研 究 速 報
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A Note on Stochastic Finite Element Method (Part 6)
— An Application in Problems of Uncertain Elastic Foundation —
確率有限要素法に関するノート (第6報)
—不確かな弾性床上の梁に対する応用—

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1. Introduction

Application of the stochastic finite element method is not limited in the field of mechanical engineering. The interaction between structure and foundation has attracted attention from both mechanical engineers and civil engineers in recent years. The foundation for liquid storage tanks and related pipelines are modeled in some cases as elastic one whose behavior is simplified and represented by the so-called modulus of foundation. However, it should be difficult to deal with the modulus as precise and definite value owing to much simplification of foundation reaction, in which creep and compaction phenomena are included in usual. This note presents another study of the stochastic finite element method [1)~5)] applied to the case of uncertain elastic foundation, and the numerical result is scrutinized in the light of the analytical solution by Bolotin[6)].

2. Stiffness matrix representation of Winkler type foundation

In the case that elastic foundation is of Winkler type, the reaction of the foundation is described by the modulus of foundation λ which is defined as the ratio of acting pressure p to settlement w. Assumption is made further that no gap takes place between the structure and foundation. Let us consider a beam element of l in length subjected to downward distributed pressure p and placed on the foundation whose modulus is λ. Then the beam faces to the upward reaction λw in proportion to the settlement w which is taken equal to the beam deflection. This furnishes the following expression of the potential energy of the beam element under interest.

$$\pi=\int_0^l \frac{1}{2}\sigma\varepsilon dvol-\int_0^l\left(p-\frac{1}{2}\lambda w\right)wdarea \qquad (1)$$

By use of the relevant interpolation function, the first term of Eq. (1) gives rise to the conventional stiffness matrix $[k]$, and the second one to the equivalent nodal force vector $\{f\}$ and the third one to the foundation reaction matrix $[g]$. Cubic polynomials of usual use are taken as the interpolation function in this note, and then the matrix $[g]$ is obtained as follows,

$$[g]=\frac{\lambda A}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ \text{SYM.} & & & 4l^2 \end{bmatrix} \qquad (2)$$

where A denotes the contact area of the beam element with the foundation. Now that the foundation reaction is represented by Eq. (2), the static problems of beam resting on Winkler type foundation can be solved by means of the stiffness equation given below,

$$([K]+[G])\{U\}=\{F\} \qquad (3)$$

where the upper case letters mean the matrices and vectors resulting from the merging of element variables into the global coordinate system. It should be noted that the matrix $[g]$ is not singular as like consistent mass matrix. It goes without saying that similar formulation holds in case of bottom plate of tanks when proper interpolation function for plate element is applied.

3. Second order perturbation solution due to uncertainty in modulus of foundation

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Only the modulus λ is considered uncertain in this note and we express it in form of Eq. (4),

$$\lambda=\bar{\lambda}(1+\varepsilon) \tag{4}$$

where $\bar{\lambda}$ is the expectation, and ε means random variable of zero in its expectation. The random variable ε is taken uniform in an element. Then the stochastic reaction matrix is given in the followings,

$$[g]=[g^0]+[g^1]\varepsilon \tag{5}$$

$$[G]=[G^0]+\sum_r[G_r^1]\varepsilon_r \tag{6}$$

$[g^0]=[g^1]=$Eq. (2) whose λ is replaced by $\bar{\lambda}$.

where the superscript 0 means deterministic term hereafter, r the element number, and Σ the merging with respect to the elements. Correspondingly, the unknown displacements are assumed as follows,

$$\{U\}=\{U^0\}+\sum_r\{U_r^1\}\varepsilon_r+\sum_r\sum_s\{U_{rs}^2\}\varepsilon_r\varepsilon_s \tag{7}$$

where s is the element number as well as r counting the number of the elements subjected to uncertainty in the modulus. Substituting Eqs. (6) and (7) into Eq. (3) and applying the second order perturbation technique to the stiffness equation, we have

$$([K]+[G^0])\{U^0\}=\{F\} \tag{8}$$

$$([K]+[G^0])\{U_r^1\}=-[G_r^1]\{U^0\} \tag{9}$$

$$([K]+[G^0])\{U_{rs}^2\}=-[G_r^1]\{U_s^1\} \tag{10}$$

and these enable us to evaluate successively $\{U^0\}$, $\{U_r^1\}$ and $\{U_{rs}^2\}$ based upon single matrix inversion of $([K]+[G^0])^{-1}$.

4. Analysis of a pipeline on Winkler type foundation

Bolotin[6] analytically derived the power spectrum of the fluctuation of deflection component $S_w(k)$ around the expectation in the case that infinitely long pipeline is placed on Winkler type foundation, the modulus of which varies along the pipeline as homogeneous one-dimensional spatial stochastic process with power spectrum $S_\lambda(k)$. The result is given as

$$S_w(k)=(p/\bar{\lambda})^2S_\lambda(k)/\{\bar{\lambda}(1+k^4/k_0^4)\}^2,\quad k_0=(\bar{\lambda}/EI)^{1/4} \tag{11}$$

where k is the angular wave number, and EI is the flexural rigidity of the pipeline. The fluctuation of λ is assumed so small that higher than second order term can be neglected in Bolotin's derivation. The autocorrelation function of w is also given as

$$R_w(\Delta x)=\int_{-\infty}^{\infty}(p/\bar{\lambda}^2)\frac{S_\lambda(k)e^{ik\Delta x}}{\{\bar{\lambda}(1+k^4/k_0^4)\}^2}\cdot dk \tag{12}$$

where Δx is taken as distance.

Using $S_\lambda(k)$ in the following form of

$$S_\lambda(k)=a^2/4\cdot\{\delta(k-k^*)+\delta(k+k^*)\} \tag{13}$$

as an example, we have the autocorrelation function of w given below.

$$R_w(\Delta x)=(p/\bar{\lambda})^2(a^2/2)\frac{\cos(k^*\Delta x)}{\{\bar{\lambda}(1+k^4/k_0^4)\}^2} \tag{14}$$

To compare with such an analytical result, a pipeline of $L=200$ m in whole length and of 0.01 m in wall thickness is analysed numerically by the present stochastic finite element method in the case of the distributed load of $p=7.795$ N/m equivalent to oil-filled condition. The autocorrelation of w between the points k and l is evaluated on the basis of the following equations by use of the first rate of change in nodal displacement $\{U_r^1\}$ correspondingly to the first order approximation analysis by Bolotin, as follows,

$$R_w(x_k,x_l)=E[\sum_r U_{kr}^1\varepsilon_r\sum_r U_{lr}^1\varepsilon_r]=\sum_r\sum_s U_{kr}^1U_{ls}^1E[\varepsilon_r,\varepsilon_s] \tag{14}$$

$$E[\varepsilon_r\varepsilon_s]=1/\bar{\lambda}^2\cdot R_\lambda(x_s-x_r)=1/\bar{\lambda}^2\cdot\int_{-\infty}^{\infty}a^2/4\cdot\{\delta(k-k^*)+\delta(k+k^*)\}e^{ik(x_s-x_r)}\cdot dk=(1/\bar{\lambda}^2)(a^2/2)\cos k^*(x_s-x_r) \tag{15}$$

where $R_\lambda(x_s-x_r)$ is the autocorrelation of λ, x_s and x_r the coordinate at the center of the beam elements s and r, as shown in Fig. 1, x_k and x_l the coordinate of the nodes k and l. In case if the stochastic process of the deflection (settlement) w is homogeneous along

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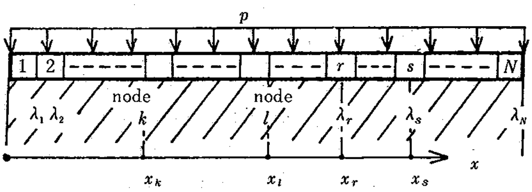

Fig. 1　Beam elements resting on Winkler type foundation with uncertain modulus

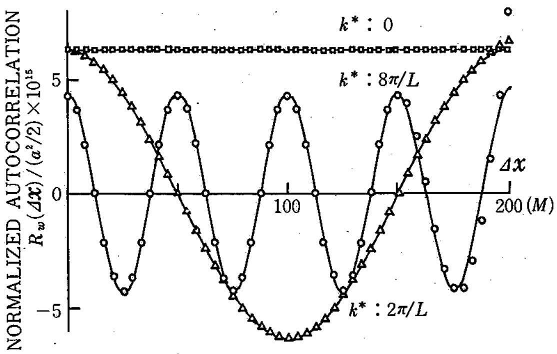

Fig. 2　Comparison between Bolotin's analytical solution and stochastic finite element solution

the x-axis, $R_w(x_k, x_l)$ is replaced by $R_w(x_l-x_k)$. Although the present example is not this case because of the finite length of pipeline, the result of Eq. (15) may be compared approximately with that of Eq. (14) when the points k and l are taken distant from the pipe ends.

Figure 2 depicts the comparison stated above in the cases of $k^*=0$, $2\pi/L$ and $8\pi/L$, showing that very good agreement is obtained between the stochastic finite element solution using 48 elements and the analytical solution excepting the narrow range around $\Delta x=200$ m where the effect of the pipe ends becomes influential. The Young's modulus E is taken equal to 2.059×10^5 MPa, and $N=48$ is the number of elements, by which the solution R_w converges sufficiently for $k^*=8\pi/L$.

Following the above verification of the validity of the present methodology, we carry out the stochastic finite element analysis taking $N=48$ and the input autocorrelation functions of Eq. (17) (case 1) and Eq. (18) (case 2), instead of Eq. (16).

$$E[\varepsilon_r\varepsilon_s]=\frac{2aB}{\bar{\lambda}^2}\frac{\sin 2\pi B(x_s-x_r)}{2\pi B(x_s-x_r)} \tag{17}$$

$$E[\varepsilon_r\varepsilon_s]=\frac{2aB}{\bar{\lambda}^2}\frac{\sin \pi B(x_s-x_r)}{\pi B(x_s-x_r)}\times\cos 2\pi f^*(x_s-x_r) \tag{18}$$

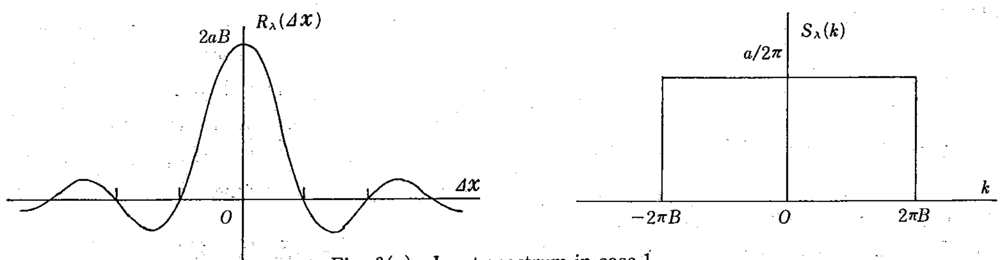

Fig. 3(a)　Input spectrum in case 1

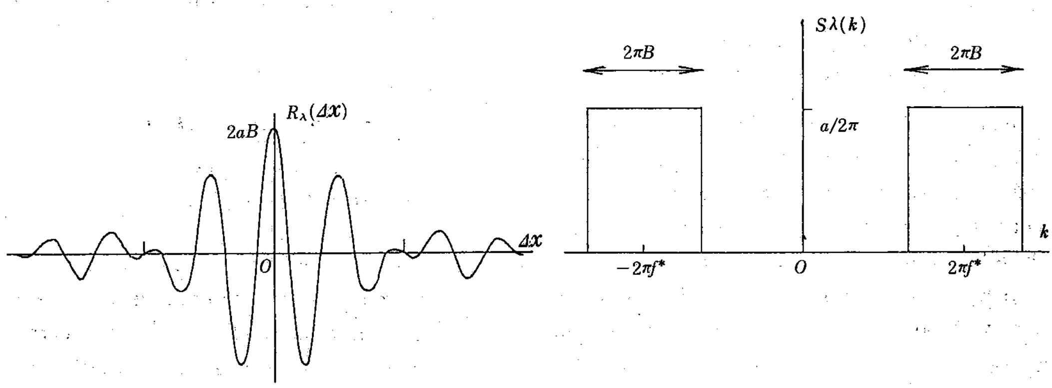

Fig. 3(b)　Input spectrum in case 2

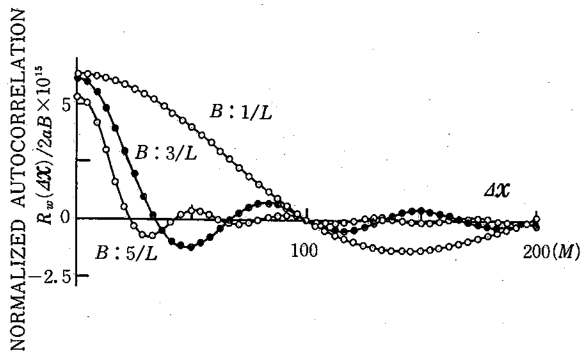

Fig. 4(a)　Output autocorrelation function of case 1

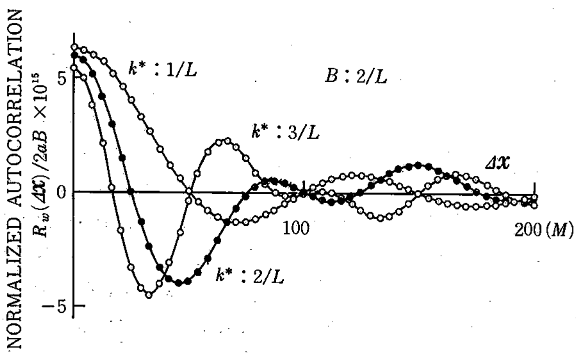

Fig. 4(b)　Output autocorrelation function of cace 2

The shapes of the used autocrrelation functions and their corresponding power spectra are illustrated in Figs. 3(a) and (b), and the output autocorrelations of the settlement w in Figs. 4(a) and (b) respectively. It is obvious that nearly equal solution to infinte beam is obtained excepting the vicinity of $\Delta x = 200$ m, since the end effect is negligibly small over the major part of the finite beam, as is expected from Fig. 2.

5. Conclusions

The behavior of a pipe resting on foundation, whose modulus λ is uncertain, is analysed by means of the proposed stochastic finite element method. The result is in very good agreement with an available analytical solution. It is also shown that the present method can output not only the expectation and variance but also the autocorrelation or the power spectrum of the solution.

This method features its simplicity in formulation and straightforward procedure versatile, which fact is of importance, in practical cases where input spectrum is not homogeneous or boundary conditions are complicated so that any analytical treatment is no longer possible.

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