研 究 速 報

A New Discrete Model for Analysis of Solid Mechanics Problems

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固体力学における新しい離散化モデル

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Summary

A new discrete model for analysis of various problems in solid mechanics is proposed in this note. This model consists of three dimensional rigid bodies with connection springs distributed over the contact area of two neighbouring bodies. It is generalization of the rigid bars or plates-spring models previously proposed by the present author. (1)

1. Theoretical Basis

Consider a set of three dimensional rigid bodies of arbitrary shape as shown in Fig. 1. They are assumed in equilibrium with external loads, and reaction forces are distributed over the spring system on the contact surface of two adjacent bodies. Taking such two rigid bodies under contact, deformation of the spring system is considered. (Fig. 2)

Displacement $\boldsymbol{u}$ of an arbitrary point in a rigid body can be given by the following vectorial equation:

$$\boldsymbol{u}=\boldsymbol{u}_G+\boldsymbol{O}\times(\boldsymbol{r}-\boldsymbol{r}_G) \quad (1)$$

where $\boldsymbol{u}_G$ is displacement vector of the centroid, $\boldsymbol{O}$ is the rotation vector and $\boldsymbol{r}$, $\boldsymbol{r}_G$ are position vectors of arbitrary point and centroid before deformation respectively, i. e.

$$\boldsymbol{u}_G=(u_g,\ v_g,\ w_g) \qquad \boldsymbol{O}=(\theta,\ \varphi,\ \chi) \quad (2)$$

Denoting the displacement vectors of arbitrary point P (x, y, z) in body (I) and (II) by $\boldsymbol{u}'$, $\boldsymbol{u}''$, they are given by the following equations:

$$\left.\begin{aligned}\boldsymbol{u}'&=\boldsymbol{u}_1+\boldsymbol{O}_1\times(\boldsymbol{r}-\boldsymbol{r}_1)\\ \boldsymbol{u}''&=\boldsymbol{u}_2+\boldsymbol{O}_2\times(\boldsymbol{r}-\boldsymbol{r}_2)\end{aligned}\right\} \quad (3)$$

More precisely,

$$\left.\begin{aligned}u'&=u_1+(z-z_1)\phi_1-(y-y_1)\chi_1\\ v'&=v_1+(x-x_1)\chi_1-(z-z_1)\theta_1\\ w'&=w_1+(y-y_1)\theta_1-(x-x_1)\phi_1\end{aligned}\right\} \quad (4\text{-a})$$

$$\left.\begin{aligned}u''&=u_2+(z-z_2)\phi_2-(y-y_2)\chi_2\\ v''&=v_2+(x-x_2)\chi_2-(z-z_2)\theta_2\\ w''&=w_2+(y-y_2)\theta_2-(x-x_2)\phi_2\end{aligned}\right\} \quad (4\text{-b})$$

Therefore denoting the point P after displacement in bodies (I) and (II) by P′ and P″, the relative displacement vector of the point P can be defined as follows:

$$\overrightarrow{P'P''}=\boldsymbol{u}''-\boldsymbol{u}' \quad (5)$$

Defining the unit normal drawn outward to

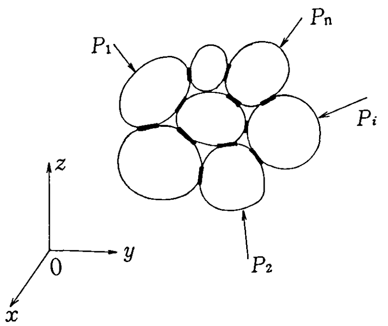

Fig. 1 Many discrete rigid bodies under contact

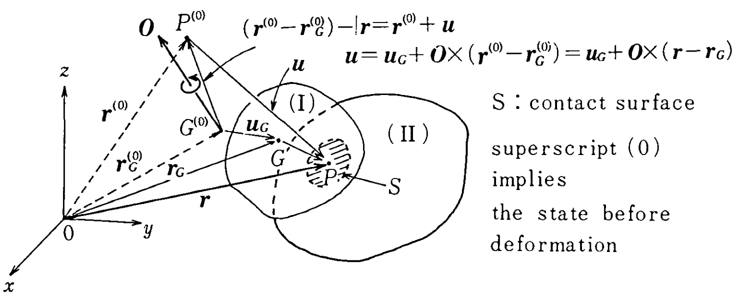

Fig. 2 Definition of rigid bodies-spring system

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the contact surface at the point P by $\boldsymbol{n}$, (See Fig. 3) the normal displacement δ_d to the surface S can be given as follows :

$$\delta_d=(\overrightarrow{P'P''},\ \boldsymbol{n})$$
$$=l(u''-u')+m(v''-v')+n(w''-w') \qquad (6-a)$$

where

$$\boldsymbol{n}=(l,\ m,\ n) \qquad (6-b)$$

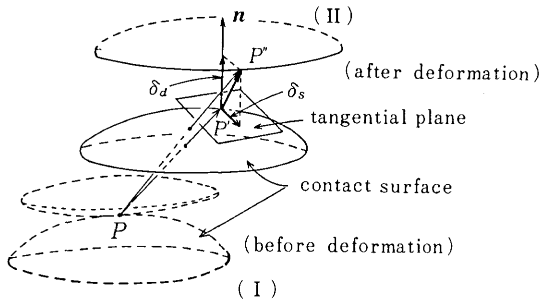


Fig. 3 Relative displacements of P on the contact surface in bodies (I) and (II)

Similarly the displacement component δ_s in the tangential plane to the surface can be given by the following equation:

$$\delta_s^2=|\boldsymbol{n}\times\overrightarrow{P'P''}|^2$$
$$=\{m(w''-w')-n(v''-v')\}^2$$
$$+\{n(u''-u')-l(w''-w')\}^2$$
$$+\{l(v''-v')-m(u''-u')\}^2 \qquad (7)$$

Basing on the above preliminaries, strain energy due to the relative displacements (δ_d, δ_s) to be stored in the spring system which is distributed over the contact surface S can be given by the following equation :

$$V=\frac{1}{2}\int(k_d\delta_d^2+k_s\delta_s^2)\,dS \qquad (8)$$

In view of eqs. (6) and (7), the strain energy V is a quadratic function of the displacement vector $\boldsymbol{u}$ of the centroids in bodies (I) and (II) as follows :

$$V(\boldsymbol{u})=\frac{1}{2}\boldsymbol{u}^T\boldsymbol{k}\boldsymbol{u} \qquad (9-a)$$

$$\boldsymbol{u}^T=\lfloor u_1,\ v_1,\ w_1,\ \theta_1,\ \phi_1,\ \chi_1;\ u_2,\ v_2,\ w_2,\ \theta_2,\ \phi_2,\ \chi_2\rfloor \qquad (9-b)$$

Applying Castigliano's theorem, the following stiffness equation can be derived

$$\boldsymbol{R}=\frac{\partial V}{\partial \boldsymbol{u}}=\boldsymbol{K}\boldsymbol{u} \qquad (10-a)$$

where $\boldsymbol{K}$ is a (12 × 12) symmetric matrix given by the following equation :

$$\boldsymbol{K}=[k_{ij}]\quad(i,\ j=1,2,\cdots\cdots12) \qquad (10-b)$$

and $\boldsymbol{R}$ is nodal reaction vector defined by the following equation :

$$\boldsymbol{R}^T=\lfloor X_1,\ Y_1,\ Z_1,\ L_1,\ M_1,\ N_1:\ X_2,\ Y_2,\ Z_2,\ L_2,\ M_2,\ N_2\rfloor \qquad (10-c)$$

2. Derivation of the Stiffness Matrix

For illustration the first low matrix is derived as follows :

$$X_1=\frac{\partial V}{\partial u_1}=\int\left(k_d\delta_d\frac{\partial\delta_d}{\partial u_1}+k_s d_s\frac{\partial\delta_s}{\partial u_1}\right)dS$$
$$=k_{11}u_1+k_{12}v_1+k_{13}w_1+k_{14}\theta_1+k_{15}\phi_1+k_{16}\chi_1$$
$$+k_{17}u_2+k_{18}v_{12}+k_{19}w_2+k_{110}\theta_2+k_{111}\phi_2$$
$$+k_{112}\chi_2 \qquad (11-a)$$

where

$$k_{11}=\int[k_dl^2+k_s(1-l^2)]\,dS=-k_{17}$$
$$k_{12}=\int(k_d-k_s)\,lm\,dS=-k_{18}$$
$$k_{13}=\int(k_d-k_s)\,ln\,dS=-k_{19}$$
$$k_{14}=\int(k_d-k_s)\,l\{n(y-y_1)-m(z-z_1)\}\,dS$$
$$k_{15}=\int[k_dl\{l(z-z_1)-n(x-x_1)\}+k_s\{(1-l^2)(z-z_1)+ln(x-x_1)\}]\,dS$$
$$k_{16}=\int[k_dl\{m(x-x_1)-l(y-y_1)\}-k_s\{(1-l^2)(y-y_1)+lm(x-x_1)\}]\,dS$$
$$k_{110}=\int[k_dl\{m(z-z_2)-n(y-y_2)\}+k_sl\{-m(z-z_2)+n(y-y_2)\}]\,dS$$
$$k_{111}=\int[k_dl\{n(x-x_2)-l(z-z_2)\}+k_s\{-(1-l^2)(z-z_2)+ln(x-x_2)\}]\,dS$$
$$k_{112}=\int[k_dl\{l(y-y_2)-m(x-x_2)\}+k_s\{(1-l^2)(y-y_2)+lm(x-x_2)\}]\,dS$$
$$(11-b)$$

It should be mentioned here that spring constants k_d and k_s can be determined systematically by using the finite difference expressions for strain components [1], and consideration of material nonlinearity can be made easily by introducing pertinent constitutive law of the materials under consideration. The stiffness equation defined by eq. (11) must be obtained for each contact surface if a given rigid body (I) has a number of contact surfaces with other rigid bodies including the body (II), and for equilibrium of a given total system of rigid bodies, they should be summed up and the final form of the stiffness equation can be given as in the standard form of the finite element method

$$\boldsymbol{KU} = \boldsymbol{F} \tag{13}$$

where

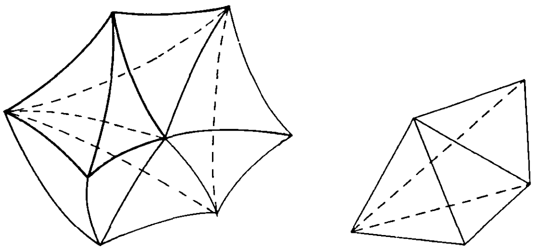
$$\boldsymbol{K} = \sum \boldsymbol{k} \ , \ \boldsymbol{U} = \sum \boldsymbol{u} \ , \ \boldsymbol{F} = \sum \boldsymbol{f} \tag{14}$$

Care must be exercised in constructing eq. (13), because in this method the centroid of each rigid body is selected as the node and therefore superposition of stiffness matrices are somewhat different from that of the standard finite element method.

In case where the body (I) is supported by other bodies through its whole boundary surface S, i.e.

$$S = S_1 + S_2 + \cdots\cdots\cdots + S_m$$

this model is idealization of three dimensional elastic continuum as shown in Fig. 4 in which the shape of each element can be chosen arbitrary.



(a) general form　　(b) standard form

Fig. 4　Tetrahedron elements in three dimensional analysis

The method outlined so far will be called hereafter as the Rigid Bodies–Spring Method (RBSM), or stiffness Lumping Method (SLM). Method). Using this method stress analysis of deformable bodies under contact will be possible in iterative way typical application of which is analysis of the rockfill dam. A series of element matrices are now under development for practical application of the RBS Method to analysis of various problems in solid mechanics. In any element total number of degrees of freedom never exceeds 6 because it is assumed rigid. In case of a beam, deformation consists of axial, bending (about two principal axes) and torsional deformation, and in bending problem effect of shear deformation can be easily taken into account. In case of plate and shell problems, memberane stiffness as well as bending stiffness can be defined again by this (6×6) stiffness matrix.

Furthermore effect of shear deformation on the bending problem and effect of in-plane rotation on the membrane problem can be easily considered. Analysis of three dimensional stress problems including crack problems is now under way.　(Manuscript received January 6, 1977)

Reference

1) T. Kawai and K. Kondou, "New Beam and Plate Bending Elements in Finite Element Analysis", Seisan Kenkyu, Vol. 28, No.9 (September 1976)