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A Note on Stochastic Finite Element Method (Part 8)
—An Application to Uncertain Intrinsic Stresses Generated in Frame Structure with Misfits—
確率有限要素法に関するノート（第8報）
—くいちがいのある骨組構造に発生する不確定初期応力への適用—

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1. Introduction

A methodology for stochastic intrinsic stress analysis is presented based on the stochastic finite element method[1]. Modern computational vehicles have enabled us to evaluate structural response against external loading so far. In addition to the residual stress generated locally by welding, intrinsic stress should be expected to prevail within a structure which is fabricated by assembling many members into integrated form, if the members are not finished to as-designed size. Such intrinsic stress cannot be ignored for the purpose to elaborate the analysis of structural safety and reliability. This note deals with the probabilistic intrinsic stress state at large which is resulted from uncertain misfits in problem of quasi-straight beam members assembled into a frame structure.

2. Stochastic FEM in assemblage of members with initial imperfections

First, it is assumed that an element stiffness matrix $[k]$ is expanded in terms of probability variables $\alpha_r (r=1,2,\cdots,m)$ which represent uncertain initial imperfections in the element.

$$[k]=[k^0]+\sum_{r=1}^{m}[k_r^1]\alpha_r+\frac{1}{2}\sum_{r=1}^{m}\sum_{s=1}^{m}[k_{rs}^2]\alpha_r\alpha_s \tag{1}$$

α_r is defined so that the ensemble average $E[\alpha_r]$ equals zero. Second, it is assumed that the initial imperfections of the element at the nodes are reformed by giving a small displacement $\{\tilde{u}\}$ to the nodes. $\{\tilde{u}\}$ is calculated simply by the geometrical relation and is approximated as Eq. (2). This provides the corresponding nodal force $\{\tilde{f}\}$ in the form of Eq. (3).

$$\{\tilde{u}\}=\sum_r\{\tilde{u}_r^1\}\alpha_r+\frac{1}{2}\sum_r\sum_s\{\tilde{u}_{rs}^2\}\alpha_r\alpha_s \tag{2}$$

$$\begin{aligned}\{\tilde{f}\}&=[k]\{\tilde{u}\}\\&=\sum_r[k^0]\{\tilde{u}_r^1\}\alpha_r+\sum_r\sum_s\left(\frac{1}{2}[k^0]\{\tilde{u}_{rs}^2\}+[k_r^1]\{\tilde{u}_s^1\}\right)\alpha_r\alpha_s\\&\equiv\sum_r\{\tilde{f}_r^1\}\alpha_r+\frac{1}{2}\sum_r\sum_s\{\tilde{f}_{rs}^2\}\alpha_r\alpha_s\end{aligned} \tag{3}$$

The reformation generates the initial strain and $\{\tilde{f}\}$ means the nodal forces which restricts this initial strain. Therefore the element stiffness equation in the assemblage of initially imperfect members is given as follows according to the context of conventional FEM for initial strains[2].

$$[k]\{u\}+\{\tilde{f}\}=\{f\} \tag{4}$$

Merging Eq. (4) after proper transformation of coordinates whose matrix is given in the form of Eq. (5), we have the global stiffness equation as Eqs. (6) to (8).

$$[T]=[T^0]+\sum_r[T_r^1]\alpha_r+\frac{1}{2}\sum_r\sum_s[T_{rs}^2]\alpha_r\alpha_s \tag{5}$$

$$[K]\{U\}+\{\tilde{F}\}=\{F\} \tag{6}$$

$$[K]=[K^0]+\sum_{k=1}^{n}[K_k^1]\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}[K_{kl}^2]\alpha_k\alpha_l \tag{7}$$

$$\{\tilde{F}\}=\sum_{k=1}^{n}\{\tilde{F}_k^1\}\alpha_k+\frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n}\{\tilde{F}_{kl}^2\}\alpha_k\alpha_l \tag{8}$$

Suffices k and l vary from one to n, the total number of probability variables involved in the structure. We re-define at this stage the mechanical boundary condition, corresponding unknown displacements, initial nodal forces and global stiffness matrix by $\{F\}$, $\{U\}$, $\{\tilde{F}\}$ and $[K]$ so that Eq. (6) represents the reduced global stiffness equation. Substituting Eqs. (7), (8) and the following (9) assumed for $\{U\}$ into Eq. (6), we have Eqs. (10) to (12) based on the principle of the

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second order perturbation method.

$$\{U\}=\{U^0\}+\sum_k\{U_k^1\}\alpha_k+\frac{1}{2}\sum_k\sum_l\{U_{kl}^2\}\alpha_k\alpha_l \qquad (9)$$

$$\{U^0\}=[K^0]^{-1}\{F\} \qquad (10)$$

$$\{U_k^1\}=-[K^0]^{-1}(\{\tilde{F}_k^1\}+[K_k^1]\{U^0\}) \qquad (11)$$

$$\{U_{kl}^2\}=-[K^0]^{-1}(\{\tilde{F}_{kl}^2\}+[K_k^1]\{U_l^1\}+[K_l^1]\{U_k^1\} \qquad (12)$$
$$+[K_{kl}^2]\{U^0\})$$

Equation (12) is obtained under the conditions ; $[K_{kl}^2]=[K_{lk}^2]$, $\{U_{kl}^2\}=\{U_{lk}^2\}$ and $\{\tilde{F}_{kl}^2\}=\{\tilde{F}_{lk}^2\}$. The unknown vectors $\{U^0\}$, $\{U_k^1\}$ and $\{U_{kl}^2\}$ can be evaluated based on the single computation of $[K^0]^{-1}$. Finally, a stress is evaluated in the form of Eq. (14) through Eq. (13) which is obtained by the substitution of $\{u\}$ of the part of Eq. (9) into Eq. (4).

3. Initial stress analysis of frame structure modeled by quasi-straight beams with uncertain length and curvature

3.1 Stochastic stiffness equation

The stiffness matrix for a curved beam of R in radius of curvature and β in central angle is given by Martin[3], in which only the bending strain energy is considered. We newly extend this stiffness matrix so as to be applicable to the case that the curvature $\rho(=1/R)$ nearly equals zero by means of taking the axial strain energy into account. Namely, the stiffness matrix is given by Eq. (17) for quasi-straight beam whose arc length, curvature, flexural rigidity and radius of gyration are, L, ρ, EI and r respectively as shown in Fig. 1.

$$\{f\}=\{f^0\}+\sum_k\{f_k^1\}\alpha_k+\frac{1}{2}\sum_k\sum_l\{f_{kl}^2\}\alpha_k\alpha_l \qquad (13)$$

$$\sigma=\sigma^0+\sum_k\sigma_k^1\alpha_k+\frac{1}{2}\sum_k\sum_l\sigma_{kl}^2\alpha_k\alpha_l \qquad (14)$$

The expectation and variance of a stress are given subsequently by Eqs. (15) and (16) based on the second order approximation method and by Eqs. (15-a) and (16-a) based on the first order one[4].

$$E[\sigma]=\sigma^0+\frac{1}{2}\sum_k\sum_l\sigma_{kl}^2E[\alpha_k\alpha_l] \qquad (15)$$

$$Var[\sigma]=\sum_k\sum_l\sigma_k^1\sigma_l^1E[\alpha_k\alpha_l]+\sum_k\sum_l\sum_p\sigma_k^1\sigma_{lp}^2E[\alpha_k\alpha_l\alpha_p]$$
$$+\frac{1}{4}\sum_k\sum_l\sum_p\sum_q\{\sigma_{kl}^2\sigma_{pq}^2(E[\alpha_k\alpha_l\alpha_p\alpha_q]$$
$$-E[\alpha_k\alpha_l]E[\alpha_p\alpha_q])\} \qquad (16)$$

$$E[\sigma]=\sigma^0 \qquad (15\text{-a})$$

$$Var[\sigma]=\sum_k\sum_l\sigma_k^1\sigma_l^1E[\alpha_k\alpha_l] \qquad (16\text{-a})$$

$$[k]=\begin{bmatrix}\boldsymbol{k}_{11} & \boldsymbol{k}_{12}\\ \boldsymbol{k}_{21} & \boldsymbol{k}_{22}\end{bmatrix} \qquad (17)$$

$$\boldsymbol{k}_{11}=\frac{EI\rho^3}{G}\begin{array}{c:c}\begin{array}{ccc} u_1 & v_1 & \theta_1\end{array} & \\ \hdashline \begin{bmatrix} A & B & C/\rho^2L\\ & D & E/\rho^2L\\ \text{SYM.} & & F/\rho^3L\end{bmatrix} & \begin{array}{c}p_1\\ q_1\\ m_1\end{array}\end{array} \qquad (17\text{-a})$$

$\boldsymbol{k}_{22}=\{\boldsymbol{k}_{11}$, (1, 2), (2, 3) entities of which have reversed sign$\}$ (17-b)

$$\boldsymbol{k}_{12}^T=\boldsymbol{k}_{21}=\boldsymbol{H}\cdot\boldsymbol{k}_{11} \qquad (17\text{-c})$$

$$\boldsymbol{H}=\begin{bmatrix}-\cos\rho L & -\sin\rho L & 0\\ \sin\rho L & -\cos\rho L & 0\\ (\cos\rho L-1)/\rho & \sin\rho L/\rho & -1\end{bmatrix} \qquad (17\text{-d})$$

$$\left.\begin{array}{l}A=e^2/\rho L-d-r^2\rho^2d,\ B=b-ae/\rho L+r^2\rho^2g\\ C=ad-be+r^2\rho^2(ad-ge),\ D=a^2/\rho L-c-r^2\rho^2f\\ E=ce-ab+r^2\rho^2(fe-ag),\ F=b^2-cd+r^2\rho^2(2bg-cd-fd)+r^4\rho^4(g^2-fd)\\ G=bB+cA+aC/\rho L+r^2\rho^2(2bg-cd-fd)+r^4\rho^4(g^2-fd)+r^2\rho/L\cdot(a^2d+e^2f-2aeg)\end{array}\right\} \qquad (17\text{-e})$$

$$\left.\begin{array}{l}a=\rho L-\sin\rho L,\ b=\cos\rho L+1/2\cdot\sin^2\rho L-1,\ c=3/2\cdot\rho L-2\sin\rho L+1/4\cdot\sin2\rho L\\ d=1/2\cdot\rho L-1/4\cdot\sin2\rho L,\ e=\cos\rho L-1,\ f=1/2\cdot\rho L+1/4\cdot\sin2\rho L,\ g=1/4\cdot(1-\cos2\rho L)\end{array}\right\} \qquad (17\text{-f})$$

The raw expressions as above are not practicable when $\rho\to0$, because the computation such as $0/0$ appears in them. To avoid this, we expand the above $\sin\rho L$ and $\cos\rho L$ and reduce to common denominators in terms of ρ. For instance, Eq. (17-d) is rewritten as Eq. (18), and Eq. (17-a) as Eq. (19).

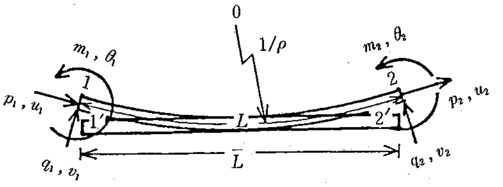

Fig. 1 Straight beam element and its initial imperfection

$$\boldsymbol{H}=\begin{bmatrix}-1+1/2\cdot\rho^2L^2 & -\rho L+1/6\cdot\rho^3L^3 & 0\\ \rho L-1/6\cdot\rho^3L^3 & -1+1/2\cdot\rho^2L^2 & 0\\ \rho L^2(-1/2+1/24\cdot\rho^2L^2) & L(1-1/6\cdot\rho^2L^2) & -1\end{bmatrix} \qquad (18)$$

$$\boldsymbol{k}_{11}=\frac{EI}{(r^2+r^4\rho^2-2/15\cdot r^2L^2\rho^2+1/720\cdot\rho^2L^4)L^3}$$

$$\times\begin{bmatrix}(1+4r^2\rho^2-3/10\cdot\rho^2L^2)L^2 & (1/2\cdot L^2-6r^2)\rho L & (1/12\cdot L^2-3r^2)\rho L^2\\ & 12r^2-4r^2\rho^2L^2+4/15\cdot\rho^2L^4 & (6r^2-3/2\cdot r^2\rho^2L^2+1/20\cdot\rho^2L^4)L\\ \text{SYM.} & & (4r^2+r^4\rho^2-19/30\cdot r^2\rho^2L^2+1/80\cdot\rho^2L^4)L^2\end{bmatrix}$$

$$\equiv\frac{EI}{J}[N] \tag{19}$$

In order to deal with the uncertainty of initial imperfections, the arc length L is expressed as $L=\bar{L}(1+\varepsilon)$ by its expectation $\bar{L}$ and small random veriable ε whose expectation is 0. On the other hand, the curvature ρ itself is re-defined as small random variable because its expectation is zero.

Then, Eq. (18), $[N]$ and J of Eq. (19) can be expanded with respect to ε and ρ, and are given by Eqs. (20) to (22) when higher order terms than the third are truncated.

$$\boldsymbol{H}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&\bar{L}&-1\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&\bar{L}&0\end{bmatrix}\varepsilon+\begin{bmatrix}0&-\bar{L}&0\\\bar{L}&0&0\\-\bar{L}^2/2&0&0\end{bmatrix}\rho$$

$$+\begin{bmatrix}0&-\bar{L}&0\\\bar{L}&0&0\\-\bar{L}^2&0&0\end{bmatrix}\varepsilon\rho+\frac{1}{2}\begin{bmatrix}\bar{L}^2&0&0\\0&\bar{L}^2&0\\0&-\bar{L}^3/3&0\end{bmatrix}\rho^2 \tag{20}$$

$$[N]=\begin{bmatrix}\bar{L}^2&0&0\\&12r^2&6r^2\bar{L}\\\text{SYM.}&&4r^2\bar{L}^2\end{bmatrix}+\begin{bmatrix}2\bar{L}^2&0&0\\&0&6r^2\bar{L}\\\text{SYM.}&&8r^2\bar{L}^2\end{bmatrix}\varepsilon+\begin{bmatrix}0&1/2\cdot\bar{L}^3-6r^2\bar{L}&1/12\cdot\bar{L}^4-3r^2\bar{L}^2\\&0&0\\\text{SYM.}&&0\end{bmatrix}\rho$$

$$+\frac{1}{2}\begin{bmatrix}2\bar{L}^2&0&0\\&0&0\\\text{SYM.}&&8r^2\bar{L}^2\end{bmatrix}\varepsilon^2+\begin{bmatrix}0&(3/2\cdot\bar{L}^2-6r^2)\bar{L}&(1/3\cdot\bar{L}^2-6r^2)\bar{L}^2\\&0&0\\\text{SYM.}&&0\end{bmatrix}\varepsilon\rho$$

$$+\frac{1}{2}\begin{bmatrix}\bar{L}^2(8r^2-3/5\cdot\bar{L}^2)&0&0\\&8\bar{L}^2(-r^2+1/15\cdot\bar{L}^2)&(-3r^2+1/10\cdot\bar{L}^2)\bar{L}^3\\\text{SYM.}&&(2r^4-19/15\cdot r^2\bar{L}^2+1/40\cdot\bar{L}^4)\bar{L}^2\end{bmatrix}\rho^2$$

$$\equiv[N_0]+[N_\varepsilon]\varepsilon+[N_\rho]\rho+1/2\cdot[N_{\varepsilon\varepsilon}]\varepsilon^2+[N_{\varepsilon\rho}]\varepsilon\rho+1/2\cdot[N_{\rho\rho}]\rho^2 \tag{21}$$

$$J=r^2\bar{L}^3+3r^2\bar{L}^3\varepsilon+3r^2\bar{L}^3\varepsilon^2+1/2\cdot\bar{L}^3(2r^4-4/15\cdot r^2\bar{L}^2+1/360\cdot\bar{L}^4)\rho^2$$

$$\equiv J_0+J_\varepsilon\varepsilon+1/2\cdot J_{\varepsilon\varepsilon}\varepsilon^2+1/2\cdot J_{\rho\rho}\rho^2 \tag{22}$$

By the use of the above equations, the final form of $\boldsymbol{k}_{11}$ is expressed as Eq. (23). When the sign of the (1, 2) and (2, 3) entities of it is reversed, $\boldsymbol{k}_{22}$ is obtained. Furthermore, the similar expansion of $\boldsymbol{k}_{12}^T=\boldsymbol{k}_{21}$ can be evaluated by substituting Eqs. (20) and (23) into Eq. (17-c).

$$\boldsymbol{k}_{11}=EI/J_0\cdot[N_0]+EI/J_0\cdot([N_\varepsilon]-J_\varepsilon/J_0\cdot[N_0])\varepsilon+EI/J_0\cdot[N_\rho]\rho+EI/2J_0^2\cdot([N_{\varepsilon\varepsilon}]J_0-[N_0]J_{\varepsilon\varepsilon}-2[N_\varepsilon]J_\varepsilon$$

$$+2J_\varepsilon^2/J_0\cdot[N_0])\varepsilon^2+EI/J_0^2\cdot([N_{\varepsilon\rho}]J_0-[N_\rho]J_\varepsilon)\varepsilon\rho+EI/2J_0^2\cdot([N_{\rho\rho}]J_0-[N_0]J_{\rho\rho}+2J_\rho^2/J_0\cdot[N_0])\rho^2 \tag{23}$$

Putting $\varepsilon=\alpha_1$ and $\rho=\alpha_2$ in the above equations, we have $[k^0]$, $[k_r^1]$ and $[k_{rs}^2]$ $(r,\ s=1,2)$ of Eq. (1). $[k^0]$ is confirmed identical with the stiffness matrix of straight beam whose length is $\bar{L}$.

3.2 Stochastic initial nodal forces

In Fig. 1 a quasi-straight beam with initial imperfections is denoted by *1-2* and as-designed one without imperfection by *1′-2′*. As mentioned in the preceding chapter, nodes *1* and *2* are superposed on *1′* and *2′* respectively in the sense of tangent as well as coordinate. This reformation is approximated by Eq. (24) and the corresponding coefficients of equivalent initial nodal force vector are given by Eq. (25).

$$\left.\begin{aligned}&u_1=-\bar{L}/2\cdot(1-\rho^2L^2/8)+L/2\cdot(1-\rho^2L^2/24),\ v_1=\rho L^2/8-\rho L\bar{L}/4,\ \theta_1=\rho L/2,\\&u_2=-u_1,\ v_2=v_1,\ \theta_2=-\theta_1\end{aligned}\right\} \tag{24}$$

$$\left.\begin{array}{l}\{\tilde{f}_1^1\}=[k^0]\lfloor \bar{L}/2\ 0\ 0\ -\bar{L}/2\ 0\ 0\rfloor^T,\ \{\tilde{f}_2^1\}=[k^0]\lfloor 0\ -\bar{L}^2/8\ -\bar{L}/2\ 0\ -\bar{L}^2/8\ -\bar{L}/2\rfloor^T,\\ \{\tilde{f}_{11}^2\}=[k_1^1]\lfloor \bar{L}/2\ 0\ 0\ -\bar{L}/2\ 0\ 0\rfloor^T,\\ \{\tilde{f}_{12}^2\}=[k^0]\lfloor 0\ 0\ \bar{L}/2\ 0\ 0\ -\bar{L}/2\rfloor^T+[k_1^1]\lfloor 0\ -\bar{L}^2/8\ -\bar{L}/2\ 0\ -\bar{L}^2/8\ -\bar{L}/2\rfloor^T\\ \qquad +[k_1^2]\lfloor \bar{L}/2\ 0\ 0\ -\bar{L}/2\ 0\ 0\rfloor^T\\ \{\tilde{f}_{22}^2\}=[k^0]\lfloor \bar{L}^3/12\ 0\ 0\ -\bar{L}^3/12\ 0\ 0\rfloor^T+2[k_2^1]\lfloor 0\ -\bar{L}^2/8\ -\bar{L}/2\ 0\ -\bar{L}^2/8\ -\bar{L}/2\rfloor^T\end{array}\right\} \quad (25)$$

3.3 Coordinate transformation matrix and evaluation of stress

The quasi-straight beam of Fig. 1 is rorated by θ through the transformation matrix in the form of

$$[T]=\begin{bmatrix} C & S & 0 & 0 & 0 & 0\\ -S & C & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & C & S & 0\\ 0 & 0 & 0 & -S & C & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}+\frac{\bar{L}}{2}\begin{bmatrix} S & -C & 0 & 0 & 0 & 0\\ C & S & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -S & C & 0\\ 0 & 0 & 0 & -C & -S & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}(\rho+\varepsilon\rho)$$
$$+\frac{\bar{L}^2}{8}\begin{bmatrix} -C & -S & 0 & 0 & 0 & 0\\ S & -C & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -C & -S & 0\\ 0 & 0 & 0 & S & -C & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\rho^2 \quad \left(\begin{array}{l}\text{where}\\ S=\sin\theta,\ C=\cos\theta\end{array}\right) \quad (26)$$

The global stiffness matrix and initial nodal force vector are evaluated in the form of Eqs. (7) and (8) through the proper merging of $[T]^T[k][T]$ and $[T]^T[\tilde{f}]$. When the nodal forces are evaluated based on $\{U\}$, this $[T]$ matrix is again introduced. Finally the bending stress σ_b is evaluated in the form of Eq. (14) through $\sigma_b=6m/th^2$ with the moment m of a component of $\{f\}$, where a rectangular cross section of t in width and h in depth are assumed.

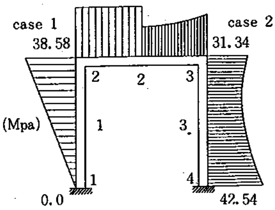

Fig. 2 Distribution of standard deviation of initial bending stress $\sqrt{V_{ar}[\sigma_b]}$(MPa) in one-storeyed portal frame

4. Numerical example

As numerical example, a portal frame is analysed which consists of three members of 1 m long (in design), 2.5 x 2.5cm^2 cross-section and 205.8GPa in Young's modulus, as shows in Fig. 2. External loading is taken equal to zero ($\{F\}=\{0\}$), and all the curvature of member is defined as positive when the center of the radius of curvature lies outside of the structure. These members are considered uncertain in length and straightness. The uncertainties are treated by the said random variables ε and ρ. The deterministic term and rates of change obtained by the present method are verified through comparison with conventional FEM conducted in regard to various values of arc length and curvature. Then, the expectation and variance of bending stress, $E[\sigma_b]$ and $Var\,[\sigma_b]$, are calculated in case of $Var\,[\varepsilon]=0.005^2$ and $Var\,[\rho]=0.01^2(m^{-2})$. As regards the correlation of the six random variables involved in this structure, perfect correlation (case 1) and perfect independency (case 2) are assumed for instance, and the distributions of $\sqrt{Var[\sigma_b]}$ are illustrated along the members in Fig. 2. In this example, $Var\,[\sigma_b]$ is evaluated by the second order approximation with the use of the third and fourth moments of ε and/or ρ given under the assumption of Gaussian distribution. The result hardly differs from the one by the first order approximation, however, within the present case. $E[\sigma_b]$ based on the second order approximation is also so small as to be negligible.

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