UDC 624.042
624.0741.075

A Discrete Element Analysis of Beam Bending Problems Including the Effects of Shear Deformation

剪断変形を考慮した梁の一離散化解析

by Tadahiko KAWAI* and Chang-Neow CHEN*
川 井 忠 彦・陳　長 鈕

Summary

In the field of structural engineering, it is a common practice to idealize real structures as composed of beam and plate elements. Ships, tall buildings, bridges, aircrafts etc. are typical examples of such structures. In these cases, the effects of shear deformation can not be neglected. In this note a new beam elements is proposed which consists of two rigid bars connected by a rotational spring and a shear spring. By simple numerical examples it will be illustrated that the effects of shear deformation on a beam bending problems can be easily handled.

1. Theoretical Basis

1. 1 Stiffness matrix

Recently one of the present authors proposed a new discrete model for analysis of solid mechanics problems. The general stiffness matrix of this model is a (12 x 12) symmetric matrix expressed by the displacement of the centroids in two rigid bodies under contact. The new beam bending element is shown in the following Fig. 1-a and the corresponding stiffness matrix which includes the effect of shear deformations is given by the Table 1.

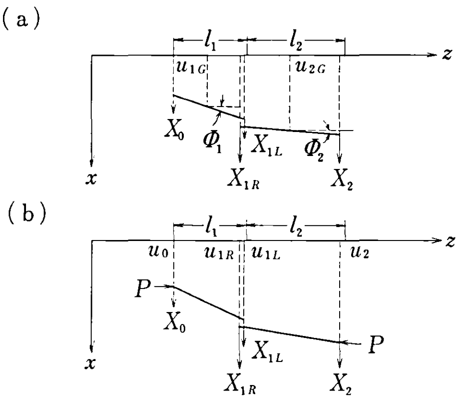

Fig. 1 A new beam element

*Dept. of Mechanical Engineering and Naval Architecture, Inst. of Industrial Science, Univ. of Tokyo.

Table 1. The bending stiffness matrix of a beam element including effect of shear deformation

	u_{1G}	ϕ_1	u_{2G}	ϕ_2
X_{1G}	K_1	$\frac{l_1}{2}K_1$	$-K_1$	$\frac{l_2}{2}K_1$
M_1		$K_{15}+\frac{l_1^2}{4}K_1$	$-\frac{l_1}{2}K_1$	$-K_{15}+\frac{l_1 l_2}{4}K_1$
X_{2G}	S.Y.M $K_1=\iint k_s dS$		K_1	$-\frac{l_2}{2}K_1$
M_2	$K_{15}=\iint k_d x^2 dS$			$K_{15}+\frac{l_2^2}{4}K_1$

For calculation of the spring constants k_d and k_s, the following formula can be used:

$$k_d=\frac{2E}{l_1+l_2}\,,\qquad k_s=\frac{2G}{l_1+l_2} \tag{1}$$

For convenience of further analysis the nodal displacement vector $\{u_G\}=\lfloor u_{1G},\ \phi_1\ ;\ u_{2G},\ \phi_2 \rfloor^T$ will be transformed by the following matrix to

$$\{u\}=\lfloor u_0,\ u_{1L}\ ;u_{1R},\ u_2\rfloor^T$$

$$\begin{Bmatrix} u_{1G} \\ \phi_1 \\ u_{2G} \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{l_1} & \frac{1}{l_1} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{l_2} & \frac{1}{l_2} \end{bmatrix} \begin{Bmatrix} u_0 \\ u_{1L} \\ u_{1R} \\ u_2 \end{Bmatrix} \tag{2}$$

or

$$\{u_G\}=[\Pi]\{u\}$$

Now $\overline{K}$ can be transformed into K by the following equation:

$$[K]=[\Pi]^T[\overline{K}]\,[\Pi] \tag{3}$$

and finally $[K]$ is given by the following equation:

$$\begin{Bmatrix} X_0 \\ X_{1L} \\ X_{1R} \\ X_2 \end{Bmatrix} = \begin{bmatrix} \frac{K_{15}}{l_1^2} & -\frac{K_{15}}{l_1^2} & -\frac{K_{15}}{l_1 l_2} & \frac{K_{15}}{l_1 l_2} \\ & K_1+\frac{K_{15}}{l_1^2} & -K_1+\frac{K_{15}}{l_1 l_2} & -\frac{K_{15}}{l_1 l_2} \\ & & K_1+\frac{K_{15}}{l_2^2} & -\frac{K_{15}}{l_2^2} \\ \text{S.Y.M} & & & \frac{K_{15}}{l_2^2} \end{bmatrix} \begin{Bmatrix} u_0 \\ u_{1L} \\ u_{1R} \\ u_2 \end{Bmatrix} \quad (4)$$

1. 2 Initial stress matrix and mass matrix

A pair of buckled beam elements under axial compression P is shown in Fig. 1-b.

Potential $\overline{W}$ of this compressive force P is obtained in the following form:

$$\overline{W} = \frac{P(u_{1L}-u_0)^2}{4l_1} + \frac{P(u_2-u_{1R})^2}{4l_2} + \frac{P(u_{1R}-u_{1L})(u_2-u_0)}{l_1+l_2} \quad (5)$$

Then, it is not difficult to derive the following initial stress matrix $\boldsymbol{K}_G$ by applying Castigliano's Theorem.

$$[\boldsymbol{K}_G] = P \begin{bmatrix} \frac{1}{2l_1} & \frac{l_1-l_2}{2l_1(l_1+l_2)} & \frac{-1}{l_1+l_2} & 0 \\ & \frac{1}{2l_1} & 0 & \frac{-1}{l_1+l_2} \\ & & \frac{1}{2l_2} & \frac{l_2-l_1}{2l_2(l_1+l_2)} \\ & & & \frac{1}{2l_2} \end{bmatrix} \quad (6)$$

In the same way the consistent mass matrix of a given beam bending element is obtained in the following form:

$$[\boldsymbol{M}] = \frac{\gamma A}{6g} \begin{bmatrix} 2l_1 & l_1 & 0 & 0 \\ l_1 & 2l_1 & 0 & 0 \\ 0 & 0 & 2l_2 & l_2 \\ 0 & 0 & l_2 & 2l_2 \end{bmatrix} \quad (7)$$

1. 3 Yielding function and constitutive equation

In elastic range the incremental form of the stress-strain relation is given as follows:

$$\{ds\} = [D^e]\{dE\}$$

where

$$[D^e] = \begin{bmatrix} k_1 & & & \\ & \ddots & & \boldsymbol{O} \\ & & k_i & \\ \boldsymbol{O} & & & \ddots \\ & & & k_n \end{bmatrix} \quad (8)$$

Let plastic potential and plastic strain increment be

$$f(s_i) = 1$$

$$\{dE^p\} = \lambda\left\{\frac{\partial f}{\partial s}\right\} df \quad (9)$$

Then the constitutive equation in the plastic range can be expressed as follows:

$$\{ds\} = [D^P]\{dE\}$$

where

$$[D^P] = [k_{ij}^P] = [D^e] - \frac{[D^e]\{\partial f/\partial s\}\lfloor \partial f/\partial s \rfloor [D^e]}{\lfloor \partial f/\partial s \rfloor [D^e]\{\partial f/\partial s\}} \quad (10)$$

Also, the unloading condition occurs when

$$\frac{\lfloor \partial f/\partial s \rfloor [D^e]\{dE\}}{\lfloor \partial f/\partial s \rfloor [D^e]\{\partial f/\partial s\}} < 0 \quad (11)$$

Observing, that the denominator of eq. (11) is always positive and using eq. (8), (9), eq. (10) and eq. (11) can be rewritten as

$$\left.\begin{aligned} k_{ij}^P &= k_i\delta_{ij} - \frac{1}{\sum_i k_i f_i^2} f_i f_j k_i k_j \\ f_i &= \partial f/\partial s_i \end{aligned}\right\} \quad (12)$$

$$\Sigma f_i k_i de_i < 0 \quad (13)$$

In the present problem,

$$f(M, F) = \left(\frac{M}{M_p}\right)^2 + \left(\frac{F}{F_y}\right)^2 \quad (14)$$

where M_p is the full plastic moment and F_y is the yield shearing force. And eqs. (12) and (13) can be given as follows:

$$\left.\begin{aligned} k_{11}^P &= K_{15} - \frac{K_{15}^2 M^2/M_p^4}{K_{15}M^2/M_p^4 + K_1 F^2/F_y^4} \\ k_{22}^P &= K_1 - \frac{K_1^2 F^2/F_y^4}{K_{15}M^2/M_p^4 + K_1 F^2/F_y^4} \\ k_{12}^P &= k_{21}^P = -\frac{K_1 K_{15} MF/M_p^2 F_y^2}{K_{15}M^2/M_p^4 + K_1 F^2/F_y^4} \end{aligned}\right\} \quad (15)$$

$$\frac{K_{15}M}{M_p^2} d\phi + \frac{K_1 F}{F_y^2} d\delta_s < 0 \quad (16)$$

1. 4 Dynamic response analysis of a cantilever subjected to a prescribed ground motion

For simplicity, effects of large deflection, rotatary inertia are not considered in this analysis. Furthermore, effects of damping was neglected for the time being. Then, the equations of motion for this analysis are expressed in the incremental form as follows:

$$[\boldsymbol{M}]\{d\ddot{u}\} + [\boldsymbol{K}]\{du\} = \{d\boldsymbol{F}(t)\} \quad (17)$$

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} d\ddot{u}_s + d\ddot{u}_d \\ d\ddot{u}_b \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} du_s + du_d \\ du_b \end{Bmatrix} = \begin{Bmatrix} dF(t) \\ dF_0(t) \end{Bmatrix} \quad (18)$$

where, u_s : the quasi-static displacement
u_d : the displacement due to the inertia force
u_b : the support displacement

It will be noticed that Newmark's β method is employed for integration of eq. (18).

2. Numerical Examples

For verification of effectiveness of the new element introduced, a series of numerical analysis have been conducted. Fig. 2 is the result of bending analysis of a cantilever beam under a concentrated load at the tip.

As will be seen in Fig. 2, the present solution is in good agreement with Timoshenko's and Kawai & Fujitani's solutions for static displacement analysis.

Fig. 3 shows the effects of shear deformation on the natural frequency of beam bending vibration, from which convergency of numerical solutions can be seen. Fig. 4 shows the schematic drawing of calculated natural modes from which effects of shear deformation can be clearly seen. Fig. 5 show comparison of the natural modes of beam bending vibration with and without effects of shear deformation.

The effect of shear deformation on the natural frequencies of a simply supported beam is shown in Fig. 6.

Fig. 7 shows buckling load analysis of a compressed column in comparison with two solutions given by S. P. Timoshenko.

Lastly, Dynamic collapse of a cantilever column subjected to a prescribed sine wave ground motion is analyzed. Fig. 8, 9, and 10 show the result of this analysis, from which it can be seen that the shearing force plays a significant role in dynamic collapse analysis. Generalization of this dynamic collapse analysis under arbitrary seismic motion of the ground is now under way.

3. Conclusion

In this note it was shown that the effects of shear deformation on the beam bending problems can be easily taken into account in the new discrete element analysis which was proposed by one of the authors. Extension of the present theory to the bending problems of plates and shells will be straightforward. Finally the authors would like to express their sincere appreciation to Mr. Yutaka Toi for his kind assistance and valuable discussions in preparation of this manuscript. (Manuscript received, February 21, 1978)

References

1) T. Kawai, "A New discrete Model for Analysis of Solid Mechanics Problems", Seisan Kenkyu, Vol. 29, No. 4, (April, 1977) 2) Y. Yamada, "Plasticity and Viscoelasticity", Baifukan 1972 (in Japanese) 3) T. Kawai, and Y. Toi, "A Discrete Analysis of Dynamic Collapse of Beam under Impulsive Transverse Load", Seisan Kenkyu, Vol. 29, No. 5. 1977 4) T. Kawai and Y. Fujitani, "Some Attempts on the Refinement of Modern Engineering Theory of Beams", Seisan Kenkyu, Vol. 25, No. 11, 1973, Vol. 26, No. 6, 1974 5) S. P. Timoshenko and J. N. Goodier, "Theory of Elasticity" Third Edition 6) S. P. Timoshenko, "Vibration Problems in Engineering" Third Edition 7) S. P. Timoshenko and J. M. Gere "Theory of Elastic Stability", Second Edition 8) Y. Kawabata and Y. Fujitani, "An Approximate Method for Analysing Tall Buildings by Beam Theory including Shear Deformation" private communication

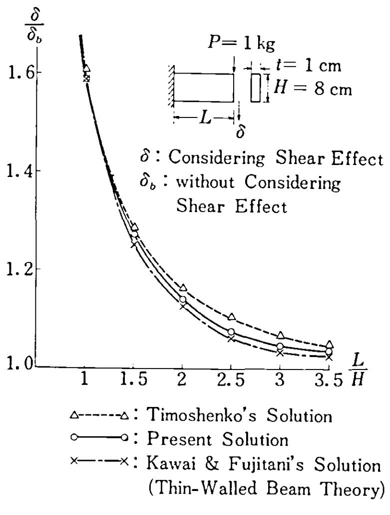

Fig. 2 The effects of shear deformations on static displacement at free end of a cantilever beam

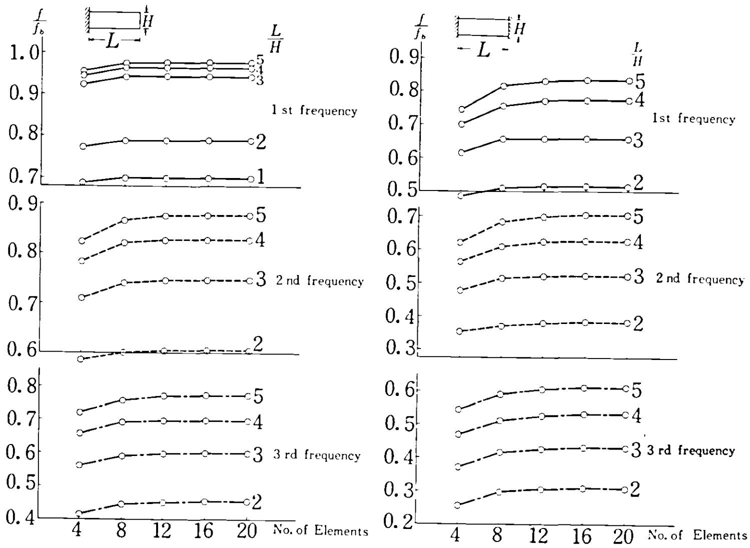

Fig. 3 Convergency characteristics of natural frequency of beam bending vibration

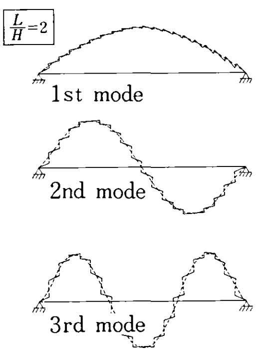

Fig. 4. The natural modes of beam bending vibration of a simply supported beam including shear deformation

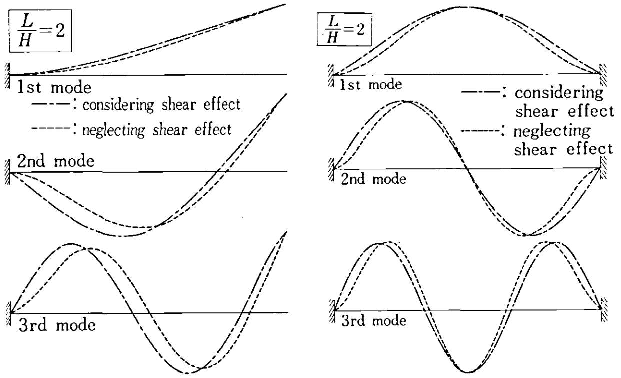

Fig. 5 Effect of shear deformation on the natural modes of beam bending vibration

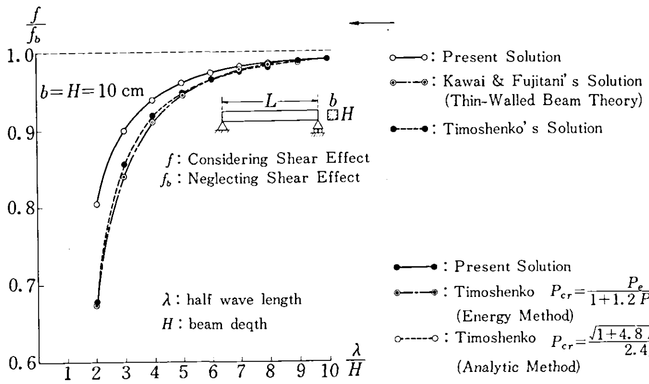

Fig. 6 The effects of shear deformations on bending vibration of a simply supported beam

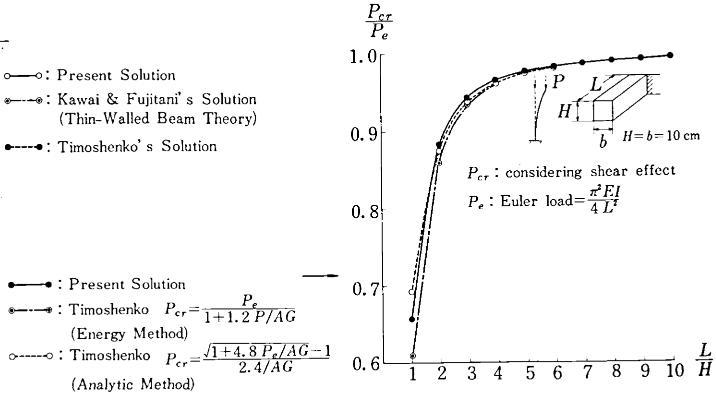

Fig. 7 The effect of shearing deformation on the critical load of a compressed cantilever column

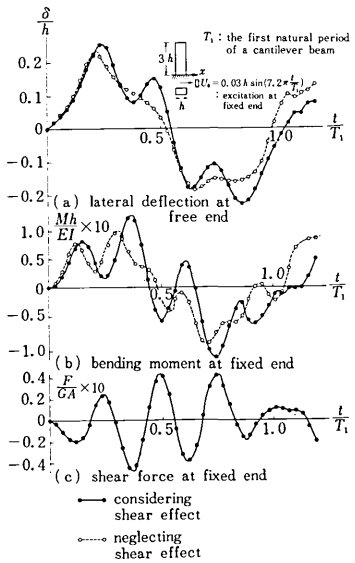

Fig. 8 Elastic response analysis of a cantilever beam under ground motion

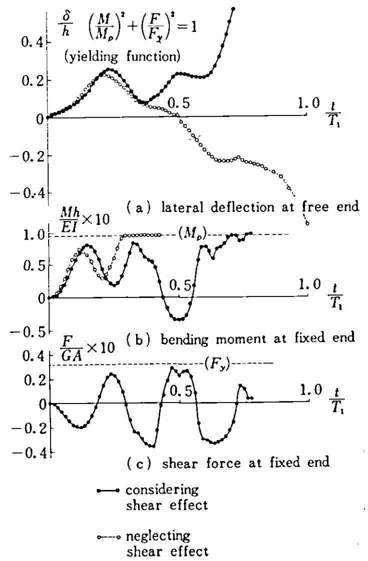

Fig. 9 Dynamic collapse analysis of a cantilever beam under ground motion

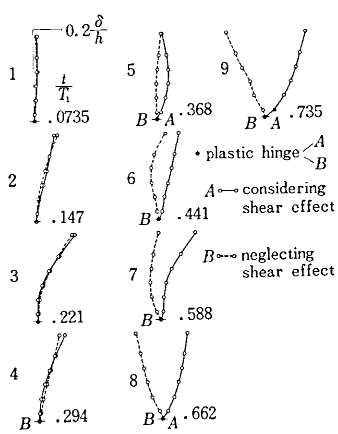

Fig. 10 Deflection shapes of a cantilever beam under ground motion