Some Consideration on the Variational Basis of Finite Element Models

有限要素モデルの変分学的一考察

by Tadahiko KAWAI*
川井忠彦

Summary

Hybrid property of the linearized displacement field in deformable continuum is pointed out first. Using the displacement field in hybrid form and employing the hybrid potential energy principle attempt is made to establish mathematical basis of the rigid body-spring elements previously proposed by the present author.

1. Introduction

It is well known that the linearized displacement field in deformable continuum has hybrid properties. It consists of two displacement fields:

the displacement field due to rigid body motion and the displacement field due to constant strain distribution.

This is the very basis for derivation of the CST element in plane stress problem and also it gives the theoretical basis for rigid body-spring elements which were recently proposed by the present author if the latter can be neglected to compare with the former.

Attempt is made to establish variational basis of the rigid body-spring elements and also to develop modified elements of the better convergency by using the displacement field in hybrid form and hybrid potential energy principle.

2. Hybrid Property of Infinitesimal Displacement Field

Consider two neighboring particles P and Q in a three dimensional continuous medium. Suppose that the medium is displaced and deformed such that they occupy new positions P' and Q' as shown Fig. 1. In case of three dimentional Cartesian space the relative displacement of Q relative to P, i.e., $du=u_P-u_Q$ can be given in the following tensor form:

$$du_i=\frac{\partial u_i}{\partial x_j}dx_j=\omega_{ij}dx_j+\varepsilon_{ij}dx_j \qquad (1)$$

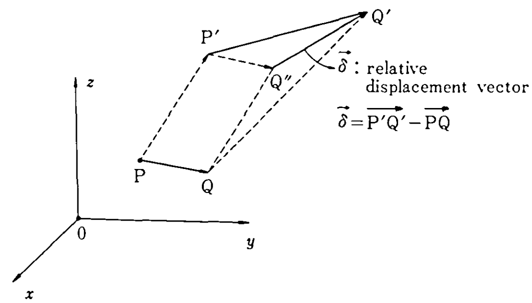


Fig. 1 Relative Displacement Vector $\vec{\delta}$ in 3D Cartesian Space

where

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \quad \text{: strain tensor} \qquad (2)$$

$$\omega_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) \quad \text{: rotation tensor} \qquad (3)$$

The integral form of eq. (1) can be given by the following equations:

$$u_i(X_k)=u_i(X_k^p)+\omega_{ij}(X_k-X_k^p)+\varepsilon_{ij}(X_k-X_k^p) \qquad (4)$$

or in matrix form:

$$\boldsymbol{u}(\boldsymbol{X})=\boldsymbol{H}_d(\boldsymbol{X})\boldsymbol{d}+\boldsymbol{H}_\varepsilon(\boldsymbol{X})\boldsymbol{\varepsilon} \qquad (5)$$

where $\boldsymbol{d}$ is the rigid body displacement vector at P and $\boldsymbol{\varepsilon}$ the strain vector defined at P.

Eq. (5) implies that the infinitesimal displacement field consists of two types of displacement fields, i.e. displacement field due to rigid body movement $\mathbf{d}$ and the displacement field due to strains $\boldsymbol{\varepsilon}$. In case of in-plane displacement field, eq. (5) can be given by the following equation:

$$\begin{Bmatrix} U(x,y)\\ V(x,y)\end{Bmatrix}=\begin{bmatrix} 1 & 0 & -(y-y_p)\\ 0 & 1 & (x-x_p)\end{bmatrix}\begin{Bmatrix}u_p\\ v_p\\ \chi_p\end{Bmatrix}+\begin{bmatrix}(x-x_p) & 0 & \frac{1}{2}(y-y_p)\\ 0 & (y-y_p) & \frac{1}{2}(x-x_p)\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} \qquad (6)$$

*Dept. of Mechanical Engineering and Naval Architecture, Inst. of Industrial Science, Univ. of Tokyo.

3. The Variational Basis of the Finite Element Method

In this section variational formulation of conventional finite element method will be reviewed by using the linear displacement field expressed by eq. (5) and also by employing the hybrid potential energy principle.

For simplicity in-plane deformation field as shown in Fig. 2 is considered.

The principle of minimum potential energy requires minimization of the following functional expressed by a trial displacement field $\boldsymbol{u}$ satisfying the prescribed geometrical boundary condition: $\boldsymbol{u}=\bar{\boldsymbol{u}}$ on C_u.

$$\Pi(\boldsymbol{u})=\int_A \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{D\varepsilon}\,dS-\int_A \boldsymbol{u}^T\bar{\boldsymbol{P}}\,dS-\int_{C_\sigma}\boldsymbol{u}^T\bar{\boldsymbol{T}}\,ds \tag{7}$$

Let's devide a given domain into a number of subdomains (finite elements) as shown in Fig. 2 and for each element the infinitesial displacement field given by eq. (6) is assumed without ensuring the inter-element compatibility.

In such a case the functional $\Pi(u)$ given by eq. (7) should be modified by adding some line integrals defined on the inter-element boundaries involving Lagrangian multipliers as follows:

$$\Pi_{PH}(\boldsymbol{u})=\Sigma\Pi^e(\boldsymbol{u})-\Sigma\Pi_1^e(\boldsymbol{u}) \tag{8}$$

where $\Pi^e(u)$ is the functional eq. (7) for a specific element. $\Pi_1^e(u)$ is line integral involving Lagrange multiplier and they are given as follows:

(i) hybrid displacement model (I)

$$\Pi_1(\boldsymbol{u})=\int_{CB}\boldsymbol{\lambda}(\boldsymbol{u}_1-\boldsymbol{u}_2)ds \tag{9}$$

(ii) hybrid displacement model (II)

$$\Pi_1(\boldsymbol{u})=\int_{CB}\{(\boldsymbol{n\sigma})_1^T(\boldsymbol{u}_1-\boldsymbol{\mu})+(\boldsymbol{n\sigma})_2^T(\boldsymbol{u}_2-\boldsymbol{\mu})\}ds \tag{10}$$

where λ is Lagrangian multiplier which represents the surface traction on the inter-element boundaries, while $\boldsymbol{\mu}$ is Lagrange multiplier which represents the boundary displacement on the inter-element surfaces.

$\boldsymbol{n}$ is normal drawn outward to the boundary curve of a given element. 1 and 2 stand for any adjoining elements. C_B implies the inter-element boundary between two adjoining elements.

As nodal parameters are only defined within a single element, the following equation can be derived by taking variation with respect to $\boldsymbol{\varepsilon}$ of a typical element :

(i) In case of hybrid displacement model (I)

$$A\boldsymbol{D\varepsilon}-\boldsymbol{f}_\varepsilon^e-\boldsymbol{I\lambda}_B=0 \quad \therefore\ \boldsymbol{\varepsilon}=\frac{\boldsymbol{C}}{A}(\boldsymbol{I\lambda}_B+\boldsymbol{f}_\varepsilon^e) \tag{11}$$

(ii) In case of hybrid displacement model (II)

$$A^*\boldsymbol{D\varepsilon}+\boldsymbol{f}_\varepsilon^e+\boldsymbol{DHd}-\boldsymbol{DJ\mu}_B=0$$

$$\therefore\ \boldsymbol{\varepsilon}=\frac{1}{A^*}(\boldsymbol{J\mu}_B-\boldsymbol{Hd}-\boldsymbol{Cf}_\varepsilon^e) \tag{12}$$

where

$$\begin{aligned}\boldsymbol{\lambda}_B^T&=\lfloor X_{B4}, X_{B5}, X_{B6}, Y_{B4}, Y_{B5}, Y_{B6}\rfloor\\ \boldsymbol{\mu}_B^T&=\lfloor \mu_{B4}, \mu_{B5}, \mu_{B6}, \mu_{B4}, \mu_{B5}, \mu_{B6}\rfloor\end{aligned} \tag{13}$$

$\boldsymbol{\lambda}_B$, $\boldsymbol{\mu}_B$ are nodal parameters defined on the inter-element boundary.

A = area of a given element

$$A_1=\oint_{CB}\boldsymbol{n}^T\boldsymbol{H}_\varepsilon ds,\quad A^*=2A_1-A$$

$$\boldsymbol{f}_\varepsilon^e=\int_{Se}\boldsymbol{H}_\varepsilon^T\bar{\boldsymbol{P}}ds+\oint_{CB}\boldsymbol{H}_\varepsilon^T\bar{\boldsymbol{T}}ds$$

$$\boldsymbol{H}=\oint_{CB}\boldsymbol{n}^T\boldsymbol{H}_d ds,\quad \boldsymbol{C}=\boldsymbol{D}^{-1}\quad :\ \text{material compliance}$$

$$\boldsymbol{I}=\left[\int_{AB}\boldsymbol{H}_\varepsilon^T ds \;\vdots\; \int_{BC}\boldsymbol{H}_\varepsilon^T ds \;\vdots\; \int_{CA}\boldsymbol{H}_\varepsilon^T ds\right]$$

$$\boldsymbol{J}=\left[\int_{AB}\boldsymbol{n}_4^T ds \;\vdots\; \int_{BC}\boldsymbol{n}_5^T ds \;\vdots\; \int_{CA}\boldsymbol{n}_6^T ds\right] \tag{14}$$

Eliminating $\boldsymbol{\varepsilon}$ from eq. (8) by using eq. (11) or eq. (12),

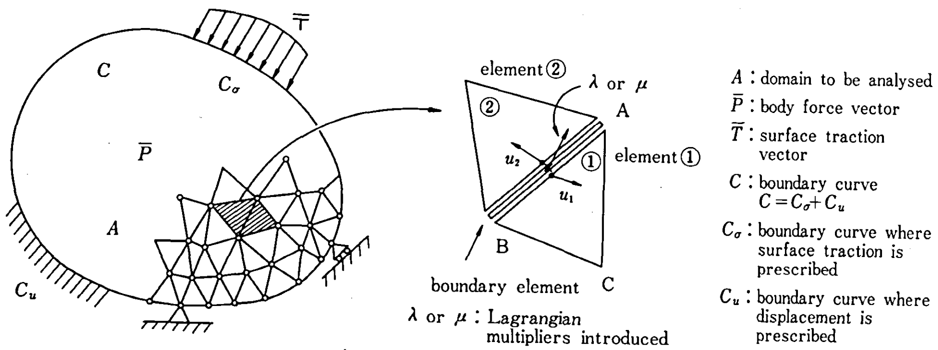


Fig. 2　Finite Element Analysis of Two Dimensional Displacement Field

the functional $\Pi_{PH}(\boldsymbol{u})$ can be written in the following form:

(i) in case of hybrid displacement models (I)

$$\Pi_{PH}(\boldsymbol{u})=\Pi_{PH}(\boldsymbol{d},\boldsymbol{\lambda}_B) \qquad (15)$$

(ii) in case of hybrid displacement model (II)

$$\Pi_{PH}(\boldsymbol{u})=\Pi_{PH}(\boldsymbol{d},\boldsymbol{\mu}_B) \qquad (16)$$

$\Pi_{PH}(\boldsymbol{d},\boldsymbol{\lambda}_B)$ is the functional of mixed type, while $\Pi_{PH}(\boldsymbol{d},\boldsymbol{\mu}_B)$ is the functional of pure displacement type, and in case of in-plane deformation problems they are illustrated in the following Fig. 3.

For calculation of eqs. (11) and (12) the following equations are used:

$$\boldsymbol{H}_d(x,y)=\left[\begin{array}{c|c|c} 1 & 0 & -(y-y_0) \\ \hline 0 & 1 & (x-x_0) \end{array}\right] \qquad (17)$$

$$\boldsymbol{H}_\varepsilon(x,y)=\left[\begin{array}{c|c|c} x-x_0 & 0 & \frac{1}{2}(y-y_0) \\ \hline 0 & y-y_0 & \frac{1}{2}(x-x_0) \end{array}\right] \qquad (18)$$

$$\boldsymbol{n}_i=\left[\begin{array}{c|c|c} l_i & 0 & m_i \\ \hline 0 & m_i & l_i \end{array}\right] \qquad (19)$$

$(i=4,5,6)$

(x_0, y_0): the centroid of a given element

Various terms in eqs. (11) and (12) are now given by the following equations:

$$A_1=\oint_{CB}\boldsymbol{n}^T\boldsymbol{H}_\varepsilon ds=\int_{AB}\boldsymbol{n}_4^T\boldsymbol{H}_\varepsilon ds+\int_{BC}\boldsymbol{n}_5^T\boldsymbol{H}_\varepsilon ds$$
$$+\int_{CA}\boldsymbol{n}_6^T\boldsymbol{H}_\varepsilon ds={}_3[\overset{3}{I_0}]A \qquad (20)$$

where ${}_3[\overset{3}{I_0}]$ is the (3 x 3) unit matrix, and

$$A=\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad \text{: area of } \Delta\text{ABC} \qquad (21)$$

Therefore $\qquad A^*=A \qquad (22)$

Similarly

$$\boldsymbol{H}=\oint_{CB}\boldsymbol{n}^T\boldsymbol{H}_d ds=\int_{AB}\boldsymbol{n}_4^T\boldsymbol{H}_d ds+\int_{BC}\boldsymbol{n}_5^T\boldsymbol{H}_d ds$$
$$+\int_{CA}\boldsymbol{n}_6^T\boldsymbol{H}_d ds={}_3[\overset{3}{0}] \qquad (23)$$

$$\left[\begin{array}{c|c|c|c|c|c} L_{BC}(x_4-x_0) & 0 & L_{BC}(x_5-x_0) & 0 & L_{CA}(x_6-x_0) & 0 \\ 0 & L_{AB}(y_4-y_0) & 0 & L_{BC}(y_5-y_0) & 0 & L_{CA}(y_6-y_0) \\ L_{AB}(y_4-y_0) & L_{AB}(x_4-x_0) & L_{BC}(y_5-y_0) & L_{BC}(y_5-y_0) & L_{CA}(y_6-y_0) & L_{CA}(x_6-x_0) \end{array}\right] \qquad (24)$$

$$\boldsymbol{J}=\left[\begin{array}{c|c|c|c|c|c} l_4L_{AB} & 0 & l_5L_{BC} & 0 & l_6L_{CA} & 0 \\ \hline 0 & m_4L_{AB} & 0 & m_5L_{BC} & 0 & m_6L_{CA} \\ \hline m_4L_{AB} & l_4L_{AB} & m_5L_{BC} & l_5L_{BC} & m_6L_{CA} & l_6L_{CA} \end{array}\right] \qquad (25)$$

where L_{AB}, L_{BC}, L_{CA} are length of three sides of ΔABC respectively.

From the results of calculation, the following equations can be obtained:

(i) hybrid displacement model (I)

$$\boldsymbol{\varepsilon}=\frac{\boldsymbol{L}}{A}\boldsymbol{\lambda}_B, \qquad \text{except } \boldsymbol{f}_\varepsilon^e$$

where $\qquad \boldsymbol{L}=\boldsymbol{C}\boldsymbol{I} \qquad (26\text{-a})$

(ii) hybrid displacement model (II)

$$\boldsymbol{\varepsilon}=\frac{\boldsymbol{J}}{A}\boldsymbol{u}_m \qquad \text{except } \boldsymbol{f}_\varepsilon^e \qquad (26\text{-b})$$

where $\qquad \boldsymbol{u}_m^T=\lfloor u_4, v_4, u_5, v_5, u_6, v_6 \rfloor$

in which $\boldsymbol{u}_m$ consists of displacement vectors at 3 midside nodes of a given triangle. (Fig. 3)

Substituting eq. (26-a) and (26-b) into eq. (7) it can be concluded that

(a) In case of hybrid displacement model (I)

$$\Pi_{PH}(\boldsymbol{u})=\Pi_{PH}(\boldsymbol{d},\boldsymbol{\lambda}_B) \qquad (27\text{-a})$$

(b) In case of hybrid displacement model (II)

$$\Pi_{PH}(\boldsymbol{u})=\Pi_{PH}(\boldsymbol{u}_m) \qquad (27\text{-b})$$

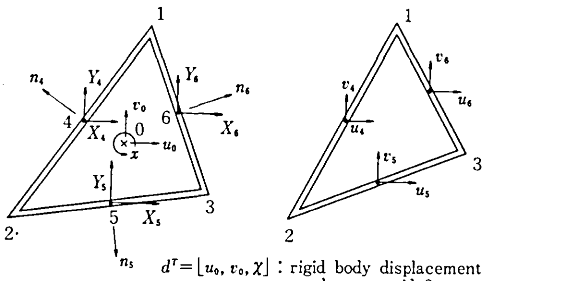
Minimization of Π_{PH} with respect to $\boldsymbol{d}$, $\boldsymbol{\lambda}_B$ or $\boldsymbol{u}_m$ yields the following matrix equations:

(i) Hybrid displacement model (I).

$$\left[\begin{array}{c|c} 0 & \boldsymbol{K}_{12} \\ \hline \boldsymbol{K}_{12}^T & \boldsymbol{K}_{22} \end{array}\right]\left\{\begin{array}{c} \boldsymbol{d} \\ \hline \boldsymbol{\lambda}_B \end{array}\right\}=\left\{\begin{array}{c} \boldsymbol{f}_d \\ \boldsymbol{f}_\lambda \end{array}\right\} \qquad (27\text{-c})$$

(ii) Hybrid displacement model (II).

$$\boldsymbol{K}_{33}\boldsymbol{\mu}_B=\boldsymbol{f}_\mu \qquad (27\text{-d})$$



$d^T=\lfloor u_0, v_0, \chi \rfloor$: rigid body displacement at the centroid 0

$\lambda_B^T=\lfloor X_4, Y_4, X_5, Y_5, X_6, Y_6 \rfloor$ $\qquad \mu_B^T=\lfloor u_4, v_4, u_5, v_5, u_6, v_6 \rfloor$

(a) hybrid displacement model (I) (b) hybrid displacement model (II)

Fig. 3 Two types of Hybrid Displacement Models for In-plane Deformation Problems.

4. Derivation of New Discrete Elements Through Simple Matrix Transformation

In this section a method of derivation of new discrete elements through simple matrix transformation will be described.

Eq. (26) was derived by using the so-called hybrid potential energy principle. This equation can be easily derived by a simple matrix transformation follows:
First displacements at the midside nodes can be obtained by using eq. (6) in the following form:

$$\{u_m\}=[G_1 \vdots G_2]\left\{\begin{matrix} d \\ \cdots \\ \varepsilon \end{matrix}\right\} \tag{28}$$

Then solving eq. (28) with respect to ε, the following matrix equation can be obtained:

$$\varepsilon=Eu_m \tag{29}$$

It is not difficult to show that eq. (29) is mathematically identical to eq. (26) except f_ε^e although physical meaning of both equations are different each other.

Once eq. (29) is obtained, the stiffness matrix of an in-plane triangular element k_m can be derived in term of nodal parameters u_m as follows:

$$V^e=\frac{1}{2}u_m^T k_m u_m$$

where

$$k_m=\int_{Ae}\frac{h}{2}E^T DE ds \tag{30}$$

This matrix was first derived by Fraeijs de Veubeke by using the complementary energy principle [4] and it is, however, equivalent to the well-known CST element. Now a set of 4 standard CST elements as shown in Fig. 4 is considered, and eq. (5) is assumed for the displacement field of each element. Continuity condition of displacement on three boundary edges $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ requires the following relation between strain ε and rigid body displacement Δ:

$$A\varepsilon=B\Delta \tag{31}$$

$$\begin{aligned} &\varepsilon^T=\lfloor \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rfloor \\ \text{where } &\varepsilon_i^T=\lfloor \varepsilon_x^i, \varepsilon_y^i, \gamma_{xy}^i \rfloor \\ &\Delta^T=\lfloor d_1, d_2, d_3, d_4 \rfloor \\ &d_i^T=\lfloor u_i, v_i, \chi_i \rfloor \end{aligned} \tag{32}$$

The subscript i implies i th element (i=1, 2, 3, 4). Solving eq. (31), ε can be now expressed in term of Δ as follows:

$$\varepsilon=A^{-1}B\Delta \tag{33}$$

and from which the following equation can be derived:

$$\varepsilon_1=G\Delta \tag{34}$$

Therefore, the strain energy V^e and stiffness matrix K for the element ① can be obtained as follows:

$$V^e=hA\cdot\frac{1}{2}\varepsilon_1 D\varepsilon_1=\frac{1}{2}\Delta^T K\Delta \tag{35}$$

$$K=hA(G^T DG) \tag{36}$$

K is (12 x 12) symmetric square matrix and it is clearly seen that this element is a reasonable generalization of the rigid body-spring elements as mentioned before.

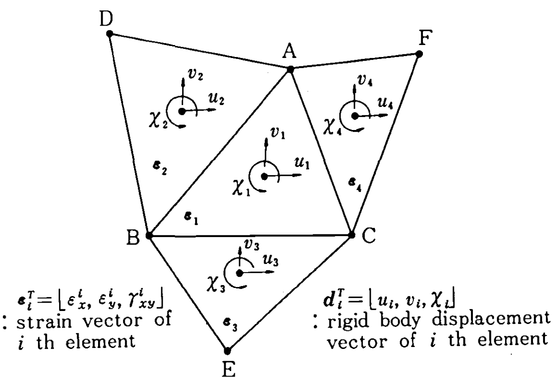


Fig. 4　A set of four CST Elements

5. Conclusion

Using the infinitesmal displacement field in hybrid form, it was shown that variational formulation of the finite element method can be made in an unified way. It was also discussed that a new discrete element of a lower order shape function can be derived by applying simple matrix transformation and partial approximation to a constant strain element for a given problem.

(Manuscript received, October 24, 1978)

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