研 究 速 報

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A Note on Stochastic Finite Element Method (Part 5)
——A Framework for Structural Safety and Reliability——

確率有限要素法に関するノート(第5報)
——構造安全性・信頼性へのフレームワーク——

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1. Introduction

The objective of the present note is to discuss the adaptation of the stochastic finite element method the authors have been developing[1),2),3),4)] to structural safety and reliability. Various uncertainties are involved in actual structures and are regarded as random variables. The behaviours of such structures are well analysed by our method without using any simulation technique, which fact is of great importance from the aspect of CPU time especially when a finite element method is concerned. Prior to the detailed formulation, a couple of concepts in structural reliability theory are briefly discussed from our standpoint.

Let's denote the random variables mentioned above by $X_k(k=1,2,\cdots,n)$ or $\{X\}$. A structural behaviour Y such as a stress, strain, displacement, force, eigenvalue and so forth is a function of $\{X\}$ and is written in the form of

$$Y=h(\{X\}) \tag{1}$$

Suppose a failure occurs when the following relation holds.

$$Z=Y_c-Y=g(\{X\})\leqq 0 \tag{2}$$

For the brevity of the description, Y_c is taken as deterministic. The integration of the joint probability density function $f(\{X\})$ of $\{X\}$ in the n-dimensional failure domain (D) which satisfies Eq. (2) is nothing but the probability of failure, ie.

$$P_f=\int_{(D)}f(\{X\})dX_1dX_2\cdots dX_n \tag{3}$$

The boundary $Z=g(\{X\})=0$ is called the limit state equation or failure surface.

According to our stochastic finite element context which is based on the second order perturbation technique, Y is estimated as follows.

$$Y=Y^0+\sum_k Y_k^1\alpha_k+\sum_k\sum_l Y_{kl}^2\alpha_k\alpha_l \tag{4}$$

$$\equiv Y^0+[Y^1]\{\alpha\}+[\alpha][Y^2]\{\alpha\} \tag{4-a}$$

where

$$\{\alpha\}=\{X\}-E[\{X\}] \tag{5}$$

Y^0, Y_k^1 and Y_{kl}^2 $(k,l=1,2,\cdots,n)$ are evaluated by solving the governing equation only once. Y^0 corresponds to the solution of the conventional finite element method.

As mentioned in the previous reports, $E[Y]$ and $\mathrm{Var}[Y]$, the expectation and variance of Y, are estimated as follows based on the second order approximation principle[5)].

$$E[Y]=Y^0+\sum_k\sum_l Y_{kl}^2E[\alpha_k\alpha_l] \tag{6}$$

$$\begin{aligned}\mathrm{Var}[Y]=&\sum_k\sum_l Y_k^1Y_l^1E[\alpha_k\alpha_l]\\&+2\sum_k\sum_l\sum_m Y_k^1Y_{lm}^2E[\alpha_k\alpha_l\alpha_m]\\&+\sum_k\sum_l\sum_m\sum_n\{Y_{kl}^2Y_{mn}^2(E[\alpha_k\alpha_l\alpha_m\alpha_n]\\&-E[\alpha_k\alpha_l]E[\alpha_m\alpha_n])\}\end{aligned} \tag{7}$$

These statistics should be made much of from engineers' intuitive point of view. The well known principle of reliability index β may be adopted as its interpretation[6)], ie.

$$\beta=E[Z]/\sqrt{\mathrm{Var}[Z]} \tag{8}$$

Because Eqs. (6) and (7) are based on the second order approximation, the problem of so-called 'lack of invariance'[7),8)] of Eq. (8) is expected to be smaller than that in the 'mean-centered first order second moment method'[6)]. The drawback of the above manipulation would be that the third and fourth moments of $\{\alpha\}$

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must be given in Eq. (7).

In contrast to the mean-centered first order second moment method, the so-called 'advanced first order second moment method'[9),8)] has been developed recently. According to its principle, the smallest distance β_{HL} from the mean value point to the failure surface is measured in terms of the standard deviation in the orthogonally transformed basic variable coordinate system, and a 'notional probability of failure' is calculated from β_{HL}. The conventional reliability index β based on the first order second moment method agrees with this new index β_{HL} only when $g(\{X\})$ is a linear function with respect to $\{X\}$. Although β_{HL} is said to be rather free from the lack of invariance problem under certain conditions[8)], it does not appear to fit the stochastic finite element method. Let's suppose a bar consists of two elements with stiffnesses K_1 and K_2 and let's assume a failure occurs when the displacement U exceeds U_c under a given tensile loading F (see Fig. 1). We can also regard this as a single element model with a equivalent stiffness K_E. When the expectations and covariance matrix of K_1 and K_2 are given, β_{HL} in two dimensional sense is determined according to the procedure described briefly in the above. On the other hand, it is obvious that the full distribution information about K_1 and K_2 is required to calculate $E[K_E]$ and $\mathrm{Var}[K_E]$ on which β_{HL} in one dimensional sense depends. This means, in general, β_{HL} in one dimensional sense differs from that in two dimensional, althogh probabilities of failure in one and two dimensional senses coincide under a given distribution of K_1 and K_2. It should be noted that the above lack of invariance occurs even though the same problem is discussed in terms of the same physical variable 'stiffness of a bar'. The context of our stochastic finite element method is slightly different from this example, however, the same trouble which in general has been called lack of dimensional invariance[10)] is obviously anticipated because of the arbitrariness of the mesh division in the finite element method.

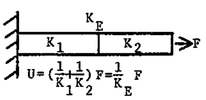

Fig. 1 An example showing the lack of invariance of β_{HL}.

It is believed that there exists a refined interpretation of Eq. (4) which takes advantage of the recent development in structural reliability. An idea is framed in the following.

2. A framework

In general α_k's given in Eq. (5) are correlated each other and $\mathrm{Var}[\alpha_k]$ is not equal to $\mathrm{Var}[\alpha_l]$ when $k \neq l$. As is done in the advanced first order second moment method, it is convenient to transform $\{\alpha\}$ into $\{\alpha'\}$ so that $[C_\alpha']$, the covariance matrix of $\{\alpha'\}$ is equal to the unit matrix $[I]$. The transformation is given as[11)]

$$\{\alpha'\}=([\Phi]\mathrm{diag}[\sqrt{\lambda_i}][\Phi]^T)^{-1}\{\alpha\} \equiv [A]^{-1}\{\alpha\} \qquad (9)$$

where $[\Phi]$ is the modal matrix of $[C_\alpha]$ as the covariance matrix of $\{\alpha\}$ and λ_i's are the corresponding eigenvalues, ie.

$$[C_\alpha][\Phi]=[\Phi]\mathrm{diag}[\lambda_i] \qquad (10)$$

$[\Phi]$ is normalized so as to satisfy the following relation.

$$[\Phi]^{-1}=[\Phi]^T \qquad (11)$$

Substitution of the inverse form of Eq. (9) into Eq. (4-a) leads to

$$Y=Y^0+[Y^1][A]\{\alpha'\}+[\alpha'][A]^T[Y^2][A]\{\alpha'\} \equiv Y^0+[Y^{1\prime}]\{\alpha'\}+[\alpha'][Y^{2\prime}]\{\alpha'\} \qquad (12)$$

It should be noted that, from Eqs. (5) and (9), $E[\{\alpha\}]$ and $E[\{\alpha'\}]$ hold $\{0\}$. As shown later, because of its quadratic expression, Eq. (12) gives a good approximation for Y over the wide range of $\{\alpha\}$ ($\{\alpha'\}$) even where fairly strong nonlinearity is observed. Therefore, in many cases, Eq. (12) may be applied to the limit state equation. In case the approximation of Eq. (12) is not good enough, the expansion point may be moved from the means towards the failure surface by a suitable algorithm[8)] or even by an engineering judgement. As mentioned in the above, the expansion point does not have to be put very close to the failure surface by virtue of the quadratic approximation used here. Therefore, in general, a limit state may be written in the form of

$$Y_c-Y^0-[Y^{1\prime}]\{\alpha'\}-[\alpha'][Y^{2\prime}]\{\alpha'\}=0 \qquad (13)$$

At this stage, one can find that the recent development of quadratic limit state is conveniently utilized. According to the literature[12)], $\{\alpha'\}$ is again transformed as Eq. (14) so that Eq. (13) is rewritten as follows.

$$\{\alpha''\}=[\Psi]\{\alpha'\} \tag{14}$$

$$\begin{aligned} &Y_c-Y^0-[Y^{1\prime}][\Psi]^{-1}\{\alpha''\} \\ &\quad -[\alpha'']([\Psi]^{-1})^T[Y^{2\prime}][\Psi]^{-1}\{\alpha''\} \\ &\equiv Y_c-Y^0-[Y^{1\prime\prime}]\{\alpha''\}-[\alpha''][Y^{2\prime\prime}]\{\alpha''\} \\ &=0 \end{aligned} \tag{15}$$

Where $[\Psi]$ is the modal matrix of $[Y^{2\prime}]$ which is normalized to satisfy the similar relation as Eq. (11), and, therefore, $[Y^{2\prime\prime}]$ is equal to the diagonal matrix whose components are the eigenvalues of $[Y^{2\prime}]$. After some calculations, it is found that Eq. (15) is reduced to the followings.

$$\begin{aligned} &([\alpha'']-[\delta])[Y^{2\prime\prime}](\{\alpha''\}-\{\delta\}) \\ &=1/4[Y^{1\prime\prime}][Y^{2\prime\prime}]^{-1}\{Y^{1\prime\prime}\}+Y_c-Y^0 \end{aligned} \tag{16}$$

with

$$\{\delta\}=-1/2[Y^{2\prime\prime}]^{-1}\{Y^{1\prime\prime}\} \tag{17}$$

$$\begin{aligned} \text{or}\quad &\sum_{i=1}^{n} Y_{ii}^{2\prime\prime}(\alpha_i''-\delta_i)^2 \\ &=1/4[Y^{1\prime\prime}][Y^{2\prime\prime}]^{-1}\{Y^{1\prime\prime}\}+Y_c-Y^0 \\ &\equiv K_1 \end{aligned} \tag{16-a}$$

$E[\{\alpha''\}]$ again equals $\{0\}$ and it should be noted that $[C_{\alpha}'']$, the covariance matrix of $\{\alpha''\}$, continues to be the unit matrix based on the following equations.

$$\begin{aligned} &E[\{\alpha''\}[\alpha'']] \\ &=E[[\Psi]\{\alpha'\}[\alpha'][\Psi]^T] \\ &=[\Psi]E[\{\alpha'\}[\alpha']][\Psi]^T \\ &=[\Psi][I][\Psi]^T \\ &=[\Psi][\Psi]^T \\ &=[\Psi][\Psi]^{-1} \\ &=[I] \end{aligned} \tag{18}$$

In case the $\{\alpha\}$ are believed to be Gaussian random variables, then so are the linearly transformed $\{\alpha''\}$ and it is found[12] that the probability of failure P_f can be calculated based on Eq. (16-a) with a linear combination of noncentral chi-squared distribution formula[13], ie.

$$\begin{aligned} P_f&=P\left(\sum_{i=1}^{n} Y_{ii}^{2\prime\prime}(\alpha_i''-\delta_i)^2\geqq K_1\right) \\ &=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\frac{\sin\theta(u)}{u\rho(u)}du \end{aligned} \tag{19}$$

where

$$\begin{aligned} \theta(u)=&\frac{1}{2}\sum_{j=1}^{n}[\tan^{-1}(Y_{jj}^{2\prime\prime}u) \\ &+\delta_j^2 Y_{jj}u(1+(Y_{jj}^{2\prime\prime}u)^2)^{-1}]-\frac{1}{2}K_1u \end{aligned} \tag{20}$$

$$\begin{aligned} \rho(u)=&\prod_{j=1}^{n}(1+(Y_{jj}^{2\prime\prime}u)^2)^{\frac{1}{4}} \\ &\times\exp\left\{\sum_{j=1}^{n}(\delta_j Y_{jj}^{2\prime\prime}u)^2/2(1+(Y_{jj}^{2\prime\prime}u)^2)\right\} \end{aligned} \tag{21}$$

when some of the components of $[Y^{2\prime}]$ are zero, Eq. (16-a) is replaced by the following equation after a similar calculation[12].

$$\sum_{i=1}^{m} Y_{ii}^{2\prime\prime}(\alpha_i''-\delta_i)^2+\sum_{i=m+1}^{n} Y_i^{1\prime\prime}\alpha_i''=K_2 \tag{22}$$

Therefore, the probability of failure P_f in this case is given as

$$\begin{aligned} P_f=\int_{-\infty}^{\infty}&\phi\left(\frac{\alpha'''}{\sigma_{\alpha'''}}\right)P\left(\sum_{i=1}^{m} Y_{ii}^{2\prime\prime}(\alpha_i''-\delta_i)^2\right. \\ &\left.\geqq K_2-\alpha'''\right)d\alpha''' \end{aligned} \tag{23}$$

where σ_{α}''' is the standard deviation of $\alpha'''\equiv\sum_{i=m+1}^{n} Y_i^{1\prime\prime}\alpha_i''$ and $\phi(\cdot)$ the standard normal density function.

In case $\{\alpha\}$ are not subjected to Gaussian distributions, the notion of 'generalized' or 'equivalent' reliability index proposed by Ditlevsen[14] may be employed as a useful criterion, ie.

$$\beta_E=-\Phi^{-1}(P_f) \tag{24}$$

where $\Phi^{-1}(\cdot)$ is the inverse of standard normal distribution function and P_f is evaluated as Eq. (19) or (23). Although Gaussian expressions are used temporarily in this manipulation, β_E obtained should be regarded as rather free from such a distribution[14].

3. Numerical example

Let's take a column buckling under an uncertain boundary condition of Fig. 2 as an example. The uncertainty of the boundary condition is simulated by the virtual spring elements whose spring constants are $(S/1-S)(EI/l^3)$ for the deflection and $(C/1-C)(EI/l)$ for the rotation[4]. S and C are expressed in the form of $S=S_0(1+\alpha_1)$ and $C=C_0(1+\alpha_2)$ with random variables α_1 and α_2. EI is the bending rigidity of the column and l the length of the finite element. Details of the stochastic finite element analysis[4] are not described here and only the results are shown in Fig. 3. The buckling loads versus S and C shown in this figure are obtained through the conventional deterministic computations. The broken lines are the estimated contour lines. The solid lines denoted 'SOA' show the variation of the buckling load along the indicated sections as approximated by our stochastic finite element method, the form is

$$\begin{aligned} Y=&(EI/(10^3 l^2)) \\ &\times(434.3+63.4\alpha_1+132.3\alpha_2 \end{aligned}$$

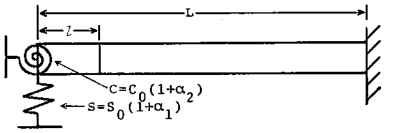

Fig. 2 Column buckling under uncertain boundary conditions.

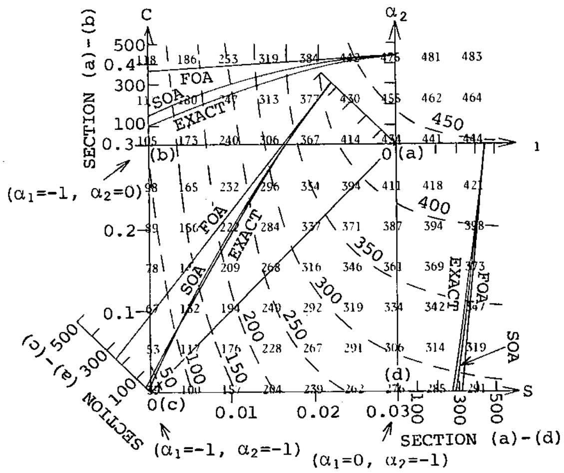

Fig. 3 Buckling load distribution (normalized by $EI/(10^3 l^2)$) and its approximation by the stochastic finite element method (number of elements $N=8$, ie. $l=L/8$).

$$-221.9\alpha_1^2+15.7\alpha_1\alpha_2-16.2\alpha_2^2) \qquad (25)$$

The mean values S_0 and C_0 are taken as 0.03 and 0.3 respectively. As is mentioned in the preceding chapter, the superiority of the second order approximation to the first order one (shown by the straight lines denoted 'FOA') is obvious. At least in this case, the mean centered second order approximation therefore seems good for the evaluation of the probability of failure or the reliability index.

Suppose that the applied compressive load is $Y_c=250\times(EI/(10^3 l^2))$ and the covariance matrix of $\{\alpha\}$ takes the following value.

$$[C_\alpha]=0.2^2\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \qquad (26)$$

Then the following values are calculated according to the procedure described in the preceding chapter.

$$P_f=P[4.860\times10^{-4}(\alpha_1''+23.559)^2 +8.724\times10^{-3}(\alpha_2''-1.485)^2\geqq0.473] =6.85\times10^{-4} \qquad (27)$$

or $\beta_E=-\Phi^{-1}(P_f)=3.2$ (28)

Even if many random variables are involved in this procedure, the CPU time is expected to be negligible, because only a single fold integral (Eq. (19)) or its convolution (Eq. (23)) appears.

4. Conclusions

According to the terminology in structural reliability, the proposed framework may be classified as a second order second moment method. As is well known it is sometimes impractical to introduce higher order moments of random variables because of the lack of information, however, there is no reason the higher order mechanical approximation should not be applied to the evaluation of the limit state if it is effective and conveniently dealt with. Our stochastic finite element method based on the second order perturbation technique fits these conditions.

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