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Dynamic Soil Reactions (Impedance Functions) Including The Effect of Dynamic Response of Surface Stratum (Part 3)

表層地盤の動反力係数(3)

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(Continued from No. 10)

5. SOIL REACTION IN ROTATION

The last case that may be treated under the assumptions adopted is indicated in Fig. 10. All soil particles are assumed to vibrate in the direction of the cylinder axis in an anti-symmetrical fashion. Thus, the horizontal displacement $u=v=0$ and the equations of motion of the surface soil stratum written in cylindrical coordinates reduce to

$$\left(G+G'\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)+\left(\lambda+2G+2G'\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial z^2}=\rho\frac{\partial^2 w}{\partial t^2} \qquad (45)$$

To solve the above equation, one puts

$$w=R(r)\cdot\cos\theta\cdot\sum_n^{\infty}\sin(h_n\cdot z),\ n=1,\ 3,\ 5,\cdots \qquad (46)$$

The above expression of w satisfies the boundary conditions that the shear stresses are zero at $z=H$ and that the displacement vanishes at $z=0$. Substitution of Eq. (46) into Eq. (45) gives the ordinary differential equation of R:

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}-(\alpha_n\cdot r)^2=1 \qquad (47)$$

Quantity α_n is given in Eq. (5). The general solution to Eq. (47) can be obtained by Eq. (20) by changing the argument β_n to α_n. Then, the displacement w is

$$w=\sum_n^{\infty}A_n\cdot K_1(\alpha_n\cdot r)\cdot\cos\theta\cdot\sin(h_n\cdot z) \qquad (48)$$

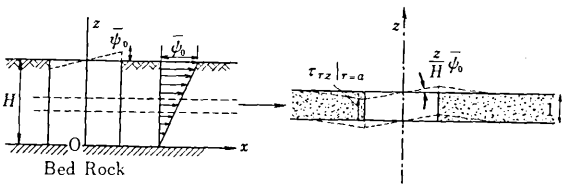

Fig. 10 Notation and Displacement in Antisymmetric Mode

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and the shear stress τ_{rz} is

$$\tau_{rz}=\left(G+G'\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial r}$$
$$=-G(1+i2D)\sum_n^{\infty}A_n\left[\alpha_nK_0(\alpha_n\cdot r)+\frac{1}{r}K_1(\alpha_nr)\right]\cdot\cos\theta\cdot\sin(h_nz) \qquad (49)$$

By using the above obtained shear stress τ_{rz}, the local moment soil reaction to the motion of the cylinder is

$$M(z)=-\int_0^{2\pi}\tau_{rz}|_{r=a}\cdot a\cdot\cos\theta\cdot a\cdot d\theta$$
$$=G\pi a^2(1+i2D)\sum_n^{\infty}A_n\left[\alpha_nK_0(\alpha_na)+\frac{1}{a}K_1(\alpha_na)\right]\sin(h_n\cdot z), \qquad (50)$$

The dynamic displacement of the cylinder may be assumed as follows:

$$W=\frac{z}{H}\cdot\overline{\Psi}_0\cdot a\cdot\cos\theta=\frac{8a\overline{\Psi}_0}{\pi^2}\sum_n^N a_n\cdot\cos\theta \qquad (51)$$

where W is the vertical displacement of the circumference of the cylinder and $\overline{\psi}_0$ is the rotational amplitude at $z=H$.

Equating Eq. (48) and Eq. (51) at $r=a$ determines the constant A_n and the local moment soil reaction becomes

$$M(z)=\frac{8Ga^2\cdot\overline{\Psi}_0}{\pi^2}(1+i2D)\sum_n^N\lambda_n\cdot a_n \qquad (52)$$

where

$$\lambda_n=1.0+\alpha_na\frac{K_0(\alpha_na)}{K_1(\alpha_na)}$$

Then, the local dynamic stiffness is written as

$$K_\psi(z)=\frac{M(z)}{\frac{z}{H}\cdot\overline{\Psi}_0}=G\pi a^2(1+i2D)K'_\psi(z) \qquad (53)$$

in which

$$K'_\psi(z)=\sum_n^N\lambda_n\cdot a_n/\sum_n^N a_n \qquad (54)$$

Figure 11 shows the vertical distribution of $K'_\psi(z)$ normalized by the static value at $z=H$. It can be seen that the local dynamic stiffness is almost constant along the height of cylinder as observed in the previous cases. Then, the

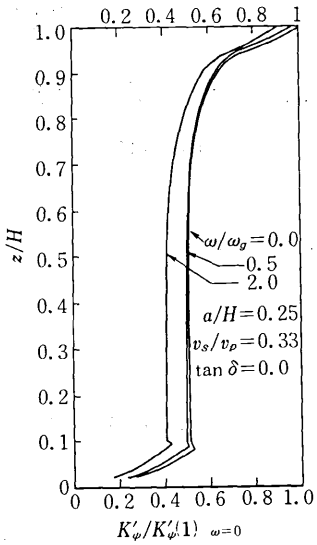

Fig. 11 Vertical Distribution of Local Dynamic Stiffness for Antisymmetric Mode (Eq. (54))

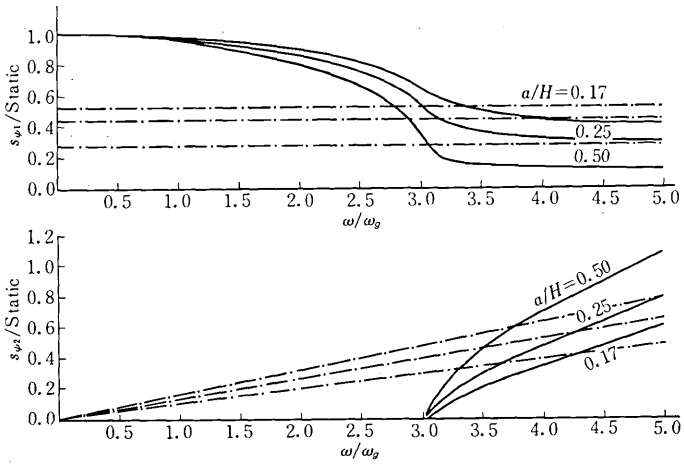

Fig. 12 Variations of Rotational Dimensionless Dynamic Stiffness with Frequency and the a/H Ratio (Eq. (55) or (56)). Full Line for the Proposed Solution and Dushed Line for the Approximate Solution by Novak

dymanic stiffness of a unit length of the cylinder is defined:

$$K_\psi\cdot\frac{\overline{\Psi_0}}{2}=\frac{1}{H}\int_0^H M(z)dz$$

The expression of K_ψ is

$$K_\psi=\frac{32Ga^2}{\pi^2}(1+i\,2D)\sum_n^N\lambda_n\frac{(-1)^{\frac{n-1}{2}}}{n^3} \qquad (55)$$

This expression for the dynamic stiffness to a unit length of the cylinder can be written as

$$K_\psi=Ga^2[s_{\psi1}(\omega/\omega_g,\tan\delta,v_s/v_p,a/H)+is_{\psi2}(\omega/\omega_g,\tan\delta,v_s/v_p,a/H)] \qquad (56)$$

where $s_{\psi1}$ and $s_{\psi2}$ are both real. The variations of $s_{\psi1}$ and $s_{\psi2}$ normalized by the static values are shown in Fig. 12.

As in the case of torsion, the variations with frequency is relatively smooth. In this case, however, a horizontally progressive wave only appears above the fundamental vertical frequency ω_p which is 3.0 ω_g in the case shown in Fig. 12.

APPENDIX I: EQUATIONS OF MOTION AND FORMING FOR FUNCTIONS, $K_{0,1}$

EQUATIONS OF MOTION:

The equations of motion of the viscoelastic medium can be obtained from the equations of an elastic medium by introducing the viscosity constants λ' and G' which are associated with the Lame's constants [8]. Thus, the equations of motion of the viscoelastic medium in cylindrical coordinate are

$$\left.\begin{aligned}
&\left[(\lambda+2G)+(\lambda'+2G')\frac{\partial}{\partial t}\right]\frac{\partial\Delta}{\partial r}-\left(G+G'\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial\omega_z}{\partial\theta}-\frac{\partial\omega_\theta}{\partial z}\right)=\rho\frac{\partial^2u}{\partial t^2}\\
&\left[(\lambda+2G)+(\lambda'+2G')\frac{\partial}{\partial t}\right]\frac{1}{r}\frac{\partial\Delta}{\partial r}-\left(G+G'\frac{\partial}{\partial t}\right)\left(\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r}\right)=\rho\frac{\partial^2v}{\partial t^2}\\
&\left[(\lambda+2G)+(\lambda'+2G')\frac{\partial}{\partial t}\right]\frac{\partial\Delta}{\partial z}-\left(G+G'\frac{\partial}{\partial t}\right)\left(\frac{1}{r}\frac{\partial(r\omega_\theta)}{\partial r}-\frac{1}{r}\frac{\partial\omega_r}{\partial\theta}\right)=\rho\frac{\partial^2w}{\partial t^2}
\end{aligned}\right\} \qquad (\text{I-1})$$

in which relative volume (dilatation)

$$\Delta=\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{\partial w}{\partial z} \qquad (\text{I-2})$$

and the components of rotational vector

$$\left.\begin{aligned}
\omega_r&=\frac{1}{r}\frac{\partial w}{\partial\theta}-\frac{\partial v}{\partial z}\\
\omega_\theta&=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\\
\omega_z&=\frac{1}{r}\frac{\partial(rv)}{\partial r}-\frac{1}{r}\frac{\partial u}{\partial\theta}
\end{aligned}\right\} \qquad (\text{I-3})$$

FORMING FOR FUNCTIONS K_0 and K_1:

The modified Bessel functions of the second kind of complex argument x can be evaluated from the following equations:

$$\left.\begin{aligned} K_0(x) &= -\left[\ln\frac{x}{2}+\gamma\right]+\sum_{s=1}^{\infty}\left(\frac{x}{2}\right)^{2s}\frac{1}{(s!)^2}\left\{\sum_{r=1}^{s}\frac{1}{r}\right. \\ &\quad \left.-\left[\ln\frac{x}{2}+\gamma\right]\right\} \\ K_1(x) &= \frac{1}{x}+\sum_{s=1}^{\infty}\left(\frac{x}{2}\right)^{2s-1}\frac{1}{(s!)^2}\left[\frac{1}{2}+s\left(\ln\frac{x}{2}\right.\right. \\ &\quad \left.\left.+\gamma-\sum_{r=1}^{s}\frac{1}{r}\right)\right] \end{aligned}\right\} \tag{I-4}$$

in which $\gamma = 0.772$.

Approximate formulae for the above equations are seen in a standard book, for example, [9].

APPENDIX II: DYNAMIC STIFFNESS FOR PLANE STRAIN CASE

Dynamic stiffness for plane strain case is obtained by Novak et al. as follows [1]:

(a) VERTICAL VIBRATION

$$K_w = 2\pi G(1+i2D)\cdot\delta_0 = G[s_{w1}(a_0, \tan\delta, v_s/v_p) + is_{w2}(a_0, \tan\delta, v_s/v_p)]$$

$$\delta_0 = \alpha_0 a\cdot\frac{K_1(\alpha_0 a)}{K_0(\alpha_0 a)}$$

where α_0 is obtained by Eq. (5) putting $n = 0$, and a_0 is the dimensionless frequency defined by

$$a_0 = \omega a/v_s$$

(b) TORSION

$$K_\theta = 2\pi Ga^2(1+i2D)\cdot\eta_0 = Ga^2[s_{\theta1}(a_0, \tan\delta, v_s/v_p) + is_{\theta2}(a_0, \tan\delta, v_s/v_p)]$$

$$\eta_0 = 2+\beta_0 a\frac{K_0(\beta_0 a)}{K_1(\beta_0 a)}$$

where β_0 is obtained by Eq. (19) by putting $n=0$

(c) HORIZONTAL VIBRATION

$$K_u = \pi Ga^2\cdot\Omega_0 = G[s_{u1}(a_0, \tan\delta, v_s/v_p) + is_{u2}(a_0, \tan\delta, v_s/v_p)]$$

$$\Omega_0 = \frac{[4K_1(\gamma_0 a)K_1(\beta_0 a)+\beta_0 aK_1(\gamma_0 a)K_0(\beta_0 a) + \gamma_0 aK_1(\beta_0 a)K_0(\gamma_0 a)]}{[K_1(\gamma_0 a)+\gamma_0 aK_0(\gamma_0 a)][K_1(\beta_0 a) + \beta_0 aK_0(\beta_0 a)]-K_1(\gamma_0 a)K_1(\beta_0 a)}$$

where γ_0 and β_0 are obtained by Eqs. (35) and (19) by putting $n=0$.

(d) ROTATION

$$K_\psi = \pi Ga^2(1+i2D)\lambda_0 = Ga^2[s_{\psi1}(a_0, \tan\delta, v_s/v_p) + is_{\psi2}(a_0, \tan\delta, v_s/v_p)]$$

$$\lambda_0 = 1.0+\alpha_0 a\cdot\frac{K_0(\alpha_0 a)}{K_1(\alpha_0 a)}$$

where α_0 is obtained by Eq. (5) by putting $n=0$.

Aprroximate values for dimensionless dynamic stiffness, s_{j1} and s_{j2} $(j=w,\theta,u,\psi)$ are tabulated in Table 1.

Table 1 Dimensionless Dynamic Stiffness for Plane Strain Case Approximated by Novak et al [1,2,3]

	s_{j1}	s_{j2}	Validity Range
Vertical	2.70	$6.70a_0$	$0 \leqq a_0 \leqq 2.0$
Torsional	12.4 10.2	$2.0a_0$ $5.4a_0$	$0 \leqq a_0 \leqq 0.2$ $0.2 \leqq a_0 \leqq 2.0$
Horizontal	4.10 3.60	$10.6a_0$ $8.20a_0$	$\nu=0.4$ $\nu=0.0$ $0 \leqq a_0 \leqq 2.0$
Rotational	2.50	$1.80a_0$	$0 \leqq a_0 \leqq 1.5$
Note	$a_0=\omega a/v_s$; Dimensionless Frequency		

(Ended)

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