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A New Element in Discrete Analysis of Plane Strain Problems

平面歪問題の離散化解析に対する新しい要素

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Summary

Recently one of the author has proposed a new physical model for bending analysis of beam and plate structures. The same idea is now extended to finite element analysis of the plane strain problems. This element consists of rigid plates connected with various types of springs and size of the stiffness matrix of this element is only 3×3 and therefore considerable reduction of computational time can be expected in nonlinear analysis of plane strain problems.

1. A New Physical Model in Plane Strain Problem

Consider two rigid plates which are connected by three different types of springs k_d, k_s, and k_r as shown in Fig. 1.

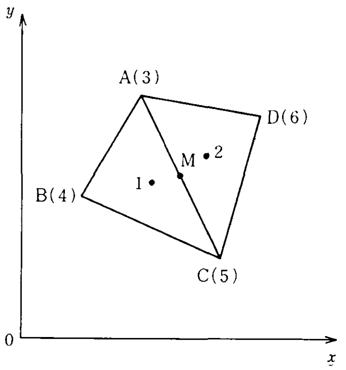

Fig.1 A New plane strain element

Centroidal displacements of each plate is denoted by (u_1, v_1, θ_1) and (u_2, v_2, θ_2) respectively.

The displacements of an arbitrary point in ΔABC can be given by the following equation

$$\left.\begin{aligned} u &= u_1 + (y - y_1)\,\theta_1 \\ v &= v_1 - (x - x_1)\,\theta_1 \end{aligned}\right. \qquad (1)$$

where the rotational displacement θ_1 is assumed to be very small. After some loading, two plates are displaced to positions as shown in Fig. 2.

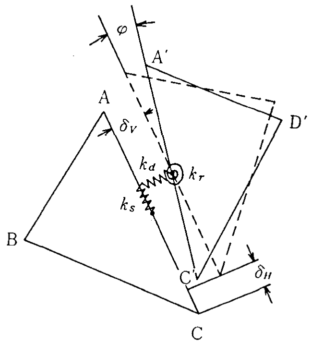

Fig.2 Two plates positions after deformation

The relative displacement of the edge $\overline{AC}$ in each plate is given by δ_V, δ_H, and φ. Coordinates of the mid point M of the edge $\overline{AC}$ is given by

$$\left(\frac{1}{2}(x_3 + x_5),\ \frac{1}{2}(y_3 + y_5)\right).$$

This point will be displaced to M' in ΔABC and M" in ΔACD respectively after loading.

Coordinates of these points are given as follows:

$$\left.\begin{aligned} x_{M'} &= \frac{1}{2}\{(x_3 + x_5) + 2u_1 + (y_{31} + y_{51})\,\theta_1\} \\ y_{M'} &= \frac{1}{2}\{(y_3 + y_5) + 2v_1 - (x_{31} + x_{51})\,\theta_1\} \\ x_{M''} &= \frac{1}{2}\{(x_3 + x_5) + 2u_2 + (y_{32} + y_{52})\,\theta_2\} \\ y_{M''} &= \frac{1}{2}\{(y_3 + y_5) + 2v_2 - (x_{32} + x_{52})\,\theta_2\} \end{aligned}\right\} \qquad (2)$$

where $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$

Therefore a vector $\overrightarrow{M'M''}$ is given by the following equations:

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$$\left.\begin{aligned}\overrightarrow{(M'M'')}_x &= \frac{1}{2}\{\,2u_{21} + (y_{32}+y_{52})\theta_2 \\ &\quad -(y_{31}+y_{51})\theta_1\} \\ \overrightarrow{(M'M'')}_y &= \frac{1}{2}\{\,2v_{21} - (x_{32}+y_{52})\theta_2 \\ &\quad +(x_{31}+x_{51})\theta_1\}\end{aligned}\right\} \quad (3)$$

again the following notations are employed.

$u_{ij} = u_i - u_j, \quad v_{ij} = v_i - v_j$

Denoting an unit vector along the edge $\overline{AC}$ by $\boldsymbol{t}$ before deformation as shown in Fig. 1 displacement component δ_H of the vector $\overrightarrow{M'M''}$ along the edge $\overline{AC}$ can can be given as follows:

$$\begin{aligned}\delta_H = (\overrightarrow{M'M''},\ \boldsymbol{t}) = \frac{1}{2\,l_{35}}\,[\,& x_{53}\{\,2u_{21} \\ & +(y_{32}+y_{53})\,\theta_2 - (y_{31}+y_{51})\,\theta_1\} \\ & + y_{53}\{\,2v_{21} - (x_{32}+x_{52})\,\theta_2 \\ & + (x_{31}+x_{51})\,\theta_1\}\,]\end{aligned} \quad (4)$$

Similarly the displacement component δ_V of the vector $\overrightarrow{M'M''}$ perpendicular to the edge $\overline{AC}$ can be given by the following equation:

$$\begin{aligned}\delta_V^2 = |\,\boldsymbol{t}\times\overrightarrow{M'M''}\,|^2 = \frac{1}{4\,l_{35}^2}\,[\,& x_{53}\{\,2v_{21} \\ & -(x_{32}+x_{52})\,\theta_2 + (x_{31}+x_{51})\theta_1\} \\ & - y_{53}\{\,2u_{21} + (y_{32}+y_{52})\,\theta_2 \\ & -(y_{31}+y_{51})\,\theta_1\}\,]^2\end{aligned} \quad (5)$$

Relative angle change φ of the edges $\overline{AC}$ and $\overline{A'C'}$ is also obtained from following equation:

$$\cos\varphi = (\boldsymbol{t},\ \boldsymbol{t}') = 1 - \frac{\varphi^2}{2}$$

where $\boldsymbol{t}'$ is the unit vector along the edge $\overline{A'C'}$ after deformation

$$\begin{aligned}\frac{\varphi^2}{2} = \frac{1}{l_{35}^2}\,[\,& (u_{53}+u'_{53})\,x_{53} + (v_{53}+v'_{53})\,y_{53} \\ & + u_{53}u'_{53} + v_{53}v'_{53}\,]\end{aligned} \quad (6)$$

Now strain energy V to be stored in the spring k_d, k_s and k_r after deformation will be given as follows:

$$V = \frac{1}{2}k_d\,\delta_V^2 + \frac{1}{2}k_s\,\delta_H^2 + \frac{1}{2}k_r\,\varphi^2 \quad (7)$$

In view of (5), (6), (7) and (8) it is clearly seen that the strain energy V is a quadratic function of u_1, v_1, θ_1) and (u_2, v_2, θ_2) and therefore applying Castigliano's theorem, the following reaction force vector $\boldsymbol{R}$ can be derived.

$$\boldsymbol{R} = \frac{\partial V}{\partial \boldsymbol{u}} = \boldsymbol{K}\boldsymbol{u} \quad (8)$$

where $\boldsymbol{K}$ is the stiffness matrix to be obtained and $\boldsymbol{u}$ is the displacement vector given as follows:

$$\boldsymbol{u}^T = \lfloor u_1\ v_1\ \theta_1\ u_2\ v_2\ \theta_2 \rfloor \quad (9)$$

Table 1.

	u_1	v_1	θ_1	u_2	v_2	θ_2
X_1	$k_d y_{53}^2 + k_s x_{53}^2$					
Y_1	$-(k_d - k_s) x_{53} y_{53}$	$k_d x_{53}^2 + k_s y_{53}$				
M_1	$k_d y_{53}\varDelta_{11} - k_s x_{53}\varDelta_{21}$	$-(k_d x_{53}\varDelta_{11} + k_s y_{53}\varDelta_{21})$	$k_d\varDelta_{11}^2 + k_s\varDelta_{21}^2 + k_r l_{35}^2$		SYM.	
X_2	$-(k_d y_{53}^2 + k_s x_{53}^2)$	$(k_d - k_s)\,x_{53}y_{53}$	$-(k_d y_{53}\varDelta_{11} - k_s x_{53}\varDelta_{21})$	$k_d y_{53}^2 + k_s x_{53}^2$		
Y_2	$(k_d - k_s)\,x_{53}y_{53}$	$-(k_d x_{53}^2 + k_s y_{53}^2)$	$k_d x_{53}\varDelta_{11} + k_s y_{53}\varDelta_{21}$	$-(k_d - k_s)\,x_{53}y_{53}$	$k_d x_{53}^2 + k_s y_{53}^2$	
M_2	$k_d y_{53}\varDelta_{22} - k_s x_{53}\varDelta_{12}$	$-(k_d x_{53}\varDelta_{22}\ k_s y_{53}\varDelta_{12})$	$k_d\varDelta_{11}\varDelta_{22} + k_s\varDelta_{21}\varDelta_{12} - k_r l_{35}^2$	$-(k_d y_{53}\varDelta_{22} - k_s x_{53}\varDelta_{12})$	$k_d x_{53}\varDelta_{22} + k_s y_{53}\varDelta_{12}$	$k_d\varDelta_{22}^2 + k_s\varDelta_{12}^2 + k_r l_{35}^2$

$$\left.\begin{aligned}2\varDelta_{11} &= x_{53}(x_{31}+x_{51}) + y_{53}(y_{31}+y_{51}) \\ 2\varDelta_{12} &= x_{53}(y_{32}+y_{52}) - y_{53}(x_{32}+x_{52}) \\ 2\varDelta_{21} &= -x_{53}(y_{31}+y_{51}) + y_{53}(x_{31}+x_{51}) \\ 2\varDelta_{22} &= -x_{53}(x_{32}+x_{52}) - y_{53}(y_{32}+y_{52})\end{aligned}\right\}$$

The final form of the stiffness matrix is given in the Table 1.

Spring constant k_d, k_s and k_r are determined in the following way. Considering two plates shown in Fig. 3 the normal strains ε_d and shearing strain γ in these springs may be given by the following formulae.

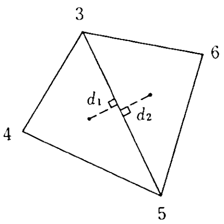

Fig. 3　Determination of the spring constant

$$\varepsilon_d = \frac{\delta_V}{d_1+d_2} = \frac{(1+\nu)(1-2\nu)\sigma_n}{E(1-\nu)}$$

$$= \frac{(1+\nu)(1-2\nu)k_d\delta_V}{E(1-\nu)l_{35}}$$

$$\gamma = \frac{\delta_H}{d_1+d_2} = \frac{\tau_n}{2G} = \frac{(1+\nu)k_s\delta_H}{E\,l_{35}}$$

or

$$\left.\begin{aligned} k_d &= \frac{E(1-\nu)l_{35}}{(1+\nu)(1-2\nu)(d_1+d_2)} \\ k_s &= \frac{E\,l_{35}}{(1+\nu)(d_1+d_2)} \end{aligned}\right\} \quad (10)$$

And k_r can be determined as follows:

The rotational moment M_φ of the spring k_r is given by the following equation:

$$M_\varphi = \int_{-\frac{l_{35}}{2}}^{\frac{l_{35}}{2}} k_d(s\varphi)s\,ds = k_r\varphi$$

$$\therefore\ k_r = \frac{k_d l_{35}^2}{12} \quad (11)$$

2. Some numerical examples on the elasto-plastic plane strain problems

To show validty of this new element, elasto-plastic analysis of the punch problem and a slit notch specimen under tensile load are made.

(1) Punch problem

Punch problem of an elasto-plastic slab as shown in Fig. 4 is considered.

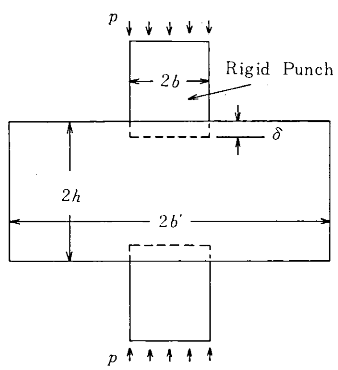

Fig. 4　Plane Strain Punch Problem

This problem is a standard plane strain problem in plasticity and it was studied by many investigators for various cases of h/b ratios using the so-called "slip line theory".

In present analysis a given material is ideal plastic and the maximum shearing stress theory is used as the yield criterion. Since the material is assumed incompressible after yielding, the plastic strain increment is purely shearing deformation. Analysis was made for three different cases of h/b ratios as follows:

(1) h/b = 1　　(2) h/b = 2　　(3) h/b = 1

The load deformation curves obtained. assumed mesh division and slip lines as well as computing time (w. r. t. "HITAC 8700−8800 approximately comparable to IBM360−195) are shown in Fig. 5, 6, and 7.

It should be mentioned here that the rotational component was neglected in this analysis so that the size of stiffness matrices used were only 2×2 and yet the ultimate loads and slip lines obtained

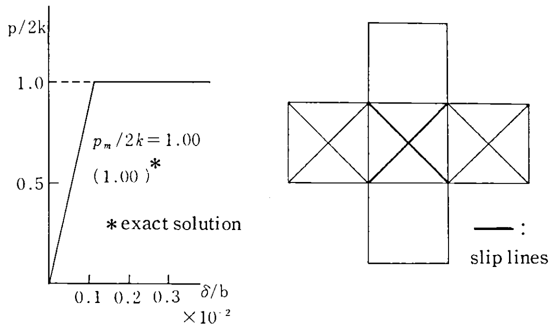

Fig. 5　Punch Problem (1) h/b = 1

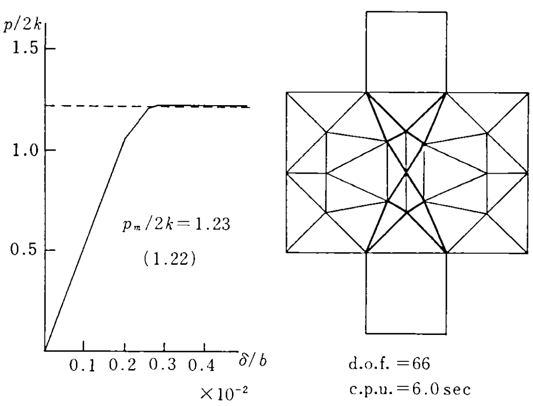

Fig. 6　Punch Problem (2) h/b = 2

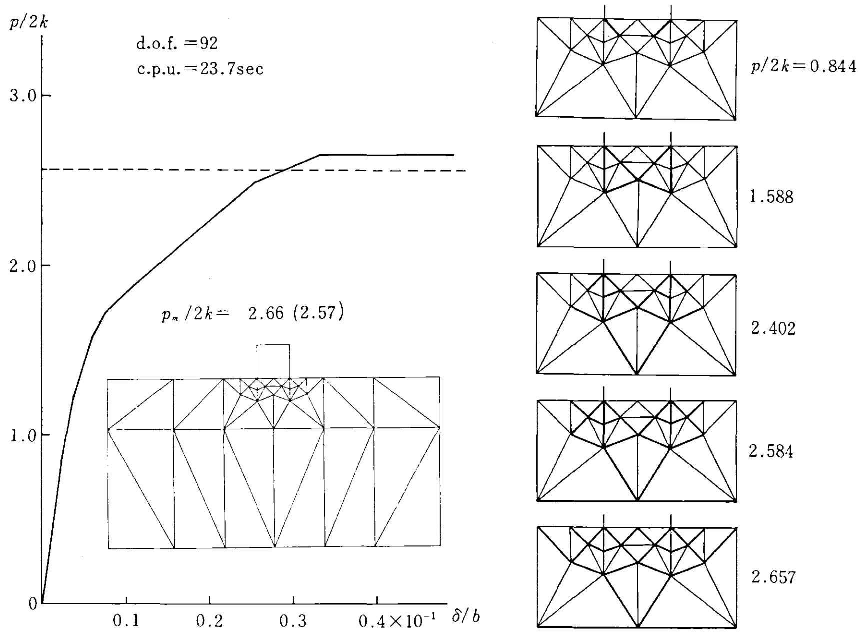

Fig.7 Punch Problem (3) h/b ≥ 8.74

were in good agreement with the results obtained by previous authors.

(2) Elasto-plastic analysis of a slit notch tensile specimen

Analysis was made for the specimen as shown in Fig.8. Elasto–plastic incremenal analysis was made under the assumption of no rotational displacement and the maximum shearing stress theory. The result obtained was extremely in good agreement with the result of previous investigators.

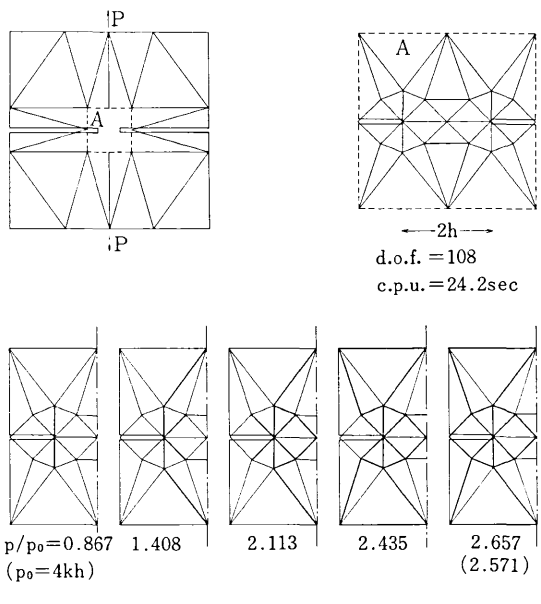

Fig.8 Elasto–plastic analysis ofaa slit notch tensile specimen

3. Conclusion

A new element suitable for nonlinear analysis of plane strain problems is proposed in this paper. Results of numerical analysis on some simple problems duly justified use of this element for elasto-plastic analysis of complex plane strain problems. The authors would like to express their thanks to Messrs. K. Kondou and M. Watanabe for their valuable discussions.

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