UDC 624.044
539.374

A New Discrete Model for Analysis of Visco-Elastic Problems

粘弾性解析における新しい離散化モデル

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Summary

A new discrete model for analysis of visco-elastic problems is proposed in this note. This model consists of rigid bodies which are connected by various types of visco-elastic system composed of some springs and dashpots. It is extension to visco-elastic stress analysis of the rigid bodies-spring model previously proposed by one of authors[1].

1. A New Physical Model in Visco-Elastic Problems

Various types of mechanical model of visco-elasticity which is composed of springs and dashpots are known, for example, Maxwell model, Kelvin (Voigt) model, Boltzmann model, generalized Kelvin model, etc. Whatever types of these mechanical models can be used to derive a new element for visco-elastic stress analysis. Here using Boltzmann model (three-parameter solid) which includes Maxwell and Kelvin model as special cases, new beam bending, plate bending, plane strain, and plane stress elements will be derived.

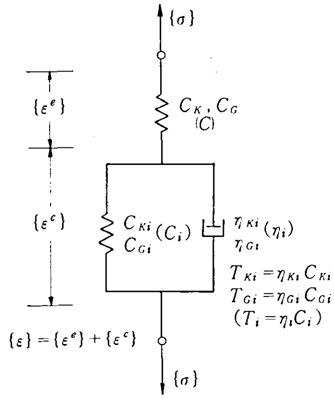
1.1 Beam bending element

One-dimensional Boltzmann model is shown in Fig. 1. It is composed of a spring and a Kelvin element which are connected in series. Stress-strain relation in this model is given as follows:

$$\dot{\sigma} = \frac{1}{C}(\dot{\varepsilon} - \dot{\varepsilon}^c) \tag{1}$$

where

$$\dot{\varepsilon}^c = \frac{\sigma}{\eta_i} - \frac{\varepsilon^c}{T_i} \tag{2}$$



C_K, C_{Ki} : compliances for volumetric deformation
C_G, C_{Gi} : compliances for shearing deformation
η_K, η_{Ki} : viscous coefficients for volumetric deformation
η_G, η_{Gi} : viscous coefficients for shearing deformation
( ) : one dimensional

Fig. 1 Boltzmann model

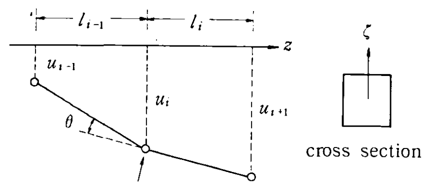


Fig. 2 A new beam bending element for visco-elastic analysis

in which ε^c is the creep strain.

Consider two rigid bars which are connected by rotational Boltzmann model as shown in Fig. 2. Relation between bending moment and relative rotational angle is expressed as follows:

$$\dot{M} = \frac{1}{C_b}(\dot{\theta} - \dot{\theta}^c) \tag{3}$$

where

$$\dot{\theta}^c = \frac{M}{\eta_{bi}} - \frac{\theta^c}{T_{bi}} \tag{4}$$

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In this model the bending strain is approximated by the finite difference formula as follows :

$$\varepsilon = \zeta \frac{\theta}{\frac{1}{2}(l_{i-1}+l_i)} \tag{5}$$

From Eq.(1) and Eq.(2) with the aid of Eq. (5), three parameters in Eq.(3) and Eq.(4) can be determined as

$$\left.\begin{aligned} \frac{1}{C_b} &= \frac{I}{\frac{1}{2}(l_{i-1}+l_i)C} \\ \frac{1}{C_{bi}} &= \frac{I}{\frac{1}{2}(l_{i-1}+l_i)C_i} \\ \eta_{bi} &= \frac{\eta_i I}{\frac{1}{2}(l_{i-1}+l_i)} \end{aligned}\right\} \tag{6}$$

The virtual work equation is expressed as

$$\begin{aligned} &\delta\dot{\theta}\cdot\dot{M} - \{\delta\dot{u}\}^t\{\dot{F}\} \\ &= \delta\dot{\theta}\frac{1}{C_b}(\dot{\theta}-\dot{\theta}^c) - \{\delta\dot{u}\}^t\{\dot{F}\} = 0 \end{aligned} \tag{7}$$

From Eq.(7) the following equilibrium equation is derived :

$$[k(1/C_b)]\{\dot{u}\} = \{\dot{F}\} + \frac{1}{C_b}\dot{\theta}^c\{R\} \tag{8}$$

where the following relation is used :

$$\left.\begin{aligned} &\theta = \lfloor R \rfloor \{u\} \\ &\lfloor R \rfloor = \left\lfloor \frac{1}{l_{i-1}} \; -\left(\frac{1}{l_{i-1}}+\frac{1}{l_i}\right) \; \frac{1}{l_i} \right\rfloor \\ &\{u\}^t = \lfloor u_{i-1} \; u_i \; u_{i+1} \rfloor \end{aligned}\right\} \tag{9}$$

Eq.(3), Eq.(4), and Eq.(8) are the basic equations for visco-elastic stress analysis of beam bending. $[k(1/C_b)]$ in Eq.(8) is the element stiffness matrix given previously in Ref. 2).

1. 2 Plane strain element

Two-dimensional Boltzmann model is shown in Fig. 1. Stress-strain relation in generalized plane strain condition is given as follows: (See Ref. 5).)

$$\{\dot{\sigma}\} = [D^e]\{\dot{\varepsilon}\} + [Q]\{\varepsilon^c\} - [P]\{\sigma\} \tag{10}$$

Consider two rigid triangular plates which are connected by three different types of one-dimensional Boltzmann models as shown in Fig. 3. Relation between internal-forces and relative displacements is expressed as follows:

$$\{\dot{S}\} = [C]^{-1}(\{\dot{d}\} - \{\dot{d}^c\}) \tag{11}$$

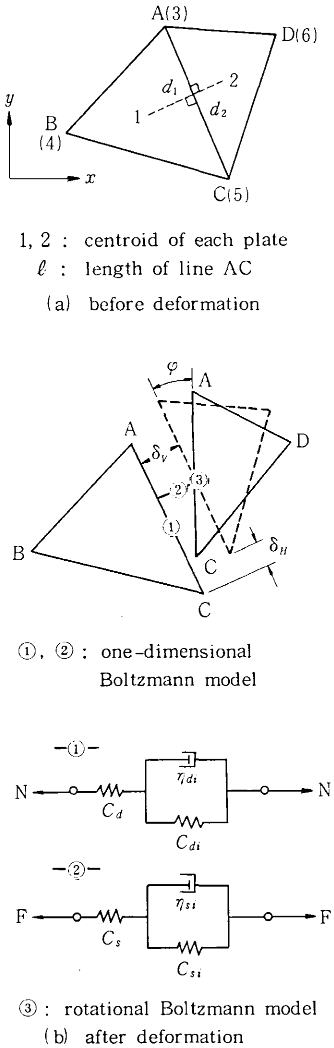


Fig. 3 A new plane element for visco-elastic analysis

$$\{\dot{d}^c\} = [\eta_i]^{-1}\{S\} - [\eta_i]^{-1}[C_i]^{-1}\{d^c\} \tag{12}$$

where

$$\left.\begin{aligned} &\{S\}^t = \lfloor N\,F\,M \rfloor \qquad [C_i] = \begin{bmatrix} C_{di} & & \text{sym.} \\ 0 & C_{si} & \\ 0 & 0 & C_{ri} \end{bmatrix} \\ &\{d\}^t = \lfloor \delta_V \, \delta_H \varphi \rfloor \\ &[C] = \begin{bmatrix} C_d & & \text{sym.} \\ 0 & C_s & \\ 0 & 0 & C_r \end{bmatrix} \quad [\eta_i] = \begin{bmatrix} \eta_{di} & & \text{sym.} \\ 0 & \eta_{si} & \\ 0 & 0 & \eta_{ri} \end{bmatrix} \end{aligned}\right\} \tag{13}$$

In this model the normal strain ε_d and shearing strain γ may be given by the following finite difference formulae:

$$\varepsilon_d = \frac{\delta_V}{d_1+d_2} \qquad \gamma = \frac{2\,\delta_H}{d_1+d_2} \tag{14}$$

From Eq.(10) in which $\dot{\varepsilon}_z=0$ is assumed with the aid of Eq.(14), nine parameters in Eq.(11) and Eq.(12) can be determined as

$$\left.\begin{aligned}
&\frac{1}{C_d}=\frac{l(3C_G+4C_K)}{3(d_1+d_2)C_K C_G}\\
&\frac{1}{C_{di}}=\frac{l(3C_G T_{Gi}+4C_K T_{Ki})}{(d_1+d_2)(C_G C_{Ki} T_{Gi}+2C_K C_{Gi} T_{Ki})}\\
&\eta_{di}=\frac{l(3C_G+4C_K)\eta_{Gi}\eta_{Ki}}{(d_1+d_2)(C_G\eta_{Gi}+2C_K\eta_{Ki})}\\
&\frac{1}{C_s}=\frac{2l}{(d_1+d_2)C_G} \qquad \frac{1}{C_r}=\frac{l^2}{12C_d}\\
&\frac{1}{C_{si}}=\frac{2l}{(d_1+d_2)C_{Gi}} \qquad \frac{1}{C_{ri}}=\frac{l^2}{12C_{di}}\\
&\eta_{si}=\frac{2l\eta_{Gi}}{(d_1+d_2)} \qquad \eta_{ri}=\frac{l^2\eta_{di}}{12}
\end{aligned}\right\} \quad (15)$$

The virtual work equation is expressed as follows:

$$\{\delta\dot{d}\}^t\{\dot{S}\}-\{\delta\dot{u}\}^t\{\dot{F}\}=\{\delta\dot{d}\}^t[C]^{-1}(\{\dot{d}\}-\{\dot{d}^c\})-\{\delta\dot{u}\}^t\{\dot{F}\}=0 \quad (16)$$

From Eq.(16) the following equilibrium equation is derived:

$$[k(1/C_d,\ 1/C_s,\ 1/C_r)]\{u\}=\{F\}+[R]^t[C]^{-1}\{d^c\} \quad (17)$$

where the following relation is used:

$$\left.\begin{aligned}
&\{d\}=[R]\{u\}\\
&[R]=\frac{1}{2l}\\
&\begin{bmatrix}
2y_{53}-2x_{53} & x_{53}(x_{31}+x_{51})+y_{53}(y_{31}+y_{51}) & -2y_{53} & 2x_{53} & -x_{53}(x_{32}+x_{52})-y_{53}(y_{32}+y_{52})\\
-2x_{53}-2y_{53} & -x_{53}(y_{31}+y_{51})+y_{53}(x_{31}+x_{51}) & 2x_{53} & 2y_{53} & x_{53}(y_{32}+y_{52})-y_{53}(x_{32}+x_{52})\\
0\quad 0 & 2l & 0 & 0 & -2l
\end{bmatrix}\\
&\{u\}^t=\lfloor u_1\ v_1\ \theta_1\ u_2\ v_2\ \theta_2\rfloor
\end{aligned}\right\} \quad (18)$$

Eq.(11), Eq.(12), and Eq.(17) are the basic equations for visco-elastic stress analysis of plane strain problems. $[k(1/C_d,\ 1/C_s,\ 1/C_r)]$ in Eq.(17) is the element stiffness matrix given previously in Ref. 3).

1.3 Plane stress element

The basic equations are the same as in case of plane strain problems except values of the following three parameters

$$\left.\begin{aligned}
&\frac{1}{C_d}=\frac{4l(3C_G+C_K)}{(d_1+d_2)C_G(3C_G+4C_K)}\\
&\frac{1}{C_{di}}=\\
&\frac{2l\eta_{Ki}\eta_{Gi}\{3C_G(T_{Gi}+T_{Ki})+2C_K T_{Ki}\}}{(d_1+d_2)T_{Ki}T_{Gi}\{2(C_G\eta_{Gi}+C_K\eta_{Ki})+3C_G\eta_{Ki}\}}\\
&\eta_{di}=\frac{4l\eta_{Ki}\eta_{Gi}(3C_G+C_K)}{(d_1+d_2)\{2(C_G\eta_{Gi}+C_K\eta_{Ki})+3C_G\eta_{Ki}\}}
\end{aligned}\right\} \quad (19)$$

These values are obtained from Eq.(10) in which $\dot{\sigma}_z=0$ is assumed with the aid of Eq.(14)

1.4 Plate bending element

Consider two rigid triangular plates which are connected by one-dimensional rotational Boltzmann model as shown in Fig. 4. Relation between bending moment and relative rotational angle is given by Eq.(3) and Eq.(4). In this model the bending strain is approximated by the following finite difference formula:

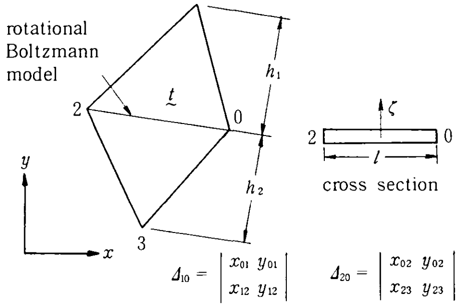


Fig. 4 A new plate bending element for visco-elastic analysis

$$\varepsilon=\zeta\frac{\theta}{\frac{1}{2}(h_1+h_2)} \quad (20)$$

From Eq.(10) in which $\dot{\sigma}_z=0$ is assumed with the aid of Eq.(20) three parameters in Eq.(3) and Eq.(4) can be determined as

$$\left.\begin{aligned}
&\frac{1}{C_b}=\frac{2l(3C_G+C_K)t^3}{3(h_1+h_2)C_G(3C_G+4C_K)}\\
&\frac{1}{C_{bi}}=\\
&\frac{l\eta_{Ki}\eta_{Gi}\{3C_G(T_{Gi}+T_{Ki})+2C_K T_{Ki}\}t^3}{3(h_1+h_2)T_{Ki}T_{Gi}\{2(C_G\eta_{Gi}+C_K\eta_{Ki})+3C_G\eta_{Ki}\}}\\
&\eta_{bi}=\frac{2l\eta_{Ki}\eta_{Gi}(3C_G+C_K)t^3}{3(h_1+h_2)\{2(C_G\eta_{Gi}+C_K\eta_{Ki})+3C_G\eta_{Ki}\}}
\end{aligned}\right\} \quad (21)$$

From the virtual work equation the following equilibrium equation is derived:

$$[k(1/C_b)]\{\dot{w}\}=\{\dot{F}\}+\frac{1}{C_b}\dot{\theta}^c\{R\} \quad (22)$$

where the following relation is used:

$$\theta=[R]\{w\}$$

$$[R]=\left\lfloor\frac{x_{02}x_{12}+y_{02}y_{12}}{l\Delta_{10}}\quad -\frac{x_{02}x_{23}+y_{02}y_{23}}{l\Delta_{20}}\quad -\frac{l}{\Delta_{10}}\right.$$

$$\left.\begin{array}{l}\dfrac{x_{01}x_{02}+y_{01}y_{02}}{l\,\Delta_{10}}+\dfrac{x_{02}x_{03}+y_{02}y_{03}}{l\,\Delta_{20}}\quad-\dfrac{l}{\Delta_{20}}\rfloor \\ \{w\}^t=\lfloor w_0\ w_1\ w_2\ w_3\ \rfloor\end{array}\right\}\quad(23)$$

2. Time Stepping Algorithm

In general a discretized system of ordinary differential equations in visco-elastic stress analysis is given by the following three equations:

$$[K]\{\dot{u}\}=\{\dot{F}\}+\{F_v(\dot{\varepsilon}^c)\}\quad(24)$$

$$\{\dot{\sigma}\}=\{\dot{\sigma}(\dot{u}\,,\dot{\varepsilon}^c)\}\quad(25)$$

$$\{\dot{\varepsilon}^c\}=\{\dot{\varepsilon}^c(\sigma\,,\varepsilon^c)\}\quad(26)$$

Using Eq.(24), Eq.(25), and Eq.(26) solution can be obtained in a time stepping manner. The procedure is as follows:

(1) Starting from known values of $\{\sigma\}_n$, $\{\varepsilon^c\}_n$, $\{u\}_n$, and $\{F\}_n$ at $t=t_n$, compute the rate of creep strain $\{\dot{\varepsilon}^c\}_n$ by Eq.(26). Using Eq.(24). the velocity $\{\dot{u}\}_n$ can be determined as

$$\{\dot{u}\}_n=[K](\{\dot{F}\}_n+\{F_v\}_n)\quad(27)$$

Compute the rate of stress $\{\dot{\sigma}\}_n$ by Eq.(25).

(2) Determine approximately the corresponding values at $t=t_{n+1}$ as

$$\left.\begin{array}{l}\{\sigma\}_{n+1}\simeq\{\sigma\}_n+\{\dot{\sigma}\}_n\Delta t\\ \{\varepsilon^c\}_{n+1}\simeq\{\varepsilon^c\}_n+\{\dot{\varepsilon}^c\}_n\Delta t\end{array}\right\}\quad(28)$$

where

$$\Delta t=t_{n+1}-t_n\quad(29)$$

(3) Using the predicted values in (2), compute $\{\dot{\varepsilon}^c\}_{n+1}$ by Eq.(26). From Eq.(24) $\{\dot{u}\}_{n+1}$ can be determined as

$$\{\dot{u}\}_{n+1}=[K](\{\dot{F}\}_{n+1}+\{F_v\}_{n+1})\quad(30)$$

Compute $\{\dot{\sigma}\}_{n+1}$ by Eq.(25)

(4) Improve the corresponding values at $t=t_{n+1}$ as

$$\left.\begin{array}{l}\{\sigma\}_{n+1}\simeq\{\sigma\}_n+(\{\dot{\sigma}\}_n+\{\dot{\sigma}\}_{n+1})\Delta t/2\\ \{\varepsilon^c\}_{n+1}\simeq\{\varepsilon^c\}_n+(\{\dot{\varepsilon}^c\}_n+\{\dot{\varepsilon}^c\}_{n+1})\Delta t/2\\ \{u\}_{n+1}\simeq\{u\}_n+(\{\dot{u}\}_n+\{\dot{u}\}_{n+1})\Delta t/2\end{array}\right\}\quad(31)$$

3. Numerical Examples of Some Simple Problems

As for convergency test of new discrete elements, numerical stationary and nonstationary creep bending of a simply-supported beam was conducted. Fig. 5 shows result of numerical analysis on the stationary creep bending of a beam, from which convergency of numerical solutions can be seen. Fig. 6 shows result of numerical analysis on the nonstationary creep bending. It can be seen that 16 elements approximate solution almost coincides with the exact solution. These two results may justify use of new elements to analysis of visco-elastic problems of solids.

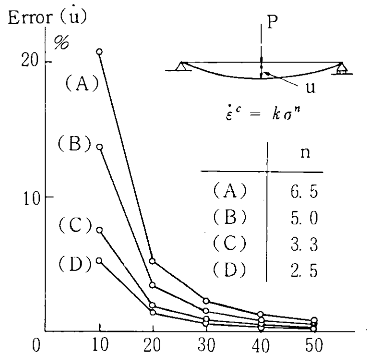


Fig. 5 Nonlinear stationary creep bending of a beam

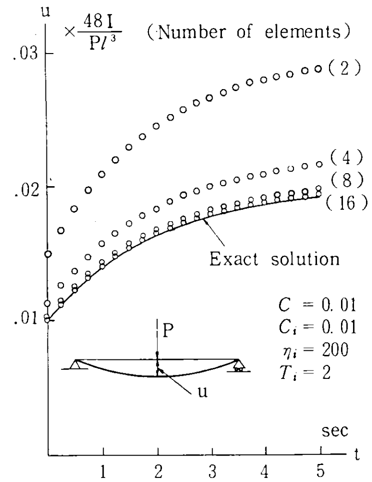


Fig. 6 Nonstationary creep bending of a beam

4. Conclusion

A family of new elements for visco-elastic stress analysis is proposed in this note. Size of stiffness matrices of these elements is equal to or even smaller than 1/2 of those of conventional finite elements and therefore considerable reduction of computing time can be expected.

These elements will effectively be applied to creep analysis of various types of materials including nonlinear and dynamic effects. Numerical Analysis of three as well as two dimensional creep problems is now under way. (Manuscript received, May 16, 1977)

— continued on p. 12 —

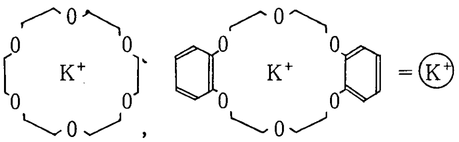
り向上しない．

活性メチレン化合物のアルキル化反応をPTC系で行った例は，これまで多数報告されている[3]．これらの場合，アルキル化されるのは，(9)式に示すように2個の活性基

$$YCH_2Z + RX \longrightarrow YCH(R)Z \qquad (9)$$

$$Y,\ Z = -Ph,\ {>}C{=}O,\ {>}SO_2,\ {>}P{=}O,\ -CN,\ -S-$$

にはさまれたメチレンまたはメチングループに限られていた．すなわち，十分大きい酸性度を有するプロトンを持つ炭素のみが，効率良くアルキル化されることを示しているが，本研究に示したように，単純ケトンのアルキル化も適当な条件下では可能であることがわかる．



$$(KOH)_s + (\bigcirc)_l \rightleftharpoons (\textcircled{K}^{+}\ OH^-)_l \qquad (10)$$

$$(\textcircled{K}^{+}\ Cl^-)_l + (KOH)_s \longrightarrow (\textcircled{K}^{+}\ OH^-)_l + (KCl^-)_s \qquad (11)$$

表1に示したように，18－クラウン－6および15－クラウン－5は，KOHまたはNaOH水溶液を用いる二液相間反応では，ほとんど触媒効果を示さないが，固体KOH－有機相間反応では有効な触媒として作用する．この場合には，クラウンエーテルのカリウムイオン錯体（上図）が，アンモニウムイオンと同様な作用を行い，(3)および(4)と類似の反応を行い，最終的には(11)式の反応でサイクルを完成するものと推定される．

以上にのべたように，比較的安価なアセトンおよびイソプレンを原料として，温和な条件下でMHを合成することが可能であることが示されたが，先にのべたような副反応の抑制による収率の向上が残された課題である．また，ケトンのアルキル化には，従来 $n-BuLi$，$NaNH_2$，NaH等の強塩基が用いられ，したがって湿気のしゃ断が必要とされていたが，一般にNaOH水溶液を用いるPTC反応が可能であれば，合成上便利な方法になると考えられるため，さらに一般的な反応への応用を検討している．

（1977年3月23日受理）

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