3 and Proposition 3.1.5, $x_{j,p}^{n_j p^{e_p}} z_p \in (x_{\lambda,p}^{n_\lambda p^{e_p}} | \lambda \in \Lambda) R_p$ for almost all p. Therefore, $y_p = F^{e_p}_* z_p \in (x_{\lambda,p}^{n_\lambda} | \lambda \in \Lambda) F^{e_p}_* R_p : x_{j,p}^{n_j}$ for almost all p. Hence, $y \in (x_\lambda^{n_\lambda} | \lambda \in \Lambda) F^{\varepsilon}_* R_\infty : x_j^{n_j}$. Similarly, we can also show the result for R^{upf}. □

Proposition 3.1.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and M be an R-module. Suppose that there exists a system of parameters $x_1, \ldots, x_d$ for R such that $x_1, \ldots, x_d$ is a* p*-standard sequence on M. Then we have*

$$H^i_{(x_1,\ldots,x_t)}(M) = H^0_{\mathfrak{m}}(H^i_{(x_1,\ldots,x_t)}(M))$$

for any $1 \leqslant t \leqslant d$ and $i < t$.

Proof. We work by induction on t. If $t = 1$ and $y \in H^0_{(x_1)}(M)$, then there exists $n \in \mathbb{N}$ such that $x_1^n y = 0$. For any $j \geqslant 1$, we have $x_1^n x_j y = 0$. Since $x_1, \ldots, x_d$ is a p-standard sequence on M, we have $y \in 0 :_M x_1^n x_j = 0 :_M x_j$. Hence, $x_j y = 0$. Since $x_1, \ldots, x_d$ is a system of parameters for R, we have $y \in H^0_{\mathfrak{m}}(H^0_{(x_1)}(M))$. Next, assume that $t > 1$. Take any $n \in \mathbb{N}$ and consider an exact sequence

$$0 \to 0 :_M x_1^n \to M \xrightarrow{\cdot x_1^n} M \to M/x_1^n \to 0.$$

Since $x_1, \ldots, x_d$ is a p-standard sequence on M, $(x_1, \ldots, x_d)(0 :_M x_1^n) = 0$. Hence, we get a long exact sequence

$$\begin{array}{l} 0 \longrightarrow 0 :_M x_1^n \longrightarrow H^0_{(x_1,\ldots,x_t)}(M) \xrightarrow{\cdot x_1^n} H^0_{(x_1,\ldots,x_t)}(M) \longrightarrow H^0_{(x_1,\ldots,x_t)}(M/x_1^n M) \\ \qquad\qquad \longrightarrow H^1_{(x_1,\ldots,x_t)}(M) \xrightarrow{\cdot x_1^n} H^1_{(x_1,\ldots,x_t)}(M) \longrightarrow H^1_{(x_1,\ldots,x_t)}(M/x_1^n M) \\ \qquad\qquad \longrightarrow \cdots \end{array}$$

For any $i < t$ and for any $\eta \in H^i_{(x_1,\ldots,x_t)}(M)$, there exists $n \in \mathbb{N}$ such that $x_1^n \eta = 0$. If $i = 0$, then we have $\eta \in H^0_{\mathfrak{m}}(H^0_{(x_1,\ldots,x_t)}(M))$ since $(x_1, \ldots, x_d)(0 :_M x_1^n) = 0$. If $i > 0$, then there exists an element $\xi \in H^{i-1}_{(x_1,\ldots,x_t)}(M/x_1^n M)$ mapped to η. By the induction hypothesis, there exists $m \in \mathbb{N}$ such that $\mathfrak{m}^m \xi = 0$. Hence, we have $\mathfrak{m}^m \eta = 0$. $\square$

The next corollary is a variant of the Nagel-Schenzel isomorphism.

Corollary 3.1.8. *With notation as in Proposition 3.1.7, for any $0 \leqslant t \leqslant d$, we have*

$$H^t_{\mathfrak{m}}(M) = H^0_{\mathfrak{m}}(H^t_{(x_1,\ldots,x_t)}(M)).$$

Proof. Considering the spectral sequence

$$E_2^{ij} = H^i_{\mathfrak{m}}(H^j_{(x_1,\ldots,x_t)}(M)) \Rightarrow E^{i+j} = H^{i+j}_{\mathfrak{m}}(M),$$

the conclusion follows from Proposition 3.1.7 and the proof of [42, Lemma 3.4]. $\square$

Proposition 3.1.9. *Let $(R, \mathfrak{m})$ be a reduced local ring essentially of finite type over $\mathbb{C}$ and $\varepsilon = \operatorname{ulim}_p e_p \in {}^*\mathbb{N}$. Then the morphisms*

$$H^i_{\mathfrak{m}}(F^{\varepsilon}_* R_\infty) \to \operatorname*{ulim}_p H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p)$$

and

$$H^i_{\mathfrak{m}}(R^{\mathrm{upf}}) \to H^i_{\mathfrak{m}_p}(R_p^{1/p^\infty})$$

are injective for any $i \geqslant 0$.

Proof. Since the proofs are similar, we will only show that $H^i_{\mathfrak{m}}(F^{\varepsilon}_* R_\infty) \to \operatorname{ulim}_p H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p)$ is injective. We may assume $0 \leqslant i \leqslant d$. Let $x_1, \ldots, x_d$ be a p-standard system of parameters for R. By Corollary 3.1.8, we have

$$H^i_{\mathfrak{m}}(F^{\varepsilon}_* R_\infty) \cong H^0_{\mathfrak{m}}(H^i_{(x_1, \ldots, x_i)}(F^{\varepsilon}_* R_\infty))$$

and

$$H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p) \cong H^0_{\mathfrak{m}_p}(H^i_{(x_1, \ldots, x_i)}(F^{e_p}_* R_p))$$

for almost all p. Considering the Čech complex, any element η of $H^i_{(x_1, \ldots, x_i)}(F^{\varepsilon}_* R_\infty)$ can be represented by

$$\left[\frac{y}{(x_1 \cdots x_i)^t}\right],$$

where $y \in F^{\varepsilon}_* R_\infty$ and $t \in \mathbb{N}$. We show that

$$H^i_{(x_1, \ldots, x_i)}(F^{\varepsilon}_* R_\infty) \to \operatorname*{ulim}_p H^i_{(x_{1,p}, \ldots, x_{i,p})}(F^{e_p}_* R_p)$$

is injective. Suppose that the image of η in $\operatorname{ulim}_p H^i_{(x_{1,p}, \ldots, x_{i,p})}(F^{e_p}_* R_p)$ equals zero. Let $x' := x_1 \cdots x_i$. Then there exists $s_p \in \mathbb{N}$ such that $(x'_p)^{s_p} y_p \in (x_{1,p}^{s_p+t}, \ldots, x_{i,p}^{s_p+t}) F^{e_p}_* R_p$. Hence, $(x'_p)^{s_p p^{e_p}} y_p^{p^{e_p}} \in (x_{1,p}^{(s_p+t)p^{e_p}}, \ldots, x_{i,p}^{(s_p+t)p^{e_p}}) R_p$ for almost all p. Since $x_{1,p}, \ldots, x_{i,p}$ is a p-standard sequence of R_p for almost all p by Proposition 3.1.3 and Proposition 3.1.5, we have $(x'_p)^{p^{e_p}} y_p^{p^{e_p}} \in (x_{1,p}^{(t+1)p^{e_p}}, \ldots, x_{i,p}^{(t+1)p^{e_p}}) R_p$ for almost all p by [31, Proposition 2.4]. Therefore we have $x'_p y_p \in (x_{1,p}^{t+1}, \ldots, x_{i,p}^{t+1}) F^{e_p}_* R_p$ for almost all p. Hence, we have $x' y \in (x_1^{t+1}, \ldots, x_i^{t+1}) F^{\varepsilon}_* R_\infty$ and $\eta = 0$ in $H^i_{(x_1, \ldots, x_i)}(F^{\varepsilon}_* R_\infty)$. Then the conclusion follows from the following commutative diagram:

$$\begin{tikzcd}
H^i_{\mathfrak{m}}(F^{\varepsilon}_* R_\infty) \arrow[r, hook] \arrow[d] & H^i_{(x_1, \ldots, x_i)}(F^{\varepsilon}_* R_\infty) \arrow[d, hook] \\
\operatorname*{ulim}_p H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p) \arrow[r, hook] & \operatorname*{ulim}_p H^i_{(x_{1,p}, \ldots, x_{i,p})}(F^{e_p}_* R_p)
\end{tikzcd}.$$

$\square$

3.2 Ultra-F-purity

We introduce a new notion, ultra-F-pure singularities, and show that dense F-pure type descends under pure ring extensions.

Setting 3.2.1. Let $(R, \mathfrak{a}^t)$ be a pair consisted of the following data:

(1) $(R, \mathfrak{m})$ a reduced local ring essentially of finite type over $\mathbb{C}$ of dimension d,

(2) $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$,

(3) $t > 0$ a real number.

Definition 3.2.2. With notation as in Setting 3.2.1, $(R, \mathfrak{a}^t)$ is said to be *sharply ultra-F-pure* if for all $\varepsilon_0 \in {}^*\mathbb{N}$, there exist $\varepsilon \geqslant \varepsilon_0$ and $f \in \mathfrak{a}^{\lceil (t\pi^\varepsilon - 1)\rceil}$ such that $fF^\varepsilon : R \to R_\infty$ is pure. We simply say that R is *ultra-F-pure* if (R, R^t) is sharply ultra-F-pure.

Remark 3.2.3. (1) This definition depends on a choice of ultrafilter on $\mathcal{P}$ and isomorphism $\operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$.

(2) R is ultra-F-pure if and only if $R \to R^{\mathrm{upf}}$ is pure by Proposition 1.5.44.

Example 3.2.4. Let $R = (\mathbb{C}[x, y, z]/(x^3 + y^3 + z^3))_{(x,y,z)}$. If $R_p = (\overline{\mathbb{F}_p}[x, y, z]/(x^3 + y^3 + z^3))_{(x,y,z)}$, then $(R_p)_p$ is an approximation of R for any non-principal ultrafilter $\mathcal{F}$ on $\mathcal{P}$ and any isomorphism $\operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$. We observe that R_p is F-pure if and only if $p \equiv 1 \pmod 3$. Then R is ultra-F-pure if and only if $\{p \in \mathcal{P} | p \equiv 1 \pmod 3\} \in \mathcal{F}$ (cf. Proposition 3.2.6 and the proof of Proposition 3.2.13).

Lemma 3.2.5. *Let F be a subfield of $\mathbb{C}$ such that $F/\mathbb{Q}$ is finitely generated field extension. Given two field homomorphisms $f, g : F \to \mathbb{C}$. Then there exists an field automorphism α of $\mathbb{C}$ such that $g = \alpha \circ f$.*

Proof. Let $\{e_i\}_{i=1}^n$ be a transcendental basis of $F/\mathbb{Q}$. Take $\{a_\lambda\}, \{b_\lambda\} \subseteq \mathbb{C}$ such that $\{f(e_i)\} \cup \{a_\lambda\}$ and $\{g(e_i)\} \cup \{b_\lambda\}$ are transcendental bases of $\mathbb{C}/\mathbb{Q}$. Let $\beta : \mathbb{C} \to \mathbb{C}$ be an automorphism of $\mathbb{C}$ such that $\beta(f(e_i)) = g(e_i)$ and $\beta(a_\lambda) = b_\lambda$ for any i and λ. We define G to be $\mathbb{Q}(g(e_1), \ldots, g(e_n))$. Then $\beta(f(F))/G$ and $g(F)/G$ are finite extensions and $g \circ (\beta \circ f)^{-1} : \beta(f(F)) \to g(F)$ is an isomorphism which fixes G. Hence, there exists an automorphism γ of $\mathbb{C}$ such that γ is an extension of $g \circ (\beta \circ f)^{-1}$. Then $\alpha := \gamma \circ \beta$ is a desired automorphism. $\square$

Proposition 3.2.6. *With notation as in Setting 3.2.1, if $(R, \mathfrak{a}^t)$ is of dense sharply F-pure type, then there exist a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{P}$ and an isomorphism $\alpha : \operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$ such that $(R_p, \mathfrak{a}_p^t)$ is sharply F-pure for almost all p.*

Proof. Suppose that R is a localization of a finitely generated $\mathbb{C}$-algebra S at a prime ideal $\mathfrak{p}$, and $\mathfrak{b} = \mathfrak{a} \cap S$. Since $(R, \mathfrak{a}^t)$ is of dense sharply F-pure type, there exists a model $(A, S_A, \mathfrak{p}_A, \mathfrak{b}_A)$ such that there exists a subset D of $\operatorname{Spm} A$ such that D is dense in $\operatorname{Spec} A$ and for all $\mu \in \operatorname{Spec} A$, a pair $(R_\mu, \mathfrak{a}_\mu^t)$ is sharply F-pure. We define $\varphi : \operatorname{Spm} A \to \mathbb{N}$ by $\varphi(\mu) := \operatorname{char} A/\mu$ and we can write $A \setminus \{0\} = \{x_i\}_{i=1}^\infty$ since A is countable. Inductively take a sequence $\{\mu_i\}_{i=1}^\infty \subseteq D$ such that $\varphi(\mu_i) > \varphi(\mu_{i-1})$ and $x_1, \ldots, x_{i-1} \notin \mu_i$. Let $p_i := \varphi(\mu_i)$ and take a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{P}$ such that $\{p_i | i \in \mathbb{N}\} \in \mathcal{F}$ (see Proposition 1.5.2). For any $i \in \mathbb{N}$, we define $\gamma_{p_i} : A \to \overline{\mathbb{F}_{p_i}}$ to be the composite morphism $A \to A/\mu_i \to \overline{\mathbb{F}_{p_i}}$. Note that γ_p is defined for almost all p in this setting. Then we have a ring homomorphism $\operatorname{ulim}_p \gamma_p : A \to \operatorname{ulim}_p \overline{\mathbb{F}_p}$. Note that $\operatorname{ulim}_p \gamma_p$ is injective since for any $x \in A \setminus \{0\}$, there exists $i_0 \in \mathbb{N}$ such that for any $i \geqslant i_0$, $x \notin \mu_i$ by construction. Since $\operatorname{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$, there exists an isomorphism α such that the diagram

$$\begin{tikzcd}
& A \arrow[dl, "\mathrm{ulim}_p \gamma_p"'] \arrow[dr, hook] & \\
\operatorname*{ulim}_p \overline{\mathbb{F}_p} \arrow[rr, "\alpha"] & & \mathbb{C}
\end{tikzcd}$$

commutes by Lemma 3.2.5. Then $(\gamma_p)_p$ defined above coincides with one in [45, Lemma 4.9]. Hence, approximation S_p of S with respect to $\mathcal{F}$ and α is isomorphic to $S_\mu \otimes \overline{\mathbb{F}_p}$ for almost all p. By Proposition 1.3.5 and the proof of Proposition 1.5.48, $(R_{p_i}, \mathfrak{a}_{p_i}^t)$ is sharply F-pure for all i. Hence, $(R_p, \mathfrak{a}_p^t)$ is sharply F-pure for almost all p. $\square$

Proposition 3.2.7. *With notation as in Setting 3.2.1, suppose that* $\mathfrak{a} = (f_1, \ldots, f_n)$ *and* $(R_p, \mathfrak{a}_p^t)$ *is sharply* F*-pure for almost all* p*. Then for any* $\varepsilon_0 \in {}^*\mathbb{N}$*, there exist* $\varepsilon \geqslant \varepsilon_0$ *and* $\mu_1, \ldots, \mu_n \in {}^*\mathbb{N}$ *such that* $\mu_1 + \cdots + \mu_n = \lceil t(\pi^\varepsilon - 1) \rceil$ *and* $fF^\varepsilon : R \to R_\infty$ *is pure, where* $f = \prod_{i=1}^n f_i^{\mu_i}$*. In particular,* $(R, \mathfrak{a}^t)$ *is sharply ultra-*F*-pure.*

Proof. Take any $\varepsilon_0 = \operatorname{ulim}_p e_{0,p} \in {}^*\mathbb{N}$. Since $(R_p, \mathfrak{a}_p^t)$ is sharply F-pure for almost all p, we have

$$\sum_{e \geqslant e_{0,p}} \sum_{\varphi} \varphi(F_*^e \mathfrak{a}_p^{\lceil t(p^e - 1) \rceil}) = R_p,$$

for almost all p, where φ runs through all elements of $\operatorname{Hom}_{R_p}(F_*^e R_p, R_p)$. Hence, for almost all p, there exist $e_p \in \mathbb{N}$ and $\varphi_p \in \operatorname{Hom}_{R_p}(F_*^{e_p} R_p, R_p)$ such that $\varphi_p(F_*^{e_p} \mathfrak{a}_p^{\lceil t(p^{e_p} - 1) \rceil}) = R_p$. Since

$$\varphi_p(F_*^{e_p} \mathfrak{a}_p^{\lceil t(p^{e_p} - 1) \rceil}) = \sum_{\substack{m_1, \ldots, m_n \\ m_1 + \cdots + m_n = \lceil t(p^{e_p} - 1) \rceil}} \varphi_p(F_*^{e_p} f_{1,p}^{m_1} \ldots f_{n,p}^{m_n} R_p),$$

there exist $m_{1,p}, \ldots, m_{n,p} \in \mathbb{N}$ such that $m_{1,p} + \cdots + m_{n,p} = \lceil t(p^{e_p} - 1) \rceil$ and

$$\varphi_p(F_*^{e_p} f_{1,p}^{m_{1,p}} \ldots f_{n,p}^{m_{n,p}} R_p) = R_p.$$

Let $f_p := f_{1,p}^{m_{1,p}} \ldots f_{n,p}^{m_{n,p}}$, $f := \operatorname{ulim}_p f_p$, $\varepsilon := \operatorname{ulim}_p e_p$ and $\mu_i := \operatorname{ulim}_p m_{i,p}$. Then we have $f_p F^{e_p} : R_p \to R_p$ is pure for almost all p. It is enough to show the cyclic purity by [22]. Take any ideal $I \subseteq R$ and $x \in R$ such that $fF^\varepsilon(x) \in fF^\varepsilon(I) R_\infty$. Then

$f_p F^{e_p}(x_p) \in f_p I_p^{[p^{e_p}]}$ for almost all p. Since $f_p F^{e_p} : R_p \to R_p$ is pure for almost all p, $x_p \in I_p$ for almost all p. Then we have $x \in I = IR_\infty \cap R$ since $R \to R_\infty$ is faithfully flat, which completes the proof. $\square$

Definition 3.2.8. Let R be a local normal $\mathbb{Q}$-Gorenstein domain and r be the minimum positive integer such that $rK_R = \operatorname{div}(f)$ is Cartier, and fix a canonical ideal $R(K_R) = \omega_R \subseteq R$. We define a canonical covering $\widetilde{R}$ of R to be a $\mathbb{Z}/r\mathbb{Z}$-graded R-algebra

$$\bigoplus_{i=0}^{r-1} \omega_R^{(i)} t^i,$$

where $\omega_R^{(i)}$ is the i-th symbolic power of ω_R and $t^r = 1/f$.

Remark 3.2.9. A canonical covering $\widetilde{R}$ of R is a local normal quasi-Gorenstein domain.

Lemma 3.2.10. *Let R be a $\mathbb{Q}$-Gorenstein normal local domain essentially of finite type over $\mathbb{C}$. Let r be the minimum positive integer such that rK_R is Cartier and let $\widetilde{R}$ be a canonical cover of R. Then $(\widetilde{R})_\infty \cong R_\infty \otimes_R \widetilde{R}$.*

Proof. Let S be a normal $\mathbb{Q}$-Gorenstein domain of finite type over $\mathbb{C}$ such that $\omega_S^{(r)}$ is free and $\mathfrak{p} \in \operatorname{Spec} S$ such that $R \cong S_\mathfrak{p}$. Let $\widetilde{S} \cong \bigoplus_{i=0}^{r-1} \omega_S^{(i)} t^i$ such that $(\widetilde{S})_\mathfrak{p} \cong \widetilde{R}$. Take a $\mathbb{Z}$-subalgebra A of $\mathbb{C}$ such that there exist models (A, S_A) and $(A, \widetilde{S}_A)$ as in [64, Theorem 3.8]. Since $(\widetilde{R})_p \cong ((\widetilde{S})_p)_{\mathfrak{p}_p}$ for almost all p, it is enough to show $(\widetilde{R})_\infty \cong R_\infty \otimes_S \widetilde{S}$. This follows from the fact that the reductions modulo $p \gg 0$ of $\widetilde{S}$ as a finite S-module coincide with those as a ring of finite type over $\mathbb{C}$. $\square$

Proposition 3.2.11. *With notation as in Setting 3.2.1, suppose that R is $\mathbb{Q}$-Gorenstein normal, $(R, \mathfrak{a}^t)$ is sharply ultra-F-pure, and $\widetilde{R}$ is a canonical covering of R. Then $(\widetilde{R}, (\mathfrak{a}\widetilde{R})^t)$ is sharply ultra-F-pure.*

Proof. For any $\varepsilon_0 \in {}^*\mathbb{N}$, there exist $\varepsilon = \operatorname{ulim}_p e_p \in {}^*\mathbb{N}$ and $f \in \mathfrak{a}^{\lceil t(\pi^\varepsilon - 1)\rceil}$ such that $fF^\varepsilon : R \to R_\infty$ is pure. Let r be the minimum positive integer such that rK_R is Cartier. Let $x \in \widetilde{S}$ be a homogeneous element with $\deg x = i$ and $\varepsilon \in {}^*\mathbb{N}$. Then $F^\varepsilon(x)$ is a homogeneous element of degree $i\pi^\varepsilon \bmod r$, where $j \equiv i\pi^\varepsilon \bmod r$ if and only if $j \equiv ip^{e_p} \bmod r$ for almost all p. Since p does not divide r for almost all p, if $i \not\equiv 0 \bmod r$, then we have $i\pi^\varepsilon \not\equiv 0 \bmod r$. Hence, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{R} & \xrightarrow{\cdot F^\varepsilon_* f} & F^\varepsilon_* (\widetilde{R})_\infty \\ {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow{\scriptstyle F^\varepsilon_* \mathrm{pr}_0} \\ R & \xrightarrow{\cdot F^\varepsilon_* f} & F^\varepsilon_* R_\infty \end{array},$$

where pr_0 are the 0-th projections with respect to $\mathbb{Z}/r\mathbb{Z}$-grading induced by the definition of $\widetilde{R}$ and Lemma 3.2.10. Tensoring the above diagram with $H^d_{\mathfrak{m}}(\omega_R)$, we have

$$\begin{array}{ccc} H^d_{\mathfrak{m}}(\omega_R) \otimes_R \widetilde{R} & \xrightarrow{\mathrm{id} \otimes (\cdot F^{\varepsilon}_* f)} & H^d_{\mathfrak{m}}(\omega_R) \otimes_R F^{\varepsilon}_*(\widetilde{R}_\infty) \\ \downarrow & & \downarrow \\ H^d_{\mathfrak{m}}(\omega_R) & \xrightarrow{\mathrm{id} \otimes (\cdot F^{\varepsilon}_* f)} & H^d_{\mathfrak{m}}(\omega_R) \otimes_R F^{\varepsilon}_* R_\infty \end{array} .$$

Note that $H^d_{\mathfrak{m}}(\omega_R) \otimes_R \widetilde{R} \cong H^d_{\mathfrak{m}}(\omega_{\widetilde{R}})$. Take $\eta \in H^d_{\mathfrak{m}}(\omega_R) \otimes_R \widetilde{R}$ such that $(\mathrm{id} \otimes (\cdot F^{\varepsilon}_* f))(\eta) = 0$. Since $\mathrm{Soc}_R H^d_{\mathfrak{m}}(\omega_R) = \mathrm{Soc}_{\widetilde{R}} H^d_{\mathfrak{m}}(\omega_{\widetilde{R}})$ by [18, Lemma 2.3], we may assume that $\eta \in \mathrm{Soc}_R H^d_{\mathfrak{m}}(\omega_R)$. Since $fF^{\varepsilon} : R \to R_\infty$ is pure, the bottom horizontal morphism is injective and we have $\eta = 0$. Hence, the top horizontal morphism is also injective. $\square$

Proposition 3.2.12. *With notation as in Setting 3.2.1, suppose that $R \to S$ is a pure local $\mathbb{C}$-algebra homomorphism between reduced local rings essentially of finite type over $\mathbb{C}$. If $(S_p, (\mathfrak{a}_p S_p)^t)$ is sharply F-pure for almost all p, then $(R, \mathfrak{a}^t)$ is sharply ultra-F-pure.*

Proof. By Proposition 3.2.7, for any $\varepsilon_0 \in {}^*\mathbb{N}$, there exist $\varepsilon \geqslant \varepsilon_0$ and $f \in \mathfrak{a}^{\lceil t(\pi^{\varepsilon}-1)\rceil}$ such that $fF^{\varepsilon} : S \to S_\infty$ is pure. Then we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{fF^{\varepsilon}} & R_\infty \\ \downarrow & & \downarrow \\ S & \xrightarrow{fF^{\varepsilon}} & S_\infty \end{array} .$$

Since $R \to S$ and $fF^{\varepsilon} : S \to S_\infty$ are pure, $fF^{\varepsilon} : R \to R_\infty$ is also pure. Therefore, $(R, \mathfrak{a}^t)$ is sharply ultra-F-pure. $\square$

Proposition 3.2.13. *With notation as in Setting 3.2.1, suppose that R is quasi-Gorenstein and $(R, \mathfrak{a}^t)$ is sharply ultra-F-pure. Then $(R, \mathfrak{a}^t)$ is of dense sharply F-pure type.*

Proof. Suppose that $(R_p, \mathfrak{a}_p^t)$ is not sharply F-pure for almost all p. Then, for almost all p, there exists $e_{0,p} \in \mathbb{N}$ such that for any $e \geqslant e_{0,p}$ and any $f \in \mathfrak{a}_p^{\lceil t(p^e-1)\rceil}$, $fF^e : R \to R$ is not pure. Let $\varepsilon_0 := \mathrm{ulim}_p\, e_{0,p}$. Since $(R, \mathfrak{a}^t)$ is sharply ultra-F-pure, there exist $\varepsilon = \mathrm{ulim}_p\, e_p \geqslant \varepsilon_0$ and $f \in \mathfrak{a}^{\lceil t(\pi^{\varepsilon}-1)\rceil}$ such that $fF^{\varepsilon} : R \to R_\infty$ is pure. Then we have a commutative diagram

$$\begin{array}{ccc} H^d_{\mathfrak{m}}(R) & \xhookrightarrow{\cdot F^{\varepsilon}_* f} & H^d_{\mathfrak{m}}(F^{\varepsilon}_* R_\infty) \\ \downarrow & & \downarrow \\ \mathrm{ulim}_p\, H^d_{\mathfrak{m}_p}(R_p) & \xrightarrow{\cdot(\mathrm{ulim}_p F^{e_p}_* f_p)} & \mathrm{ulim}_p\, H^d_{\mathfrak{m}_p}(F^{e_p}_* R_p) \end{array} ,$$

where the injectivity of the right vertical morphism follows from Proposition 3.1.9. Let η be a nonzero element of $\mathrm{Soc}_R H^d_{\mathfrak{m}}(R)$. Then $\eta_p \in \mathrm{Soc}_{R_p} H^d_{\mathfrak{m}_p}(R_p)$ for almost all p. Since $f_p F^{e_p} : R_p \to R_p$ is not pure for almost all p, the image of $\mathrm{ulim}_p \eta_p$ in $\mathrm{ulim}_p H^d_{\mathfrak{m}_p}(F^{e_p}_* R_p)$ is zero. This is a contradiction. Hence, $(R_p, \mathfrak{a}_p^t)$ is sharply F-pure for almost all p. Comparing approximations with reductions modulo p, $(R, \mathfrak{a}^t)$ is of dense sharply F-pure type (cf. Proposition 1.3.5 and the proof of Proposition 3.2.6). □

Theorem 3.2.14. *With notation as in Setting 3.2.1, suppose that R is $\mathbb{Q}$-Gorenstein normal, S is a reduced local ring essentially of finite type $\mathbb{C}$, and $R \to S$ is a pure local $\mathbb{C}$-algebra homomorphism. If $(S, (\mathfrak{a}S)^t)$ is of dense sharply F-pure type, then $(R, \mathfrak{a}^t)$ is also of dense sharply F-pure type.*

Proof. If $(S, (\mathfrak{a}S)^t)$ is of dense sharply F-pure type, then there exist a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{P}$ and an isomorphism $\alpha : \mathrm{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$ such that $(S_p, (\mathfrak{a}_p S_p)^t)$ is sharply F-pure for almost all p by Proposition 3.2.6. Since $R \to S$ is a pure local $\mathbb{C}$-algebra homomorphism, $(R, \mathfrak{a}^t)$ is ultra-F-pure by Proposition 3.2.12. Let $\widetilde{R}$ be a canonical covering of R. Then $(\widetilde{R}, (\mathfrak{a}\widetilde{R})^t)$ is sharply ultra-F-pure by Proposition 3.2.11. Since $\widetilde{R}$ is quasi-Gorenstein and $(\widetilde{R}, (\mathfrak{a}\widetilde{R})^t)$ is sharply ultra-F-pure, $(\widetilde{R}, (\mathfrak{a}\widetilde{R})^t)$ is of dense sharply F-pure type by Proposition 3.2.13. Since $R \to \widetilde{R}$ is finite and split, $(R, \mathfrak{a}^t)$ is also of dense sharply F-pure type. □

3.3 Ultra-F-injectivity

In this section, we discuss a non-standard variant of F-injectivity in the same setting as Setting 3.2.1.

Definition 3.3.1. With notation as in Setting 3.2.1, $(R, \mathfrak{a}^t)$ is said to be *sharply ultra-F-injective* if for any integer i, for any nonzero element $\eta \in H^i_{\mathfrak{m}}(R)$ and for any $\varepsilon_0 \in {}^*\mathbb{N}$, there exist $\varepsilon \geqslant \varepsilon_0$ and $f \in \mathfrak{a}^{\lceil t(\pi^\varepsilon - 1) \rceil}$ such that the image of η under the following composite morphism

$$H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F^\varepsilon_* R_\infty) \xrightarrow{\cdot F^\varepsilon_* f} H^i_{\mathfrak{m}}(F^\varepsilon_* R_\infty)$$

is nonzero.

Remark 3.3.2. If $\mathfrak{a} = R$, then $(R, \mathfrak{a}^t)$ is sharply ultra-F-injective if and only if

$$H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R^{\mathrm{upf}})$$

is injective for all $i \in \mathbb{Z}$. When this condition holds, we say that R is ultra-F-injective.

Proposition 3.3.3. *With notation as in Setting 3.2.1, if $(R, \mathfrak{a}^t)$ is of dense sharply F-injective type, then there exist a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{P}$ and an isomorphism $\alpha : \mathrm{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$ such that $(R_p, \mathfrak{a}_p^t)$ is sharply F-injective for almost all p.*

Proof. By Proposition 1.3.8, the conclusion follows from an argument similar to Proposition 3.2.6. $\square$

Proposition 3.3.4. *With notation as in Setting 3.2.1, assume that $\mathfrak{a} = (f_1, \dots, f_n)$ and $(R_p, (\mathfrak{a}_p)^t)$ is sharply F-injective for almost all p. Then for any $i \in \mathbb{Z}$, for any nonzero element $\eta \in H^i_{\mathfrak{m}}(R)$ and for any $\varepsilon_0 \in {}^*\mathbb{N}$, there exist $\varepsilon \geqslant \varepsilon_0$ and $\mu_1, \dots, \mu_n \in {}^*\mathbb{N}$ such that $\mu_1 + \dots + \mu_n = \lceil t(\pi^\varepsilon - 1) \rceil$ and the image of η under the following composite morphism*

$$H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F^\varepsilon_* R_\infty) \xrightarrow{\cdot F^\varepsilon_* f} H^i_{\mathfrak{m}}(F^\varepsilon_* R_\infty)$$

is nonzero, where $f = \prod_{i=1}^n f_i^{\mu_i}$. In particular, $(R, \mathfrak{a}^t)$ is sharply ultra-F-injective.

Proof. Take any $i \in \mathbb{Z}$, any nonzero element $\eta \in H^i_{\mathfrak{m}}(R)$ and any $\varepsilon_0 = \operatorname{ulim}_p e_{0,p} \in {}^*\mathbb{N}$. Since $H^i_{\mathfrak{m}}(R) \to \operatorname{ulim}_p H^i_{\mathfrak{m}_p}(R_p)$ is injective by Proposition 3.1.9, $\eta_p \in H^i_{\mathfrak{m}_p}(R_p)$ is nonzero for almost all p. By the assumption, for almost all p, there exist $e_p \geqslant e_{0,p}$ and $m_{1,p}, \dots, m_{n,p} \in \mathbb{N}$ such that $m_{1,p} + \dots + m_{n,p} = \lceil t(p^{e_p} - 1) \rceil$ and the image of η_p under the morphism $H^i_{\mathfrak{m}_p}(R_p) \xrightarrow{\cdot F^{e_p}_* f_p} H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p)$ is nonzero, where $f_p := f_{1,p}^{m_{1,p}} \cdots f_{n,p}^{m_{n,p}}$. Let $\varepsilon = \operatorname{ulim}_p e_p$ and $f = \operatorname{ulim}_p f_p$. Then the image of η under the morphism

$$H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot F^\varepsilon_* f} H^i_{\mathfrak{m}}(F^\varepsilon_* R_\infty)$$

is nonzero, which completes the proof. $\square$

Definition 3.3.5 ([8, Section 2])**.** Let R be a ring and S be an R-algebra. A ring homomorphism $R \to S$ is said to be *strongly pure* if for any $\mathfrak{q} \in \operatorname{Spec} S$, $R_{\mathfrak{q} \cap R} \to S_{\mathfrak{q}}$ is pure.

Remark 3.3.6. If $R \to S$ is faithfully flat, then $R \to S$ is strongly pure.

Since strongly pure morphisms are a somewhat limited class, we consider the following condition enough strong to show the descent of ultra-F-injectivity.

Definition 3.3.7. Let $(R, \mathfrak{m})$ be a local ring and S be an R-algebra. A ring homomorphism $R \to S$ is said to satisfy the condition (*) if there exists a prime ideal $\mathfrak{q}$ of S minimal among primes of S lying over $\mathfrak{m}$ such that $R \to S_{\mathfrak{q}}$ is pure.

Example 3.3.8.

(1) Let $R = (\mathbb{C}[xy, xz])_{(xy,xz)}$, $S = (\mathbb{C}[x, y, z])_{(x,y,z)}$, $\mathfrak{q}_1 = (x)S$ and $\mathfrak{q}_2 = (y, z)$. Then $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = (xy, xz)R$. $R \to S_{\mathfrak{q}_1}$ is not pure and $R \to S_{\mathfrak{q}_2}$ is pure. Hence, $R \to S$ satisfies the condition (*) but is not strongly pure.

(2) Let $R = (\mathbb{C}[xz, xw, yz, yw])_{(xz,xw,yz,yw)}$, $S = (\mathbb{C}[x, y, z, w])_{(x,y,z,w)}$. Let $\mathfrak{q}$ be a prime ideal of S minimal among primes of S lying over $\mathfrak{m}$. Then we have $\mathfrak{q} = (x, y)R$ or $\mathfrak{q} = (z, w)R$. In both cases, $R \to S_{\mathfrak{q}}$ is not pure. Hence, $R \to S$ is pure but does not satisfy the condition (*).

Proposition 3.3.9. *With notation as in Setting 3.2.1, suppose that S is a reduced local ring essentially of finite type over $\mathbb{C}$, and a local $\mathbb{C}$-algebra homomorphism $R \to S$ satisfies the condition (*). If $(S_p, (\mathfrak{a}_p S_p)^t)$ is sharply F-injective for almost all p, then $(R, \mathfrak{a}^t)$ is sharply ultra-F-injective.*

Proof. We argue similarly to [11, Theorem 3.8]. Let $\mathfrak{q}$ be a prime ideal that is minimal among primes of S lying over $\mathfrak{m}$ and such that $R \to S_{\mathfrak{q}}$ is pure. Then $R \to S_{\mathfrak{q}}$ is pure and $S_{\mathfrak{q}}/\mathfrak{m}S_{\mathfrak{q}}$ is of dimension zero. Take any $i \in \mathbb{Z}$, any nonzero element $\eta \in H^i_{\mathfrak{m}}(R)$ and any $\varepsilon_0 \in {}^*\mathbb{N}$. Since $R \to S_{\mathfrak{q}}$ is pure, the image of η under the morphism $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}S_{\mathfrak{q}}}(S_{\mathfrak{q}}) \cong H^i_{\mathfrak{q}S_{\mathfrak{q}}}(S_{\mathfrak{q}})$ is nonzero. By Proposition 3.3.4, there exists $\varepsilon \geqslant \varepsilon_0$ and $f \in \mathfrak{a}^{\lceil t(\pi^{\varepsilon}-1)\rceil}$ such that the image of η under the composite morphism

$$H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{q}S_{\mathfrak{q}}}(S_{\mathfrak{q}}) \xrightarrow{\cdot F^{\varepsilon}_* f} H^i_{\mathfrak{q}S_{\mathfrak{q}}}(F^{\varepsilon}_*(S_{\mathfrak{q}})_{\infty})$$

is nonzero. Considering a commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, "\cdot F^{\varepsilon}_* f"] & H^i_{\mathfrak{m}S_{\mathfrak{q}}}(S_{\mathfrak{q}}) \arrow[r, "\cong"] \arrow[d, "\cdot F^{\varepsilon}_* f"] & H^i_{\mathfrak{q}S_{\mathfrak{q}}}(S_{\mathfrak{q}}) \arrow[d, "\cdot F^{\varepsilon}_* f"] \\
H^i_{\mathfrak{m}}(F^{\varepsilon}_* R_{\infty}) \arrow[r] & H^i_{\mathfrak{m}S_{\mathfrak{q}}}(F^{\varepsilon}_*(S_{\mathfrak{q}})_{\infty}) \arrow[r, "\cong"] & H^i_{\mathfrak{q}S_{\mathfrak{q}}}(F^{\varepsilon}_*(S_{\mathfrak{q}})_{\infty})
\end{tikzcd},$$

the image of η under the morphism $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot F^{\varepsilon}_* f} H^i_{\mathfrak{m}}(F^{\varepsilon}_* R_{\infty})$ is nonzero. Therefore, $(R, \mathfrak{a})^t$ is sharply ultra-F-injective. ☐

Proposition 3.3.10. *With notation as in Setting 3.2.1, assume that $(R, \mathfrak{a}^t)$ is sharply ultra-F-injective and $R/\mathfrak{m} \cong \mathbb{C}$. Then $(R, \mathfrak{a}^t)$ is dense sharply F-injective type.*

Proof. Suppose that $(R_p, (\mathfrak{a}_p)^t)$ is not sharply F-injective for almost all p. Then there exists $i \geqslant 0$ such that for almost all p, there exist $e_p \in \mathbb{N}$ and $f_p \in \mathfrak{a}_p^{\lceil t(p^{e_p}-1)\rceil}$ such that $H^i_{\mathfrak{m}_p}(R_p) \xrightarrow{\cdot F^{e_p}_* f_p} H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p)$ is not injective. Let $\varepsilon = \operatorname{ulim}_p e_p \in {}^*\mathbb{N}$ and $f = \operatorname{ulim}_p f_p$. Then we have a commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{m}}(R) \arrow[r] \arrow[d, hook] & H^i_{\mathfrak{m}}(F^{\varepsilon}_* R_{\infty}) \arrow[d, hook] \\
\operatorname*{ulim}_p H^i_{\mathfrak{m}_p}(R_p) \arrow[r] & \operatorname*{ulim}_p H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p)
\end{tikzcd},$$

where the vertical maps are injective by Proposition 3.1.9.

Claim. $\operatorname{Soc}_R H^i_{\mathfrak{m}}(R) \cong \operatorname{Soc}_{R_{\infty}} \Big(\operatorname{ulim}_p H^i_{\mathfrak{m}_p}(R_p)\Big)$.

Proof of Claim. Take a regular local ring $(S, \mathfrak{n})$ essentially of finite type over $\mathbb{C}$ such that R is a homomorphic image of S and let $t = \dim S$. Then

$$H^i_{\mathfrak{m}}(R) \cong H^i_{\mathfrak{n}}(R) \cong \operatorname{Hom}_S(\operatorname{Ext}^{t-i}_S(R, S), E_S),$$

where E_S is the injective hull of $R/\mathfrak{m} \cong S/\mathfrak{n}$ as an S-module. Hence, we have

$$\begin{aligned} \mathrm{Soc}_S\, H^i_{\mathfrak{m}}(R) &\cong \mathrm{Soc}_S(\mathrm{Hom}_S(\mathrm{Ext}^{t-i}_S(R,S), E_S)) \\ &\cong \mathrm{Hom}_S(\mathrm{Ext}^{t-i}_S(R,S), S/\mathfrak{n}). \end{aligned}$$

Therefore,

$$l_R(\mathrm{Soc}_R\, H^i_{\mathfrak{m}}(R)) = l_R(\mathrm{Hom}_S(\mathrm{Ext}^{t-i}_S(R,S), S/\mathfrak{n})).$$

On the other hand,

$$l_R(\mathrm{Hom}_S(\mathrm{Ext}^{t-i}_S(R,S), S/\mathfrak{n})) = l_{R_p}(\mathrm{Hom}_{S_p}(\mathrm{Ext}^{t-i}_{S_p}(R_p,S_p), S_p/\mathfrak{n}_p)$$

for almost all p. By a similar argument, we have

$$l_R(\mathrm{Soc}_R\, H^i_{\mathfrak{m}}(R)) = l_{R_p}(\mathrm{Soc}_{R_p}\, H^i_{\mathfrak{m}_p}(R_p))$$

for almost all p. Hence, $\mathrm{Soc}_{R_\infty}(\mathrm{ulim}_p\, H^i_{\mathfrak{m}_p}(R_p)) \cong \mathrm{ulim}_p(\mathrm{Soc}_{R_p}\, H^i_{\mathfrak{m}_p}(R_p))$ is a finite R_∞-module of length $l_R(\mathrm{Soc}_R\, H^i_{\mathfrak{m}}(R))$. Since $R/\mathfrak{m} \cong \mathbb{C}$ by the assumption, we have $R_\infty/\mathfrak{m}R_\infty \cong \mathbb{C}$. Therefore, $\mathrm{Soc}_{R_\infty}(\mathrm{ulim}_p\, H^i_{\mathfrak{m}_p}(R_p))$ is also a finite R-module of length $l_R(\mathrm{Soc}_R\, H^i_{\mathfrak{m}}(R))$. By Proposition 3.1.9, $H^i_{\mathfrak{m}}(R) \to \mathrm{ulim}_p\, H^i_{\mathfrak{m}_p}(R_p)$ is injective. Hence, the morphism

$$\mathrm{Soc}_R\, H^i_{\mathfrak{m}}(R) \to \mathrm{Soc}_{R_\infty}(\operatorname*{ulim}_p H^i_{\mathfrak{m}_p}(R_p))$$

is an isomorphism.

Since $H^i_{\mathfrak{m}}(R) \xrightarrow{\cdot F^\varepsilon_* f} H^i_{\mathfrak{m}}(F^\varepsilon_* R_\infty)$ is injective by the assumption,

$$\operatorname*{ulim}_p H^i_{\mathfrak{m}_p}(R_p) \xrightarrow{\cdot F^{e_p}_* f_p} \operatorname*{ulim}_p H^i_{\mathfrak{m}_p}(F^{e_p}_* R_p)$$

is injective by the above claim. However, this is a contradiction. Hence, $(R_p, (\mathfrak{a}_p)^t)$ is sharply F-injective for almost all p. Comparing approximations with reductions modulo $p > 0$, $(R, \mathfrak{a}^t)$ is of dense sharply F-injective type. $\square$

Combining above propositions, we get the following theorem.

Theorem 3.3.11. *With notation as in Setting 3.2.1, suppose that S is a reduced local ring essentially of finite type over $\mathbb{C}$, a local $\mathbb{C}$-algebra homomorphism $R \to S$ satisfies the condition (*) and $R/\mathfrak{m} \cong \mathbb{C}$. If $(S, (\mathfrak{a}S)^t)$ is of dense sharply F-injective type, then $(R, \mathfrak{a}^t)$ is of dense sharply F-injective type.*

Proof. By Proposition 3.3.3, there exist a non-principal ultrafilter $\mathcal{F}$ on $\mathcal{P}$ and an isomorphism $\mathrm{ulim}_p \overline{\mathbb{F}_p} \cong \mathbb{C}$ such that $(S_p, (\mathfrak{a}_p S_p)^t)$ is sharply F-injective for almost all p. By Proposition 3.3.9, $(R, \mathfrak{a}^t)$ is sharply ultra- F-pure. Since $R/\mathfrak{m} \cong \mathbb{C}$, by Proposition 3.3.10, $(R, \mathfrak{a}^t)$ is of dense sharply F-injective type. $\square$

Remark 3.3.12. We expect that the conclusion holds even if we only suppose that $R \to S$ is pure because it follows from the main result of Godfrey and Murayama [17] and the weak ordinarity conjecture (see [3]).

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