UDC 624. 04 : 539. 3
624. 07

A Note on Stochastic Finite Element Method (Part 2)
— Variation of Stress and Strain Caused by Fluctuations of Material Properties and Geometrical Boundary Condition —

確率有限要素法に関するノート（第2報）
— 材料特性と幾何学的境界条件のゆらぎによる応力と歪の変動 —

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1. Introduction

A basic concept and formulation of stochastic finite element method have been constructed on the problem of shape fluctuation in the previous report.[1] It is worthy to emphasize that the expectations and dispersions of displacement, strain and stress at arbitrary points in structure under interest are obtained by solving the equilibrium equation only once.

The methodology proposed is, however, applicable to the treatment of stochastic stiffness matrices raised by properties of material distributed randomly, of which the phenomena are encountered in the field of rock mechanics,[2] composit materials[3] and even usual materials.[4] Hence, the extension of the methodology is of practical importance, and is primary concern of this study.

Secondly, stochastic finite element method on the problem of fluctuation of geometrical boundary condition is formulated. Nevertheless this kind of problem can be dealt with more easily than the above problem, the characteristic of analysis result will be of great interest from engineering point of view.

For simplicity, the following formulation is made on a two dimensional linear elastic problem as depicted Fig. 1, however, it is not difficult to extend it to three dimensional problems.

2. Formulation for Fluctuations of Material Properties

In the present chapter, Young's modulus E and Poisson's ratio ν are introduced as a two-variate stochastic

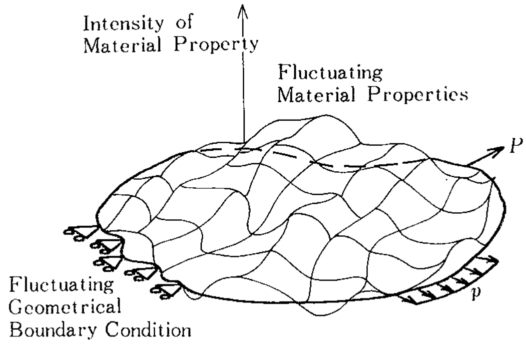

Fig. 1 Image of a Sample Illustrating Random Material Properties and Random Geometrical Boundary Condition

process. The element stiffness matrix of plane stress or plane strain state is obtained through the integration of the following equation[5] over the relevant quadrature points.

$$[\tilde{k}]=\xi\begin{bmatrix}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\nu'\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} & \bar{\nu}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}+\nu'\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\\ \bar{\nu}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}+\nu'\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y} & \nu'\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\end{bmatrix} \tag{1}$$

$$\xi=\begin{cases}\frac{E}{1-\nu^2}\\ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\end{cases},\quad \nu'=\begin{cases}\frac{1-\nu}{2}\\ \frac{1-2\nu}{2(1-\nu)}\end{cases},\quad \bar{\nu}=\begin{cases}\nu & :\text{plane stress} \quad (1\text{-a})\\ \frac{\nu}{1-\nu} & :\text{plane strain} \quad (1\text{-b})\end{cases}$$

It is therefore obvious that the ij-th entity of the global stiffness matrix K_{ij}^* has the following general form through

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usual merging procedure in case E and ν can be regarded as constant in a element.

$$K_{ij}=\begin{cases}\sum_l \dfrac{E_l}{1-\nu_l^2}(\nu_l Q_l^{ij}+R_l^{ij}) : \text{plane stress} & \text{(2-a)}\\ \sum_l \dfrac{E_l(1-\nu_l)}{(1+\nu_l)(1-2\nu_l)}\left(\dfrac{Q_l^{ij'}}{1-\nu_l}+\dfrac{\nu_l}{1-\nu_l}R_l^{ij'}+S_l^{ij'}\right) & \\ \equiv \sum_l \dfrac{E_l}{(1+\nu_l)(1-2\nu_l)}(\nu_l Q_l^{ij}+R_l^{ij}) & \\ : \text{plane strain} & \text{(2-b)}\end{cases}$$

where suffix l represents the elements concerned with merging procedure, in which Q_l^{ij} and R_l^{ij} are numerically evaluated.

For convenience, the authors express the stochastic process as sum of deterministic term and stochastic one which indicates small fluctuations.

$$E_l=E_l^0+\eta_l \qquad (3)$$

$$\nu_l=\nu_l^0+\varphi_l \qquad (4)$$

where deterministic terms E_l^0 and ν_l^0 correspond to the mean values of E_l and ν_l, which reduce the mean values of η_l and φ_l to nil.

Substituting Eqs. (3) and (4) into Eqs. (2-a) or (2-b), we derive K_{ij} as follows through the approximation $1/(1+x)=1-x+x^2$.

$$K_{ij}=\sum_l \frac{E_l^0}{1-(\nu_l^0)^2}\Bigg[(\nu_l^0 Q_l+R_l)+\frac{\nu_l^0 Q_l+R_l}{E_l^0}\eta_l+\left\{\frac{2\nu_l^0}{1-(\nu_l^0)^2}(\nu_l^0 Q_l+R_l)+Q_l\right\}\left\{\varphi_l+\frac{1}{E_l^0}\eta_l\varphi_l\right\}+\left\{\left\{\frac{1}{1-(\nu_l^0)^2}+\left(\frac{2\nu_l^0}{1-(\nu_l^0)^2}\right)^2\right\}(\nu_l^0 Q_l+R_l)+\frac{2\nu_l^0 Q_l}{1-(\nu_l^0)^2}\right\}\varphi_l^2+\cdots\cdots\Bigg] : \text{plane stress} \quad \text{(5-a)}$$

$$K_{ij}=\sum_l \frac{E_l^0}{1-\nu_l^0-2(\nu_l^0)^2}\Bigg[(\nu_l^0 Q_l+R_l)+\frac{\nu_l^0 Q_l+R_l}{E_l^0}\eta_l+\left\{\frac{(1+4\nu_l^0)(\nu_l^0 Q_l+R_l)}{1-\nu_l^0-2(\nu_l^0)^2}+Q_l\right\}\left\{\varphi_l+\frac{1}{E_l^0}\eta_l\varphi_l\right\}+\left\{\left\{\frac{2}{1-\nu_l^0-2(\nu_l^0)^2}+\frac{(1+4\nu_l^0)^2}{(1-\nu_l^0-2(\nu_l^0)^2)^2}\right\}\cdot(\nu_l^0 Q_l+R_l)+\frac{1+4\nu_l^0}{1-\nu_l^0-2(\nu_l^0)^2}Q_l\right\}\varphi_l^2+\cdots\cdots\Bigg] : \text{plane strain} \quad \text{(5-b)}$$

It will be reasonable that the terms greater than third order with respect to η_l and/or φ_l are neglected based on the assumption that the fluctuations are small. As shown in Eq. (5-a) or (5-b), K_{ij} has the following general expansion form regardless of plane stress state or plane strain state.

$$K_{ij}=K_{ij}^0+\sum_l(K_{ijl}^1\eta_l+K_{ijl}^{1'}\varphi_l)+\sum_l(K_{ijl}^2\eta_l\varphi_l+K_{ijl}^{2'}\varphi_l^2) \qquad \text{(5-c)}$$

At the same time, the authors assume Eq. (6) as regards displacement, which is to be subsituted with Eq. (5-c) into Eq. (7) of equilibrium equation in order to carry out the second-order perturbation method.

$$U_i=U_i^0+\sum_k(U_{ik}^1\eta_k+U_{ik}^{1'}\varphi_k)+\sum_k\sum_l(U_{ikl}^2\eta_k\eta_l+U_{ikl}^{2'}\eta_k\varphi_l+U_{ikl}^{2''}\varphi_k\varphi_l) \qquad (6)$$

$$[\boldsymbol{K}]\{\boldsymbol{U}\}=\{\boldsymbol{F}\} \qquad (7)$$

As described in detail in the previous report [1], unknown coefficients are evaluated as follows.

$$\{U_j^0\}_j=[K_{ij}^0]^{-1}\{F_i^0\}_i \qquad (8)$$

$$\{U_{jk}^1\}_j=-[K_{ij}^0]^{-1}\{\sum_j U_j^0 K_{ijk}^1\}_i \qquad (9)$$

$$\{U_{jk}^{1'}\}_j=-[K_{ij}^0]^{-1}\{\sum_j U_j^0 K_{ijk}^{1'}\}_i \qquad (10)$$

$$\{U_{jkl}^2\}_j=-[K_{ij}^0]^{-1}\{\sum_j K_{ijk}^1 U_{jl}^1\}_i \qquad (11)$$

$$\{U_{jkl}^{2'}\}_j=-[K_{ij}^0]^{-1}\{\sum_j(\delta_{kl}U_j^0 K_{ijl}^2+K_{ijk}^1 U_{jl}^{1'}+K_{ijl}^{1'}U_{jk}^1)\}_i \qquad (12)$$

$$\{U_{jkl}^{2''}\}_j=-[K_{ij}^0]^{-1}\{\sum_j(\delta_{kl}U_j^0 K_{ijk}^{2'}+K_{ijk}^{1'}U_{jl}^{1'})\}_i \qquad (13)$$

where δ_{kl} is Kronecker's delta, $\{\cdot\}_i$ means column vector with respect to i, and i and j vary from 1 to p of the degrees of freedom of unknown displacements. In the above equations, it is pointed out that unknown coefficients are evaluated successively on the basis of Eq. (8) which agrees with the solution of conventional finite element equilibrium equation. Then as shown in the previous report, the expectation $E[U_i]$ and dispersion $Var[U_i]$ are obtained as functions of the correlations of η_k and/or φ_k.

The statistics of stress and strain are derived as follows.

Firstly, strain is given as Eq. (14) through the appropriate strain-nodal displacement matrix $[B]$.

$$e_j=\sum_m B_{jm}U_m \qquad (14)$$

where e_1, e_2 and e_3 correspond to ε_x, ε_y and γ_{xy} respectively, and B_{jm} represents the jm-th entity of $[B]$. Consequently, Eq. (14) is reduced to the form of Eq. (14-a) on the basis of Eq. (6) because B_{jm} is evaluated deterministically as a spatial derivative of a shape function within the present problem.

$$e_j=\sum_m B_{jm}U_m^0+\sum_m\sum_k B_{jm}(U_{mk}^1\eta_k+U_{mk}^{1'}\varphi_k)+\sum_m\sum_k\sum_l B_{jm}(U_{mkl}^2\eta_k\eta_l+U_{mkl}^{2'}\eta_k\varphi_l+U_{mkl}^{2''}\varphi_k\varphi_l) \qquad \text{(14-a)}$$

$E[e_j]$ and $Var[e_j]$ can be evaluated in the straitforward manner as $E[U_i]$ and $Var[U_i]$.

Secondly, stress is given through the stress-strain matrix $[D]$ of the element n under interest, of which the form is

$$[D]=\xi\begin{bmatrix}1 & \overline{\nu} & 0\\ \overline{\nu} & 1 & 0\\ 0 & 0 & \nu'\end{bmatrix} \tag{15}$$

It is apparent that D_{ij}, the entity of $[D]$, generally has the same form as Eqs. (2-a) or (2-b) except the summation with respect ot l and that D_{ij} is expanded as following in a similar manner as in Eqs. (5-a), (5-b) or (5-c).

$$D_{ij}=D_{ij}^0+D_{ij}^1\eta_n+D_{ij}^{1'}\varphi_n+D_{ij}^2\eta_n\varphi_n+D_{ij}^{2'}\varphi_n^2 \tag{15-a}$$

Therefore, stress is summarized as follows.

$$\begin{aligned}\sigma_i &=\sum_j D_{ij}e_j\\ &=\sum_j D_{ij}\sum_m B_{jm}U_m^0\\ &\quad+\sum_j [\sum_k \{D_{ij}^0\sum_m B_{jm}U_{mk}^1+\delta_{kn}\sum_m B_{jm}U_m^0 D_{ij}^1\}\eta_k\\ &\quad+\sum_k \{D_{ij}^0\sum_m B_{jm}U_{mk}^{1'}+\delta_{km}\sum_m B_{jm}U_m^0 D_{ij}^{1'}\}\varphi_k]\\ &\quad+\sum_j [\sum_k\sum_l \{D_{ij}^0\sum_m B_{jm}U_{mkl}^2+\delta_{ln}\sum_m B_{jm}U_{mk}^1 D_{ij}^1\}\eta_k\eta_l\\ &\quad+\sum_k\sum_l \{D_{ij}^0\sum_m B_{jm}U_{mkl}^{2'}+\delta_{kn}\delta_{ln}\sum_m B_{jm}U_m^0 D_{ij}^2\\ &\quad+\delta_{kn}\sum_m B_{jm}U_{mk}^{1'}D_{ij}^1+\delta_{ln}\sum_m B_{jm}U_{mk}^1 D_{ij}^{1'}\}\eta_k\varphi_l\\ &\quad+\sum_k\sum_l \{D_{ij}^0\sum_m B_{jm}U_{mkl}^{2''}+\delta_{kn}\delta_{ln}\sum_m B_{jm}U_m^0 D_{ij}^{2'}\\ &\quad+\delta_{ln}\sum_m B_{jm}U_{mk}^{1'}D_{ij}^{1'}\}\varphi_k\varphi_l]\\ &\equiv\sigma_i^0+\sum_k(\sigma_{ik}^1\eta_k+\sigma_{ik}^{1'}\varphi_k)+\sum_k\sum_l(\sigma_{ikl}^2\eta_k\eta_l+\\ &\quad+\sigma_{ikl}^{2'}\eta_k\varphi_l+\sigma_{ikl}^{2''}\varphi_k\varphi_l)\end{aligned} \tag{16}$$

where σ_1, σ_2 and σ_3 correspond to σ_x, σ_y and τ_{xy} respectively. Finally, the expectation and dispersion of σ_i are derived as follows.

$$\begin{aligned}E[\sigma_i]&=\sigma_i^0+\sum_k\sum_l\{\sigma_{ikl}^2 E[\eta_k\eta_l]\\ &\quad+\sigma_{ikl}^{2'}E[\eta_k\varphi_l]+\sigma_{ikl}^{2''}E[\varphi_k\varphi_l]\}\end{aligned} \tag{17}$$

$$\begin{aligned}Var[\sigma_i]&=E[\sigma_i^2]-\{E[\sigma_i]\}^2\\ &=(\sigma_i^0)^2+\sum_k\sum_l\{(\sigma_{ik}^1\sigma_{il}^1+2\sigma_i^0\sigma_{ikl}^2)E[\eta_k\eta_l]\\ &\quad+2(\sigma_{ik}^1\sigma_{il}^{1'}+\sigma_i^0\sigma_{ikl}^{2'})E[\eta_k\varphi_l]\\ &\quad+(\sigma_{ik}^{1'}\sigma_{il}^{1'}+2\sigma_i^0\sigma_{ikl}^{2''})E[\varphi_k\varphi_l]\}\\ &\quad-\{E[\sigma_i]\}^2\end{aligned} \tag{18}$$

3. Formulation for Fluctuation of Geometrical Boundary Condition

In the present chapter, geometrical boundary condition $\{U^\beta\}$ in the following equilibrium equation is taken as a stochastic process.

$$\begin{bmatrix}K^{\alpha\alpha} & K^{\alpha\beta}\\ K^{\beta\alpha} & K^{\beta\beta}\end{bmatrix}\begin{Bmatrix}U^\alpha\\ U^\beta\end{Bmatrix}=\begin{Bmatrix}F^\alpha\\ F^\beta\end{Bmatrix} \tag{19}$$

where $\{U^\alpha\}$ represents unknown displacement vector, and $\{F^\alpha\}$ and $\{F^\beta\}$ indicate known and unknown nodal force vectors respectively. The stochastic process U_i^β, the i -th entity of $\{U^\beta\}$, is represented as following in the same way in Eq. (3) and (4).

$$U_i^\beta=U_{0i}^\beta+\gamma_i \tag{20}$$

where U_{0i}^β is a deterministic term as the expectation of U_i^β, and γ_i indicates the stochastic process of which the mean value is zero.

Applying Eq. (20) to Eq. (21) derived from Eq. (19), we obtain Eq. (22) as the i -th entity of the unknown displacement vector $\{U^\alpha\}$.

$$\{U^\alpha\}=[K^{\alpha\alpha}]^{-1}(\{F^\alpha\}-[K^{\alpha\beta}]\{U^\beta\}) \tag{21}$$

$$\begin{aligned}U_i^\alpha&=\sum_i\overline{K}_{ij}^{\alpha\alpha}F_j^\alpha-\sum_j\sum_k\overline{K}_{ij}^{\alpha\alpha}K_{jk}^{\alpha\beta}U_{0k}^\beta\\ &\quad-\sum_j\sum_k\overline{K}_{ij}^{\alpha\alpha}K_{jk}^{\alpha\beta}\gamma_k\equiv U_{0i}^\alpha+\sum_k U_{1ik}^\alpha\gamma_k\end{aligned} \tag{22}$$

where $K_{ij}^{\alpha\alpha}$ represents the ij -th entity of $[K^{\alpha\alpha}]^{-1}$, and it goes without saying that a linear relation holds between U_i^α and γ_k.

Then, the expectation and dispersion of U_i^α are given as follows.

$$E[U_i^\alpha]=U_{0i}^\alpha \tag{23}$$

$$Var[U_i^\alpha]=\sum_k\sum_l U_{1ik}^\alpha U_{1il}^\alpha E[\gamma_k\gamma_l] \tag{24}$$

In case γ_k and γ_l represent a horizontal fluctuation and a vertical fluctuation respectively as an example, it should be pointed out that $E[\gamma_k\gamma_l]$ corresponds to a cross-correlation function, giving rise to the fact the use of two-variate stochastic process is generally required to deal with the present problem.

Through appropriate $[B]$ matrix, strain e_l is calculated as Eq. (26) from the corresponding nodal displacements which can be written in the general form of Eq. (25).

$$U_m=U_{0m}+\sum_k U_{1mk}\gamma_k \tag{25}$$

$$e_l=\sum_m B_{lm}U_m=\sum_m B_{lm}U_{0m}+\sum_m B_{lm}\sum_k U_{1mk}\gamma_k \tag{26}$$

In the above equations, U_{1mk} is reduced to Kronecker's delta δ_{mk} when U_m represents U_m^β, and the definitions of e_l and B_{lm} are the same as in the preceding chapter. It will be unnecessary to show the simple forms of $E[e_l]$ and $Var[e_l]$.

Stress σ_i, defined also in the preceding chapter, and the statistics are then derived as follows.

$$\sigma_i = \sum_l D_{il} e_l$$
$$= \sum_l D_{il} \sum_m B_{lm} U_{0m} + \sum_l D_{il} \sum_m B_{lm} \sum_k U_{1mk} \gamma_k$$
$$\equiv \sigma_{0i} + \sum_k \sigma_{1ik} \gamma_k \tag{27}$$
$$E[\sigma_i] = \sigma_{0i} \tag{28}$$
$$Var[\sigma_i] = \sum_k \sum_h \sigma_{1ik} \sigma_{1ih} E[\gamma_k \gamma_h] \tag{29}$$

It should be noted in the present chapter that any approximation techniques such as second-order perturbation method is not utilized and the exact solutions are derived by virtue of the linear relations among displacement, strain and stress.

The stochastic process is assumed to be homogeneous in the present paper. The expectations in the form of $E[\alpha_i \beta_j]$ emerging in the present and preceding chapter are compatible with spectrum representation and evaluated as follows.

$$E[\alpha_i \beta_j] = R_{\alpha\beta}(\boldsymbol{X}_j - \boldsymbol{X}_i)$$
$$= 2\int_0^\infty S_{\alpha\beta}(\boldsymbol{\lambda}) \cos 2\pi \boldsymbol{\lambda} \cdot (\boldsymbol{X}_j - \boldsymbol{X}_i) d\boldsymbol{\lambda} \tag{30}$$

where $R_{\alpha\beta}(\cdot)$ and $S_{\alpha\beta}(\cdot)$ denote cross-correlation function and two-sided cross-spectral density respectively, $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ indicate space vectors, and $\boldsymbol{\lambda}$ represents the wave number vector.

4. Conclusions

Original concept of stochastic finite element method is found to be expansible into the problems of fluctuations of material properties and geometrical boundary condition.

The present attempt not only diversifies the finite element methods but also is compatible with structural risk analysis or reliability-based design such as second moment method.[6]

(Manuscript received, February 4, 1980)

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