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UDC 519. 2 : 624. 075. 2 : 530. 162

A Note on Stochastic Finite Element Method (Part 4)
Eigenvalue problem of column buckling under uncertain boundary conditions
確率有限要素法に関するノート（第4報）
—不確定境界条件下の柱の座屈問題への応用—

中　桐　　滋・久　田　俊　明
Shigeru NAKAGIRI* and Toshiaki HISADA*

1. Introduction

This note discusses how to estimate the buckling eigenvalue of a straight column under compression, the boundary conditions of which are uncertain, in order to present a methodology based on the second order perturbation method incorporated with the finite element displacement method and to examine its applicability to linear eigenvalue problems.

Buckling load is governed by the imposed boundary conditions and initially existing imperfections, if any, besides mechanical properties of material used and dimensions of structure under consideration. Initial deflection is likely to develop with increasing external load, and, as the result, linear eigenvalue analysis is made out of place in the presence of initial deflection. The fluctuations in the mechanical properties of material and dimensions, which can be incorporated with the finite element stiffness matrix in the manner reported previously [1),2)], seem to be less significant compared with the effect of the fluctuating boundary conditions.

We have chosen to investigate the effect of uncertain boundary conditions on buckling load by the use of finite element modeling equipped with virtual spring elements which represent elastic restraints against deflection and angle of rotation at column end. The spring constants of which are considered varying as random variable to meet uncertainty in the boundary conditions. Then the probabilistic nature of the fluctuating buckling load is evaluated on the basis of the second order perturbation method, instead of the first order one for the linear statistical model [3),5),6)]. The present method is exemplified by straight column under conservative compressive force, whose bending rigidety EI and cross-sectional area S remain definite and uniform through the whole column length L.

* Dept. of Appiled Physics and Applied Mechanics, Institute of Industrial Science, University of Tokyo.

2. Stiffness matrices

Figure 1 illustrates the elastic restraints for deflection and angle of rotation at node 1 of an end simulated by virtual spring elements. We assume that the spring constants are expressed in form of $(s/1-s)(EI/l^3)$ and $(c/1-c)(EI/l)$ for the deflection and angle of rotation respectively, where l is the length of element given by the nodes 1 and 2. The stiffness matrix $[k]$ and geometrical stiffness matrix $[k_G]$ [4)] are given below;

$$[k]=\frac{EI}{l^3}\begin{bmatrix} 12+\frac{s}{1-s} & 6l & -12 & 6l \\ & \left(4+\frac{c}{1-c}\right)l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ \text{SYM.} & & & 4l^2 \end{bmatrix} \quad (1)$$

$$\lambda[k_G]=\lambda\frac{EI}{30l^3}\begin{bmatrix} 36 & 3l & -36 & 3l \\ & 4l^2 & -3l & -l^2 \\ & & 36 & -3l \\ \text{SYM.} & & & 4l^2 \end{bmatrix} \quad (2)$$

where $\lambda=Pl^2/EI$ is the buckling eigenvalue of the column (compressive force P taken positive). The second

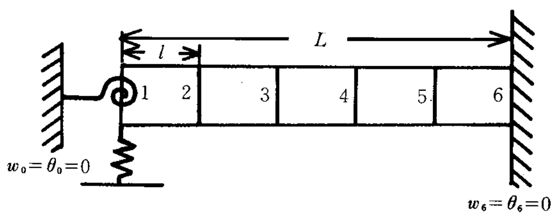

Figure 1　Virtual spring elements and finite element division

terms of k_{11} and k_{22}, which arise from the contribution of spring reaction, are discarded for any element not subjected to such boundary conditions.

The parameters representing the spring constants, s and c, are taken as the sum of the expectations (s_0 and c_0) and random variable terms as $s=s_0(1+\mu)$ and $c=c_0(1+\varepsilon)$. It goes without saying that the expectations of μ and ε, $E[\mu]$ and $E[\varepsilon]$, are equal to zero. Then we have the following Taylor expansion of the stiffness matrix $[k]$ with respect to small values of μ and ε up to the second order products.

$$[k]=[k_0]+[k_\mu]\mu+[k_\varepsilon]\varepsilon+[k_{\mu\mu}]\mu^2+[k_{\varepsilon\varepsilon}]\varepsilon^2 \qquad (3)$$

where $[k_0]$ is expressed by Eq. (1), in which s and c are replaced with s_0 and c_0, and

$$[k_\mu],[k_{\mu\mu}]=\frac{EI}{l^3}\left(\frac{s_0}{(1-s_0)^2},\frac{s_0^2}{(1-s_0)^3}\right)\begin{bmatrix}1&0&0&0\\&0&0&0\\&&0&0\\\mathrm{SYM.}&&&0\end{bmatrix} \qquad (4)$$

$$[k_\varepsilon],[k_{\varepsilon\varepsilon}]=\frac{EI}{l^3}\left(\frac{c_0}{(1-c_0)^2},\frac{c_0^2}{(1-c_0)^3}\right)\begin{bmatrix}0&0&0&0\\&l^2&0&0\\&&0&0\\\mathrm{SYM.}&&&0\end{bmatrix} \qquad (5)$$

Such matrix as $[k_{\mu\varepsilon}]$ is not involved since there is no coupling of μ and ε in Eq. (1).

3. Second order perturbation applied to governing equation

The governing equation of the buckling eigenvalue problem is expressed usually as

$$([K]-\lambda[K_G])\{U\}=\{0\} \qquad (6)$$

where the overall stiffness matrices $[K]$ and $[K_G]$ are generated by means of merging $[k]$ and $[k_G]$, and $\{U\}$ denotes generalized displacement vector consisting of n unknown components. Dealing with such a simple case that only the node 1 is subjected to fluctuating elastic restraints as in Fig. 1, we make use of the same polynomial expansion of $[K]$, λ and $\{U\}$ regarding μ and ε up to the second order, as given in the following.

$$[K]=[K_0]+[K_\mu]\mu+[K_\varepsilon]\varepsilon+[K_{\mu\mu}]\mu^2+[K_{\varepsilon\varepsilon}]\varepsilon^2+[K_{\mu\varepsilon}]\mu\varepsilon \qquad (7)$$

$$\lambda=\lambda_0+\mu\lambda_\mu+\varepsilon\lambda_\varepsilon+\mu^2\lambda_{\mu\mu}+\varepsilon^2\lambda_{\varepsilon\varepsilon}+\mu\varepsilon\lambda_{\mu\varepsilon} \qquad (8)$$

$$\{U\}=\{U_0\}+\{U_\mu\}\mu+\{U_\varepsilon\}\varepsilon+\{U_{\mu\mu}\}\mu^2+\{U_{\varepsilon\varepsilon}\}\varepsilon^2+\{U_{\mu\varepsilon}\}\mu\varepsilon \qquad (9)$$

$[K_G]$ remains definite, and $[K_{\mu\varepsilon}]=[0]$. Substituting Eqs. (7), (8) and (9) into the governing equation (6) and sorting out pertinent terms of μ, ε, μ^2, ε^2 and $\mu\varepsilon$, we have the following equations based on the principle of the second order perturbation method, in other words, by means of neglecting all the higher terms of μ and/or ε than third order product.

$$([K_0]-\lambda_0[K_G])\{U_0\}=\{0\} \qquad (10)$$

$$([K_\mu]-\lambda_\mu[K_G])\{U_0\}+([K_0]-\lambda_0[K_G])\{U_\mu\}=\{0\} \qquad (11)$$

$$([K_\varepsilon]-\lambda_\varepsilon[K_G]\{U_0\}+([K_0]-\lambda_0[K_G])\{U_\varepsilon\}=\{0\} \qquad (12)$$

$$([K_{\mu\mu}]-\lambda_{\mu\mu}[K_G])\{U_0\}+([K_\mu]-\lambda_\mu[K_G])\{U_\mu\}+([K_0]-\lambda_0[K_G])\{U_{\mu\mu}\}=\{0\} \qquad (13)$$

$$([K_{\varepsilon\varepsilon}]-\lambda_{\varepsilon\varepsilon}[K_G])\{U_0\}+([K_\varepsilon]-\lambda_\varepsilon[K_G])\{U_\varepsilon\}+([K_0]-\lambda_0[K_G])\{U_{\varepsilon\varepsilon}\}=\{0\} \qquad (14)$$

$$([K_{\mu\varepsilon}]-\lambda_{\mu\varepsilon}[K_G])\{U_0\}+([K_\varepsilon]-\lambda_\varepsilon[K_G])\{U_\mu\}+([K_\mu]-\lambda_\mu[K_G])\{U_\varepsilon\}+([K_0]-\lambda_0[K_G])\{U_{\mu\varepsilon}\}=\{0\} \qquad (15)$$

Equation (10) equals ordinary characteristic equation of eigenvalue problem, by which the eigen value λ_0 and eigenvector $\{U_0\}$ are determined for given structural data and s_0 and c_0.

4. Rates of changes of eigenvalues and eigenvectors

Equations (11) through (15) are used to calculate the rates of change of eigenvalues and eigenvectors, now that λ_0 and $\{U_0\}$ are given. Premultiplying Eq. (11) of the i th order by $\lfloor U_0\rfloor^i$ and taking advantage of the symmetry of $[K_0]-\lambda_0^i[K_G]$, we can easily determine the rate of change of eigenvalue as

$$\lambda_\mu^i=\frac{\lfloor U_0\rfloor^i[K_\mu]\{U_0\}^i}{\lfloor U_0\rfloor^i[K_G]\{U_0\}^i} \qquad (16)$$

where superscript i means the order of eigenvalue and $\lfloor\ \rfloor$ the transpose of $\{\ \}$. The rate of change of eigenvector $\{U_\mu\}^i$ cannot be obtained by simple manipulation of the matrix inverse of $[K_0]-\lambda_0^i[K_G]$ because of its singularity. A few methods have been proposed by Collins [5] and Fox [6] to determine such rate of change of eigenvector. We use the method proposed by Collins, though it takes considerable CPU time since it requires all the orders of eigenvalues and eigenvectors, as Fox's method was reported to encounter numerical difficulty.

Following Collins, we assume $U_{\mu 1}$, $U_{\varepsilon 1}$, $U_{\mu\mu 1}$, $U_{\varepsilon\varepsilon 1}$, $U_{\mu\varepsilon 1}=0$, that is, $U_1\equiv U_{01}$ without losing generality, where subscript 1 means the first component of the vector $\{U\}$. This gives rise to the decrease of number of unknowns $\{U_\mu\}$, $\{U_\varepsilon\}$, $\{U_{\mu\mu}\}$, $\{U_{\varepsilon\varepsilon}\}$, $\{U_{\mu\varepsilon}\}$ by unity. Premultiplying Eq. (11) of the i th order by $\lfloor U_0\rfloor^j (j\neq i)$ and subtract-

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ing $\lfloor U_0 \rfloor^j([K_0]-\lambda_0^j[K_G])\{U_\mu\}^i=0$ from the result, we have

$$\lfloor U_0 \rfloor^j[K_G]\{U_\mu\}^i=\frac{\lfloor U_0 \rfloor^j}{\lambda_0^i-\lambda_0^j}([K_\mu]-\lambda_\mu^i[K_G])\{U_0\}^i \quad (17)$$

Arrangement of $n-1$ equations (17) for $j=1,2,\cdots,n(j \neq i)$ results in $n-1$ simultaneous equations in form of

$$[A]\{U_\mu{}^*\}^i=\{B\} \quad (18)$$

with respect to $\{U_\mu{}^*\}^i$ which stands for $\lfloor U_{\mu 2}, U_{\mu 3},\cdots U_{\mu n} \rfloor^T$. The matrix $[A]$ in size of $n-1 \times n-1$ is no longer singular, which is generated from $\lfloor U_0 \rfloor^j[K_G]$ for $j=1,2,\cdots,n\ (j \neq i)$. Thus $\lfloor U_\mu \rfloor^i$ can be calculated by solving the simultaneous equations (18). The other rates of change of eigenvalues and eigenvectors are detemined successively in the similar manner by the use of Eqs. (12) to (15). It should be noted that we have

$$\lambda_{\mu\varepsilon}^i=\frac{\lfloor U_0 \rfloor^i}{\lfloor U_0 \rfloor^i[K_G]\{U_0\}^i}([K_{\mu\varepsilon}]\{U_0\}^i + ([K_\varepsilon]-\lambda_\varepsilon^i[K_G])\{U_\mu\}^i+([K_\mu]-\lambda_\mu^i[K_G])\{U_\varepsilon\}^i) \quad (19)$$

$$\lfloor U_0 \rfloor^j[K_G]\{U_{\mu\varepsilon}\}^i=\frac{\lfloor U_0 \rfloor^j}{\lambda_0^i-\lambda_0^j}\{([K_{\mu\varepsilon}]-\lambda_{\mu\varepsilon}^i[K_G])\{U_0\}^i +([K_\varepsilon]-\lambda_\varepsilon^i[K_G])\{U_\mu\}^i+([K_\mu]-\lambda_\mu^i[K_G])\{U_\varepsilon\}^i\} \quad (20)$$

It is seen from the above equations that $\lambda_{\mu\varepsilon}^i$ and $\{U_{\mu\varepsilon}\}^i$ are not zero in spite of the fact that $[K_{\mu\varepsilon}]=[0]$.

5. Numerical example

Let us consider a straight column and divide it into five finite elements. In the case that the column is built-in at the both ends, the analytical buckling load is given as $P=4\pi^2EI/L^2$, and $P=(\pi^2/4)EI/L^2$ in the case of an end built-in and the other free. Then the analytical eigenvalues $\lambda=Pl^2/EI$ are $\lambda=1.5791$ and 0.098696 respectively in the above two cases, because $l=L/5$. To estimate the buckling eigenvalue under nearly built-in condition and nearly free condition at an end to uncertain extent, we apply the aforementioned formulation to such cases as $s_0=c_0=0.999$, 0.960 and 0.001. The eigenvalues and their rates of change thus calculated are listed in Table 1. Table 2 lists the eigenvector and its rates of change in case of $s_0=c_0=0.999$. In this calculation, the eigenvector $\{U_0\}^i$ is normalized as $\lfloor U_0 \rfloor^i[K_G]\{U_0\}^i=1$. It is noted that only the results about the primary mode $(i=1)$ are shown here. Owing to the small number of elements used, the numerically calculated eigenvalues remain rather larger than the analytical solutions.

Table 1 also shows that the effect of the restraint against angle of rotation is greater than that against deflection in the case of nearly built-in condition, and is smaller in the case of nearly free condition. Figure 2 depicts the comparison between the result obtained by the present method and those

Table 1　Eigenvalues and their rates of change (Primary mode: $i=1$)

$s_0=c_0$	λ_0	λ_μ	λ_ε	$\lambda_{\mu\mu}$	$\lambda_{\varepsilon\varepsilon}$	$\lambda_{\mu\varepsilon}$
0.001	0.10315	0.406×10^{-2}	0.401×10^{-3}	0.131×10^{-5}	-0.610×10^{-8}	0.118×10^{-6}
0.960	1.5579	0.482×10^{-7}	0.648×10^{0}	-0.639×10^{-7}	-0.400×10^{0}	-0.447×10^{-5}
0.999	1.5836	0.163×10^{-13}	0.640×10^{0}	-0.265×10^{-13}	-0.442×10^{0}	-0.104×10^{-7}

Table 2　Eigenvector and its rates of change in case of $s_0=c_0=0.999$ (Primary mode: $i=1$)

	$\{U_0\}$	$\{U_\mu\}$	$\{U_\varepsilon\}$	$\{U_{\mu\mu}\}$	$\{U_{\varepsilon\varepsilon}\}$	$\{U_{\mu\varepsilon}\}$
w_1	0.30856×10^{-10}	0.0	0.0	0.0	0.0	0.0
	-0.48410×10^{-5}	-0.484×10^{-2}	-0.484×10^{-2}	-0.484×10^{1}	-0.484×10^{1}	-0.484×10^{1}
w_2	-0.84527×10^{-1}	-0.846×10^{2}	-0.169×10^{3}	-0.845×10^{5}	-0.253×10^{6}	-0.169×10^{6}
	-0.36487×10^{-2}	-0.365×10^{1}	-0.730×10^{1}	-0.365×10^{4}	-0.109×10^{5}	-0.730×10^{4}
w_3	-0.22104×10^{0}	-0.221×10^{3}	-0.442×10^{3}	-0.221×10^{6}	-0.663×10^{6}	-0.442×10^{6}
	-0.22519×10^{-2}	-0.225×10^{1}	-0.451×10^{1}	-0.225×10^{4}	-0.676×10^{4}	-0.451×10^{4}
w_4	-0.22095×10^{0}	-0.221×10^{3}	-0.442×10^{3}	-0.230×10^{6}	-0.663×10^{6}	-0.442×10^{6}
	0.22558×10^{-2}	0.226×10^{1}	0.451×10^{1}	0.226×10^{4}	0.676×10^{4}	0.451×10^{4}
w_5	-0.84381×10^{-1}	-0.844×10^{2}	-0.169×10^{3}	-0.844×10^{5}	-0.253×10^{6}	-0.169×10^{6}
	0.36472×10^{-2}	0.365×10^{1}	0.729×10^{1}	0.365×10^{4}	0.109×10^{5}	0.730×10^{4}

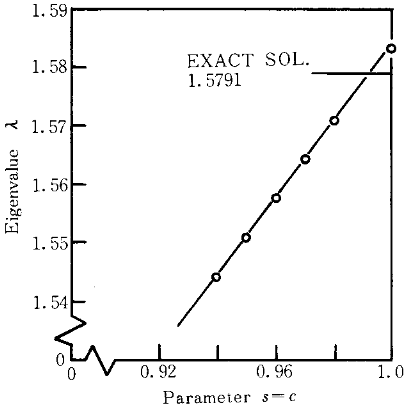

Figure 2 Comparison between second order perturbation estimation and deterministic solutios (Primary eigenvalue: $i=1$)

obtained in deterministic manner. The blank circles in Fig. 2 stand for the values calculated by Eq.(10) while $s_0=c_0$ are varied deterministically. The solid line in the figure is drawn in accordance with

$$\lambda=1.5579+(0.48168\times10^{-7}+0.64807)\varepsilon +(-0.63874\times10^{-7}-0.40025\times10^{0}-0.44715 \times10^{-5})\varepsilon^2 \tag{21}$$

by use of the result given in Table 1 for $s_0=c_0=0.960$ and $\mu=\varepsilon$. The good agreement of the solid line with the circles proves the validity of the present stochastic treatment. It is seen from Table 2 that the rates of change of eigenvector take almost similar form with the fundamental (deterministic) mode.

It goes without saying that the expectations and variances of eigenvalues and eigenvectors, $E[\lambda^i], Var[\lambda^i], E[\{U\}^i]$ and $Var[\{U\}^i]$, are easily evaluated as the functions of moments about μ and/or ε which represent the uncertainty of the boundary condition.

6. Concluding remarks

We present herein a method based on the second order perturbation to evaluate the statistics of the eigenvalues and eigenvectors. The uncertainty about boundary condition is well taken into account by the introduction of virtual spring elements of which the rigidity is stochastic. Although the numerical example given is a simple one to show the effect of uncertain boundary conditions on the eigenvalues in regard to column buckling under compression, it should be emphasized that more complex structures are analized in a similar manner without difficulty.

It is needless to say that present method can be easily modified for application to the other problems of mechanical vibration under uncertain boundary conditions.

Acknowledgement

The authors wish to express their sincere thanks to Assoc. Prof. A.P. Kabaila of Civil Engineering, the University of New South Wales, who enabled one of the authors to carry out a part of this work and related computation by use of Computer Services Unit of the said universiry during his stay in Sydney. (Manuscript received, April 8, 1981)

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