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Stochastic Stress Analysis of an Infinite Plate Weakened by a Circular Hole with Shape Fluctuation (Part 1)
—Formulation—
形状に不確かさを有する円こう周辺の確率応力解析（第1報）—定　式　化—

Toshiaki HISADA,* Wataru SATOH* and Shigeru NAKAGIRI*
久　田　俊　明・佐　藤　　亘・中　桐　　滋

1. Introduction

Let us suppose the case in which a very small circular hole in a large plate in uniform stress state is a cause of crack initiation. The time to crack initiation, as is well known, is subjected to a probability distribution such as Weibull's. This fact has been an issue of random nature of material strength, but on one side, this implies that the hole might not be circular precisely, or in other words, the stress concentration factor might fluctuate from hole to hole. It turns out then that even elastic stress analysis around a hole requires probabilistic treatment in some applications from the standpoint of stochastic structural mechanics.

We authors newly propose a stochastic stress analysis procedure which enables us to evaluate the probabilistic nature of stress distribution around a hole, the shape of which homogeneously fluctuates. The present theory is based on a stochastic conformal mapping, the function of which involves probabilistic variables. If the power spectrum of shape fluctuation is given, such spectrum has once been observed for a particular case[1], the covariance matrix of aforementioned probabilistic variables is evaluated so as to characterize the probabilistic nature of stress functions. Then the expectation and dispersion of stress at a point on the contour are obtained by the means of the first-order approximation principle.

In this paper, only the formulation is described, and the numerical results and discussions will be presented in the subsequent paper which follows shortly later.

2. Formulation

* Dept. of Appiled Physics and Applied Mechanics, Institute of Industrial Science, University of Tokyo.

2.1 Stochastic Conformal Mapping

Consider a randomly shaped hole in an infinite plate (Fig. 1), whose contour is defined in form of $r+f(\theta)$ where r is a constant radius of reference circle and $f(\theta)$ a function of angular position θ of arbitrary point A. The shape fluctuation is described through $f(\theta)$. A relation $z=Z(\zeta)$ is taken to represent the conformal mapping of the external of $r+f(\theta)$ onto that of the circle r. We assume $Z(\zeta)$ as follows;

$$Z(\zeta)=\zeta+\sum_{n=1}^{N}\frac{s_n}{\zeta^n}=\zeta+\frac{s_1}{\zeta}+\frac{s_2}{\zeta^2}+\cdots+\frac{s_N}{\zeta^N} \qquad (1)$$

where $s_n (n=1, 2, \cdots, N)$ are unknown complex coefficients, and $\zeta=\xi+i\eta=\rho e^{i\alpha} (i=\sqrt{-1})$. By the use of the values chosen at M points, the following simultaneous equations are derived to determine N coefficients.

$$\{r+f(\theta_k)\}e^{i\theta_k}=re^{i\theta_k}+\sum_{n=1}^{N}\frac{s_n}{r^n}e^{-in\theta_k} \qquad (2)$$
$$(k=1, 2, \cdots, M\ ;\ M\geqq N)$$

Making use of the partition into real and imaginary parts of s_n as $s_n=s'_n+is''_n$, we have

$$[C]\{\tilde{s}\}=\{\tilde{f}\} \qquad (3)$$

where the vectors $\{\tilde{s}\}(2N\times 1)$, $\{\tilde{f}\}(2M\times 1)$ and matix $[C](2M\times 2N)$ are defined by

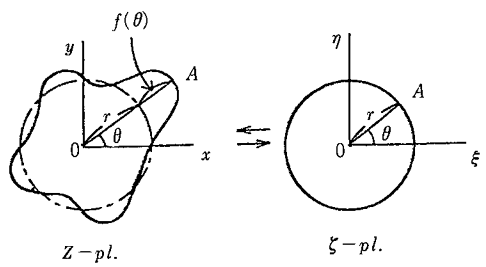

Fig. 1　Present Conformal Transformation Representing a Randomly Shaped Hole on a Regular Circle

$$\begin{cases} \{\tilde{s}\} = \lfloor s_1' \; s_1'' \; s_2' \; s_2'' \cdots s_N' \; s_N'' \rfloor^T, \\ \{\tilde{f}\} = \lfloor f(\theta_1) \; 0 \; f(\theta_2) \; 0 \cdots f(\theta_M) \; 0 \rfloor^T, \\ [C] = \begin{bmatrix} C_1^2 & C_1^3 & \cdots & C_1^{N+1} \\ \vdots & \vdots & & \vdots \\ C_M^2 & C_M^3 & \cdots & C_M^{N+1} \end{bmatrix} \end{cases}$$

$$C_k^{n+1} = \frac{1}{r^n} \begin{bmatrix} \cos(n+1)\theta_k & \sin(n+1)\theta_k \\ -\sin(n+1)\theta_k & \cos(n+1)\theta_k \end{bmatrix}, \quad (4)$$

$$(n = 1, 2, \cdots, N;\; k = 1, 2, \cdots, M).$$

The solution $\{\tilde{s}\}$ for $\{\tilde{f}\}$ is used to determine the conformal mapping Eq. (1). In the case of $M=N$, Eq. (3) is solved in usual manner. If $M>N$, Eq. (3) is solved in the sense of the least square approximation. Generally the solution of Eq. (3) is expressed as follows.

$$\{\tilde{s}\} = [[C]^T[C]]^{-1} \cdot [C]^T\{\tilde{f}\} = [C^*]\{\tilde{f}\} \quad (5)$$

When the function is assumed to be a homogeneous random process with $E[f]=0$, the coefficients s_n are also random with $E[s_n']=E[s_n'']=0$. The covariances $Cov(s_i', s_j')$, $Cov(s_i', s_j'')$ and $Cov(s_i'', s_j'')$, denoted by $Cov(\tilde{s}_i, \tilde{s}_j)$ commonly, are defined by taking the expectation of the following matrix

$$\{s\}\{s\}^T = \begin{bmatrix} s_1's_1' & s_1's_1'' & \cdots & s_1's_N' & s_1's_N'' \\ & s_1''s_1'' & \cdots & s_1''s_N' & s_1''s_N'' \\ & & \ddots & \vdots & \vdots \\ & & & s_N's_N' & s_N's_N'' \\ \text{sym.} & & & & s_N''s_N'' \end{bmatrix}$$

$$= [C^*]\{\tilde{f}\}\{\tilde{f}\}^T[C^*]^T, \quad (6)$$

that is

$$[Cov(\tilde{s}_i, \tilde{s}_j)] = [C^*] \cdot E[\{\tilde{f}\}\{\tilde{f}\}^T] \cdot [C^*]^T. \quad (7)$$

The matrix $E[\{\tilde{f}\}\{\tilde{f}\}^T]$ is written as

$$\begin{bmatrix} E[f(\theta_1)f(\theta_1)] & 0 & E[f(\theta_1)f(\theta_2)] & 0 & \cdots & E[f(\theta_1)f(\theta_M)] & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ E[f(\theta_N)f(\theta_1)] & 0 & E[f(\theta_M)f(\theta_2)] & 0 & \cdots & E[f(\theta_M)f(\theta_M)] & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (8)$$

where $E[f(\theta_m)f(\theta_k)]$ $(m, k = 1, 2, \cdots, M)$ equals the autocorrelation of the random process $f(\theta)$ denoted by $R(\theta_m - \theta_k)$. Introducing the discrete Wiener-Khintchine relation, we have

$$E[f(\theta_m)f(\theta_k)] = R(\theta_m - \theta_k) = \sum_{n=-\infty}^{\infty} S(\lambda_n) e^{i2\pi\lambda_n(\theta_m - \theta_k)} \quad (9)$$

where $S(\lambda_n)$ is the line spectrum for the n-th wave number $\lambda_n = n/2\pi$. This means that the covariance of the coefficients s_n for the stochastic conformal mapping function can be determined through the power spectrum $S(\lambda_n)$ of the given shape fluctuation.

2.2 Expectation and Dispersion of Stresses

We deal with an infinite plate made of isotropic elastic material and weakened by a nearly circular hole. The Goursat's stress functions $\varphi(z)$ and $\psi(z)$[2] are used to calculate stresses σ_x, σ_y and τ_{xy} in the following form,

$$\begin{cases} \dfrac{\sigma_x + \sigma_y}{2} = 2Re[\Phi'(\zeta)] \\ \dfrac{\sigma_y - \sigma_x}{2} + i\tau_{xy} = \overline{Z(\zeta)} \cdot \Phi''(\zeta) + \Psi'(\zeta) \end{cases} \quad (10)$$

where $\Phi(\zeta) = \varphi\{Z(\zeta)\}$, $\Psi(\zeta) = \psi\{Z(\zeta)\}$. The superscripts $'$ and $''$ denote the first and second derivatives with respect to z, and $\overline{Z(\zeta)}$ is the conjugate of $Z(\zeta)$. Since $Z(\zeta)$ represents the stochastic conformal mapping function aforementioned, the stresses given by Eq. (10) are of stochastic nature also. Taking into account uniform stress state prescribed by σ_x^∞, σ_y^∞ and τ_{xy}^∞ at infinity, we have the following form of $\Phi(\zeta)$, $\Psi(\zeta)$ for $Z(\zeta)$ defined by Eq. (1).

$$\begin{cases} \Phi(\zeta) = A\zeta - 2\overline{B}\dfrac{r^2}{\zeta} - A\left(\dfrac{s_1}{\zeta} + \dfrac{s_2}{\zeta^2} + \cdots + \dfrac{s_N}{\zeta^N}\right) \\ \Psi(\zeta) = X(\zeta) - \overline{Z}\left(\dfrac{r^2}{\zeta}\right) \cdot \Phi'(\zeta) \end{cases} \quad (11)$$

where an auxilliary function $X(\zeta)$, a real constant A, and a complex constant B are given as blow

$$\begin{cases} X(\zeta) = -A\dfrac{r^2}{\zeta} + 2B\zeta + A\left\{\overline{s}_1\dfrac{\zeta}{r^2} + \overline{s}_2\left(\dfrac{\zeta}{r^2}\right)^2 + \cdots + \overline{s}_N\left(\dfrac{\zeta}{r^2}\right)^N\right\} \\ A = \dfrac{\sigma_x^\infty + \sigma_y^\infty}{4} \qquad 2B = \dfrac{\sigma_y^\infty - \sigma_x^\infty}{2} + i\tau_{xy}^\infty \end{cases} \quad (12)$$

where $\overline{s_i}$ means the conjugate of s_i.

Solving Eqs. (10) for σ_x, σ_y and τ_{xy}, we have

$$\begin{cases} \sigma_x = Re[2\Phi'(\zeta) - \overline{Z(\zeta)}\Phi''(\zeta) - \Psi'(\zeta)] \equiv g_2(\zeta, \tilde{s}) \\ \sigma_y = Re[2\Phi'(\zeta) + \overline{Z(\zeta)}\Phi''(\zeta) + \Psi'(\zeta)] \equiv g_1(\zeta, \tilde{s}) \\ \tau_{xy} = Im[\overline{Z(\zeta)}\Phi''(\zeta) + \Psi'(\zeta)] \equiv g_3(\zeta, \tilde{s}) \end{cases} \quad (13)$$

where $\tilde{s}$ represents the conformal mapping coefficients s_n for every n, by which the stochastic nature of the stresses is governed.

The stress values $g_1(\zeta, \tilde{s})$, $g_2(\zeta, \tilde{s})$ and $g_3(\zeta, \tilde{s})$ are derived by substituting Eqs. (1) and (11) into (13), as follows

$$\left\{\begin{aligned}&\left\{\begin{matrix}g_1(\zeta,\tilde{s})\\ g_2(\zeta,\tilde{s})\end{matrix}\right\}=2q_1q_2+\left\{\begin{matrix}1\\ -1\end{matrix}\right\}q_3(q_4q_5-q_6q_7+q_8q_9-q_{10}q_{11})\\ &g_3(\zeta,\tilde{s})=q_3(q_6q_5+q_4q_7+q_{10}q_9+q_8q_{11})\end{aligned}\right. \tag{14}$$

in which $q_1 \sim q_{11}$ are calculated as given below,

$$\left\{\begin{aligned}&q_1=\frac{1}{|h_3|^2}\quad q_2=Re[h_1]Re[h_3]+Im[h_1]Im[h_3]\\ &q_3=\frac{1}{|h_3|^6}\quad q_4=Re[h_5-h_6]\quad q_5=R_{24}\hat{R}+I_{24}\hat{I}\\ &q_6=Im[h_5-h_6]\quad q_7=I_{24}\hat{R}-R_{24}\hat{I}\\ &q_8=Re[h_3]\quad q_9=R_{87}\hat{R}+I_{87}\hat{I}\quad q_{10}=Im[h_3]\\ &q_{11}=I_{87}\hat{R}-R_{87}\hat{I}\\ &\hat{R}=Re[(h_3)^3]\quad \hat{I}=Im[(h_3)^3]\end{aligned}\right. \tag{15}$$

where R_{24} and I_{24} are the real and imaginary parts of $h_2h_3-h_1h_4$, and R_{87} and I_{87} of $h_8h_3-h_1h_7$, respectively. The complex functions $h_1 \sim h_8$ are given by

$$\left\{\begin{aligned}&h_1=\Phi'(\zeta)\quad h_2=\Phi''(\zeta)\quad h_3=Z'(\zeta)\\ &h_4=Z''(\zeta)\quad h_5=\overline{Z(\zeta)}\\ &h_6=\overline{Z}\left(\frac{r^2}{\zeta}\right)\quad h_7=\overline{Z}'\left(\frac{r^2}{\zeta}\right)\quad h_8=X'(\zeta)\end{aligned}\right. \tag{16}$$

where $(\cdot)'=d(\cdot)/d\zeta$, $(\cdot)''=d^2(\cdot)/d\zeta^2$, but their explicit expressions are tabulated in Appendix 1.

The stress expectations and variations $E[\sigma_x]$, $Var[\sigma_x]$ and the likes are calculated on the basis of the first-order approximation[3].

$$E\left\{\begin{matrix}\sigma_y\\ \sigma_x\\ \tau_{xy}\end{matrix}\right\}\approx\left\{\begin{matrix}g_1(\zeta,\tilde{s})\\ g_2(\zeta,\tilde{s})\\ g_3(\zeta,\tilde{s})\end{matrix}\right\}_{\tilde{s}=E[\tilde{s}]=0}$$

$$Var\left\{\begin{matrix}\sigma_y\\ \sigma_x\\ \tau_{xy}\end{matrix}\right\}\approx\sum_{i=1}^{2N}\sum_{j=1}^{2N}\frac{\partial}{\partial\tilde{s}_i}\left\{\begin{matrix}g_1(\zeta,\tilde{s})\\ g_2(\zeta,\tilde{s})\\ g_3(\zeta,\tilde{s})\end{matrix}\right\}\times\frac{\partial}{\partial\tilde{s}_j}\left\{\begin{matrix}g_1(\zeta,\tilde{s})\\ g_2(\zeta,\tilde{s})\\ g_3(\zeta,\tilde{s})\end{matrix}\right\}\cdot Cov(s_i,s_j)\Bigg|_{\tilde{s}=E[\tilde{s}]=0} \tag{17}$$

where $\tilde{s}_i$ denotes for the i-th element of $\tilde{s}$, namely, $\tilde{s}_i=s^{\cdot}_{(i+1)/2}$ for odd number i and $s'_{i/2}$ for even.

The derivatives of $g_1(\zeta,\tilde{s})$, $g_2(\zeta,\tilde{s})$, $g_3(\zeta,\tilde{s})$ as regards $\tilde{s}_i$ can be rewritten in the following form

$$\frac{\partial}{\partial\tilde{s}_i}\left\{\begin{matrix}g_1(\zeta,\tilde{s})\\ g_2(\zeta,\tilde{s})\\ g_3(\zeta,\tilde{s})\end{matrix}\right\}=\sum_{j=1}^{16}\frac{\partial}{\partial\tilde{h}_j}\left\{\begin{matrix}g_1(\tilde{h})\\ g_2(\tilde{h})\\ g_3(\tilde{h})\end{matrix}\right\}\frac{\partial\tilde{h}_j}{\partial\tilde{s}_i} \tag{18}$$

Appendix 1

	Re	Im
h_1	$A+\frac{2r^2}{\rho^2}(B^{\cdot}C_2-B'S_2)+A\sum_n d_1(s^{\cdot}_nC_{n+1}+s'_nS_{n+1})$	$\frac{2r^2}{\rho^2}(-B^{\cdot}S_2-B'C_2)+A\sum_n d_1(-s^{\cdot}_nS_{n+1}+s'_nC_{n+1})$
h_2	$-4\frac{r^2}{\rho^3}(B^{\cdot}C_3-B'S_3)-A\sum_n d_2(s^{\cdot}_nC_{n+2}+s'_nS_{n+2})$	$4\frac{r^2}{\rho^3}(B^{\cdot}S_3+B'C_3)-A\sum_n d_2(-s^{\cdot}_nS_{n+2}+s'_nC_{n+2})$
h_3	$1-\sum_n d_1(s^{\cdot}_nC_{n+1}+s'_nS_{n+1})$	$-\sum_n d_1(-s^{\cdot}_nS_{n+1}+s'_nC_{n+1})$
h_4	$\sum_n d_2(s^{\cdot}_nC_{n+2}+s'_nS_{n+2})$	$\sum_n d_2(-s^{\cdot}_nS_{n+2}+s'_nC_{n+2})$
h_5	$\rho C_1+\sum_n d_3(s^{\cdot}_nC_n+s'_nS_n)$	$-\rho S_1+\sum_n d_3(s^{\cdot}_nS_n-s'_nC_n)$
h_6	$\frac{r^2}{\rho}C_1+\sum_n d_4(s^{\cdot}_nC_n+s'_nS_n)$	$-\frac{r^2}{\rho}S_1+\sum_n d_4(s^{\cdot}_nS_n-s'_nC_n)$
h_7	$-\frac{r^2}{\rho^2}C_2+\sum_n d_5(s^{\cdot}_nC_{n-1}+s'_nS_{n-1})$	$\frac{r^2}{\rho^2}S_2+\sum_n d_5(s^{\cdot}_nS_{n-1}-s'_nC_{n-1})$
h_8	$A\frac{r^2}{\rho^2}C_2+2B^{\cdot}+A\sum_n d_5(s^{\cdot}_nC_{n-1}+s'_nS_{n-1})$	$-A\frac{r^2}{\rho^2}S_2+2B'+A\sum_n d_5(s^{\cdot}_nS_{n-1}-s'_nC_{n-1})$

where $S_n=\sin n\alpha$, $C_n=\cos n\alpha$, $\sum_n=\sum_{n=1}^{N}$, $d_1=\frac{n}{\rho^{n+1}}$, $d_2=\frac{n(n+1)}{\rho^{n+2}}$, $d_3=\frac{1}{\rho^n}$, $d_4=\frac{\rho^n}{r^{2n}}$, $d_5=n\frac{\rho^{n-1}}{r^{2n}}$.

where $\tilde{h}$ means all of the 16 functions arranged such as $Re[h_1]$, $Im[h_1]$, $\cdots$, $Im[h_8]$, and $\tilde{h}_j$ the j-th entity of arranged $\tilde{h}$. The derivatives of $g_1(\tilde{h})$, $g_2(\tilde{h})$ and $g_3(\tilde{h})$ with respect to $\tilde{h}_j$ are summerized in Appendix 2.

3. Conclusion

A stochastic procedure to estimate the probabilistic stress state around a hole with uncertain shape is presented, provided that the power spectrum of shape fluctuation is given. Once the probabilistic conformal mapping is established in conjunction with the power spectrum, the stress state is calculated by the use of Grousat's stress functions which are subjected to the conformal mapping. The first-order approximation is taken to evaluate the expectation and dispersion of the stresses.

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Appendix 2

$q_{1,5}=-2\tilde{h}_5/|h_3|^4$　　$q_{1,6}=-2\tilde{h}_6/|h_3|^4$

$q_{2,1}=\tilde{h}_5$　　$q_{2,2}=\tilde{h}_6$　　$q_{2,5}=\tilde{h}_1$　　$q_{2,6}=\tilde{h}_2$

$q_{3,5}=-6\tilde{h}_5/|h_3|^8$　　$q_{3,6}=-6\tilde{h}_6/|h_3|^8$

$q_{5,1}=-\tilde{h}_7\hat{R}-\tilde{h}_8\hat{I}$　　$q_{5,2}=\tilde{h}_8\hat{R}-\tilde{h}_7\hat{I}$

$q_{5,3}=\tilde{h}_5\hat{R}+\tilde{h}_6\hat{I}$　　$q_{5,4}=-\tilde{h}_6\hat{R}+\tilde{h}_5\hat{I}$

$q_{5,5}=\tilde{h}_3\hat{R}+R_{24}P+\tilde{h}_4\hat{I}+I_{24}Q$

$q_{5,6}=-\tilde{h}_4\hat{R}-R_{24}Q+\tilde{h}_3\hat{I}+I_{24}P$

$q_{5,7}=-\tilde{h}_1\hat{R}-\tilde{h}_2\hat{I}$　　$q_{5,8}=\tilde{h}_2\hat{R}-\tilde{h}_1\hat{I}$

$q_{7,1}=-\tilde{h}_8\hat{R}+\tilde{h}_7\hat{I}$　　$q_{7,2}=-\tilde{h}_7\hat{R}-\tilde{h}_8\hat{I}$

$q_{7,3}=\tilde{h}_6\hat{R}-\tilde{h}_5\hat{I}$　　$q_{7,4}=\tilde{h}_5\hat{R}+\tilde{h}_6\hat{I}$

$q_{7,5}=\tilde{h}_4\hat{R}+I_{24}P-\tilde{h}_3\hat{I}-R_{24}Q$

$q_{7,6}=\tilde{h}_3\hat{R}-I_{24}Q+\tilde{h}_4\hat{I}-R_{24}P$

$q_{7,7}=-\tilde{h}_2\hat{R}+\tilde{h}_1\hat{I}$　　$q_{7,8}=-\tilde{h}_1\hat{R}-\tilde{h}_2\hat{I}$

$q_{9,1}=-\tilde{h}_{13}\hat{R}-\tilde{h}_{14}\hat{I}$　　$q_{9,2}=\tilde{h}_{14}\hat{R}-\tilde{h}_{13}\hat{I}$

$q_{9,5}=\tilde{h}_{15}\hat{R}+R_{87}P+\tilde{h}_{16}\hat{I}+I_{87}Q$

$q_{9,6}=-\tilde{h}_{16}\hat{R}-R_{87}Q+\tilde{h}_{15}\hat{I}+I_{87}P$

$q_{9,13}=-\tilde{h}_1\hat{R}-\tilde{h}_2\hat{I}$　　$q_{9,14}=\tilde{h}_2\hat{R}-\tilde{h}_1\hat{I}$

$q_{9,15}=\tilde{h}_5\hat{R}+\tilde{h}_6\hat{I}$　　$q_{9,16}=-\tilde{h}_6\hat{R}+\tilde{h}_5\hat{I}$

$q_{11,1}=-\tilde{h}_{14}\hat{R}+\tilde{h}_{13}\hat{I}$　　$q_{11,2}=-\tilde{h}_{13}\hat{R}-\tilde{h}_{14}\hat{I}$

$q_{11,5}=\tilde{h}_{16}\hat{R}+I_{87}P-\tilde{h}_{15}\hat{I}-R_{87}Q$

$q_{11,6}=\tilde{h}_{15}\hat{R}-I_{87}Q+\tilde{h}_{16}\hat{I}-R_{87}P$

$q_{11,13}=-\tilde{h}_2\hat{R}+\tilde{h}_1\hat{I}$　　$q_{11,14}=-\tilde{h}_1\hat{R}-\tilde{h}_2\hat{I}$

$q_{11,15}=\tilde{h}_6\hat{R}-\tilde{h}_5\hat{I}$　　$q_{11,16}=\tilde{h}_5\hat{R}+\tilde{h}_6\hat{I}$

$q_{4,9}=q_{6,10}=q_{8,5}=q_{10,6}=1$

$q_{4,11}=q_{6,12}=-1$

All the others are nil.

Note; $P=3\ (\tilde{h}_5^2-\tilde{h}_6^2)$, $Q=6\tilde{h}_5\tilde{h}_6$, $q_{i,j}=\partial q_i/\partial\tilde{h}_j$.