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Stability Analysis of Time Integration Operator in Problems of Visco-elastic Wave Propagation

粘弾性波伝ぱ解析における時間積分法の安定性解析

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1. Introduction

A scheme to incorporate rate-dependent constitutive equation into wave propagation problems is proposed. This scheme is based on the application of the Newmark's relation to the derivation of incremental stress-strain relation, and enables us to get the incremental stiffness matrix used in the finite element method. The analysis method is exemplified with wave propagation in Voigt solid, and the result is verified in comparison with the relevant analytical solution. Direct time integration operator should be examined in the case of elastic system with damping, since the Newmark's operator gives rise to instability in certain circumstances. The stable limit of the time increment is given herein as the result of the stability analysis, in which both the properties of visco-elastic material and the time integration operator are taken into account.

2. Solution procedure of visco-elastic wave propagation

Visco-elastic wave propagation is treated with the aid of conventional procedure of the incremental finite element analysis by means of incorporating the rate-dependent constitutive equation into the equation of motion and the direct time integration. Newmark gave the following formulae[1] for the relation between variable increment Δf and its time derivatives of $\Delta \dot{f}$ and $\Delta \ddot{f}$,

$$\Delta \dot{f} = \frac{1}{2\beta_C}\left(\frac{\Delta f}{\Delta t} - \dot{f}\right) + \left(1 - \frac{1}{4\beta_C}\right)\Delta t \ddot{f} \qquad (1)$$

$$\Delta \ddot{f} = \frac{1}{\beta_C}\left(\frac{\Delta f}{\Delta t^2} - \frac{\dot{f}}{\Delta t} - \frac{\ddot{f}}{2}\right) \qquad (2)$$

where $(\dot{\ })$ means the differentiation with time. When f is taken as stress and β_C as $1/4$, the relation of Eq. (1) coincides with the case of the linear change of stress within time increment Δt, which was used successfully in the visco-elastic stress analysis by use of the compliance expression of the material property[2]. In the cases other than $1/4$, the derivative to the second order is taken into the computation.

The constitutive equation of linear visco-elastic materials is expressed in a general form as

$$\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + \cdots = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + \cdots \qquad (3)$$

and, in a particular case of Voigt solid, as

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} \qquad (4)$$

where σ and ε are stress and strain, E the Young's modulus and η the coefficient of viscosity. Substituting Eq. (1) applied to $\Delta\dot{\varepsilon}$ into the incremental form of Eq. (4) and expressing the rate increment by the stress and strain increments, we have

$$\begin{aligned} \Delta\sigma &= D\Delta\varepsilon - F \\ D &= E\left(1 + \tau/2\beta_C\Delta t\right) \\ F &= \eta\left\{\dot{\varepsilon}/2\beta_C - \left(1 - 1/4\beta_C\right)\Delta t\ddot{\varepsilon}\right\} \end{aligned} \qquad (5)$$

where $\tau = \eta/E$ is the characteristic retardation time of the material. Based upon the stress-strain relation in form of Eq. (5), the equation of motion is given below, for example, when one-dimensional constant strain element of ℓ in length and consistent mass matrix $[m]$ are applied,

$$\frac{D}{\ell}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\left\{\Delta u\right\} - \begin{Bmatrix}-1\\ 1\end{Bmatrix}F + \frac{\rho\ell}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\left\{\Delta\ddot{u}\right\} = \left\{\Delta f\right\} \qquad (6)$$

where ρ is the material mass density, and $\{\Delta f\}$ the incremental external force vector per cross-sectional area of the bar element. Differently from usual stiffness equation, D of the stiffness matrix $[k]$ and F are not material variable, but involve the strain rate and acceleration in the particulr case of Voigt solid and of the stress rate and

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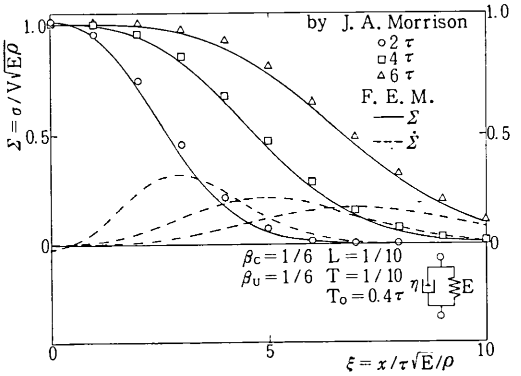

Fig. 1 Stress wave propagation in Voigt solid from impacted end

acceleration in cases of Maxwell body and so on. Merging the above Eq. (6) into the overall system and solving the resultant equation by means of timewise step-by-step integration, we analyse the one-dimensional wave propagation in the boundary and impact conditions under interest. In doing so, we use again the Newmark's integration operator and transform Eq. (6) into the following form,

$$\left(\left[k\right]+\frac{1}{\beta_U \Delta t^2}\left[m\right]\right)\left\{\Delta \ddot{u}\right\}=\frac{1}{\beta_U \Delta t^2}\left(\left\{\Delta f\right\}+\left\{\begin{matrix}-1\\1\end{matrix}\right\}F\right)-\frac{[k]}{2\beta_U}\left\{2\frac{\dot{u}}{\Delta t}+\ddot{u}\right\} \quad (7)$$

in which the acceleration increment is taken as the principal unkown variable. The parameter β of the operator is distinguished by β_C in Eqs. (1) and (2), and β_U in Eq. (7).

Figure 1 shows the good agreement between the analytical solution[3] of the stress propagation in Voigt solid caused by the stepwise impact of constant velocity V and the present analysis. Use is made of $\beta_C=\beta_U=1/6$ so as to assume the linear change of stress and strain rates within Δt, and $T=\Delta t/\tau$ and $L=\ell/\tau\sqrt{E/\rho}$ are taken equal to 1/10 in the numerical solution. The broken line in the figure is the stress rate that was not given in the analytical solution. The computer program is so advantageous that the arbitrary velocity profile of the impact can be dealt with. The profile up to V is given in approximate form of the error function for $T_0=0.4\tau$ in the above example.

The decay of the elastic stress wave front is shown in Fig. 2 in the case that the viscous damping of steel is represented by Voigt model of E=

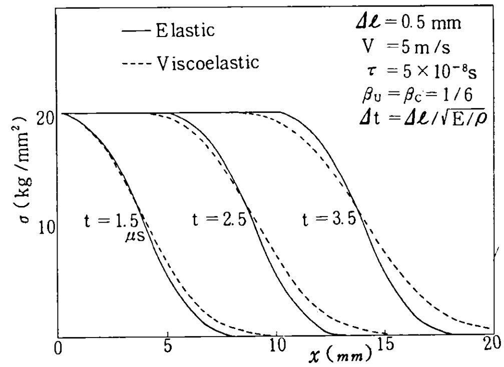

Fig. 2 Wave front decay due to viscous damping in steel compared with elastic stress wave

2.15×10^4 kg/mm^2 and $\tau=5\times10^{-8}$ sec[4]. The solid line in the figure corresponds to the elastic wave front without damping, caused by low impact velocity of V = 5 m/sec attained after a short rise time also in profile of the error function[4].

3. Stability analysis of time integration

As seen in the preceding section, the present scheme succeeds to trace numerically the wave propagation in Voigt solid when the parameters and time increment are chosen appropriately. If β_C is taken larger than 1/6, the scheme is rendered instable as shown in Fig. 3. This figure depicts that the computation diverges sharply in the cases of larger β_C even when the same time increment is used. The instability appears early with the increasing value of β_C to 1/4. It means that this simple scheme requires stability analysis[5), 6)] with respect to the property of the element, through

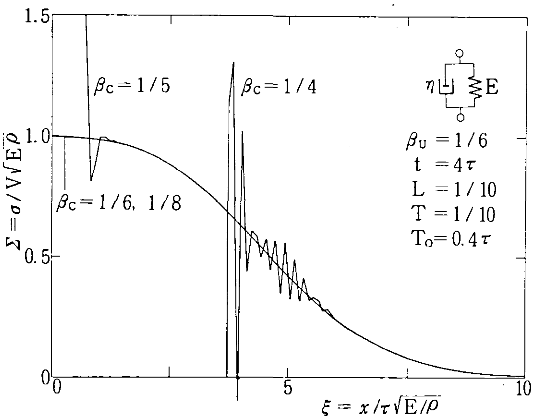

Fig. 3 Instability caused by inadequate choice of parameter β_C and time increment

which the visco-elastic wave propagates.

For the purpose of the stability analysis, we convert the nodal displacement vector $\{u\}$ into the eigen displacement vector $\{v\}$ with the aid of the transformation of $\{v\}=[R]^{-1}\{u\}$, where $[R]$ is the orthonormal eigenvector matrix determined from $[m]^{-1}[k]$. After such a conversion, we have the following individual equation for the i-th component of $\{v\}$,

$$(\omega_i^2+\frac{1}{\beta_U\Delta t^2})\Delta\ddot{v}_i=-\frac{\omega_i^2}{2\beta_U}\Big[(2-\frac{\eta}{\beta_C D\Delta t})\frac{\dot{v}_i}{\Delta t}+\{1+\frac{2\eta}{D\Delta t}(1-\frac{1}{4\beta_C})\}\ddot{v}_i\Big] \quad (8)$$

where ω_i is the pertinent i-th natural circular frequency determined from the eigenvalues of $[m]^{-1}[k]$. Furthermore use is made of the strain-displacement relation below in order to rewrite the

$$\varepsilon=\frac{1}{\ell}\llcorner -1 \quad 1 \lrcorner\{u\} \quad (9)$$

vector term of F in Eq. (7) into the consistent form to the stiffness matrix. Thus we have the fundamental relation between the acceleration increment and the current values of the rate and acceleration. The amplification matrix of the displacement and its time derivatives for a time step is derived on the basis of Eq. (8) and the relation of Eqs. (1) and (2). Then equation (10) can be obtained, where $\alpha=\omega_i^2\Delta t^2/(1+\beta_U\omega_i^2\Delta t^2)$. If any error is involved in v_i, $\dot{v}_i$ and $\ddot{v}_i$, the error is magnified accordingly to the amplification matrix of Eq. (10). The von Neumann condition[7] states that the integration scheme is stable, if the absolute value of every eigenvalue of the amplification matrix is equal to or less than unity. As seen in Eq. (10) the three eigenvalues λ are a function of the parameter β_C used in the approximation of the incremental constitutive equation and β_U in the direct time integration, in addition to the time increment and material constants, as follows,

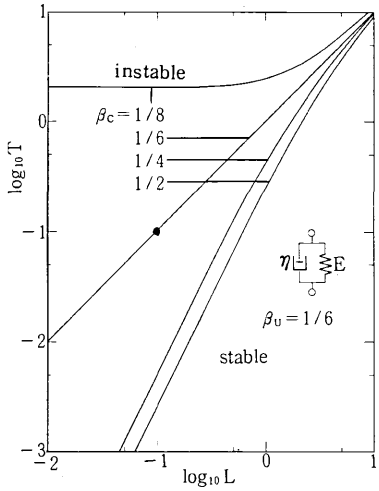

Fig. 4 Stable domain for various values of β_C in case of $\beta_U=1/6$

$$\lambda=1 \text{ and } A\pm\sqrt{A^2-(1-\eta\alpha/D\Delta t)} \quad (11)$$

where $A=1-(\alpha/2)\{1+(\eta/D\Delta t)(1-1/2\beta_C)\}$. Figures 4 and 5 show the stable domain of Δt calculated so that the condition of $|\lambda|\leqq 1$, corresponding to the largest ω_i, is kept for $\beta_U=1/6$ and $1/8$, respectively. It is seen that, for a certain value of $L=\ell/\tau\sqrt{E/\rho}$ decided by the element length in choice and the material constants, larger value of $T=\Delta t/\tau$ than that on the limit line indicated by the parameter β_C results in the instability of the time integration, because λ is larger than unity in the domain above the limit lines. The stable domain for $\beta_U=1/8$ is narrower compared

$$\begin{Bmatrix} v_i \\ \dot{v}_i \\ \ddot{v}_i \end{Bmatrix}^{n+1}=\begin{bmatrix} 1 & \Delta t\{1-\frac{\alpha}{2}\beta_U(2-\frac{\eta}{\beta_C D\Delta t})\} & \Delta t^2[\frac{1}{2}-\frac{\alpha}{2}\beta_U\{1+\frac{2\eta}{D\Delta t}(1-\frac{1}{4\beta_C})\}] \\ 0 & 1-\frac{\alpha}{2}(1-\frac{\eta}{2\beta_C D\Delta t}) & \Delta t\,[1-\frac{\alpha}{4}\{1+\frac{2\eta}{D\Delta t}(1-\frac{1}{4\beta_C})\}] \\ 0 & -\frac{\alpha}{2\Delta t}(2-\frac{\eta}{\beta_C D\Delta t}) & 1-\frac{\alpha}{2}\{1+\frac{2\eta}{D\Delta t}(1-\frac{1}{4\beta_C})\} \end{bmatrix}\begin{Bmatrix} v_i \\ \dot{v}_i \\ \ddot{v}_i \end{Bmatrix}^{n} \quad (10)$$

with that of $\beta_U=1/6$. In case of elastic material of D=E and F=0, it is well known that the time integration is stable unconditionally when β_U is taken equal to or larger than 1/4. Similar stable domain is obtained in case of $\beta_U=1/4$ with these figures, and the integration is also stable unconditionally when $\beta_C \leqq 1/4$, but is stable conditionally when $\beta_C > 1/4$ in the case of visco-elastic wave propagation analysis by use of $\beta_U=1/4$. This is different from the case of elastic wave propagation. The numerical examples illustrated in Figs. 1 and 3 are carried out correspondingly to the time increment shown by the solid circle in Fig. 4, giving rise to stable integration for $\beta_C=1/6$ and instability for $\beta_C=1/5$ and $\beta_C=1/4$, because that the point lies above the limit lines for the larger β_C.

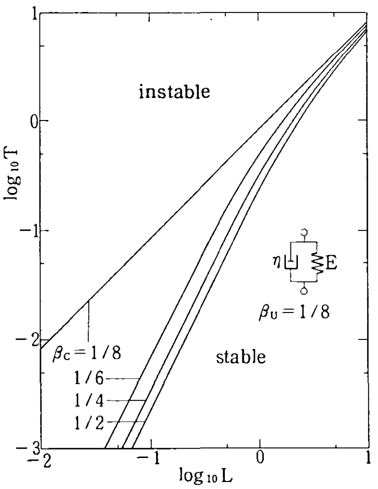

Fig. 5 Stable domain for various values of β_C in case of $\beta_U=1/8$

4. Concluding remarks

The stability of the time integration related with the wave propagation in Voigt solid can be analysed as described in the preceding section. As for the other visco-elastic materials represented by Maxwell body and three-parameter models, the present scheme is applicable to the wave propagation analysis, since all the time derivatives of stress and strain are bounded to the second order, and Eqs. (1) and (2) are satisfactory for the treatment.

When we proceed to deal with four-parameter models and so on, it is desirable to devise the similar incremental relation with Eq. (2) to the higher order. The stability analysis with respect to Voigt solid is rather simple, because that the additional term F, which appears in the incremental stress-strain relation, is expressed by strain rate and acceleration only. On the other hand, when Maxwell body and three-parameter models are concerned, the existence of stress rate and acceleration in the term F necessitates further consideration in the derivation of the amplification matrix from the viewpoint of the stability analysis. It should be noted that error, that creeps in from the external force vector $\{\Delta f\}_r$ is omitted in Eq. (10), but the contribution is included implicitly in form of the error in the displacement terms.

Voigt solid lacks spontaneous elastic resilience. This means that any stepwise wave front is not formed in the solid as shown in Fig. 1, and that the apparent propagation velocity is infinite. In the sequel, there is difficulty in the treatment of the wave front in Voigt solid, when Laplace transform is applied to the analysis, on the contrary to the case of the other visco-elastic materials having the spontaneous elasticity. The simple scheme presented herein overcomes the difficulty and results in unified treatment of the wave reflection and transmission on the boundary between adjacent different materials, together with the ability to take into account various profiles of the impulsive forces and impact velocities.

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References

1) Newmark, N. M.; Proc. ASCE, **85** EM 3 (1959), p. 67.
2) Yamada, Y.; J. of JSME, **74** (1971), p. 631.
3) Morrison, J. A.; Quart. Appl. Math., **14** (1956), p. 153.
4) Tanaka, K. and Kurokawa, T.; Bull. of JSME, **38** (1972), p. 1217.
5) Nickell, R. E.; Int. J. Solids Structures, **7** (1971), p. 301.
6) Geradin, M.; AIAA Paper No. 74-105, AIAA 12th Aerospace Meeting 1974.
7) Richtmeyer, R. D. and Morton, K. W.; Difference Methods for Initial-Value Problems, Interscience Publisher (1967)