研　究　速　報

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A Note on Stochastic Finite Element Method (Part 7)
—Time-history Analysis of Structural Vibration with Uncertain Proportional Damping—
確率有限要素法に関するノート（第7報）
—不確かな比例減衰を有する構造の時刻歴解析—

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1. Introduction

Time-history analyses have been carried out as a part of dynamic analysis of structure subjected to ground motion. Damping influences dynamic response, and the followings are two major means to treat damping. The first one is to introduce damping elements which simulate actual structural components, and is adapted to the study of structural configuration related to the arrangement of damper as shown in Fig. 1. The second one is to take damping into account in form of the ratio to critical damping, which is appropriate to handle damping related to the mode of vibration. In either case, it seems difficult to determine precisely the intensity of damping, as the accuracy of damping identification is dependent on the way of analysis and measurement.

Consequently it might be useful to extend Hoshiya's probabilistic study of single-degree-of-freedom vibration with the first order perturbation[1] to a more general case by our stochastic finite element method[2),3)], which enables us to evaluate accurately and efficiently the response statistics of structure with uncertain parameters. This note proposes a method to deal with probabilistic time-history analysis of linear vibration system on the basis of the second order perturbation technique by regarding uncertain proportional damping as random variable.

2. General equation of motion of continuum with multiple-support excitation

The equation of motion is expressed in matrix form by Eq. (1) with the absolute (total) displacements $\{V^t\}$ of unrestricted n-degree-of-freedom and $\{V_g\}$ of m-degree-of-freedom which represents support-

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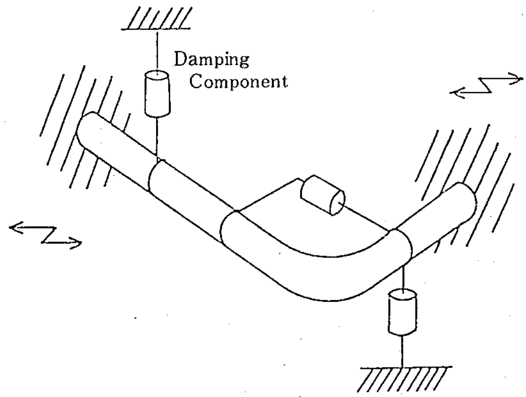

Fig. 1　Structure with multiple-support excitation and damping components

point-motion,

$$[M]\{\ddot{V}^t\}+[C]\{\dot{V}^t\}+[K]\{V^t\} =\{P\}-[M_g]\{\ddot{V}_g\}-[C_g]\{\dot{V}_g\}-[K_g]\{V_g\} \quad (1)$$

where $\{P\}$ is external force vector (taken equal to zero hereafter), the suffix g denotes n x m matrix corresponding to $\{V_g\}$, and $[M]$, $[C]$ and $[K]$ are n x n square mass, damping and stiffness matrices. ($\dot{}$) means differentiation with time. $\{V^t\}$ is given as sum of the dynamic (relative) displacements[4] $\{V\}$ and pseudostatic ones $\{V_s\}$ which are caused by equivalent force $-[K_g]\{V_g\}$ as follows.

$$\{V_s\}=-[K]^{-1}[K_g]\{V_g\}\equiv[R]\{V_g\} \quad (2)$$

The equation of motion is rewritten for the dynamic displacements $\{V\}$ as given below.

$$[M]\{\ddot{V}\}+[C]\{\dot{V}\}+[K]\{V\} =-([M][R]+[M_g])\{\ddot{V}_g\}-([C][R]+[C_g])\{\dot{V}_g\} \quad (3)$$

In the case the damping components are assumed as shown in Fig. 1, $[C]$ in Eq. (3) can be given in concrete form. The second term with $\{\dot{V}_g\}$ on the righthanded side of Eq. (3) remains in the formulation, excepting such cases as the proportionality of $[C]=b[K]$ and $[C_g]=b[K_g]$ holds, but is neglected usually in computation.[4]

3. Modal decoupling of the equation of motion

The eigenvalue ω_i (circular frequency) and eigenvector $\{\phi_i\}$ of ith order are evaluated firstly in the eigenvalue problem Eq. (4) without damping under the condition that all the excitation points are fixed.

$$([K]-\omega_i^2[M])\{\phi_i\}=\{0\} \quad (4)$$

The modal matrix $[\Phi]$ is generated by use of n vectors of $\{\phi_i\}$, which is normalized as $[\Phi]^T[M][\Phi]=[I]$. This normalization gives rise to $[\Phi]^T[K][\Phi]=[K^*]=[\diagdown\omega_i^2\diagdown]$ ($[\diagdown A_i\diagdown]$ means diagonal matrix with diagonal components A_i hereafter). The dynamic displacements $\{V\}$ (as a function of time) is expressed in from of Eq. (5) with the generalized coordinates $\{q^i\}$.

$$\{V\}=[\Phi]\{q^i\} \quad (5)$$

We obtain the equation of motion (6) expressed by $\{q^i\}$, substituting Eq. (5) into Eq. (3) and premultiplying it with $[\Phi]^T$,

$$\{\ddot{q}^i\}+[\Phi]^T[C][\Phi]\{\dot{q}^i\}+[K^*]\{q^i\}$$
$$=-[\Phi]^T([M][R]+[M_g])\{\ddot{V}_g\}-[\Phi]^T([C][R]+[C_g])\{\dot{V}_g\} \quad (6)$$

In case if $[C]$ can be diagonalized by the manipulation with $[\Phi]$ as follows,

$$[\Phi]^T[C][\Phi]=[C^*]=[\diagdown 2\xi_i\omega_i\diagdown] \quad (7)$$

then the vectorical equation (6) is decoupled into the scalar equation (8) with respect to the ith mode.

$$\ddot{q}^i+2\xi_i\omega_i\dot{q}^i+\omega_i^2q^i=-\llcorner\phi_i\lrcorner([M][R]+[M_g])\{\ddot{V}_g\}-\llcorner\phi_i\lrcorner([C][R]+[C_g])\{\dot{V}_g\} \quad (8)$$

This sort of problem with multiple-support excitation may also be solved by superposing the solutions for single excitation point (the other points fixed).[4] When damping is given in terms of the damping ratio ξ_i of ith mode, the damping matrix $[C]$ can be calculated reversely from Eq. (7) as given below.

$$[C]=([\Phi]^T)^{-1}[C^*][\Phi]^{-1}=[M][\Phi][C^*][\Phi]^T[M] \quad (9)$$

4. Perturbation solutions in the generalized coordinate system with uncertain damping ratios

A way of general modal decoupling is formulated in the case of deterministic structural system in the preceding sections. This section describes how the fluctuation of the solution of Eq. (8) in the decoupled form can be evaluated by means of the second order perturbation technique in the case that damping can be diagonalized as Eq. (7), and that the ith damping ratio ξ_i fluctuates in the vicinity of its expectation $\bar{\xi}_i$ through random variable α_i ($E[\alpha_i]=0$ is assumed hereafter) as Eq. (10). In this case the fluctuation of $[C]$ is expressed in form of Eqs. (11) to (13)

$$\xi_i=\bar{\xi}_i(1+\alpha_i) \quad (10)$$

$$[C]=[\bar{C}]+\sum_i[C_i^1]\alpha_i \quad (11)$$

$$[\bar{C}]=[M][\Phi][\bar{C}^*][\Phi]^T[M] \quad (12)$$

$$[C_i^1]=[M][\Phi][C_i^{1*}][\Phi]^T[M] \quad (13)$$

where $[\bar{C}^*]$ is given by Eq. (7) whose ξ_i is replaced with $\bar{\xi}_i$, and $[C_i^{1*}]$ denotes the matrix, the ii-th component of which is $2\bar{\xi}_i\omega_i$ while all the other components are zero. Putting the fluctuation of q^i with respect to ξ_i as Eq. (14) and substituting it together with Eq. (10) into Eq. (8), we obtain the governing equations for the deterministic component $\bar{q}^i$ and the first and second order rates of change, q^{1i} and q^{2i}, on the basis of the second order perturbation technique as follows.

$$q^i=\bar{q}^i+q^{1i}\alpha_i+q^{2i}\alpha_i^2 \quad (14)$$

$$\ddot{\bar{q}}^i+2\bar{\xi}_i\omega_i\dot{\bar{q}}^i+\omega_i^2\bar{q}^i=-\llcorner\phi_i\lrcorner([M][R]+[M_g])\{\ddot{V}_g\} \quad (15)$$

$$\ddot{q}^{1i}+2\bar{\xi}_i\omega_i\dot{q}^{1i}+\omega_i^2q^{1i}=-2\bar{\xi}_i\omega_i\dot{\bar{q}}^i \quad (16)$$

$$\ddot{q}^{2i}+2\bar{\xi}_i\omega_i\dot{q}^{2i}+\omega_i^2q^{2i}=-2\bar{\xi}_i\omega_i\dot{q}^{1i} \quad (17)$$

The term multiplied with $\{\dot{V}_g\}$ in Eq. (8) is neglected herein as usual, and perturbation technique does not hold any more if this term remains, The left-handed sides of Eqs. (15) to (17) are of the same pattern, implying that the same time integration scheme can be applied to the calculation of $\bar{q}^i$, q^{1i} and q^{2i}.

5. Mode decoupling in case of uncertain [C] matrix

In general, $[M]$, $[C]$ and $[K]$ matrices cannot be diagonalized by the manipulation of pre-and post-multiplication with $[\Phi]^T$ and $[\Phi]$, when fluctuation is involved in them. However, the mode decoupling as shown by Eqs. (6) to (8) still is possible with the aid of perturbation technique. An example is given in the follwing in the case that only $[C]$ fluctuates. When $[C]$ takes the following from

$$[C]=a[M]+b[K] \quad (18)$$

of Rayleigh (proportional) damping, $[C]$ can be diagonalized by the product manipulation of $[\Phi]$, since $[M]$ and $[K]$ are deterministic. On the other hand, Eq. (18) does not hold, if $[C]$ is not dependent of $[M]$ and/or $[K]$ and fluctuates through $[C_k^1]$ and α_k corresponding to components as follows.

$$[C]=[\bar{C}]+\sum_k [C_k^1]\alpha_k \quad (19)$$

The matrix $[C]$ cannot be diagonalized in this case. Expressing the generalized coordinate vector $\{q^i\}$ as Eq. (20) for the damping fluctuation α_k of Eq. (10) and substituting it into Eq. (6), we obtain the governing equations for the deterministic component and rates of change with the aid of the second order perturbation technique as Eqs. (21) to (23),

$$\{q^i\}=\{\bar{q}^i\}+\sum_k\{q_k^{1i}\}\alpha_k+\frac{1}{2}\sum_k\sum_l\{q_{kl}^{2i}\}\alpha_k\alpha_l \quad (20)$$

$$\{\ddot{\bar{q}}^i\}+[\Phi]^T[\bar{C}][\Phi]\{\dot{\bar{q}}^i\}+[K^*]\{\bar{q}^j\} = -[\Phi]^T([M][R]+[M_g])\{\ddot{V}_g\}-[\Phi]^T([\bar{C}][R]+[\bar{C}_g])\{\dot{V}_g\} \quad (21)$$

$$\{\ddot{q}_k^{1i}\}+[\Phi]^T[\bar{C}][\Phi]\{\dot{q}_k^{1i}\}+[K^*]\{q_k^{1i}\} = -[\Phi]^T[C_k^1][\Phi]\{\dot{\bar{q}}^i\} -[\Phi]^T([C_k^1][R]+[C_{gk}^1])\{\dot{V}_g\} \quad (22)$$

$$\{\ddot{q}_{kl}^{2i}\}+[\Phi]^T[\bar{C}][\Phi]\{\dot{q}_{kl}^{2i}\}+[K^*]\{q_{kl}^{2i}\} = -[\Phi]^T([C_k^1][\Phi]\{\dot{q}_l^{1i}\}+[C_l^1][\Phi]\{\dot{q}_k^{1i}\}) \quad (23)$$

where k and l vary from unity to the total number of fluctuation origins, and the equality $\{q_{kl}^{2i}\}=\{q_{lk}^{2i}\}$ is assumed in the above.

It can be seen in the above that modal decoupling into the same form as Eq. (8) is made possible for the governing equations, if only the matrix $[\bar{C}]$ is diagonalized as $[\Phi]^T[\bar{C}][\Phi]=[\diagdown C_i \diagdown]$, while $[C]$ cannot be diagonalized. Also the same scheme of time integral is applicable to the determination of the deterministic component and rates of change. In this case, q^i does not fluctuate with ξ_i of the section 4, but with the causes of the fluctuation k and l.

6. Numerical example

The fluctuation of time-history response is calculated by use of the Newmark β method ($\beta=1/4$) based on the technique stated in the section 4 in the case that ξ_i fluctuates. The structure under interest is a tower[5] which is modeled as a serial assembly of fourteen beam elements of hollow cylinder whose data are given in Table 1. The bottom is fixed to the base, to which El Centro 1940 NS acceleration wave $\ddot{V}_g$ is applied directly. Consistent mass matrix is employed, and therefore $[M_g]$ is non-zero matrix in this example. The Young's modulus and mass density of the beam material are taken equal to 205.9 GPa and 0.7959×10^3 kgf·sec²/m⁴, respectively.

Table 1 Element division and its data of tower-like structure

Elem.	Length (m)	Dia. (m)	Wall thickness (cm)
1	2.0	2.656	1.4
2	3.4	2.6	4.2
3	2.6	2.6	3.8
4	3.0	2.6	3.8
5	3.0	2.6	3.8
6	0.7	2.1	4.5
7	1.0	1.6	4.5
8	2.0	1.6	1.6
9	3.0	1.6	1.6
10	3.0	1.6	1.6
11	3.0	1.6	1.6
12	3.0	1.6	1.6
13	3.0	1.6	1.6
14	3.0	1.6	1.6

Eigenvalue analysis of this example gives the eigen circular frequencies up to sixth order as 17.05, 56.36, 160.81, 505.09 and 822.57 rad/sec and the related eigenvectors. The time-history analysis is carried out by use of the first six modes and by taking $\bar{\xi}_1=\bar{\xi}_2=\bar{\xi}_3=\bar{\xi}_4=\bar{\xi}_5=\bar{\xi}_6=0.05$. This is because the displacement solution is converged sufficiently when the modes up to sixth are employed. The pth degree of freedom of actual dislacements V_p is expressed by Eq. (24),

$$V_p=\sum_{i=1}^{6}\phi_p^i q^i \quad (24)$$

and its expectation and variance are evaluated based on the second order approximation as Eqs. (25) and (26).

$$E[V_p]=\sum_{i=1}^{6}\phi_p^i(\bar{q}^i+q^{2i}E[\alpha_i^2]) \quad (25)$$

$$Var[V_p]=\sum_{i=1}^{6}\sum_{j=1}^{6}\phi_p^i(\bar{q}^i\bar{q}^j+\bar{q}^i q^{2i}E[\alpha_i^2]+q^{1i}q^{1j}E[\alpha_i\alpha_j]+\bar{q}^j q^{2j}E[\alpha_j^2]+q^{2i}q^{1j}E[\alpha_i^2\alpha_j]+q^{1i}q^{2j}E[\alpha_i\alpha_j^2]+q^{2i}q^{2j}E[\alpha_i^2\alpha_j^2])-(E[V_p])^2 \quad (26)$$

If normal dintribution is assumed for the random variable α_i, the third and fourth moments are given by the following formulae.[6]

$$E[\alpha_i\alpha_j\alpha_p]=0 \quad (27)$$

$$E[\alpha_i\alpha_j\alpha_p\alpha_l]=E[\alpha_i\alpha_j]E[\alpha_p\alpha_l]+E[\alpha_i\alpha_p]E[\alpha_j\alpha_l]+E[\alpha_i\alpha_l]E[\alpha_j\alpha_p] \quad (28)$$

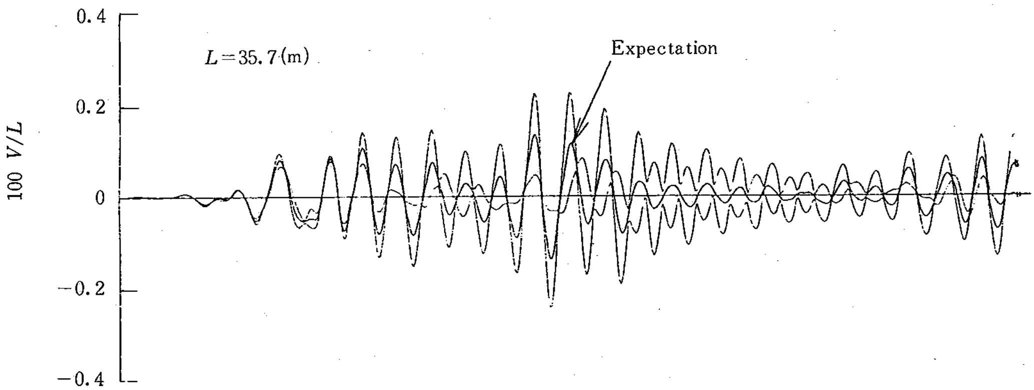

Fig. 2　Expectation and 3-σ bounds of top deflection of example structure

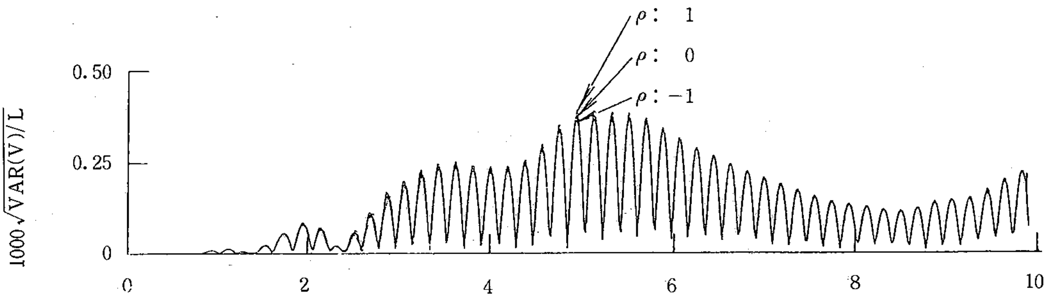

Fig. 3　Effect of correlation coefficient of α_i on standard deviation of top deflection of example structure

Thus the variance of Eq. (26) can be computed when the covariance matrix of α_i is given once, and the assumption is made as above. Figure 2 shows the time-history of the expectation and 3-σ bounds of the top deflection of the example tower in the case of 0.5 for the standard deviations of α_i and $\rho=1$ of the correlation coefficient between α_i and α_j ($i, j=1\sim6$). The standard deviations σ of the top deflection are compared in Fig. 3 in the cases of $\rho=1$, 0 and -1. As is shown, the difference of the correlation coefficient ρ hardly influences the response standard deviation, and the reason why seems to be the conspiquousness of the effect of α_1 over the others, because the primary mode is predominant in this case.

7. Conclusions

The general equation of motion of linear vibration system with multiple-support exctation is prenented. The relationship between the fluctuation of damping ratio and that of response is formulated based on the modal decoupling and the second order perturbation technique. The modal decoupling is discussed in the case that damping matrix is not diagonalized. Through the numerical example of the time-history analysis of a tower-like structure subjected to seismic wave, the effect of the uncertainty of damping ratio is evaluated quantitatively to evidence the applicability of the stochastic finite element method to time-history analyses. (Manuscript received. March 17, 1983)

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