研　究　速　報
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A Note on Stochastic Finite Element Method (Part 1)

—Variation of Stress and Strain Caused by Shape Fluctuation—

確率有限要素法に関するノート(第1報)

—形状のゆらぎによる応力と歪の変動—

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1. Introduction

An attempt is made in this note to extend the versatility of the finite element method to the degree of stochastic modelling. The finite element stress analysis has been well established in engineering practice. Nevertheless, there still remains a disadvantage of conventional finite element method, in which element modelling is carried out in deterministic manner.

When we need to estimate the expectation and dispersion of stress at an arbitrary point in structure under interest, expensive calculation should be repeated supposedly many times, in case Monte Carlo technique is applied in order to simulate any uncertainties caused by distributing properties of material,[1] fluctuation of loads, variations in boundary condition and so on.

It is therefore desirable that finite element modelling itself involves stochastic nature so that the analysis results can be expressed in form fitted with stochastic treatment. Authors examine the possibility whether the finite element method can be incorporated into stochastic formulation and report herein the concept of stochastic finite element method and the formulations based on the perturbation method[2] that is also applied in the study of random vibration.[3] For brevity, only the formulation is described that holds for the case that stiffness matrix is characterized in stochastic manner as the result of fluctuation of nodal coordinates alone. The methodology proposed is applicable, however, to the treatment of stochastic stiffness matrices raised by properties of material distributed randomly.

2. Stochastic Finite Element Equilibrium Equations and Solution

As a starting point, it is assumed that the ij-th entity of the global stiffness matrix is expanded in terms of random variables α_k and β_k representing small fluctuations of k-th nodal coordinates. Taking the terms up to the second order products as regards α_k and/or β_k, we have

$$K_{ij}=K^0_{ij}+\sum_k(K^1_{ijk}\alpha_k+K^{1\prime}_{ijk}\beta_k)+\sum_k\sum_l(K^2_{ijkl}\alpha_k\alpha_l+K^{2\prime}_{ijkl}\alpha_k\beta_l+K^{2\prime\prime}_{ijkl}\beta_k\beta_l) \quad (1)$$

where K^0_{ij}, K^1_{ijk}, …… are the coefficients to be obtained in chapter 3.

In case three dimensional problems are dealt with, the third random variable γ_k corresponding to z_k is to be introduced in the same manner.

At the same time, authors put Eq. (2) as regards displacement.

$$U_i=U^0_i+\sum_k(U^1_{ik}\alpha_k+U^{1\prime}_{ik}\beta_k)+\sum_k\sum_l(U^2_{ikl}\alpha_k\alpha_l+U^{2\prime}_{ikl}\alpha_k\beta_l+U^{2\prime\prime}_{ikl}\beta_k\beta_l) \quad (2)$$

If the following straight-forward expressions were put instead of Eqs. (1) and (2), formulations in the latter part of this chapter become so complicated that the CPU time is expected enormous prohibitingly.

$$K_{ij}=\tilde{K}^0_{ij}+\kappa_{ij} \quad (1\text{-a})$$

where

$$\kappa_{ij}=\sum_k(\kappa^1_{ijk}\alpha_k+\kappa^{1\prime}_{ijk}\beta_k)+\sum_k\sum_l(\kappa^2_{ijkl}\alpha_k\alpha_l+\kappa^{2\prime}_{ijkl}\alpha_k\beta_l+\kappa^{2\prime\prime}_{ijkl}\beta_k\beta_l) \quad (1\text{-b})$$

$$U_i=\tilde{U}^0_i+\sum_k\sum_l\tilde{U}^1_{ikl}\kappa_{kl}+\sum_k\sum_l\sum_m\sum_n\tilde{U}^2_{iklmn}\kappa_{kl}\kappa_{mn} \quad (2\text{-a})$$

Emphasis can be placed on that a non-linear relation still holds between K_{ij} and U_i as seen from Eqs. (1) and (2).

Substituting Eqs. (1) and (2) into Eq. (3) of the ordinary form of equilibrium equation, we have Eq. (4) as regards the i-th row with unknown displacement.

$$[\boldsymbol{K}]\{\boldsymbol{U}\}=\{\boldsymbol{F}\} \quad (3)$$

$$\sum_j\{K^0_{ij}U^0_j+K^0_{ij}\sum_k(U^1_{jk}\alpha_k+U^{1\prime}_{jk}\beta_k)+U^0_j\sum_k(K^1_{ijk}\alpha_k+K^{1\prime}_{ijk}\beta_k)+K^0_{ij}\sum_k\sum_l(U^2_{jkl}\alpha_k\alpha_l+U^{2\prime}_{jkl}\alpha_k\beta_l+U^{2\prime\prime}_{jkl}\beta_k\beta_l)+\sum_k(K^1_{ijk}\alpha_k+K^{1\prime}_{ijk}\beta_k)\cdot\sum_k(U^1_{jk}\alpha_k+U^{1\prime}_{jk}\beta_k)+U^0_j\sum_k\sum_l(K^2_{ijkl}\alpha_k\alpha_l+K^{2\prime}_{ijkl}\alpha_k\beta_l+K^{2\prime\prime}_{ijkl}\beta_k\beta_l)+\cdots\cdots\}=F^0_i \quad (4)$$

where F^0_i is the i-th nodal force which is known and the superfix 0 in F^0_i is added in order to identify that F_i is not

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stochastic. According to the principle of second-order perturbation method, U_i^0, U_{ik}^1, $\cdots$ in Eq. (2) are derived from Eq. (4) as follows.

$$\{U_j^0\}_j=[K_{ij}^0]^{-1}\{F_i^0\}_i \quad (5)$$

$$\{U_{jk}^1\}_j=-[K_{ij}^0]^{-1}\{\sum_j U_j^0 K_{ijk}^1\}_i \quad (6)$$

$$\{U_{jk}^{1'}\}_j=-[K_{ij}^0]^{-1}\{\sum_j U_j^0 K_{ijk}^{1'}\}_i \quad (7)$$

$$\{U_{jkl}^2\}_j=-[K_{ij}^0]^{-1}\{\sum_j (K_{ijk}^1 U_{jl}^1+U_j^0 K_{ijkl}^2)\}_i \quad (8)$$

$$\{U_{jkl}^{2'}\}_j=-[K_{ij}^0]^{-1}\{\sum_j (K_{ijk}^1 U_{jl}^{1'}+U_{jk}^1 K_{ijl}^{1'} +U_j^0 K_{ijkl}^{2'})\}_i \quad (9)$$

$$\{U_{jkl}^{2''}\}_j=-[K_{ij}^0]^{-1}\{\sum_j (K_{ijk}^{1'} U_{jl}^{1'}+U_j^0 K_{ijkl}^{2''})\}_i \quad (10)$$

where $\{\cdot\}_i$ means column vector with respect to i, and i and j vary from 1 to p of the degrees of freedom of unknown displacements. It is noted that Eqs. (6) ~ (10) are constructed by the deterministic components $[K_{ij}^0]^{-1}$ and $\{U_j^0\}_j$ given in Eq. (5) which agree with that of conventional finite element method.

Then, the expectation of U_i is given below from Eq. (2).

$$E[U_i]=U_i^0+\sum_k\sum_l\{U_{ikl}^2 E[\alpha_k\alpha_l] +U_{ikl}^{2'}E[\alpha_k\beta_l]+U_{ikl}^{2''}E[\beta_k\beta_l]\} \quad (11)$$

where $E[\cdot]$ represents expectation. In this Eq. (11), $E[\alpha_k]=E[\beta_k]=0$ is put tacitly or in other words α_k and β_k are defined so as to satisfy $E[\alpha_k]=E[\beta_k]=0$.

On the other hand, the dispersion of U_i, $Var[U_i]$, is derived as follows.

$$\begin{aligned}Var[U_i]&=E[U_i^2]-\{E[U_i]\}^2\\ &=(U_i^0)^2+\sum_k\sum_l\{(U_{ik}^1U_{il}^1+2U_i^0U_{ikl}^2)E[\alpha_k\alpha_l]\\ &\quad+2(U_{ik}^1U_{il}^{1'}+U_i^0U_{ikl}^{2'})E[\alpha_k\beta_l]\\ &\quad+(U_{ik}^{1'}U_{il}^{1'}+2U_i^0U_{ikl}^{2''})E[\beta_k\beta_l]\\ &\quad-\{U_i^0+\sum_k\sum_l(U_{ikl}^2E[\alpha_k\alpha_l]+U_{ikl}^{2'}E[\alpha_k\beta_l]\\ &\quad+U_{ikl}^{2''}E[\beta_k\beta_l]\}^2\end{aligned} \quad (12)$$

where higher than third order terms in products of α_k and/or β_k are neglected.

For reference, $E[U_i]$ on the basis of Eqs. (1−a), (1−b) and (2−a) is calculated in a similer manner and summerized as Eq. (11−a).

$$\begin{aligned}E[U_i]&=\tilde{U}_i^0+\sum_q\sum_r\{\tilde{U}_{iqr}^1\sum_k\sum_l(\kappa_{qrkl}^2E[\alpha_k\alpha_l]\\ &\quad+\kappa_{qrkl}^{2'}E[\alpha_k\beta_l]+\kappa_{qrkl}^{2''}E[\beta_k\beta_l])\}\\ &\quad+\sum_q\sum_r\sum_s\sum_t[\tilde{U}_{iqrst}^2\sum_l\sum_m\{\kappa_{qrl}^1\kappa_{stm}^1E[\alpha_l\alpha_m]\\ &\quad+(\kappa_{qrl}^1\kappa_{stm}^{1'}+\kappa_{stl}^1\kappa_{qrm}^{1'})E[\alpha_l\beta_m]\\ &\quad+\kappa_{qrl}^{1'}\kappa_{stm}^{1'}E[\beta_l\beta_m]\}]\end{aligned} \quad (11-a)$$

Remaining interest is to estimate the expectation and dispersion of strain and stress at an arbitrary point. A strain ε_{Ui} caused by U_i is represented by $\frac{\partial N_i}{\partial x}U_i$ etc., where $\frac{\partial N_i}{\partial x}$ is to be given in the form of Eq. (23) in chapter 3. Then, neglecting higher product terms than third order again, ε_{Ui} can be summarized in the general form of

$$\begin{aligned}\varepsilon_{Ui}&=(\text{product of } N_i\text{'s spatial derivatives and } U_i)\\ &=e_i^0+\sum_k(e_{ik}^1\alpha_k+e_{ik}^{1'}\beta_k)+\sum_k\sum_l(e_{ikl}^2\alpha_k\alpha_l\\ &\quad+e_{ikl}^{2'}\alpha_k\beta_l+e_{ikl}^{2''}\beta_k\beta_l)\end{aligned} \quad (13)$$

Consequently, the expectation and disperison of strain are reduced to Eqs. (14) and (15) respectively in the same way with Eqs. (11) and (12).

$$E[\varepsilon]=\sum_i e_i^0+\sum_i\sum_k\sum_l\{e_{ikl}^2E[\alpha_k\alpha_l] +e_{ikl}^{2'}E[\alpha_k\beta_l]+e_{ikl}^{2''}E[\beta_k\beta_l]\} \quad (14)$$

$$\begin{aligned}Var[\varepsilon]&=(\sum_i e_i^0)^2+\sum_i\sum_k\sum_j\sum_l\{(e_{ik}^1e_{jl}^1+2e_j^0e_{ikl}^2)E[\alpha_k\alpha_l]\\ &\quad+2(e_{ik}^1e_{jl}^{1'}+e_j^0e_{ikl}^{2'})E[\alpha_k\beta_l]+(e_{ik}^{1'}e_{jl}^{1'}\\ &\quad+2e_j^0e_{ikl}^{2''})E[\beta_k\beta_l]\}\\ &\quad-\{\sum_i e_i^0+\sum_i\sum_k\sum_l(e_{ikl}^2E[\alpha_k\alpha_l]\\ &\quad+e_{ikl}^{2'}E[\alpha_k\beta_l]+e_{ikl}^{2''}E[\beta_k\beta_l])\}^2\end{aligned} \quad (15)$$

where $\sum_i$ and $\sum_j$ denote summation with respect to all nodes concerning the element under interest. It is a matter of course that stress is easily calculated through the appropriate stress-strain matrix.

3. Derivation of Stochastic Stiffness Matrix

The element stiffness matrix used in elastic, small displacement analysis is calculated in usual by the following equation.[4]

$$[\boldsymbol{k}]=\iint[B(L_1,L_2)]^T[D][B(L_1,L_2)] \det|J|dL_1dL_2 \quad (16)$$

where $[D]$ denotes the stress- strain matrix, $[B]$ the strain-nodal displacement matrix, and T means matrix transpose. The isoparametric displacement function N_i, the arguments of which are the area coordinates L_1 and L_2, is borne in mind in relation to a particular case of triangular element and node number i varies from 1 to 6, if quadratic displacement function is taken into account. Regardless of plane stress state or plane strain state, any elastic constant included in $[D]$ is assumed deterministic in the subsequent formulation, and as mentioned earlier, attention is paid to the case that stochastic nature of stiffness matrix is caused by small fluctuation of the nodal coordinates. In the sequel, the influence of the fluctuating nodal cooridnates appears through $[B]$ and the Jacobian matrix related to the coordinate transformation, as defined as follows.

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial L_1}-\frac{\partial x}{\partial L_3} & \frac{\partial y}{\partial L_1}-\frac{\partial y}{\partial L_3}\\ \frac{\partial x}{\partial L_2}-\frac{\partial x}{\partial L_3} & \frac{\partial y}{\partial L_2}-\frac{\partial y}{\partial L_3}\end{bmatrix} =\begin{bmatrix}J^0_{11} & J^0_{12}\\ J^0_{21} & J^0_{22}\end{bmatrix}+\begin{bmatrix}J^p_{11} & J^p_{12}\\ J^p_{21} & J^p_{22}\end{bmatrix} \quad (17)$$

The nodal coordinates are assumed to be expressed in the form of sum of deterministic term as expectation and stochastic one as $x_i=x^0_i+\alpha_i$ and $y_i=y^0_i+\beta_i$.Consequently, J^0_{ij} and J^p_{ij} are given as linear functions of x_i^0, α_i and so on. The superfices 0 and p mean deterministic and stochastic respectively. The entities of $[B]$ are calculated through the use of appropriate strain-displacement relation and following terms are needed in doing so.

$$\begin{Bmatrix}\frac{\partial N_i}{\partial x}\\ \frac{\partial N_i}{\partial y}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}b_{i1}=\frac{\partial N_i}{\partial L_1}-\frac{\partial N_i}{\partial L_3}\\ b_{i2}=\frac{\partial N_i}{\partial L_1}-\frac{\partial N_i}{\partial L_3}\end{Bmatrix} \quad (18)$$

The aim of this formulation is to evaluate the stiffness matrix to the extent of second-order perturbation due to α_i and β_i in order to be compatible with Eq. (1) in the preceding chapter. As the essential components of the stiffness matrix comprise

$$\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x},\ \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y},\ \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x},\ \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) det|J| \quad (19)$$

$det|J|$ and $\frac{\partial N_i}{\partial x}$ are evaluated at first. By the use of the definition given in Eq. (17), and neglecting higher product terms of α_i and/or β_i than third order, we have

$$det|J|=det|J^0|+D_1+D_2 \quad (20)$$

where

$$D_0=det|J^0|=J^0_{11}J^0_{22}-J^0_{12}J^0_{21}$$
$$D_1=J^0_{11}J^p_{22}-J^0_{12}J^p_{21}-J^0_{21}J^p_{12}+J^0_{22}J^p_{11} =\sum_i A_i\alpha_i+\sum_i B_i\beta_i$$
$$D_2=J^p_{11}J^p_{22}-J^p_{12}J^p_{21}=\sum_i\sum_j C_{ij}\alpha_i\beta_i$$

It is assumed that $\frac{D_2}{D_0}<\frac{D_1}{D_0}<1$ holds on the basis of the postulate of small fluctuation. This enables us to have $[J]^{-1}$ given below by means of the approximation of $1/(1+x)=1-x+x^2$ and omitting higher than third order terms of α_i and/or β_i.

$$D_0[J]^{-1}=\left(1-\frac{D_1}{D_0}-\frac{D_2}{D_0}+\frac{D_1^2}{D_0^2}\right)\begin{bmatrix}J^0_{22} & -J^0_{12}\\ -J^0_{21} & J^0_{11}\end{bmatrix} +\left(1-\frac{D_1}{D_0}\right)\begin{bmatrix}J^p_{22} & -J^p_{12}\\ -J^p_{21} & J^p_{11}\end{bmatrix} \quad (21)$$

The entities of the matrix in Eq. (21) consist of the first deterministic term and those affected by α_i and/or β_i as given below.

$$D_0J^{-1}_{ij}=D_0(J^0)^{-1}_{ij}-\sum_k m_{ijk}\alpha_k-\sum_k n_{ijk}\beta_k +\sum_k\sum_l f_{ijkl}\alpha_k\alpha_l+\sum_k\sum_l g_{ijkl}\alpha_k\beta_l +\sum_k\sum_l h_{ijkl}\beta_k\beta_l \quad (22)$$

where i and j take 1 to 2. Substituting Eq. (22) into Eq. (18), then we have the following expression for the partial derivatives of the displacement function N_i as an example.

$$D_0\frac{\partial N_i}{\partial x}=b_{i1}J^0_{22}-b_{i2}J^0_{12}-\sum_k(b_{i1}m_{11k}+b_{i2}m_{12k})\alpha_k -\sum_k(b_{i1}n_{11k}+b_{i2}n_{12k})\beta_k +\sum_k\sum_l(b_{i1}f_{11kl}+b_{i2}f_{12kl})\alpha_k\alpha_l +\sum_k\sum_l(b_{i1}g_{11kl}+b_{i2}g_{12kl})\alpha_k\beta_l +\sum_k\sum_l(b_{i1}h_{11kl}+b_{i2}h_{12kl})\beta_k\beta_l \quad (23)$$

Any term in Eq. (19) can be calculated by the cyclic use of Eq. (23) and results in

$$\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}det|J|=\frac{1}{D_0}(b_{i1}J^0_{22}-b_{i2}J^0_{12})(b_{j1}J^0_{22}-b_{j2}J^0_{12}) +\sum_k m'_k\alpha_k+\sum_k n'_k\beta_k+\sum_k\sum_l f'_{kl}\alpha_k\alpha_l +\sum_k\sum_l g'_{kl}\alpha_k\beta_l+\sum_k\sum_l h'_{kl}\beta_k\beta_l \quad (24)$$

and so on, while the higher products of α_k and/or β_k than third are again omitted. When these terms are defined, they are arranged in accordance with the usual procedures of the introduction of $[D]$ matrix, numerical integration over the relevant quadrature points and merging of element matrix into global matrix, giving rise to the general result.

$$K_{ij}=K^0_{ij}+\sum_k(K^1_{ijk}\alpha_k+K^{1'}_{ijk}\beta_k) +\sum_k\sum_l(K^2_{ijkl}\alpha_k\alpha_l+K^{2'}_{ijkl}\alpha_k\beta_l+K^{2''}_{ijkl}\beta_k\beta_l) \quad (25)$$

Naturally the first deterministic term of the above expression agree with the stiffness matrix used in conventional finite element method, and the second term and the followings are the embodiment of the stochastic characteristics due to small fluctuation of the nodal coordinates which is taken into account to exemplify the concept of stochastic finite elements.

4. On Relation between Spectral Representation of Randomness and Isoparametric Finite Element

In the present methodology, spatial randomness on nodal coordinates is taken as stochastic process defined by power spectrum. When the stochastic process is able to be taken as homogeneous, the well known Wiener-Khintchine relation is applicable and correlations $E[\alpha_i \alpha_j]$, $E[\alpha_i \beta_j]$ and $E[\beta_j \beta_j]$ emerging in Eqs. (11), (12), (14) and (15) are determined as given below in the case of Fig. 1 as a simple example.

$$E[\alpha_i \alpha_j]=E[\alpha_i \beta_j]=0 \tag{26}$$

$$E[\beta_i \beta_j]=R_{yy}(|x_j-x_i|) = 2\int_0^\infty S_{yy}(\lambda)\cos 2\pi\lambda|x_j-x_i|\,d\lambda \tag{27}$$

where $R_{yy}(\cdot)$ and $S_{yy}(\cdot)$ denote autocorrelation function and two-sided power spectrum respectively and λ represents wave number.

The random process is bound to be interpolated sequentially by paraboras in case quadratic isoparametric finite element is applied. Authors have investigated this problem and have assessed the nodal interval which well simulates the random process with given power spectrum in the case of Fig. 1. The principle of the assessment is based on Fourier expansion of quasi-cosine wave interpolated sequentially by paraboras. The Fourier coefficients are calculated according to Eqs. (28) and (29) against various interpolation intervals and are compared with the amplitude and phase of the original cosine wave of period 2π.

$$a_{\bar{n}}=-\frac{\bar{m}}{4\pi^2\bar{n}^2}\sum_i(-y_{i+2}+4y_{i+1}-6y_i+4y_{i-1}-y_{i-2})\cos\left\{\bar{n}\left(\frac{i\cdot 2\pi}{\bar{m}}-\xi_0\right)\right\}+\frac{\bar{m}^2}{4\pi^3\bar{n}^3}\sum_i(y_{i+2}-2y_{i+1}+2y_{i-1}-y_{i-2})\sin\left\{\bar{n}\left(\frac{i\cdot 2\pi}{\bar{m}}-\xi_0\right)\right\} \tag{28}$$

$$b_{\bar{n}}=-\frac{\bar{m}}{4\pi^2\bar{n}^2}\sum_i(-y_{i+2}+4y_{i+1}-6y_i+4y_{i-1}-y_{i-2})\sin\left\{\bar{n}\left(\frac{i\cdot 2\pi}{\bar{m}}-\xi_0\right)\right\}-\frac{\bar{m}^2}{4\pi^3\bar{n}^3}\sum_i(y_{i+2}-2y_{i+1}+2y_{i-1}-y_{i-2})\cos\left\{\bar{n}\left(\frac{i\cdot 2\pi}{\bar{m}}-\xi_0\right)\right\} \tag{29}$$

$$y=\frac{a_0}{2}+\sum_{\bar{n}=1}^{\infty}(a_{\bar{n}}\cos\bar{n}x+b_{\bar{n}}\sin\bar{n}x) \tag{30}$$

where $4\pi/\bar{m}$ and ξ_0 denote interpolation interval and phase respectively. $y_{i+2}, y_{i+1}, \ldots\ldots$ are evaluated exactly at the points on original cosine wave.

Authors' conclusion is that the nodal interval should be taken to be less than $1/4\lambda_u$, where λ_u is the highest wave number which can not be disregarded in $S_{yy}(\lambda)$.

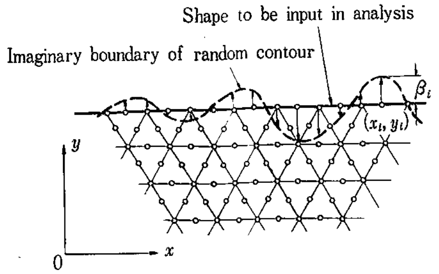

Fig. 1 Image illustrating random shape on boundary in case $\alpha_i=0$ as a simple example.

5. Conclusion

Original concept of stochastic finite element method is found to be embodied into the formulation with the aid of second-order perturbation technique.

It is worthy to emphasize that the expectations and dispersions of displacement, strain and stress at an arbitrary point in the structure under interest are obtained by solving the equilibrium equation only once.

Present methodology has wide varieties for the use in the field of structural safety and reliability, and in addition, it is supposed that the effect of different discretizations of finite elements also can be estimated along the present formulation.

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