研　究　速　報

UDC 624.07 : 519.2
530.18

A Note on Stochastic Finite Element Method (Part 3)
— An Extension of the Methodology to Nonlinear Problems —
確率有限要素法に関するノート（第3報）
—非線形問題への方法論の拡張—

Toshiaki HISADA* and Shigeru NAKAGIRI *
久 田 俊 明・中 桐　滋

1. Introduction

Our previous reports[1), 2)] presented basic concepts and formulations for linear elastic problems with fluctuating shape, fluctuating material properties or fluctuating geometrical boundary condition. The authors would like to emphasize that the expectations and dispersions of displacement, strain and stress at arbitrary points in structure have been found to be obtained by solving the essential equilibrium equation only once.

Nonlinear problems are treated usually by incremental methods or iterative methods, in which laborious computational work is required. This fact implies that the statistics mentioned above are hard to be evaluated by usual means. It is therefore natural to extend the present method to nonlinear field. Thus the objective of this note is to examine the possibility to apply the stochastic finite element method to a problem of material nonlinearity. In order to be compatible with the previous formulas, only two dimensional problems are dealt with and redundant explanations are omitted because of the limitation of space.

2. Formulation for Fluctuations of Material Properties

Nonlinear elasticity in which elastic constants are not constant but dependent on the strain state is dealt with in this note. Plasticity is sometimes treated approximately as nonlinear elasticity, for example, in case of J-integral calculation on the basis of the deformation theory.

Since the stochastic finite element method cannot manage at present the change of material constitutive equation from elasticity to plasticity, an appropriate formula is desired which covers wide range of strain. Although many stress-strain formulas[3)] have been proposed such as Ramberg-Osgood model, most of them are given in the form of $\varepsilon = g(\sigma)$. From the standpoint of finite element displacement method, it is convenient to express constitutive equation as Eq. (1) for the purpose to describe explicitly the stress-strain matrix $\boldsymbol{D}$.

$$\sigma = f(\varepsilon) \tag{1}$$

In this chapter, any two parameters $\widetilde{\alpha}$ and $\widetilde{\beta}$ in the right hand side of Eq. (1) are taken as spatially stochastic over the structure such as the following expression which is devised so as to approximate desired constitutive equation.

$$\sigma = E\varepsilon - E\widetilde{\beta}\left[n\left\{1+\left(\frac{\varepsilon}{\widetilde{\beta}}\right)^2\right\}^{\widetilde{\alpha}} - 1\right] \tag{2}$$

In this equation, $\widetilde{\alpha}$ and $\widetilde{\beta}$ are introduced in order to control the magnitude of work hardening and yield strain respectively, but the result is not so satisfactory compared with the model[4)] stated above.

Anyway, if appropriate expression is obtained for a common constitutive equation which holds for both elastic and plastic regions, $\boldsymbol{D}$ in the basic stress-strain relation $\boldsymbol{\sigma} = \boldsymbol{De}$ is given as follows in two-dimensional case of the deformation theory.

$$\boldsymbol{D} = \begin{cases} \dfrac{2G}{(1+2G\phi)\{3(1-\nu)+E\phi\}} \cdot \begin{bmatrix} 3+2E\phi & 3\nu+E\phi & 0 \\ 3\nu+E\phi & 3+2E\phi & 0 \\ 0 & 0 & \{3(1-\nu)+E\phi\}/2 \end{bmatrix} & \underline{\textit{plane stress}} \\[2ex] \dfrac{2G}{3(1+2G\phi)(1-2\nu)} \cdot \begin{bmatrix} 3(1-\nu)+E\phi & 3\nu+E\phi & 0 \\ 3\nu+E\phi & 3(1-\nu)+E\phi & 0 \\ 0 & 0 & 3(1-2\nu)/2 \end{bmatrix} & \textit{plane strain} \end{cases} \tag{3}$$

Where $\boldsymbol{\sigma} = \lfloor \sigma_x\ \sigma_y\ \tau_{xy} \rfloor^T$, $\boldsymbol{e} = \lfloor \varepsilon_x\ \varepsilon_y\ \gamma_{xy} \rfloor^T$ and $\phi = 3/2 \cdot \bar{\varepsilon}_p/\bar{\sigma}$ is the proportional constant of the stresses and strains in the case of pure plasticity, which is approximated in the case of nonlinear elasticity approximation for plasticity as follows.

$$\phi = \frac{3}{2}\left\{\frac{\bar{\varepsilon} - \bar{\sigma}/E}{\bar{\sigma}}\right.$$

$$= \frac{3}{2}\left\{\frac{\sqrt{\frac{2}{3}\left(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2+\frac{1}{2}\gamma_{yz}^2+\frac{1}{2}\gamma_{zx}^2+\frac{1}{2}\gamma_{xy}^2\right)}}{f\left(\sqrt{\frac{2}{3}\left(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2+\frac{1}{2}\gamma_{yz}^2+\frac{1}{2}\gamma_{zx}^2+\frac{1}{2}\gamma_{xy}^2\right)}\right)} - \frac{1}{E}\right. \tag{3-a}$$

* Dept. of Applied Physics and Applied Mechanics, Institute of Industrial Science, University of Tokyo.

With the aid of the strain-nodal displacement matrix $\boldsymbol{B}=\lfloor\{\boldsymbol{B}\}_1\{\boldsymbol{B}\}_2\{\boldsymbol{B}\}_3\rfloor^T$, $\boldsymbol{D}$ is expressed in terms of the nodal displacement vector $\boldsymbol{U}$ or the set of its entities U_* of the element under interest.

$$\boldsymbol{D}(\varepsilon_x,\varepsilon_y,\gamma_{xy})=\boldsymbol{D}(\lfloor\boldsymbol{B}\rfloor_1\boldsymbol{U},\lfloor\boldsymbol{B}\rfloor_2\boldsymbol{U},\lfloor\boldsymbol{B}\rfloor_3\boldsymbol{U})\equiv\boldsymbol{D}'(U_*) \quad (4)$$

$$\boldsymbol{D}'(U_*)=\begin{bmatrix} d_{11}(U_*) & d_{12}(U_*) & d_{13}(U_*) \\ d_{21}(U_*) & d_{22}(U_*) & d_{23}(U_*) \\ d_{31}(U_*) & d_{32}(U_*) & d_{33}(U_*) \end{bmatrix} \quad (4\text{-a})$$

It is assumed that $d_{kl}(U_*)$ is expanded as a power series in α and β which represent the varing portions of $\widetilde{\alpha}$ and $\widetilde{\beta}$ as $\alpha=\widetilde{\alpha}-E[\widetilde{\alpha}]$ and $\beta=\widetilde{\beta}-E[\widetilde{\beta}]$, where $E[\cdot]$ indicates expectation. Additionally, it is assumed at present that the terms of which the orders are greater than third with respect to α and/or β are negligible based on the postulate that α and β are small. Therefore, in case α and β hold constant within an element 'p', we have:

$$d_{klp}(U_*)=d^0_{klp}(U_*)+d^1_{klp}(U_*)\alpha_p+d^{1'}_{klp}(U_*)\beta_p +d^2_{klp}(U_*)\alpha^2_p+d^{2'}_{klp}(U_*)\alpha_p\beta_p+d^{2''}_{klp}(U_*)\beta^2_p \quad (5)$$

As well as in the previous reports[1), 2)], the following expansion is used for displacements.

$$\boldsymbol{U}=\boldsymbol{U}^0+\sum_r(\boldsymbol{U}^1_r\alpha_r+\boldsymbol{U}^{1'}_r\beta_r) +\sum_r\sum_s(\boldsymbol{U}^2_{rs}\alpha_r\alpha_s+\boldsymbol{U}^{2'}_{rs}\alpha_r\beta_s+\boldsymbol{U}^{2''}_{rs}\beta_r\beta_s) \quad (6)$$

or

$$U_i=U^0_i+\sum_r(U^1_{ir}\alpha_r+U^{1'}_{ir}\beta_r) +\sum_r\sum_s(U^2_{irs}\alpha_r\alpha_s+U^{2'}_{irs}\alpha_r\beta_s+U^{2''}_{irs}\beta_r\beta_s) \quad (6\text{-a})$$

where r and s are integer varying from unity to the total number of elements. By applying Eq. (6-a) into Eq. (5), $d_{klp}(U_*)$ is summarized as follows when d^0_{klp}, d^1_{klp}, --- are analytic in U_* [5)].

$$\begin{aligned} d_{klp}(U_*)=\,&d^0_{klp}(U^0_*)\\ &+\sum_r\{{}^1d^0_{klpr}(U^0_*,U^1_{*r})+d^1_{klp}(U^0_*)\delta_{rp}\}\alpha_r\\ &+\sum_r\{{}^{1'}d^0_{klpr}(U^0_*,U^{1'}_{*r})+d^{1'}_{klp}(U^0_*)\delta_{rp}\}\beta_r\\ &+\sum_r\sum_s\{{}^2d^0_{klprs}(U^0_*,U^1_{*r},U^1_{*s},U^2_{*rs})\\ &+{}^1d^1_{klpr}(U^0_*,U^1_{*r})\delta_{sp}+d^2_{klp}(U^0_*)\delta_{rp}\delta_{sp}\}\alpha_r\alpha_s\\ &+\sum_r\sum_s\{{}^{2'}d^0_{klprs}(U^0_*,U^1_{*r},U^{1'}_{*s},U^{2'}_{*rs})\\ &+{}^{1'}d^1_{klps}(U^0_*,U^{1'}_{*s})\delta_{rp}+{}^1d^{1'}_{klpr}(U^0_*,U^1_{*r})\delta_{sp}\\ &+d^{2'}_{klp}(U^0_*)\delta_{rp}\delta_{sp}\}\alpha_r\beta_s\\ &+\sum_r\sum_s\{{}^{2''}d^0_{klprs}(U^0_*,U^{1'}_{*r},U^{1'}_{*s},U^{2''}_{*rs})\\ &+{}^{1'}d^{1'}_{klpr}(U^0_*,U^{1'}_{*r})\delta_{sp}+d^{2''}_{klp}(U^0_*)\delta_{rp}\delta_{sp}\}\beta_r\beta_s \quad (7)\end{aligned}$$

Where U^0_*, U^1_{*r}, --- represent the set of the entities of $\boldsymbol{U}^0, \boldsymbol{U}^1_r$, ---, and δ_{ij} is Kronecker's delta.

The element stiffness matrix is obtained through the integration of the following equation over the relevant quadrature points.

$$\begin{aligned}[\widetilde{k}]=\,&\boldsymbol{B}^T_i\boldsymbol{D}'(U_*)\boldsymbol{B}_j\\ =&\begin{bmatrix}\{b_id_{11p}(U_*)+c_id_{31p}(U_*)\}b_j+\{b_id_{13p}(U_*)+c_id_{33p}(U_*)\}c_j & \{b_id_{12p}(U_*)+c_id_{32p}(U_*)\}c_j+\{b_id_{13p}(U_*)+c_id_{33p}(U_*)\}b_j\\ \{c_id_{21p}(U_*)+b_id_{31p}(U_*)\}b_j+\{c_id_{23p}(U_*)+b_id_{33p}(U_*)\}c_j & \{c_id_{22p}(U_*)+b_id_{32p}(U_*)\}c_j+\{c_id_{23p}(U_*)+b_id_{33p}(U_*)\}b_j\end{bmatrix} \quad (8)\end{aligned}$$

where

$$\boldsymbol{B}_i=\begin{bmatrix} b_i & 0\\ 0 & c_i\\ c_i & b_i\end{bmatrix} \quad (8\text{-a})$$

It is therefore obvious that the ij-th entity $K_{ij}(U_*)$ of the global stiffness matrix $\boldsymbol{K}(U_*)$ has the following general form through usual merging procedure represented by $\sum_p$

$$\begin{aligned} K_{ij}(U_*)=\,&\sum_p\sum_g W_g\Big[\sum_k\sum_l a^{ij}_{klp}d_{klp}(U_*)\Big]_{\boldsymbol{X}=\boldsymbol{X}_g}\\ \equiv\,&K^0_{ij}(U^0_*)+\sum_r\{K^1_{ijr}(U^0_*,U^1_{*r})\alpha_r\\ &+K^{1'}_{ijr}(U^0_*,U^{1'}_{*r})\beta_r\}\\ &+\sum_r\sum_s\{K^2_{ijrs}(U^0_*,U^1_{*r},U^1_{*s},U^2_{*rs})\alpha_r\alpha_s\\ &+K^{2'}_{ijrs}(U^0_*,U^1_{*r},U^{1'}_{*s},U^{2'}_{*rs})\alpha_r\beta_s\\ &+K^{2''}_{ijrs}(U^0_*,U^{1'}_{*r},U^{1'}_{*s},U^{2''}_{*rs})\beta_r\beta_s\} \quad (9)\end{aligned}$$

in which a^{ij}_{klp} denotes one of the products such as b_ib_j, b_ic_j, c_ib_j or c_ic_j, and $\sum_g$ corresponds to Gauss quadrature with the weight coefficient W_g at space vector $\boldsymbol{X}_g$.

Applying the global stiffness matrix $\boldsymbol{K}(U_*)$ characterized above, and the displacement vector $\boldsymbol{U}$ of the form of Eq. (6) into Eq. (10) as equilibrium equation, Eq. (11) is obtained, in which the second order perturbation method is to be carried out.

$$\boldsymbol{K}(U_*)\boldsymbol{U}=\boldsymbol{F} \quad (10)$$

$$\begin{aligned}\Big[&\boldsymbol{K}^0(U^0_*)+\sum_r\{\boldsymbol{K}^1_r(U^0_*,U^1_{*r})\alpha_r+\boldsymbol{K}^{1'}_r(U^0_*,U^{1'}_{*r})\beta_r\}\\ &+\sum_r\sum_s\{\boldsymbol{K}^2_{rs}(U^0_*,U^1_{*r},U^1_{*s},U^2_{*rs})\alpha_r\alpha_s\\ &+\boldsymbol{K}^{2'}_{rs}(U^0_*,U^1_{*r},U^{1'}_{*s},U^{2'}_{*rs})\alpha_r\beta_s\\ &+\boldsymbol{K}^{2''}_{rs}(U^0_*,U^{1'}_{*r},U^{1'}_{*s},U^{2''}_{*rs})\beta_r\beta_s\}\Big]\\ &\times\Big\{\boldsymbol{U}^0+\sum_r(\boldsymbol{U}^1_r\alpha_r+\boldsymbol{U}^{1'}_r\beta_r)+\sum_r\sum_s(\boldsymbol{U}^2_{rs}\alpha_r\alpha_s+\boldsymbol{U}^{2'}_{rs}\alpha_r\beta_s\\ &+\boldsymbol{U}^{2''}_{rs}\beta_r\beta_s)\Big\}=\boldsymbol{F} \quad (11)\end{aligned}$$

In the sequel, the following equations are obtained in the similar manner with the first report[1)].

$$\boldsymbol{K}^0(U^0_*)\boldsymbol{U}^0=\boldsymbol{F} \quad (12)$$

$$\boldsymbol{K}^1_r(U^0_*,U^1_{*r})\boldsymbol{U}^0+\boldsymbol{K}^0(U^0_*)\boldsymbol{U}^1_r=0 \quad (13)$$

$$\boldsymbol{K}^{1'}_{r}(U^{0}_{*},U^{1'}_{*r})\boldsymbol{U}^{0}+\boldsymbol{K}^{0}(U^{0}_{*})\boldsymbol{U}^{1'}_{r}=0 \tag{14}$$

$$\boldsymbol{K}^{1}_{r}(U^{0}_{*},U^{1}_{*r})\boldsymbol{U}^{1}_{s}+\boldsymbol{K}^{2}_{rs}(U^{0}_{*},\boldsymbol{U}^{1}_{*r},\boldsymbol{U}^{1}_{*s},\boldsymbol{U}^{2}_{*rs})\boldsymbol{U}^{0} +\boldsymbol{K}^{0}(U^{0}_{*})\boldsymbol{U}^{2}_{rs}=0 \tag{15}$$

$$\boldsymbol{K}^{1}_{r}(U^{0}_{*},U^{1}_{*r})\boldsymbol{U}^{1'}_{s}+\boldsymbol{K}^{1'}_{s}(U^{0}_{*},U^{1'}_{*s})\boldsymbol{U}^{1}_{r}+\boldsymbol{K}^{0}(U^{0}_{*})\boldsymbol{U}^{2'}_{rs} +\boldsymbol{K}^{2'}_{rs}(U^{0}_{*},U^{1}_{*r},U^{1'}_{*s},U^{2'}_{*rs})\boldsymbol{U}^{0}=0 \tag{16}$$

$$\boldsymbol{K}^{1'}_{r}(U^{0}_{*},U^{1'}_{*r})\boldsymbol{U}^{1'}_{s}+\boldsymbol{K}^{2''}_{rs}(U^{0}_{*},U^{1'}_{*r},U^{2'}_{*s},U^{2''}_{*rs})\boldsymbol{U}^{0} +\boldsymbol{K}^{0}(U^{0}_{*})\boldsymbol{U}^{2''}_{rs}=0 \tag{17}$$

This means that Eq. (12) is to be solved at first as regards $\boldsymbol{U}^0$ which exactly agrees with that of deterministic finite element analysis. Secondly, application of $\boldsymbol{U}^0$ into Eqs. (13) and (14) enables us to evaluate $\boldsymbol{U}^1_r$ and $\boldsymbol{U}^{1'}_r$ for every r. Thirdly, according to Eqs. (15), (16) and (17), $\boldsymbol{U}^2_{rs}, \boldsymbol{U}^{2'}_{rs}$ and $\boldsymbol{U}^{2''}_{rs}$ are determined for every r and s with the aid of $\boldsymbol{U}^0$, $\boldsymbol{U}^1_r$ and $\boldsymbol{U}^{1'}_r$. In order to solve the above nonlinear equations, a proper conventional iterative method such as Newton-Raphson's is utilized.

Then, the expectation and dispersion of the displacement under discussion are given as follows.

$$E[U_i]=U^0_i+\sum_r\sum_s\{U^2_{irs}E[\alpha_r\alpha_s]+U^{2'}_{irs}E[\alpha_r\beta_s] +U^{2''}_{irs}E[\beta_r\beta_s]\} \tag{18}$$

$$\mathrm{Var}[U_i]=\sum_r\sum_s\{U^1_{ir}U^1_{is}E[\alpha_r\alpha_s]+2U^1_{ir}U^{1'}_{is}E[\alpha_r\beta_s] +U^{1'}_{ir}U^{1'}_{is}E[\beta_r\beta_s]\} \tag{19}$$

Eq. (19) is derived based on the first order approximation[6].

In a similar manner, the statistics of strain are easily obtained based on the following equation.

$$e_j=\sum_m B_{jm}U^0_m+\sum_m\sum_r B_{jm}(U^1_{mr}\alpha_r+U^{1'}_{mr}\beta_r) +\sum_m\sum_r\sum_s B_{jm}(U^2_{mrs}\alpha_r\alpha_s+U^{2'}_{mrs}\alpha_r\beta_s+U^{2''}_{mrs}\beta_r\beta_s) \tag{20}$$

where e_1, e_2 and e_3 correspond to $\varepsilon_x, \varepsilon_y$ and γ_{xy} respectively, and B_{jm} denotes the m-th entity of $\lfloor \boldsymbol{B} \rfloor_j$. Then, the following equation is to be calculated according to Eqs.(7) and (20) so as to evaluate those of the stress of element " p ".

$$\sigma_i=\sum_j d_{ijp}(U_*)e_j \equiv\sigma^0_i+\sum_r(\sigma^1_{ir}\alpha_r+\sigma^{1'}_{ir}\beta_r)+\sum_r\sum_s(\sigma^2_{irs}\alpha_r\alpha_s+\sigma^{2'}_{irs}\alpha_r\beta_s +\sigma^{2''}_{irs}\beta_r\beta_s) \tag{21}$$

Where σ_1, σ_2 and σ_3 correspond to σ_x, σ_y and τ_{xy} respectively.

In case $\widetilde{\alpha}$ and $\widetilde{\beta}$ take the same values not only in an element but also in a whole structure, the summations $\sum_r$, $\sum_s$ and $\sum_r\sum_s$ appearing in this chapter can be reduced to $r=s=1$

3. Formulation for Fluctuation of Shape

In this chapter, α and β are taken as varing portions of the r-th nodal cordinate (x_r , y_r). It is assumed that Eq. (6) or (6-a) holds for displacement, and for brevity, discussions are made in case $d_{kl}(U_*)$ can be expanded as follows.

$$d_{kl}(U_*)=d_{kl}(U^0_*)+\sum_r\{{}^1d_{klr}(U^0_*,U^1_{*r})\alpha_r +{}^{1'}d_{klr}(U^0_*,U^{1'}_{*r})\beta_r\} +\sum_r\sum_s\{{}^2d_{klrs}(U^0_*,U^1_{*r},U^1_{*s},U^2_{*rs})\alpha_r\alpha_s +{}^{2'}d_{klrs}(U^0_*,U^1_{*r},U^{1'}_{*s},U^{2'}_{*rs})\alpha_r\beta_s +{}^{2''}d_{klrs}(U^0_*,U^{1'}_{*r},U^{1'}_{*s},U^{2''}_{*rs})\beta_r\beta_s\} \tag{22}$$

For the purpose to construct element stiffness matrix $[\widetilde{\boldsymbol{k}}]$, $d_{kl}(U_*)$ is multiplied by $\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}det|J|$ in case the isoparametric finite element is dealt with[7]. Where N_i and $|J|$ denote displacement functions and the determinant of Jacobian matrix respectively. Since $\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}det|J|$ has been already expanded like Eq. (23) as a power series in α and β in the first report (Eq. (24))[1], it is apparent that the global stiffness matrix $K_{ij}(U_*)$ can be summarized through the integration and merging procedure in a similar fashion as the right hand side of Eq. (9).

$$\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\det|J|=(\text{deterministic term})+\sum_r(m'_r\alpha_r +n'_r\beta_r)+\sum_r\sum_s(f'_{rs}\alpha_r\alpha_s+g'_{rs}\alpha_r\beta_s+h'_{rs}\beta_r\beta_s) \tag{23}$$

Then, $\boldsymbol{U}^0, \boldsymbol{U}^1_r$, … are solved based on Eqs. (12) to (17), so that $E[U_i]$ and $\mathrm{Var}[U_i]$ are also evaluated as Eqs. (18) and (19). On the occasion of the evaluation of $E[e_i]$ and $\mathrm{Var}[e_i]$, the equations in the first report (Eqs. (13) and (23))[1] are unchanged in application. Finally, σ_i is obtained by carring out the following equation so as to evaluate $E[\sigma_i]$ and $\mathrm{Var}[\sigma_i]$.

$$\sigma_i=\sum_j d_{ij}(U_*)e_j \tag{24}$$

4. Formulation for Fluctuation of Geometrical Boundary Condition

According to the formulation of the second report[2], $\{U^\beta\}$ in the following equilibrium equation is taken as a stochastic process in this chapter.

$$\begin{bmatrix}K^{\alpha\alpha} & K^{\alpha\beta}\\ K^{\beta\alpha} & K^{\beta\beta}\end{bmatrix}\begin{Bmatrix}U^\alpha\\ U^\beta\end{Bmatrix}=\begin{Bmatrix}F^\alpha\\ F^\beta\end{Bmatrix} \tag{25}$$

Where $\{U^\alpha\}$ represents unknown displacement vector, and $\{F^\alpha\}$ and $\{F^\beta\}$ indicate known and unknown nodal force vectors respectively. The stochastic process U^β_i, the i-th entity of $\{U^\beta\}$, is represented as the following equation with the variational portion γ_i.

$$U^\beta_i=E[U^\beta_i]+\gamma_i\equiv{}^0U^\beta_i+\gamma_i \tag{26}$$

At the same time, U^α_i, the i-th entity of $\{U^\alpha\}$, is assumed to be expanded as a power series in γ_i.

$$U^\alpha_i={}^0U^\alpha_i+\sum_r {}^1U^\alpha_{ir}\gamma_r+\sum_r\sum_s {}^2U^\alpha_{irs}\gamma_r\gamma_s \tag{27}$$

Since Eq. (27) involves Eq. (26), displacement is formally represented as follows regardless of the difference between U_i^α and U_i^β.

$$U_i = {}^0U_i + \sum_r {}^1U_{ir}\gamma_r + \sum_r\sum_s {}^2U_{irs}\gamma_r\gamma_s \qquad (28)$$

Where ${}^1U_{ir} = \delta_{ir}$ and ${}^2U_{irs} = 0$ in case U_i stands for U_i^β.

Based on the same principle as that of chapter 2, $d_{kl}(U_*)$ in Eq. (4-a) is consequently expanded as follows when analytic in U_*.

$$d_{kl}(U_*) = d_{kl}({}^0U_*) + \sum_r {}^1d_{klr}({}^0U_*, {}^1U_{*r})\gamma_r + \sum_r\sum_s {}^2d_{klrs}({}^0U_*, {}^1U_{*r}, {}^1U_{*s}, {}^2U_{*rs})\,\gamma_r\gamma_s \quad (29)$$

The integration and merging procedure to construct the global stiffness matrix $K_{ij}(U_*)$ corresponds to the linear operation of the product of $d_{kl}(U_*)$ and a_{klp}^{ij} appearing in Eq. (9) or $\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\det|J|$ of Eq. (23). Within this chapter, however, a_{klp}^{ij} or $\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\det|J|$ is taken as deterministic, or in other words $\alpha = \beta = 0$, so that it is obvious that $K_{ij}(U_*)$ is summarized as follows.

$$K_{ij}(U_*) = K_{ij}^0({}^0U_*) + \sum_r K_{ijr}^1({}^0U_*, {}^1U_{*r})\gamma_r + \sum_r\sum_s K_{ijrs}^2({}^0U_*, {}^1U_{*r}, {}^1U_{*s}, {}^2U_{*rs})\;\gamma_r\gamma_s \quad (30)$$

Eq. (25) is to be solved at first by examining Eq. (31), which can be expressed as Eq. (31-a) according to the treatment that U_i^α and U_i^β are formally identical.

$$[K^{\alpha\alpha}\ \ K^{\alpha\beta}]\begin{Bmatrix}U^\alpha\\U^\beta\end{Bmatrix} = \{F^\alpha\} \qquad (31)$$

$$\boldsymbol{KU} = \boldsymbol{F} \qquad (31\text{-a})$$

Substituting the vector representation of Eq. (28) and (30) into Eq. (31-a), the following relation is obtained.

$$\{\boldsymbol{K}^0({}^0U_*) + \sum_r \boldsymbol{K}_r^1({}^0U_*, {}^1U_{*r})\gamma_r + \sum_r\sum_s \boldsymbol{K}_{rs}^2({}^0U_*, {}^1U_{*r}, {}^1U_{*s}, {}^2U_{*rs})\gamma_r\gamma_s\} \times \{{}^0\boldsymbol{U} + \sum_r {}^1\boldsymbol{U}_r\gamma_r + \sum_r\sum_s {}^2\boldsymbol{U}_{rs}\gamma_r\gamma_s\} = \boldsymbol{F} \qquad (32)$$

Then, on the basis of second order perturbation principle, the set of nonlinear equations are derived as follows.

$$\boldsymbol{K}^0({}^0U_*){}^0\boldsymbol{U} = \boldsymbol{F} \qquad (33)$$

$$\boldsymbol{K}_r^1({}^0U_*, {}^1U_{*r}){}^0\boldsymbol{U} + \boldsymbol{K}^0({}^0U_*){}^1\boldsymbol{U}_r = 0 \qquad (34)$$

$$\boldsymbol{K}_r^1({}^0U_*, {}^1U_{*r}){}^1\boldsymbol{U}_s + \boldsymbol{K}_{rs}^2({}^0U_*, {}^1U_{*r}, {}^1U_{*s}, {}^2U_{*rs}){}^0\boldsymbol{U} + \boldsymbol{K}^0({}^0U_*){}^2\boldsymbol{U}_{rs} = 0 \qquad (35)$$

Finally, ${}^0\boldsymbol{U}$, ${}^1\boldsymbol{U}_r$ and ${}^2\boldsymbol{U}_{rs}$ are solved for every r and s by a proper iterative method so as to evaluate $E[\boldsymbol{U}]$ and $\mathrm{Var}[\boldsymbol{U}]$. It will be unnecessary to show the procedure to evaluate $\boldsymbol{E}$ and its statistics owing to the deteminstic nature of $\boldsymbol{B}$ matrix. On the occasion of the evaluation of $\boldsymbol{\sigma}$, Eq. (29) is again introduced and multiplied by $\boldsymbol{e}$ as a power series in γ.. Thus the statistics of $\boldsymbol{\sigma}$ are obtained in a similar manner as described in chapter 1.

It should be noted again that all the terms whose orders are greater than third with respect to the varing portions are neglected throughout this paper, and as regards second moments of these varing portions, spectral interpretation is made through Wiener-Khintchine relation such as

$$E[\alpha_i\,\beta_j] = R_{\alpha\beta}(\boldsymbol{X}_j - \boldsymbol{X}_i) = \int_{-\infty}^{\infty} S_{\alpha\beta}(\boldsymbol{\lambda})\exp i\,2\pi\boldsymbol{\lambda}\cdot(\boldsymbol{X}_j - \boldsymbol{X}_i)\,d\boldsymbol{\lambda} \quad (36)$$

Where $R_{\alpha\beta}(\cdot)$ and $S_{\alpha\beta}(\cdot)$ denote crosscorrelation function and crossspectral density respectively, $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ indicate space vectors, and $\boldsymbol{\lambda}$ represents the wave number vector.

5. Conclusions

Original concept of stochastic finite element method is found to be expansible into the nonlinear elastic problems.

Since Monte Carlo approach is hardly applicable to this sort of problem from the aspect of CPU time, the present methodology is believed to have wide varieties for the use in structural safety and reliability.

Aknowledgement

The authors wish to express their gratitude to Prof. C. Allin Cornell of M.I.T. who gave us stimulating advice and a report, MIT CE R-78-13, 'Settlement Uncertainty by Second Moment FEM' by T.S. Ingra and G.B. Baecher also of M.I.T.

(Manuscript received, July 30, 1980)

References

1) Nakagiri, S. and Hisada, T.; A Note on Stochastic Finite Element Method (Part 1, Variation of Stress and Strain Caused by Shape Fluctuation), SEISAN-KENKYU, Vol. 32, No. 2, 1980, pp. 39

2) Hisada, T. and Nakagiri, S.; A Note on Stochastic Finite Element Method (Part 2, Variation of Stress and Strain Caused by Fluctuations of Material Properties and Geometrical Boundary Condition), SEISAN-KENKYU, Vol. 32, No. 5, 1980, pp. 28

3) Osgood, W.R.; Stress-Strain Formulas, J. AERON. Sci, Vol. 13, No. 1, 1946, pp. 45

4) Ramberg, W. and Miller, J.A.; Determination and Presentation of Compressive Stress-Strain Data for Thin Sheet Metal, J. AERON. Sci, Vol. 13, No. 11, 1946, pp. 569

5) Bellman, R.E.; Perturbation Techniques in Mathematics, Physics and Engineering, Holt, New York, 1964

6) Benjamin, J.R. and Cornell, C.A.; Probability, Statistics, and Decision for Civil Engineers, McGraw-Hill, New York, 1970

7) Zienkiewicz, O.C.; The Finite Element Method in Engineering Science, McGraw-Hill, England, 1971