フラクタル格子上のパーコレーション

篠　田　正　人

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第1章　概説

1.1　Percolation

この論文では, フラクタル格子 (fractal lattices) におけるパーコレーション (percolation) の問題を考える.

パーコレーションは統計力学の確率モデルとして, 1957 年に Broadbent-Hammerslay [2] によって定式化された. $G = (V, E)$ を連結な無限グラフとする. ここで V は頂点集合, E は V の 2 つの元を結ぶ辺の集合である. V, E は高々可算, また各点から出る辺の本数は有限とする. $[0,1]$ 区間に値を取るパラメータ p を決め, E の元が独立にそれぞれ確率 p で open, 確率 $1-p$ で closed であるとする. open である辺はつながっており自由に通ることが可能であり, closed な辺はつながっておらず通ることが不可能であるとする. 各辺の open-closed が定まると, 頂点 $v \in V$ から open な辺を通って到達可能な点の集合 $C(v)$ が定まる. p が大きいほど, $C(v)$ は大きくなる傾向にあると考えられる. ある点 v を固定し, そこから到達できる点が無限個あるという事象の確率を

$$\theta(p) = P_p(|C(v)| = \infty)$$

と表す. ここで P_p は各辺の open-closed を上記の法則で定める確率測度である. θ は p に関して単調非減少である. 臨界確率 p_c $(= p_c(G))$ を

$$p_c = \inf\{p \mid \theta(p) > 0\}$$

で定める. $p > p_c$ では確率 1 で無限大のクラスターが現れ, $p < p_c$ では確率 1 ですべてのクラスターは有限である. この意味で, $p = p_c$ で相転移が起こると考えられる. このようにパーコレーションは統計力学における相転移を含む簡単なモデルとして盛んに研究されている.

こうした相転移現象を調べるための最初の問題として, $p_c < 1$, すなわち自明でない相転移が起こるかどうか, がある. $\mathbb{Z}^d$ 格子においては, $d \geq 2$ で $p_c(\mathbb{Z}^d) < 1$ であることが知られている. (Grimmett [7] に詳しい.) 以下 $\mathbb{Z}^d$ を例にとる. $\mathbf{0} = (0, 0, \dots, 0)$ とする. p_c の近くで θ やその他の関数, 例えば

$$\chi(p) = E_p(|C(\mathbf{0})|; |C(\mathbf{0})| < \infty),$$

$$\xi(p) = \lim_{n\to\infty}\left\{-\frac{1}{n}\log P_p(\mathbf{0} \leftrightarrow (n, 0, \dots, 0))\right\}$$

($\xi(p)$ は correlation length と呼ばれている.) がどのような振る舞いをしているか, というのが臨界現象の研究である. こうした関数が p_c の近くで

$$\theta(p) \approx (p - p_c)^{\beta} \quad \text{as} \quad p \downarrow p_c,$$

$$\chi(p) \approx |p - p_c|^{-\gamma} \quad \text{as} \quad p \to p_c,$$

$$\xi(p) \approx |p_c - p|^{-\nu} \quad \text{as} \quad p \uparrow p_c, \tag{1.1}$$

$$\frac{E_p(|C(\mathbf{0})|^{k+1}; |C(\mathbf{0})| < \infty)}{E_p(|C(\mathbf{0})|^{k}; |C(\mathbf{0})| < \infty)} \approx |p_c - p|^{-\Delta} \quad \text{as} \quad p \to p_c$$

のように収束または発散されると予想されている．ここで $f \approx g$ は 両辺の対数を取った比が 1 に収束することを表す．このような臨界指数 β, γ, ν, Δ の存在は d が十分大のとき (Hara-Slade [9]) を除いてまだ証明されていない．これらの指数においては関係式

$$d\nu = 2\Delta - \gamma \tag{1.2}$$

が成り立つと言われている．これを hyperscaling relation と言う．

次元 d が小さいとき，こうした問題の大半が未解決である．上記の指数則などを考えるには，グラフの平行移動不変性よりも自己相似性が重要である．そこで我々はフラクタル的な構造を持つグラフでのパーコレーションを考えることとした．こうしたグラフでのパーコレーションの研究は本研究以前にはほとんど行われていなかった．この新しい観点からの研究によって，パーコレーション現象の解明，およびフラクタル図形の性質の研究への貢献を目標とした．これらの意義および得られた成果について以下に詳しく述べる．

1.2 Finite-ramified fractals

フラクタル格子上のパーコレーションの最初の研究として，pre-Sierpinski gasket と呼ばれるグラフでのパーコレーションを考えた．$\mathbf{O} = (0,0)$, $a_0 = (1/2, \sqrt{3}/2)$, $b_0 = (1,0)$ とする．F_0 を $\triangle \mathbf{O} a_0 b_0$ の 3 頂点およびそれらを結ぶ辺からなるグラフとする．$\{F_n\}_{n=0,1,2,\cdots}$ を

$$F_{n+1} = F_n \cup (F_n + a_n) \cup (F_n + b_n)$$

で与えられるグラフの列とする．ここで $A + a = \{x + a \mid x \in A\}$, $kA = \{kx \mid x \in A\}$, $a_n = 2^n a_0$, $b_n = 2^n b_0$ である．$F = \bigcup_{n=0}^{\infty} F_n$ とする．この F を pre-Sierpinski gasket と言う．このグラフは，2 点 a_n, b_n を取り除くと不連結となる．このような性質を finite-ramified と言う．この中の長さ 1 の辺がそれぞれ独立に確率 p で open, 確率 $1-p$ で closed とする bond percolation を考える．$\theta(p)$, p_c は $\mathbb{Z}^d$ と同様に定義すると，finite-ramified であることからこのグラフでは $p_c = 1$ となる．そこで，correlation length

$$\xi(p) = \lim_{n \to \infty} \left\{ -\frac{1}{2^n} \log P_p(\mathbf{O} \leftrightarrow a_n) \right\}^{-1}$$

を定義し，$p \uparrow 1$ での発散の様子を調べた．

Theorem 1.1 (= Theorem 2.1)

$$\lim_{p \to 1} -\frac{\log \xi(p)}{\log(1-p)} = \infty, \tag{1.3}$$

さらに

$$\lim_{p \to 1} \frac{\log(\log \xi(p))}{\log(1-p)} = -2. \tag{1.4}$$

この (1.3), (1.4) の式は Gefen-Aharony-Shapir-Mandelbrot [6] によって形式的な計算により導かれているが，この論文ではこれを証明した．(1.3) を $\mathbb{Z}^d$ の場合の予想 (1.1) と比べると，pre-Sierpinski gasket では 通常の意味の臨界指数は ∞ となると言える．したがって hyperscaling relation (1.2) もそのままでは意味を持たないので，次の形

$$\{\xi(p)\}^d \approx \frac{E_p|C|^3}{\{\chi(p)\}^2} \quad \text{as} \quad p \uparrow 1 \tag{1.5}$$

に置き換えて考えることにする．

Theorem 1.2 (= Theorem 2.2) $D = \log 3/\log 2$ とする．このときすべての $k \geq 1$ に対し

$$E_p|C|^k \approx \{\xi(p)\}^{Dk} \quad \text{as} \quad p \uparrow 1.$$

この定理から，(1.5) が次元を D とみなすことで成立していることがわかる．ここで，D は Sierpinski gasket のフラクタル次元と一致している．

これらの結果については，site percolation （各辺でなく各頂点に open-closed を定めるモデル）でも成り立つことが確かめられる．

さらに，$\xi(p)$ についてはより詳細な発散のオーダーが計算できることがわかった．pre-Sierpinski gasket を一般化し，d 次元 pre-Sierpinski gasket を考える．これは上記の pre-Sierpinski gasket の構成における $\triangle \mathbf{O} a_0 b_0$ の代わりに d 次元単体を用いるものである．このグラフにおいて $\xi(p)$ を同様に定義すると，グラフの自己相似性および各点の周りの local な構造の情報から次のような結果を得ることができた．

Theorem 1.3 (= Theorem 3.2, *d*-dimensional pre-Sierpinski gasket)

$$\xi(p) \approx \exp\left\{\frac{\log 2}{2^d(d-1)}(1-p)^{-(d^2-d)}\right\} \quad \text{as} \quad p \uparrow 1. \tag{1.6}$$

このオーダーの計算方法は他の finite ramified fractal のグラフにも適用できる．その例として snowflake lattice (Figure 3.3), pentakun lattice (Figure 3.4) において計算した．

Theorem 1.4 (= Theorem 3.11, the pentakun lattice)

$$\xi(p) \approx \exp\left\{\frac{\log(1+\sqrt{3})}{64}(1-p)^{-2}\right\} \quad \text{as} \quad p \uparrow 1.$$

Theorem 1.5 (= Theorem 3.12, the snowflake lattice)

$$\xi(p) \approx \exp\left\{\frac{\log 3}{256}(1-p)^{-4}\right\} \quad \text{as} \quad p \uparrow 1.$$

上記の内容に関しては，本論文では第 2 章において pre-Sierpinski gasket における correlation length の存在，臨界指数の発散，hyperscaling relation の成立を recursion formula を用いて証明し，第 3 章において correlation length の発散のさらに詳細なオーダーを計算する方法を述べ，d 次元 pre-Sierpinski gasket や他の finite ramified fractal の場合での実際の計算を行っている．

1.3 Infinite-ramified fractals

次に，有限個の点だけでは切断されない (infinite ramified) フラクタル格子について考える．このグラフでは前節で扱った場合と異なり，必ずしも $p_c = 1$ とはならない．infinite ramified であるフラクタルとして最も有名であるものの一つに Sierpinski carpet がある．1997 年に Kumagai [10] によって，pre-Sierpinski carpet においては $p_c < 1$ であることが証明された．この後，異なる方法で Lü [12] によっても同様の結果が示されている．また，Murai [13] では d 次元に拡張された carpet 上での $d \to \infty$ としたときの p_c の漸近挙動が調べられている．

一般化された Sierpinski carpet 格子を $\mathbb{Z}^2$ の部分グラフとして定義する．$L \geq 2$, $T \subset \{0, 1, \ldots, L-1\}^2$ とする．ただし $(0,0) \in T$ を仮定する．グラフ $G_T = (V_T, E_T)$ を以下のように構成する．

$$\begin{aligned}
V_T^0 &= \mathbf{Z}^2 \cap \{(x,y) \mid 0 \leq x, y \leq 1\}, \quad V_T^{n+1} = \bigcup_{(i,j)\in T} (V_T^n + (iL^n, jL^n)) \quad (n \geq 0), \\
V_T &= \bigcup_{n=0}^{\infty} V_T^n, \quad E_T = \{\langle u, v\rangle \mid u, v \in V_T, ||u - v||_1 = 1\}.
\end{aligned}$$

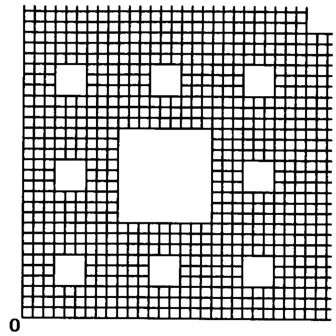

Figure 1.1: the Sierpinski carpet lattice

$L = 3$, $T = \{(i,j)|0 \le i,j \le 2, (i,j) \ne (1,1)\}$ のときの G_T が最も知られた Sierpinski carpet 格子である (Figure 1.1). また $L = 2$, $T = \{(i,j)|0 \le i,j \le 1, (i,j) \ne (1,1)\}$ のときは Sierpinski gasket 格子に対応することにも注意しておく. この G_T において bond percolation を考える. このとき, T にどのような条件があれば $p_c(G_T) < 1$ となるか, を考える. 知られている [10] の結果は $\{(i,j) \mid i \in \{0, L-1\} \text{ or } j \in \{0, L-1\}\} \subset T$ ならば $p_c(G_T) < 1$ というものである. この結果を拡張し, さらに一般的な結果を得ようとするのがこの研究の意義である. なお, bond percolation と site percolation, Ising model の相転移の有無が同値であることは Häggström [8] によって示されており, このグラフでの Ising model を考える契機にもなる. もう一つの問題意識について述べておく. Benjamini-Schramm [1] において, 次の問題が提起された.

Problem. G の等周次元を

$$\mathrm{Dim}(G) = \sup\Big\{D > 0 : \inf \frac{|\partial S|}{|S|^{\frac{D-1}{D}}} > 0\Big\}$$

で定義する. ここで S は有限かつ連結な E の部分集合, ∂S は S の outer boundary とする. このとき $\mathrm{Dim}(G) > 1$ ならば $p_c(G) < 1$ と言えるか.

この命題が Sierpinski carpet 格子で成り立つか調べられないか, というのも [10] を拡張するための動機となっている.

T に関する十分条件として, 以下のものが得られた. ここで $T_l = \{j|(0,j) \in T\}$, $T_r = \{j|(L-1,j) \in T\}$, $T_d = \{i|(i,0) \in T\}$, $T_u = \{i|(i,L-1) \in T\}$ とかくことにする.

Theorem 1.6 (= Theorem 4.2)

$$\text{任意の } t \in T \quad \text{に対し} \quad T \setminus \{t\} \quad \text{は連結}, \tag{1.7}$$

および

$$|T_l \cap T_r| \ge 2 \text{ かつ } |T_d \cap T_u| \ge 2. \tag{1.8}$$

を仮定する. このとき $p_c(G_T) < 1$ である.

この結果は [10] の真の拡張となっている. 証明から $p_c(G_T)$ についての以下の評価式が得られる.

Corollary 1.7 (= Corollary 4.5) $f_T(x) = x^{|T|} + |T|x^{|T|-1}(1-x)$ とする. Theorem 4.2 の仮定の下で,

$$p_c(G_T) \le \sqrt{\alpha}$$

である. ここで α は方程式 $f_T(x) = x$ の $(0,1)$ 区間内にある最も大きい解である.

$p_c(G_T) < 1$ となるための必要条件については, 以下のことがわかる.

Theorem 1.8 (Corollary of Proposition 4.6) 以下の (i), (ii) のいずれかが満たされているとする.
(i) ある j_0 に対し $|\{i \mid (i, j_0) \in T\}| \leq 1$.
(ii) $|T_l \cap T_r| \leq 1$.
このとき $p_c(G_T) = 1$.

証明は Proposition 4.6 の場合と同様である. この結果より, 自明でない相転移が存在することの必要十分条件を得るためには (1.7) をどこまで弱めることが出来るかが課題となる. 実際もう少し弱めることは可能であるが, (1.7), (1.8) のように T に関するチェックしやすい条件を掲げるという目的のため定理ではこのような形をとった. Sierpinski carpet 格子の等周次元と相転移の存在との関係については, 今のところ $\mathrm{Dim}(G_T) > 1$ かつ $P_c(G_T) = 1$ である例はないが, すべての T に対して命題を証明するにはこの部分の条件の精密化, および等周次元が 1 より大きいことを T の言葉でうまく表現する必要があり, さらなる課題となっている. なお, Sierpinski carpet 格子の場合でも $\mathrm{Dim}(G_T) > 1$ が $p_c(G_T) < 1$ のために必要というわけではない. この一例として $T = \{0,1,2,3,4\}^2 \setminus \{(1,3),(1,4),(2,1),(2,3),(3,0),(3,1)\}$ の場合 (Example 4.11) がある.

次に, Sierpinski carpet 格子での oriented percolation を考える. 上記と同じグラフ G_T において, bond の通過できる方向に制限をつけ, 右向きまたは上向きにしか進めないものとする. この oriented percolation も $\mathbb{Z}^d$ 上で盛んに研究にされている (詳しくは Durrett [5], [7] など). 特に contact process との関連が深い. oriented percolation での相転移点 $\vec{p_c}$ が真に 1 より小さいかどうかを調べる. $\vec{p_c}(\mathbb{Z}^2) < 1$ であることはよく知られている. 現在では $\vec{p_c}(\mathbb{Z}^2) \leq 2/3$ (Liggett [11]) という評価が得られている. ここでは Sierpinski carpet 格子のうち, 特に T が対称性を持つ場合について考えた. $L = 2a + b$ $(a, b > 0)$, $T_{a,b} = \{0, 1, \ldots, L-1\}^2 \setminus \{a, a+1, \ldots, a+b-1\}^2$ とする.

Theorem 1.9 (= Theorem 5.6) $a \leq b$ ならば $\vec{p_c}(G^2_{a,b}) = 1$.

この結果は, ある程度穴が大きければ oriented percolation においては自明でない相転移が起こらないことを示している. この点において Sierpinski carpet 格子と $\mathbb{Z}^d$ には大きな違いがあることがわかる. なお, こうした oriented percolation での相転移の消滅は $[0,1]^2$ での fractal percolation において報告されている ([3], [4]) が, 通常の percolation においては知られていなかった. なお残念ながら, 現段階では穴が小さい場合に自明でない相転移があるかどうかはわかっていない.

この問題はまた d 次元空間においても考えられる. 2 次元の場合の拡張として例えば d 次元 pre-Sierpinski carpet, すなわち $T^d_{sc} = \{0,1,2\}^d \setminus \{(1,1,\ldots,1)\}^d$ のときを考えるとこれは $\mathbb{Z}^{d-1}$ を部分グラフとして含むので $\vec{p_c}(G_{T^d_{sc}}) \leq \vec{p_c}(\mathbb{Z}^{d-1}) < 1$ は明らかである. そこで

$$T^d_{a,b} = \left\{ (i_1, i_2, \ldots, i_d) \in \{0, 1, \ldots, L-1\}^d \;\middle|\; \left|\{j \mid a \leq i_j \leq a+b-1\}\right| \leq 1 \right\}$$

を考える. これは Menger Sponge と呼ばれるフラクタルに対応するものである. この $T^d_{a,b}$ から $G_{T^d_{a,b}}$ を構成し oriented percolation を考えると, 以下の結果が得られた.

Theorem 1.10 (= Theorem 5.7) $2 \leq d \leq b$ とする. このとき $\vec{p_c}(G^d_{1,b}) = 1$.

この結果は, どんなに高次元内の格子であっても十分穴が大きければやはり相転移が消滅してしまうことを示す. 高次元の場合も, 今のところ $T \neq \{0, 1, \ldots, L-1\}^d$ で $\vec{p_c}(G_T) < 1$ となることがあるかどうかはわかっていない.

上記の内容に関しては, 本論文では第 4 章で percolation の自明でない相転移に関する T の必要条件と十分条件についての定理を証明し, 必要十分条件を得るための考察, およびグラフの等周次元との関係について述べた. 第 5 章では oriented percolation の場合の相転移の消滅についての定理を証明した.

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第2章 Percolation on the pre-Sierpinski gasket

2.1 Introduction and statements of results

In this paper, we regard percolation as a model of phase transitions. We are especially interested in problems near the *critical point*, where the phase transition occurs. We call these problems *critical behaviors.* Our purpose in this paper is to clarify the critical behaviors of percolation on the pre-Sierpinski gasket which has self-similarity.

Until now, studies of percolation are restricted on *periodic* graphs, such as $\mathbf{Z}^d$. (An exact definition of periodic graph is mentioned in Kesten[1].) There are lots of conjectures and hypotheses about critical behaviors, but many of them are still unsolved rigorously (see Grimmett[2] and references therein). In high dimension lattices $\mathbf{Z}^d$, rigorous results for critical behaviors were obtained by Hara-Slade[3]. But in low dimensions, except a work on $\mathbf{Z}^2$ by Kesten[4], few rigorous results have been proved about the existence of *critical exponents* and justification of the *scaling, hyperscaling relations.*

For critical behaviors, *self-similarity* of the graph plays more important role than periodicity. This is a motivation to consider percolation problems on the pre-Sierpinski gasket.

We now define the pre-Sierpinski gasket. Let $\mathbf{O} = (0,0), a_0 = (1/2, \sqrt{3}/2)$, $b_0 = (1,0)$. Let F_0 be the graph which consists of the vertices and edges of the triangle $\triangle \mathbf{O} a_0 b_0$. Let $\{F_n\}_{n=0,1,2,\cdots}$ be the sequence of graphs given by

$$F_{n+1} = F_n \cup (F_n + a_n) \cup (F_n + b_n)$$

where $A + a = \{x + a \mid x \in A\}$, $kA = \{kx \mid x \in A\}$, $a_n = 2^n a_0$ and $b_n = 2^n b_0$. Let $F = \bigcup_{n=0}^{\infty} F_n$. We call F the *pre-Sierpinski gasket.* (Fig. 2.1) Note that $\tilde{F} = \bigcup_{n=0}^{\infty} 2^{-n} F$ become the Sierpinski gasket. Let V be the set of all vertices in F, and E the set of all edges with length 1.

We consider the Bernoulli bond percolation on the pre-Sierpinski gasket; each edges in E are *open* with probability p and *closed* with probability $1-p$ independently. Let P_p denote its distribution. We think of open bonds as permitting to go along the bond. We write $x \leftrightarrow y$ if there is an open path from x to y. Let $C(x) = \{y \in V : x \leftrightarrow y\}$. $C(x)$ is called the *open cluster* containing x. We denote by C the open cluster containing the origin.

We define two functions in a similar way as percolations on $\mathbf{Z}^d$.

$$\theta(p) = P_p(|C| = \infty), \quad \chi(p) = E_p(|C|; |C| < \infty),$$

where $|C|$ denotes the number of vertices contained in C, and E_p denotes the expectation with respect to P_p. $\theta(p)$ is called the *percolation probability,* and $\chi(p)$ is called the *mean cluster size.*

Let p_c denote the *critical point* p_c; that is

$$p_c = \inf\{p : \theta(p) > 0\}.$$

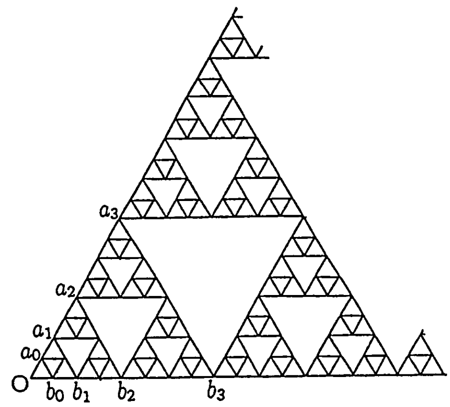

Figure 2.1: the pre-Sierpinski gasket

Then $p_c = 1$ for the pre-Sierpinski gasket because it is finitely ramified. We note that $\chi(p) = E_p|C|$ for $p < 1$.

The *correlation length* is defined by

$$\xi(p) = \lim_{n\to\infty}\left\{-\frac{1}{2^n}\log P_p(\mathrm{O}\leftrightarrow a_n)\right\}^{-1}. \tag{2.1}$$

The existence of the limit in (2.1) will be proved in Section 2.

We write $f(p) \approx g(p)$ as $p \to p_0$ if $\log f(p)/\log g(p) \to 1$ as $p \to p_0$.

We now state our main theorems:

Theorem 2.1 $\lim_{p\to 1} -\dfrac{\log\xi(p)}{\log(1-p)} = \infty,$ *and* $\lim_{p\to 1}\dfrac{\log(\log\xi(p))}{\log(1-p)} = -2.$

Theorem 2.2. *Let* $D = \log 3/\log 2$. *Then*

$$E_p|C|^k \approx \{\xi(p)\}^{Dk} \quad \textit{as} \quad p \to 1 \quad \textit{for all } k \geq 1.$$

Remark. Our results are quite different from the results on $\mathbf{Z}^d$ (see below). In physical literture, Theorem 2.1 was known by Gefen et al.[5] by using formal renormalization arguments. Our contribution is that we prove Theorem 2.1 rigorously.

We collect results and conjectures of the percolation on $\mathbf{Z}^d$. It is conjectured (see [2])

$$\xi(p) \approx |p_c - p|^{-\nu(d)} \quad \text{as} \quad p \to p_c. \tag{2.2}$$

The value $\nu(d)$ is called the *critical exponent.* It is proved that $\nu(d) = 1/2$ for sufficiently large d (Hara-Slade[3]), and conjectured $\nu(2) = 4/3$ (see [4]).

Other critical exponents considered in $\mathbf{Z}^d$ are as follows:

$$\chi(p) \approx |p_c - p|^{-\gamma}, \quad \frac{E_p(|C|^{k+1}; |C| < \infty)}{E_p(|C|^k; |C| < \infty)} \approx |p_c - p|^{-\Delta} \quad \text{as} \quad p \to p_c.$$

It is conjectured for $\mathbf{Z}^d$ that $d\nu = 2\Delta - \gamma$. This relation is one of hyperscaling relations. We note $\gamma = \Delta = \infty$ on the pre-Sierpinski gasket. So the relation $d\nu = 2\Delta - \gamma$ does not make sense on the pre-Sierpinski gasket. Accordingly we modify the hyperscaling relation as follows:

$$\{\xi(p)\}^d \approx \frac{E_p|C|^3}{\{\chi(p)\}^2} \quad \text{as} \quad p \to p_c. \tag{2.3}$$

If finite critical exponents ν, γ, Δ exist, then (2.3) is equivalent to $d\nu = 2\Delta - \gamma$.

Remark. By Theorem 1.2, we have $E_p|C|^3 \approx \{\xi(p)\}^{3D}$ and $\chi(p) \approx \{\xi(p)\}^D$. Hence the above hyper-scaling relation (2.3) holds when we regard D as the dimension of the pre-Sierpinski gasket. The value $D = \log 3/\log 2$ coincides with the fractal dimension of the Sierpinski gasket.

In addition, we mention site percolation on the pre-Sierpinski gasket: each vertices in V are determined to be open or closed independently. (Details will be given in Section 5.) We define the correlation length $\hat{\xi}(p)$ in the same manner as (2.1). We have the result below;

Theorem 2.3. $\lim_{p\to 1} -\frac{\log \hat{\xi}(p)}{\log(1-p)} = \infty,$ *and* $\lim_{p\to 1} \frac{\log(\log \hat{\xi}(p))}{\log(1-p)} = -1.$

The critical exponent in a usual sense is also infinite in this case. But $\hat{\xi}(p) \approx \log(1-p)^{-1}$, which is different from Theorem 2.1. We cannot see the universality of this exponent on the pre-Sierpinski gasket.

We refer to the self-avoiding walks on the Sierpinski gasket, as related works of phase transitions; Hattori-Hattori[6] and Hattori-Hattori-Kusuoka[7] construct the self-avoiding paths on two- and three-dimensional Sierpinski gasket. Before [6], Hattori-Hattori-Kusuoka[8] constructed them on the pre-Sierpinski gasket. These works also gave us a motivation to study percolation on the Sierpinski gasket.

The organization of this paper is as follows: In Section 2 we prepare for the proof of our main theorems; we construct recursion formulas of relations between events in F_n and ones in F_{n+1}. In the reminder of Section 2, we prove the existence of the correlation length. We prove Theorem 2.1 in Section 3 and Theorem 2.2 in Section 4. In Section 5 we study site percolation and prove Theorem 2.3.

2.2 Recursion formulas and the existence of $\xi(p)$

We introduce two connectivity functions as follows.

$$\begin{aligned}\Phi_n(p) &= P_p(\mathbf{O} \leftrightarrow a_n \text{ in } \triangle \mathbf{O} a_n b_n),\\ \Theta_n(p) &= P_p(\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow b_n \text{ in } \triangle \mathbf{O} a_n b_n).\end{aligned}$$

We write $\mathbf{O} \leftrightarrow a_n$ *in* $\triangle \mathbf{O} a_n b_n$ if there is an open path from $\mathbf{O}$ to a_n in $\triangle \mathbf{O} a_n b_n$ (contains its perimeter). We easily calculate $\Phi_0(p) = p + p^2 - p^3, \Theta_0(p) = 3p^2 - 2p^3$. Note that (i) $\Phi_n(p) \geq \Theta_n(p)$ by definition, (ii) if $\mathbf{O} \leftrightarrow a_n$ and $\mathbf{O} \leftrightarrow b_n$ then we have $a_n \leftrightarrow b_n$ automatically.

Proposition 2.4. *For each* $n \geq 0$ *and* $0 \leq p \leq 1$,

$$\Phi_{n+1}(p) = \{\Phi_n(p)\}^2 + \{\Phi_n(p)\}^3 - \Phi_n(p)\{\Theta_n(p)\}^2, \tag{2.4}$$

$$\Theta_{n+1}(p) = 3\{\Phi_n(p)\}^2\Theta_n(p) - 2\{\Theta_n(p)\}^3. \tag{2.5}$$

Proof. Recall $\triangle \mathbf{O} a_n b_n = F_n$. Let $F'_n = F_n + a_n, F''_n = F_n + b_n$, and $c_n = (3\cdot 2^{n-1}, \sqrt{3}\cdot 2^{n-1})$. Let A_n^1 and A_n^2 be events given by

$$\begin{aligned}A_n^1 &= \{\mathbf{O} \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F'_n\},\\ A_n^2 &= \{\mathbf{O} \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow c_n \text{ in } F''_n\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F'_n\}.\end{aligned}$$

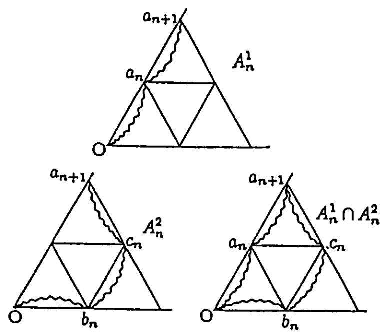

Figure 2.2:

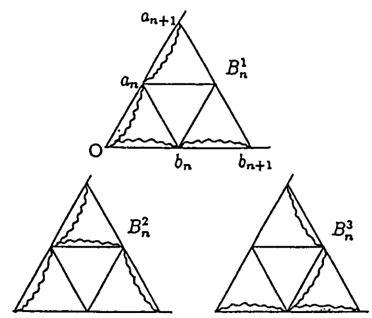

Figure 2.3:

Then we have

$$\Phi_{n+1}(p) = P_p(A_n^1) + P_p(A_n^2) - P_p(A_n^1 \cap A_n^2). \tag{2.6}$$

Here we used the fact that a path from $\mathbf{O}$ to a_{n+1} goes through a_n or b_n. Since the events in F_n , F'_n , F''_n are mutually independent, $P_p(A_n^1) = \{\Phi_n(p)\}^2$, $P_p(A_n^2) = \{\Phi_n(p)\}^3$, $P_p(A_n^1 \cap A_n^2) = \{\Theta_n(p)\}^2\Phi_n(p)$ (Fig. 2.2). Combining these with (2.6) yields (2.4).

We proceed to the proof of (2.5). Let B_n^1, B_n^2, B_n^3 be events given by

$$\begin{aligned} B_n^1 &= \{\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow b_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F'_n\} \\ &\qquad \cap \{b_n \leftrightarrow b_{n+1} \text{ in } F''_n\}, \\ B_n^2 &= \{\mathbf{O} \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ and } a_n \leftrightarrow c_n \text{ in } F'_n\} \\ &\qquad \cap \{c_n \leftrightarrow b_{n+1} \text{ in } F''_n\}, \\ B_n^3 &= \{\mathbf{O} \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow b_{n+1} \text{ and } b_n \leftrightarrow c_n \text{ in } F''_n\} \\ &\qquad \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F'_n\} \end{aligned}$$

(see Fig. 2.3). Then we have

$$\begin{aligned} \Theta_{n+1}(p) = P_p(B_n^1) + P_p(B_n^2) + P_p(B_n^3) - P_p(B_n^1 \cap B_n^2) - P_p(B_n^2 \cap B_n^3) \\ - P_p(B_n^3 \cap B_n^1) + P_p(B_n^1 \cap B_n^2 \cap B_n^3). \end{aligned}$$

We see easily

$$P_p(B_n^1) = P_p(B_n^2) = P_p(B_n^3) = \{\Phi_n(p)\}^2\Theta_n(p),$$

$$P_p(B_n^1 \cap B_n^2) = P_p(B_n^2 \cap B_n^3) = P_p(B_n^3 \cap B_n^1) = P_p(B_n^1 \cap B_n^2 \cap B_n^3) = \{\Theta_n(p)\}^3.$$

(2.5) follows from this immediately. □

From now on, we assume $0 < p < 1$. We prove the existence of the limit (2.1), correlation length $\xi(p)$, by using these recursions.

Proposition 2.5. *There exists* $\xi(p) > 0$ *such that*

$$\lim_{n\to\infty} \frac{\Phi_n(p)}{\exp\{-2^n/\xi(p)\}} = 1.$$

Remark. The convergence as $n \to \infty$ in Proposition 2.2 is stronger than the convergence in (1).

Proof. By (2.4) and $\Theta_n(p) \le \Phi_n(p)$, we have

$$\{\Phi_n(p)\}^2 \le \Phi_{n+1}(p) \le \{\Phi_n(p)\}^2 + \{\Phi_n(p)\}^3.$$

Hence

$$1 \le \frac{\Phi_{n+1}(p)}{\{\Phi_n(p)\}^2} \le 1 + \Phi_n(p).$$

Let $h_n(p) = \Phi_{n+1}(p)/\{\Phi_n(p)\}^2$. Then $1 \le h_n(p) \le 2$ and $\lim_{n\to\infty} h_n(p) = 1$ because $\lim_{n\to\infty} \Phi_n(p) = 0$. Now

$$\begin{aligned}
&\frac{1}{2^n}\log\Phi_n(p)\\
=\ &\frac{1}{2^n}\log\left(\{\Phi_0(p)\}^{2^n}\cdot\frac{\{\Phi_1(p)\}^{2^{n-1}}}{\{\Phi_0(p)\}^{2^n}}\cdot\frac{\{\Phi_2(p)\}^{2^{n-2}}}{\{\Phi_1(p)\}^{2^{n-1}}}\cdots\frac{\Phi_n(p)}{\{\Phi_{n-1}(p)\}^2}\right)\\
=\ &\log\Phi_0(p) + \frac{1}{2}\log h_0(p) + \frac{1}{2^2}\log h_1(p) + \cdots + \frac{1}{2^n}\log h_{n-1}(p)\\
\le\ &\log\Phi_0(p) + \log 2.
\end{aligned}$$

Hence $\{\log\Phi_n(p)/2^n\}_{n=0,1,2,\cdots}$ is increasing and $\lim_{n\to\infty}\log\Phi_n(p)/2^n$ exists. Let $-\{\xi(p)\}^{-1} = \lim_{n\to\infty}\log\Phi_n(p)/2^n$. Then

$$\begin{aligned}
-\frac{1}{\xi(p)} \ge \frac{1}{2^n}\log\Phi_n(p) &= -\frac{1}{\xi(p)} - \left(\frac{1}{2^{n+1}}\log h_n(p) + \frac{1}{2^{n+2}}\log h_{n+1}(p) + \cdots\right)\\
&\ge -\frac{1}{\xi(p)} - \frac{1}{2^n}\log H_n(p),
\end{aligned}$$

where $H_n(p) = \sup_{m\ge n} h_m(p)$. Therefore

$$\exp\left\{-\frac{2^n}{\xi(p)}\right\} \ge \Phi_n(p) \ge \frac{1}{H_n(p)}\exp\left\{-\frac{2^n}{\xi(p)}\right\}. \tag{2.7}$$

Since $\lim_{n\to\infty} H_n(p) = 1$, we complete the proof. □

Remark. Note that the function $\xi(p)$ is continuous and increasing on $(0,1)$ from the proof above.

Lemma 2.6. $\displaystyle\lim_{n\to\infty}\frac{P_p(\mathbf{O} \leftrightarrow a_n)}{\exp\{-2^n/\xi(p)\}} = 1.$

Proof. Recall that $\Phi_n(p) = P_p(\mathbf{O} \leftrightarrow a_n \text{ in } F_n)$. Then

$$\begin{aligned}
&P_p(\mathbf{O} \leftrightarrow a_n) - P_p(\mathbf{O} \leftrightarrow a_n \text{ in } F_n)\\
\le\ &P_p(\mathbf{O} \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow c_n \text{ in } F''_n, c_n \leftrightarrow a_n \text{ in } F'_n)\\
&+P_p(\mathbf{O} \leftrightarrow b_n \text{ in } F_n, b_n \leftrightarrow b_{n+1} \text{ in } F''_n, a_n \leftrightarrow a_{n+1} \text{ in } F'_n) \quad \text{(Fig. 2.4)}\\
=\ &2\{\Phi_n(p)\}^3.
\end{aligned}$$

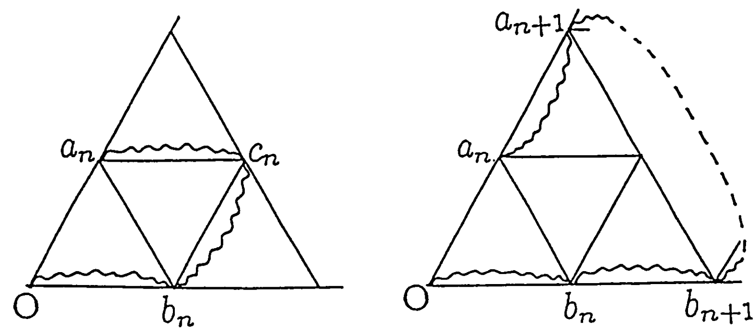

Figure 2.4:

So

$$1 \leq \frac{P_p(\mathbf{O} \leftrightarrow a_n)}{\Phi_n(p)} \leq 1 + 2\{\Phi_n(p)\}^2,$$

which implies

$$\lim_{n\to\infty} \frac{P_p(\mathbf{O} \leftrightarrow a_n)}{\Phi_n(p)} = 1.$$

Combining this with Proposition 2.5 completes the proof. □

2.3 Proof of Theorem 2.1

The next lemma is a key of the proof.

Lemma 2.7. *There exists $\varepsilon > 0$ such that*

$$2 \leq \frac{\xi(p + 3(1-p)^3)}{\xi(p)} \leq 4 \quad \textit{for} \quad 1 - \varepsilon < p < 1.$$

Proof. We introduce

$$\begin{aligned} \Psi_n(p) &= 1 - P_p(\mathbf{O} \not\leftrightarrow a_n, \mathbf{O} \not\leftrightarrow b_n, a_n \not\leftrightarrow b_n \text{ in } F_n) \\ &= 3\Phi_n(p) - 2\Theta_n(p). \end{aligned} \tag{2.8}$$

Here $\mathbf{O} \not\leftrightarrow a_n$ *in* F_n means that there exists no open path from $\mathbf{O}$ to a_n in F_n. By (2.4) and (2.5),

$$\begin{aligned} \Theta_{n+1}(p) &= S(\Theta_n(p), \Psi_n(p)), \\ \Psi_{n+1}(p) &= T(\Theta_n(p), \Psi_n(p)), \end{aligned}$$

where S, T: $\mathbf{R}^2 \to \mathbf{R}$ are functions defined by

$$\begin{aligned} S(x,y) &= -\frac{2}{3}x^3 + \frac{4}{3}x^2y + \frac{1}{3}xy^2, \\ T(x,y) &= \frac{2}{9}x^3 + \frac{4}{3}x^2 - \frac{7}{3}x^2y + \frac{4}{3}xy + \frac{1}{9}y^3 + \frac{1}{3}y^2. \end{aligned}$$

Let D be a subset of $\mathbf{R}^2$ defined by $D = \{(x,y) : 0 < x \leq y < 1\}$. We see $\partial S/\partial x, \partial S/\partial y, \partial T/\partial x, \partial T/\partial y > 0$ for $(x,y) \in D$. Indeed,

$$\begin{aligned}
\frac{\partial S}{\partial x} &= -2x^2 + \frac{8}{3}xy + \frac{1}{3}y^2 = 2x(y-x) + \frac{2}{3}xy + \frac{1}{3}y^2 > 0, \\
\frac{\partial S}{\partial y} &= \frac{4}{3}x^2 + \frac{2}{3}xy > 0, \\
\frac{\partial T}{\partial x} &= \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{3}xy + \frac{4}{3}y \geq \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{3}xy + \frac{2}{3}y^2 + \frac{2}{3}y \\
&= \frac{2}{3}(y-x)^2 + \frac{8}{3}x(1-y) + \frac{2}{3}y(1-x) > 0, \\
\frac{\partial T}{\partial y} &= -\frac{7}{3}x^2 + \frac{4}{3}x + \frac{1}{3}y^2 + \frac{2}{3}y \\
&= \frac{4}{3}x(1-x) + \frac{1}{3}(y^2 - x^2) + \frac{2}{3}(y - x^2) > 0.
\end{aligned}$$

Therefore if $(x_1, y_1), (x_2, y_2) \in D$ and $x_1 < x_2$ and $y_1 < y_2$, then

$$S(x_1,y_1) < S(x_2,y_2), \quad T(x_1,y_1) < T(x_2,y_2). \tag{2.9}$$

Note that $\Psi_n(p) = \Theta_n(p) + 3\{\Phi_n(p) - \Theta_n(p)\} \geq \Theta_n(p)$ for all n by (2.8). Hence $(\Theta_n(p), \Psi_n(p)) \in D$. Calculating $\Theta_n(p)$ and $\Psi_n(p)$ directly from the recursions, we have

$$\begin{aligned}
\Theta_n(p) &= 1 - 3(1-p)^2 - (12n-6)(1-p)^4 + 6(1-p)^5 \\
&\quad + (-48n^2 + 120n - 15)(1-p)^6 + \cdots, & (2.10) \\
\Psi_n(p) &= 1 - 3(1-p)^4 - 24n(1-p)^6 + \cdots & (2.11)
\end{aligned}$$

for $n \geq 2$. For $1 - 1/\sqrt{3} < p < 1$, let $\tilde{p} = p + 3(1-p)^3$. Then we have

$$\begin{aligned}
\Theta_3(\tilde{p}) - \Theta_2(p) &= 6(1-p)^4 + 213(1-p)^6 + \cdots, \\
\Psi_3(\tilde{p}) - \Psi_2(p) &= 12(1-p)^6 + \cdots.
\end{aligned}$$

Note that $\Theta_2(p), \Psi_2(p), \Theta_3(\tilde{p})$, and $\Psi_3(\tilde{p})$ are polynomials of finite degree. Hence we can take $\varepsilon_1 > 0$ in such a way that $\Theta_2(p) < \Theta_3(\tilde{p})$ and $\Psi_2(p) < \Psi_3(\tilde{p})$ for $1 - \varepsilon_1 < p < 1$. By (2.9), We have

$$\begin{aligned}
&\Theta_3(p) = S(\Theta_2(p), \Psi_2(p)) < S(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Theta_4(\tilde{p}), \\
&\Psi_3(p) = T(\Theta_2(p), \Psi_2(p)) < T(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Psi_4(\tilde{p}).
\end{aligned}$$

Estimating repeatedly as above, we have $\Theta_n(p) < \Theta_{n+1}(\tilde{p}), \Psi_n(p) < \Psi_{n+1}(\tilde{p})$ for $n \geq 2$. Combining this with (2.8) yields $\Phi_n(p) < \Phi_{n+1}(\tilde{p})$. So

$$\frac{\log \Phi_n(p)}{2^n} < 2 \cdot \frac{\log \Phi_{n+1}(\tilde{p})}{2^{n+1}}.$$

This implies $\xi(p)^{-1} \geq 2 \cdot \xi(\tilde{p})^{-1}$, that is $\xi(\tilde{p})/\xi(p) \geq 2$ for $1 - \varepsilon_1 < p < 1$.

We now proceed to the estimate from the opposite side. By using (2.10) and (2.11) again, we see

$$\begin{aligned}
\Theta_4(\tilde{p}) - \Theta_2(p) &= -6(1-p)^4 + 141(1-p)^6 + \cdots \\
\Psi_4(\tilde{p}) - \Psi_2(p) &= -12(1-p)^6 + \cdots.
\end{aligned}$$

Hence we can take $\varepsilon_2 > 0$ such that $\Theta_4(\tilde{p}) < \Theta_2(p)$ and $\Psi_4(\tilde{p}) < \Psi_2(p)$ for $1 - \varepsilon_2 < 1$. So we have $\Theta_{n+2}(\tilde{p}) < \Theta_n(p)$ and $\Psi_{n+2}(\tilde{p}) < \Psi_n(p)$. Therefore $\xi(\tilde{p})/\xi(p) \leq 4$ for $1 - \varepsilon_2 < p < 1$, which completes the proof. □

Proof of Theorem 2.1. Let $g(p) = \log \xi(p)$. Since $\xi(p)$ is an increasing function, $g(p)$ is also increasing. Suppose that p is sufficiently large to satisfy $g(p) > 0$. Let

$$m = \liminf_{p \to 1} -\frac{\log g(p)}{\log(1-p)} \geq 0, \quad M = \limsup_{p \to 1} -\frac{\log g(p)}{\log(1-p)}.$$

First, we prove $m \geq 2$. Suppose $m < 2$, and pick $\delta > 0$ with $m + \delta < 2$. Let

$$h(x) = \frac{1}{(x - 3x^3)^{m+\delta}} - \frac{1}{x^{m+\delta}}.$$

Applying the L'Hospital's theorem, we see $\lim_{x \to 0} h(x) = 0$. So we take p_0 such that

$$h(1-p) < \frac{1}{2} \log 2 \quad \text{for} \quad 0 < 1 - p < 1 - p_0 \tag{2.12}$$

and $1 - p_0 < \varepsilon$. (ε is given in Lemma 2.7.)

Let

$$f(p) = p + 3(1-p)^3. \tag{2.13}$$

We define $\{p_n\}_{n=1,2,\cdots}$ by $f(p_0) = p_1, f(p_n) = p_{n+1}$ inductively. Then $p_0 < p_1 < \cdots < p_n < 1$, and $\lim_{n \to \infty} p_n = 1$. By (2.13) and Lemma 2.7, we have

$$\log 2 \leq g(p_{n+1}) - g(p_n),$$

and hence

$$g(p_0) + n \log 2 \leq g(p_n). \tag{2.14}$$

Take $N = N(p_0) \in \mathbf{N}$. By assumption, there exists t such that $p_N < t < 1$ and

$$-\frac{\log g(t)}{\log(1-t)} < m + \delta. \tag{2.15}$$

For this t, there exists unique $N' = N'(t)$ such that $p_{N'} \leq t < p_{N'+1}$. By (2.15) and $1 - p_{N'+1} < 1 - t$, we have

$$\begin{aligned} g(t) &< \frac{1}{(1 - p_{N'+1})^{m+\delta}} \\ &= \left\{ \frac{1}{(1 - p_{N'+1})^{m+\delta}} - \frac{1}{(1 - p_{N'})^{m+\delta}} \right\} \\ &\qquad + \left\{ \frac{1}{(1 - p_{N'})^{m+\delta}} - \frac{1}{(1 - p_{N'-1})^{m+\delta}} \right\} + \cdots + \frac{1}{(1 - p_0)^{m+\delta}} \\ &= h(1 - p_{N'}) + h(1 - p_{N'-1}) + \cdots + h(1 - p_0) + \frac{1}{(1 - p_0)^{m+\delta}} \\ &< \frac{1}{2}(N' + 1) \log 2 + \frac{1}{(1 - p_0)^{m+\delta}}. \end{aligned} \tag{2.16}$$

The last inequality follows from (2.12). On the other hand, $g(p_0) + N' \log 2 \leq g(p_{N'}) \leq g(t)$ by (2.14). Combining this with (2.16) yields

$$\frac{1}{2}(N - 1) \log 2 < \frac{1}{2}(N' - 1) \log 2 < \frac{1}{(1 - p_0)^{m+\delta}} - g(p_0). \tag{2.17}$$

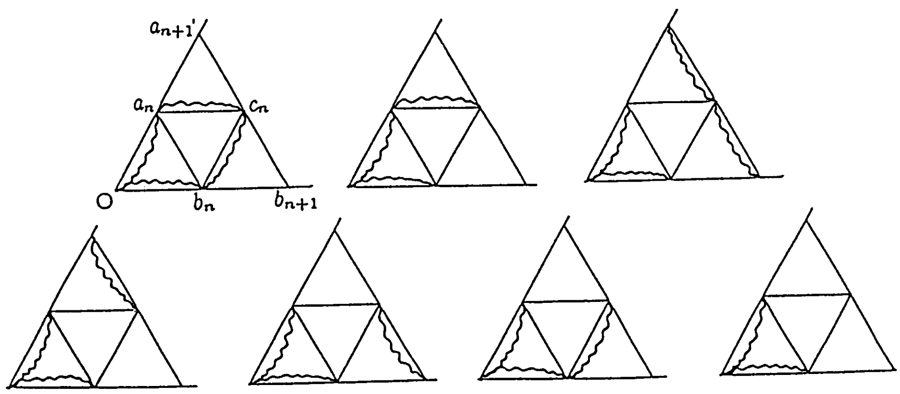

Figure 2.5:

Here we used $N < N'$ for the first inequality. We can pick $N(p_0)$ so large that (2.17) does not hold. This yields a contradiction. Hence we have $m \geq 2$.

We proceed to prove $M \leq 2$. Suppose $M > 2$. Pick $\delta > 0$ such that $M - \delta > 2$. Let

$$h(x) = \frac{1}{(x - 3x^3)^{M-\delta}} - \frac{1}{x^{M-\delta}}.$$

Note that $\lim_{x \to 0} h(x) = \infty$. Then by a similar argument as above, we lead a contradiction. Hence $M \leq 2$, Which concludes $m = M = 2$. □

2.4 Proof of Theorem 2.2.

First, we estimate the probability $P_p(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n)$. Let $M = \sup\{m : \mathrm{O} \leftrightarrow a_m \text{ or } b_m\}$. We define two conditional probabilities

$$\begin{aligned} U_n(p) &= P_p(\mathrm{O} \leftrightarrow a_n, \mathrm{O} \not\leftrightarrow b_n \text{ in } F_n | M = n), \\ V_n(p) &= P_p(\mathrm{O} \leftrightarrow a_n, \mathrm{O} \leftrightarrow b_n \text{ in } F_n | M = n). \end{aligned}$$

Clearly

$$2U_n(p) + V_n(p) = 1, \tag{2.18}$$

and

$$V_n(p) = \frac{P_p(\mathrm{O} \leftrightarrow a_n, \mathrm{O} \leftrightarrow b_n \text{ in } F_n, \mathrm{O} \not\leftrightarrow a_{n+1}, \mathrm{O} \not\leftrightarrow b_{n+1})}{P_p(M = n)}. \tag{2.19}$$

We consider the event of the numerator of (2.19), $\{\mathrm{O} \leftrightarrow a_n, \mathrm{O} \leftrightarrow b_n \text{ in } F_n, \mathrm{O} \not\leftrightarrow a_{n+1}, \mathrm{O} \not\leftrightarrow b_{n+1}\}$. We divide the case into seven parts as Fig. 2.5. Since the events in F_n, F'_n, F''_n are independent, we have

$$V_n(p) = \frac{\Theta_n(1 - 2\Phi_n - \Phi_n^2 + 4\Phi_n\Theta_n - 2\Theta_n^2)}{P_p(M = n)}. \tag{2.20}$$

Here we denoted $\Phi_n = \Phi_n(p), \Theta_n = \Theta_n(p)$ briefly. Note that

$$\begin{aligned} & P_p(M = n) \\ = \;& P_p(M \geq n) - P_p(M \geq n + 1) \\ = \;& 2\Phi_n - \Theta_n - (2\Phi_{n+1} - \Theta_{n+1}) \\ = \;& 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n^2 + 3\Phi_n^2\Theta_n - 2\Theta_n^3 \end{aligned} \tag{2.21}$$

by (2.4). Hence by (2.18),

$$\begin{aligned} U_n(p) &= \frac{1}{2}\{1 - V_n(p)\} \\ &= \frac{(\Phi_n - \Theta_n)(1 - \Phi_n - \Phi_n^2 + \Phi_n\Theta_n)}{P_p(M = n)}. \end{aligned} \tag{2.22}$$

Let

$$n_0 = n_0(p) = \sup\{n : \Theta_n(p) \geq \frac{2}{3}\}. \tag{2.23}$$

Lemma 2.8. $V_n(p) \geq \frac{2}{9}$ *if* $n < n_0$.

Proof. From (2.18), it is enough to show

$$\frac{V_n(p)}{2U_n(p)} \geq \frac{2}{7}. \tag{2.24}$$

Let

$$\kappa(x, y) = \frac{y(1 - 2x - x^2 + 4xy - 2y^2)}{2(x - y)(1 - x - x^2 + xy)}.$$

By (2.20) and (2.22), (2.24) follows from the following:

$$\kappa(x, y) \geq \frac{2}{7} \quad \text{for} \quad \frac{2}{3} \leq x < 1, \frac{1}{2}(3x - 1) < y < x. \tag{2.25}$$

The second condition in (2.25) comes from the fact that

$$3\Phi_n(p) - 2\Theta_n(p) = \Psi_n(p) < 1. \tag{2.26}$$

Let $y/x = t$. Then the domain of (2.25) is $2/3 \leq x < 1/(3 - 2t), 2/3 \leq t < y < 1$. And

$$\kappa(x, tx) = \frac{t}{2(1 - t)}\{1 - \frac{x + (-3t + 2t^2)x^2}{1 - x - (1 - t)x^2}\}.$$

Now let

$$\lambda(x) = \frac{x + (-3t + 2t^2)x^2}{1 - x - (1 - t)x^2}.$$

From a direct calculation,

$$\lambda'(x) = \frac{(1 + 2t - 2t^2)x^2 + 2(-3t + 2t^2)x + 1}{\{1 - x - (1 - t)x^2\}^2}.$$

We see that if $2/3 \leq t < 1, \lambda'(x) > 0$ for $2/3 \leq x < 1/(3 - 2t)$. Therefore

$$\kappa(x, tx) > \kappa(\frac{1}{3 - 2t}, \frac{t}{3 - 2t}) = \frac{t}{5 - 4t} \geq \frac{2}{7}. \quad \square$$

Next, we estimate the expectation of $|C|$ on condition that $M = n \quad (n < n_0)$.

Lemma 2.9. $E_p(|C||M = n) \geq \frac{2}{9} \cdot 3^n$ *if* $n < n_0$.

To prove the above Lemma, we use the following inequality:

Lemma 2.10. *For all $a \in F_n$,*

$$P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) \geq \Phi_n(p). \tag{2.27}$$

Proof. Besides (2.27), we introduce a similar inequality:

$$P_p(a \leftrightarrow a_n \text{ or } a \leftrightarrow b_n) \geq P_p(\mathbf{O} \leftrightarrow a_n \text{ or } \mathbf{O} \leftrightarrow b_n) \text{ for all } a \in F_n. \tag{2.28}$$

We prove (2.27) and (2.28) by induction at the same time. If $n = 0$, clearly both of them hold. Suppose (2.27) and (2.28) for $n = k$.

We prove (2.27) for $n = k + 1$ at first. By symmetry, it is sufficient to prove the cases (i) $a \in F_k$ and (ii) $a \in F'_k$.

(i) Suppose $a \in F_k$. By using (2.4), we see $\Phi_k(p) \geq \Phi_{k+1}(p)$. Indeed, suppose $\Phi_k(p) \geq 1/3$, then

$$\begin{aligned}
\frac{\Phi_{k+1}}{\Phi_k} &= \Phi_k + \{\Phi_k\}^2 - \{\Theta_k\}^2 \\
&\leq \Phi_k + \{\Phi_k\}^2 - (\frac{3\Phi_k - 1}{2})^2 \\
&\leq -\frac{5}{4}(1 - \Phi_k)^2 + 1 \\
&\leq 1.
\end{aligned} \tag{2.29}$$

Here we used (2.26). Combining this with assumption, we see (2.27) for $n = k + 1$ in this case.

(ii) Suppose $a \in F'_k$. Let C_n^1, C_n^2, C_n^3 be events given by

$$\begin{aligned}
C_n^1 &= \{\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \not\leftrightarrow c_n \text{ in } F_n \cup F''{}_n\}, \\
C_n^2 &= \{\mathbf{O} \not\leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow c_n \text{ in } F_n \cup F''{}_n\}, \\
C_n^3 &= \{\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow c_n \text{ in } F_n \cup F''{}_n\}.
\end{aligned}$$

We see

$$\begin{aligned}
& P_p(\mathbf{O} \leftrightarrow a \text{ in } F_{k+1}) \\
= \; & P_p(C_k^1)P_p(a_k \leftrightarrow a \text{ in } F'_k) + P_p(C_k^2)P_p(c_k \leftrightarrow a \text{ in } F'_k) \\
& \qquad + P_p(C_k^3)P_p(a_k \leftrightarrow a \text{ or } c_k \leftrightarrow a \text{ in } F'_k) \\
\geq \; & (\Phi_k - \Phi_k\Theta_k) \cdot \Phi_k + (\Phi_k - \Theta_k)\Phi_k \cdot \Phi_k + \Phi_k\Theta_k \cdot (2\Phi_k - \Theta_k) \\
= \; & \Phi_k^2 + \Phi_k^3 - \Phi_k\Theta_k^2 = \Phi_{k+1}.
\end{aligned}$$

Here we used assumption for the inequality. We thus obtain (2.27) for $n = k + 1$.

We proceed to prove (2.28) for $n = k + 1$.

(i) Suppose $a \in F_k$. Let $D_n^1, D_n^2, \ldots, D_n^5$ be events given by

$$\begin{aligned}
D_n^1 &= \{a_n \leftrightarrow a_{n+1} \text{ or } a_n \leftrightarrow b_{n+1} \text{ in } F'_k \cup F''_k\}, \\
D_n^2 &= \{b_n \leftrightarrow a_{n+1} \text{ or } b_n \leftrightarrow b_{n+1} \text{ in } F'_k \cup F''_k\},
\end{aligned}$$

$D_n^3 = D_n^1 \cap (D_n^2)^c, D_n^4 = (D_n^1)^c \cap D_n^2, D_n^5 = D_n^1 \cap D_n^2$. We see

$$\begin{aligned}
& P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1}) \\
= \;& P_p(D_k^3)P_p(a \leftrightarrow a_k \text{ in } F_k) + P_p(D_k^4)P_p(a \leftrightarrow b_k \text{ in } F_k) \\
& \qquad + P_p(D_k^5)P_p(a \leftrightarrow a_k \text{ or } a \leftrightarrow b_k \text{ in } F_k) \\
\geq \;& P_p(D_k^3)P_p(\mathbf{O} \leftrightarrow a_k \text{ in } F_k) + P_p(D_k^4)P_p(\mathbf{O} \leftrightarrow b_k \text{ in } F_k) \\
& \qquad + P_p(D_k^5)P_p(\mathbf{O} \leftrightarrow a_k \text{ or } \mathbf{O} \leftrightarrow b_k \text{ in } F_k) \\
= \;& P_p(\mathbf{O} \leftrightarrow a_{k+1} \text{ or } \mathbf{O} \leftrightarrow b_{k+1})
\end{aligned}$$

by assumption.
(ii) Suppose $a \in F'_k$. We see

$$\begin{aligned}
& P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1}) \\
\geq \;& P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k) \\
& + P_p(a \not\leftrightarrow a_{k+1} \text{ and } a \leftrightarrow c_k \text{ in } F'_k)P_p(c_k \leftrightarrow b_{k+1} \text{ in } F''_k).
\end{aligned} \tag{2.30}$$

Here we note that

$$\begin{aligned}
& P_p(a \not\leftrightarrow a_{k+1} \text{ and } a \leftrightarrow c_k \text{ in } F'_k) \\
= \;& P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow c_k \text{ in } F'_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k) \\
\geq \;& (2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k)
\end{aligned}$$

by assumption. Using this and (2.30), we have

$$\begin{aligned}
& P_p(a \leftrightarrow a_{k+1} \text{ or } a \leftrightarrow b_{k+1}) \\
\geq \;& P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k) \\
& + \{(2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k)\}P_p(c_k \leftrightarrow b_{k+1} \text{ in } F''_k) \\
= \;& P_p(a \leftrightarrow a_{k+1} \text{ in } F'_k)(1 - \Phi_k) + (2\Phi_k - \Theta_k)\Phi_k \\
\geq \;& \Phi_k(1 - \Phi_k) + 2\Phi_k^2 - \Phi_k\Theta_k = \Phi_k + \Phi_k^2 - \Phi_k\Theta_k.
\end{aligned}$$

Here we used assumption again. Now it is enough to show

$$\Phi_k + \Phi_k^2 - \Phi_k\Theta_k - P_p(\mathbf{O} \leftrightarrow a_{k+1} \text{ or } \mathbf{O} \leftrightarrow b_{k+1}) \geq 0. \tag{2.31}$$

The left-hand side of (2.31) equals

$$\begin{aligned}
& \Phi_k + \Phi_k^2 - \Phi_k\Theta_k - (2\Phi_{k+1} - \Theta_{k+1}) \\
= \;& (\Phi_k + \Phi_k^2 - \Phi_k\Theta_k) - 2(\Phi_k^2 + \Phi_k^3 - \Phi_k\Theta_k^2) + (3\Phi_k^2\Theta_k - 2\Theta_k^3) \\
= \;& \Phi_k(1 - \Theta_k)(1 - 3\Phi_k + 2\Theta_k) + 2(\Phi_k - \Theta_k)^2(1 - \Phi_k) \\
& \qquad + 2\Theta_k(\Phi_k - \Theta_k)(1 - 2\Phi_k + \Theta_k).
\end{aligned}$$

By (26), we see all terms above are nonnegative. Hence the proof is completed. □

Proof of Lemma 2.9.

$$\begin{aligned} E_p(|C||M=n) &= \sum_{a\in V} P_p(\mathbf{O} \leftrightarrow a|M=n) \\ &\geq \sum_{a\in F_n} P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n|M=n) \\ &\geq \sum_{a\in F_n} \frac{P_p(\mathbf{O} \leftrightarrow a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, M=n)}{P_p(M=n)}. \end{aligned}$$

Let $D_n^6 = (D_n^1)^c \cap (D_n^2)^c$. Note that if $M=n$ and $\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n$, then $(D_n^6)^c$ occurs. For $a \in F_n$, we see

$$\begin{aligned} & P_p(\mathbf{O} \leftrightarrow a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, M=n) \\ = \;& P_p(\mathbf{O} \leftrightarrow \; a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, D_n^6 \text{ occurs }) \\ = \;& P_p(\mathbf{O} \leftrightarrow a, \mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \\ \geq \;& P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6) \\ = \;& P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n, M=n). \end{aligned}$$

Here we used FKG inequality for the forth line. Therefore

$$\begin{aligned} E_p(|C||M=n) &\geq \sum_{a\in F_n} P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) P_p(\mathbf{O} \leftrightarrow a_n, \mathbf{O} \leftrightarrow b_n \text{ in } F_n|M=n) \\ &\geq \frac{2}{9} \sum_{a\in F_n} P_p(\mathbf{O} \leftrightarrow a \text{ in } F_n) \end{aligned}$$

by Lemma 2.8. Note that $|\{a \in V : a \in F_n\}| = \frac{3}{2}(3^n+1)$. By virtue of Lemma 2.10, we see

$$\begin{aligned} E_p(|C||M=n) &\geq \frac{2}{9} \cdot \frac{3}{2} \cdot 3^n \Phi_n(p) \\ &\geq \frac{2}{9} \cdot 3^n \quad \text{for } n < n_0. \end{aligned}$$

We used (2.23) and the fact that $\Phi_n(p) \geq \Theta_n(p)$ for the last inequality. □

We proceed to the estimate of $P_p(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n)$.

Lemma 2.11 $P_p(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n) \geq \frac{2}{79} P_p(M=n)$ *if* $n < n_0$.

Proof. Note that $|C| \leq \frac{9}{2} \cdot 3^n$ if $M=n$. Then we see the following.

$$\begin{aligned} & E_p(|C||M=n) \\ = \;& E_p(|C|; |C| \geq \frac{1}{9} \cdot 3^n |M=n) + E_p(|C|; |C| < \frac{1}{9} \cdot 3^n |M=n) \\ \leq \;& \frac{9}{2} \cdot 3^n P_p(|C| \geq \frac{1}{9} \cdot 3^n |M=n) + \frac{1}{9} \cdot 3^n P_p(|C| < \frac{1}{9} \cdot 3^n |M=n). \end{aligned}$$

By Lemma 2.9, we have

$$P_p(|C| \geq \frac{1}{9} \cdot 3^n |M=n) \geq \frac{2}{79},$$

thus the proof is completed. □

Lemma 2.12. $P_p(M=n) > \Phi_n(p)\{1-\Phi_n(p)\}^2$ *if* $n < n_0$.

Proof. Recall (2.21), that is

$$P_p(M=n) = 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n^2 + 3\Phi_n^2\Theta_n - 2\Theta_n^3.$$

Let $\pi(y) = 2x - y - 2x^2 - 2x^3 + 2xy^2 + 3x^2y - 2y^3$. It is enough to show that $\pi(y) > x(1-x)^2$ if $2/3 \le x < 1, (3x-1)/2 < y < x$. Note that

$$\pi'(y) = -6y^2 + 4xy + 3x^2 - 1,$$

and that

$$\pi'(\frac{3x-1}{2}) = \frac{1}{2}(1-x)(9x-5) > 0, \pi'(x) = x^2 - 1 < 0.$$

Hence $\pi(y) > \min\{\pi((3x-1)/2), \pi(x)\}$. $\pi((3x-1)/2) = (1-x)^2(x+3)/4$ and $\pi(x) = x(1-x)^2$, so $\pi((3x-1)/2) > \pi(x)$ for $2/3 \le x < 1$. This completes the proof. □

Proof of Theorem 2.2. First, we estimate $E_p|C|^k$ from below. By using Lemma 2.11 and 2.12, we see

$$\begin{aligned} E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C| = l) \\ &\ge \sum_{n=4,8,12\ldots} (\frac{1}{9}\cdot 3^n)^k P_p(\frac{1}{9}\cdot 3^n \le |C| \le \frac{9}{2}\cdot 3^n) \\ &\ge \frac{1}{9^k}\cdot\frac{2}{79} \sum_{\substack{m \in \mathbf{N} \\ 4m < n_0}} 3^{4km}\Phi_{4m}(p)\{1 - \Phi_{4m}(p)\}^2. \end{aligned}$$

Let p be sufficiently large. Note that the function $\iota(x) = x(1-x)^2$ is decreasing in $2/3 \le x < 1$, and $\Phi_{4m}(p) \le \mathrm{e}^{-2^{4m}/\xi(p)}$ by (2.7). We can see

$$\begin{aligned} &\sum_{\substack{m \in \mathbf{N} \\ 4m < n_0}} 3^{4km}\Phi_{4m}(p)\{1 - \Phi_{4m}(p)\}^2 \\ \ge& \sum_{\substack{m \in \mathbf{N} \\ 4m < n_0}} 3^{4km}\mathrm{e}^{-2^{4m}/\xi(p)}(1 - \mathrm{e}^{-2^{4m}/\xi(p)})^2 \\ \ge& \int_1^{\frac{n_0}{4}-1} 3^{4kx}\mathrm{e}^{-2^{4x}/\xi(p)}(1 - \mathrm{e}^{-2^{4x}/\xi(p)})^2 dx \\ =& \frac{\{\xi(p)\}^{Dk}}{4\log 2}\int_{2^4/\xi(p)}^{2^{n_0-4}/\xi(p)} y^{Dk-1}\mathrm{e}^{-y}(1 - \mathrm{e}^{-y})^2 dy. \end{aligned}$$

Here we set $y = 2^x/\xi(p)$ in the last line. Note that $\Theta_{n_0+1}(p) < 2/3$, hence $\Phi_{n_0+1}(p) < (1 + 2\Theta_{n_0+1}(p))/3 < 7/9$ by (2.24). From (2.29), if $\Phi_k(p) < 7/9$, then $\Phi_{k+1}(p)/\Phi_k(p) < 76/81$. We see

$$\Phi_{n_0+12}(p) < (\frac{76}{81})^{11}\cdot\frac{7}{9} < \frac{1}{2}\cdot\frac{7}{9}.$$

Combining this with (2.7), we have

$$\frac{1}{2}\mathrm{e}^{-2^{n_0+12}/\xi(p)} \le \Phi_{n_0+12}(p) < \frac{1}{2}\cdot\frac{7}{9}.$$

Hence $2^{n_0-4}/\xi(p) > 2^{-16}\log(9/7)$. Since $\xi(p) \to \infty$ as $p \to 1$, $E_p|C|^k > K_1\{\xi(p)\}^{Dk}$ holds if we take

$$K_1(k) = \int_{2^{-17}\log(9/7)}^{2^{-16}\log(9/7)} y^{Dk-1}\mathrm{e}^{-y}(1 - \mathrm{e}^{-y})^2 dy > 0.$$

Now we proceed to estimate from above. Note that $P_p(M \geq n) \leq 2\Phi_n(p) \leq 2\mathrm{e}^{-2^n/\xi(p)}$, and we can see easily $P_p(\frac{3}{2}\cdot 3^n < |C| \leq \frac{3}{2}\cdot 3^{n+1}) \leq P_p(M \geq n) \leq 2\mathrm{e}^{-2^n/\xi(p)}$. Hence

$$
\begin{aligned}
E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C| = l) \\
&\leq 1 + \sum_{n=0}^{\infty} (\frac{3}{2}\cdot 3^{n+1})^k P_p(\frac{3}{2}\cdot 3^n < |C| \leq \frac{3}{2}\cdot 3^{n+1}) \\
&\leq 1 + 2\cdot(\frac{9}{2})^k \sum_{n=0}^{\infty} 3^{kn}\mathrm{e}^{-2^n/\xi(p)}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\int_0^\infty 3^{kx}\mathrm{e}^{-2^x/\xi(p)}dx &= \frac{\{\xi(p)\}^{Dk}}{\log 2}\int_{\xi(p)^{-1}}^{\infty} y^{Dk-1}\mathrm{e}^{-y}dy \\
&\leq \frac{\Gamma(Dk)}{\log 2}\cdot\{\xi(p)\}^{Dk}.
\end{aligned}
$$

So we can take $K_2(k) < \infty$ such that $E_p|C|^k < K_2\{\xi(p)\}^{Dk}$. $\square$

2.5 Site percolation on the pre-Sierpinski gasket

We define the Bernoulli site percolation on the pre-Sierpinski gasket; each vertices in V are open with probability p and closed with $1-p$ independently. Let $\widetilde{P}_p$ denote its distribution. We write $x \leftrightarrow y$ if there exists a sequence of open vertices $x = x_0, x_1, \cdots, x_{n-1}, x_n = y$ such that there is a bond in E which connects x_j with x_{j+1} for $0 \leq j \leq n-1$. We define another notations in the same manner as before. We introduce connectivity functions;

$$
\begin{aligned}
\widetilde{\Phi}_n(p) &= \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ in } \triangle\mathbf{O}a_nb_n), \\
\widetilde{\Theta}_n(p) &= \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow b_n \text{ in } \triangle\mathbf{O}a_nb_n).
\end{aligned}
$$

We see $\widetilde{\Phi}_0(p) = p^2$ and $\widetilde{\Theta}_0(p) = p^3$ by definition.

Proposition 2.13. *For each $n \geq 0$ and $0 \leq p \leq 1$,*

$$
\begin{aligned}
\widetilde{\Phi}_{n+1}(p) &= p^{-1}\{\widetilde{\Phi}_n(p)\}^2 + p^{-2}\{\widetilde{\Phi}_n(p)\}^3 - p^{-3}\widetilde{\Phi}_n(p)\{\widetilde{\Theta_n}(p)\}^2, && (2.32)\\
\widetilde{\Theta}_{n+1}(p) &= 3p^{-2}\{\widetilde{\Phi}_n(p)\}^2\widetilde{\Theta}_n(p) - 2p^{-3}\{\widetilde{\Theta}_n(p)\}^3. && (2.33)
\end{aligned}
$$

Proof. We prove (2.32). Let $\widetilde{A_n^1}$ and $\widetilde{A_n^2}$ be events given by

$$
\begin{aligned}
\widetilde{A_n^1} &= \{\mathbf{O} \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } {F'}_n\}, \\
\widetilde{A_n^2} &= \{\mathbf{O} \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow c_n \text{ in } {F''}_n\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } {F'}_n\}.
\end{aligned}
$$

Then we have

$$
\widetilde{\Theta}_{n+1}(p) = \widetilde{P}_p(\widetilde{A_n^1}) + \widetilde{P}_p(\widetilde{A_n^2}) - \widetilde{P}_p(\widetilde{A_n^1} \cap \widetilde{A_n^2}). \tag{2.34}
$$

Remark that $F_n \cap {F'}_n = \{a_n\}$. So we see $\widetilde{P}_p(\widetilde{A_n^1}) = p^{-1}\{\widetilde{\Phi}_n(p)\}^2$. Similarly, we have $\widetilde{P}_p(\widetilde{A_n^2}) = p^{-2}\{\widetilde{\Phi}_n(p)\}^3, \widetilde{P}_p(\widetilde{A_n^1} \cap \widetilde{A_n^2}) = p^{-3}\{\widetilde{\Theta}_n(p)\}^2\widetilde{\Phi}_n(p)$. Thus (2.32) follows from (2.34) immediately. (2.33) is proved in the same way. $\square$

Let $\widehat{\Phi}_n(p) = p^{-1}\widetilde{\Phi}_n(p)$ and $\widehat{\Theta}_n(p) = p^{-\frac{3}{2}}\widetilde{\Theta}_n(p)$. Then we have the same recursions as (2.4), (2.5):

$$\begin{aligned} \widehat{\Phi}_{n+1}(p) &= \{\widehat{\Phi}_n(p)\}^2 + \{\widehat{\Phi}_n(p)\}^3 - \widehat{\Phi}_n(p)\{\widehat{\Theta}_n(p)\}^2, & (2.35) \\ \widehat{\Theta}_{n+1}(p) &= 3\{\widehat{\Phi}_n(p)\}^2\widehat{\Theta}_n(p) - 2\{\widehat{\Theta}_n(p)\}^3. & (2.36) \end{aligned}$$

Hence we see that there exists $\hat{\xi}(p) > 0$ such that

$$\lim_{n\to\infty} \frac{\widehat{\Phi}_n(p)}{\exp\{-2^n/\hat{\xi}(p)\}} = 1, \quad \text{that is} \quad \lim_{n\to\infty} \frac{\widetilde{P}_p(\mathbf{O} \leftrightarrow a_n)}{p\exp\{-2^n/\hat{\xi}(p)\}} = 1.$$

Lemma 2.14 *Let* $\sqrt{\tilde{p}} = \sqrt{p} + 6(1-\sqrt{p})^2$*. Then there exists* $\varepsilon > 0$ *such that*

$$2 \le \frac{\hat{\xi}(\tilde{p})}{\hat{\xi}(p)} \le 4 \quad \textit{for} \quad 1-\varepsilon < p < 1.$$

Proof. We use the same method as in Section 3 again. Let

$$\widehat{\Psi}_n(p) = 3\widehat{\Phi}_n(p) - 2\widehat{\Theta}_n(p). \tag{2.37}$$

To apply (2.9), first we prove $(\widehat{\Theta}_n(p), \widehat{\Psi}_n(p)) \in D$. (Recall $D = \{(x,y) : 0 < x \le y < 1\}$.) Since $\widehat{\Psi}_n(p) = \widehat{\Theta}_n(p) + 3\{\widehat{\Phi}_n(p) - \widehat{\Theta}_n(p)\}$, it is enough to prove $\widehat{\Phi}_n(p) \ge \widehat{\Theta}_n(p)$. Now

$$\begin{aligned} \widehat{\Phi}_n(p) &= p^{-1} \times \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ in } F_n) \\ &= \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ in } F_n | a_n \text{ is open}) \\ &= \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ in } F_n | a_n, b_n \text{ are open}), \\ \widehat{\Theta}_n(p) &= p^{-\frac{3}{2}} \times \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow b_n \text{ in } F_n) \\ &\le p^{-2} \times \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow b_n \text{ in } F_n) \\ &= \widetilde{P}_p(\mathbf{O} \leftrightarrow a_n \text{ and } \mathbf{O} \leftrightarrow b_n \text{ in } F_n | a_n, b_n \text{ are open}). \end{aligned}$$

Hence we have $\widehat{\Phi}_n(p) \ge \widehat{\Theta}_n(p)$, which implies $(\widehat{\Theta}_n(p), \widehat{\Psi}_n(p)) \in D$.

A direct calculation from (2.35) and (2.36) shows

$$\begin{aligned} \widehat{\Theta}_2(\tilde{p}) - \widehat{\Theta}_1(p) &= 6(1-\sqrt{p})^2 + 204(1-\sqrt{p})^3 + \cdots, \\ \widehat{\Psi}_2(\tilde{p}) - \widehat{\Psi}_1(p) &= 12(1-\sqrt{p})^3 + \cdots, \\ \widehat{\Theta}_3(\tilde{p}) - \widehat{\Theta}_1(p) &= -6(1-\sqrt{p})^2 + 204(1-\sqrt{p})^3 + \cdots, \\ \widehat{\Psi}_3(\tilde{p}) - \widehat{\Psi}_1(p) &= -12(1-\sqrt{p})^3 + \cdots. \end{aligned}$$

We can take $\varepsilon > 0$ such that

$$\widehat{\Theta}_3(\tilde{p}) < \widehat{\Theta}_1(p) < \widehat{\Theta}_2(\tilde{p}), \;\; \widehat{\Psi}_3(\tilde{p}) < \widehat{\Psi}_1(p) < \widehat{\Psi}_2(\tilde{p})$$

for $1-\varepsilon < p < 1$.

Now we apply (2.9). We have for $n \ge 1$ and $1-\varepsilon < p < 1$,

$$\widehat{\Theta}_{n+2}(\tilde{p}) < \widehat{\Theta}_n(p) < \widehat{\Theta}_{n+1}(\tilde{p}) \text{ and } \widehat{\Psi}_{n+2}(\tilde{p}) < \widehat{\Psi}_n(p) < \widehat{\Psi}_{n+1}(\tilde{p}).$$

We see $\widehat{\Phi}_{n+2}(\tilde{p}) < \widehat{\Phi}_n(p) < \widehat{\Phi}_{n+1}(\tilde{p})$ by (2.37), so we have the conclusion. □

Proof of Theorem 2.3. Note that $\tilde{p} = \{\sqrt{p} + 6(1-\sqrt{p})^2\}^2 = p + 3(1-p)^2 + o((1-p)^2)$ as $p \to 1$. We have Theorem 2.3 in the same way as in Section 3.

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第3章 Asymptotic behavior of the correlation length

3.1 Introduction

Percolation is a model of disordered media. It is very attractive because it is one of the simplest model to observe phase transitions. In recent years percolation has been studied well, most of the studies are on *periodic* graphs such as $\mathbf{Z}^d$. See [2], [3], [4] and references there in. The definition of the periodic graph is mentioned in [3].

In this paper, we study percolation on *fractal lattices*, which are not in the class of the periodic graphs. There are some reasons why we consider percolation on the fractal lattices. First, many objects in nature has fractal shapes. For instance, imagine water and nourishment percolating in the roots or branches of a tree. Second, we want to justify *scaling relations* of percolation. To applicate the renormalization methods, self-similarity of the graph is more important than periodicity. Third, we have mathematical interests on fractals. Most of all, studies of self-avoiding walk on Sierpinski gaskets ([5],[6],[7]) gave us good motivation.

To state problems, first we mention about bond percolation on 2-dimensional pre-Sierpinski gasket as in [1]. Set $\mathbf{O} = (0,0), \mathbf{a} = (1,0), \mathbf{b} = (1/2, \sqrt{3}/2)$. Set G^0 be the graph which consists of the vertices and edges of the regular triangle $\triangle\mathbf{Oab}$. Let $\{G^n\}_{n=0,1,2,\cdots}$ be the sequence of graphs given by

$$G^{n+1} = G^n \cup (G^n + 2^n\mathbf{a}) \cup (G^n + 2^n\mathbf{b})$$

where $A + \mathbf{a} = \{\mathbf{x} + \mathbf{a} \ : \mathbf{x} \in A\}$. Let $G = \bigcup_{n=0}^{\infty} G^n$. We call G the *pre-Sierpinski gasket.* (Figure 3.1) Note that $\tilde{G} = cl\Big(\bigcup_{n=0}^{\infty} 2^{-n}G\Big)$ become the Sierpinski gasket. Let V be the set of the vertices in G, and E the set of the edges in G with length 1.

Let us define the Bernoulli bond percolation on the pre-Sierpinski gasket. Each edge in E is *open* with probability p and *closed* with probability $1-p$ independently. Let P_p denote its distribution. More precise definition of the probability space will be mentioned in Section 2. We think of open bonds as permitting to go along the bond. We write $\mathbf{v} \leftrightarrow \mathbf{v}'$ if there is an open path from $\mathbf{v}$ to $\mathbf{v}'$. We define *open cluster* $C = \{\mathbf{v} \in V : \mathbf{O} \leftrightarrow \mathbf{v}\}$.

We define the *percolation probability*

$$\theta(p) = P_p(|C| = \infty) \tag{3.1}$$

where $|C|$ denotes the number of vertices contained in C. Let p_c denote the *critical point*; that is

$$p_c = \inf\{p : \theta(p) > 0\}. \tag{3.2}$$

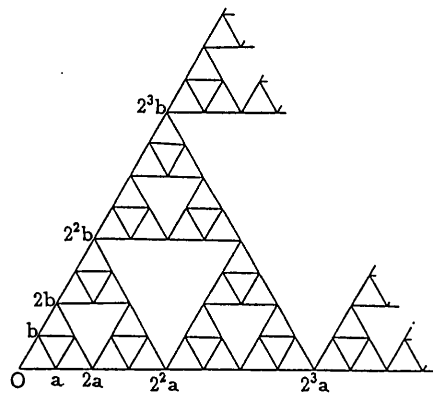

Figure 3.1: 2-dimensional pre-Sierpinski gasket

$p_c = 1$ for the pre-Sierpinski gasket because it is finitely ramified.

Remark. All graphs we treat in this paper are finitely ramified and $p_c = 1$.

The *correlation length* is defined by

$$\xi(p) = \lim_{n\to\infty}\left\{-\frac{1}{2^n}\log P_p(\mathbf{O} \leftrightarrow 2^n\mathbf{a})\right\}^{-1}. \tag{3.3}$$

The existence of the limit is proved in [1]. Note that the definition above is equivalent to

$$\xi(p) = \lim_{n\to\infty}\left\{-\frac{1}{2^n}\log P_p(\mathbf{O} \leftrightarrow 2^n\mathbf{a} \text{ in } G^n)\right\}^{-1} \tag{3.4}$$

or

$$\xi(p) = \lim_{n\to\infty}\left\{-\frac{1}{2^n}\log P_p(\mathbf{O} \leftrightarrow 2^n\mathbf{a} \text{ or } \mathbf{O} \leftrightarrow 2^n\mathbf{b} \text{ in } F^n)\right\}^{-1}. \tag{3.5}$$

It is clear that $\xi(p) \to \infty$ as $p \to 1$. We observe the asymptotic behavior of $\xi(p)$, how fast it diverges to infinity. We write $f(p) \approx g(p)$ as $p \to p_0$ if $\log f(p)/\log g(p) \to 1$ as $p \to p_0$.

Theorem 3.1 (2-dimensional pre-Sierpinski gasket)

$$\xi(p) \approx \exp\left\{\frac{\log 2}{4}(1-p)^{-2}\right\} \quad as \quad p \to 1. \tag{3.6}$$

This result is not contained in [1].

We mention about results and conjectures of percolation on $\mathbf{Z}^d$. It is conjectured (see [2])

$$\xi(p) \approx |p_c - p|^{-\nu(d)} \quad \text{as} \quad p \to p_c.$$

The value $\nu(d)$ is called the *critical exponent.* It is proved that $\nu(d) = 1/2$ for sufficiently large d ([8]), and conjectured $\nu(2) = 4/3$ (see [9]). Our result is quite different from results on $\mathbf{Z}^d$. In physical literature ([10]), this remarkable difference between on $\mathbf{Z}^d$ and on Sierpinski gaskets was suggested by using formal renormalization arguments. Our contribution is that we prove Theorem 3.1 rigorously. And we apply our method to another fractal lattices. We obtain similar results, Theorem 3.2, Theorem 3.11 and Theorem 3.12.

The organization of this paper is as follows: we state the precise definition of bond percolation on d-dimensional pre-Sierpinski gasket in Section 2 and observe the asymptotic behavior in Section 3. In Section 4 we study percolation on the pentakun lattice and the snowflake lattice, which are also in the class of fractal lattices.

3.2 Definition of bond percolation on d-dimensional pre-Sierpinski gasket

3.2.1 Precise definition and the main theorem

In this section we state the definition of percolation on d-dimensional pre-Sierpinski gasket for $d \geq 2$. It is well-known that there is a compact set K of $\mathbf{R}^d$ such that

$$K = \bigcup_{i=1}^{N} f_i(K) \tag{3.7}$$

where $f_1, f_2, \dots, f_N : \mathbf{R}^d \to \mathbf{R}^d$ are contraction mappings. K is called *self-similar set.*

Sierpinski gasket is an example of the self-similar sets. Let $\mathbf{a}_0 = \mathbf{O}$ be the origin of $\mathbf{R}^d$, and let $\mathbf{a}_i$ $(i = 0, 1, \dots, d)$ be vertices of the d-dimensional simplex with $|\mathbf{a}_i - \mathbf{a}_j| = 1$ for $i \neq j$. Set contraction mappings

$$f_i(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{a}_i) + \mathbf{a}_i \tag{3.8}$$

for $i = 0, 1, \dots, d$. The solution of equation (3.7) for (3.8) is d-dimensional Sierpinski gasket.

Remark. $\bigcup_{i \neq j} \Big(f_i(K) \cap f_j(K) \Big)$ consists of $\begin{pmatrix} d+1 \\ 2 \end{pmatrix}$ points. In this sense, Sierpinski gasket is classified into *finitely ramified fractal.* Notions of finitely ramification are defined rigorously in [11], [12].

Let $\tilde{V}^0 = \{\mathbf{O}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_d\}$, and let $\tilde{E}^0 = \{\overline{\mathbf{a}_i \mathbf{a}_j} : 0 \leq i < j \leq d\}$. Set

$$\begin{aligned} \tilde{V}^n &= \Big\{ (f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n})\mathbf{v} : \mathbf{v} \in \tilde{V}^0, (i_1, i_2, \dots, i_n) \in \{0, 1, \dots, d\}^n \Big\}, & (3.9) \\ \tilde{E}^n &= \Big\{ (f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n})\mathbf{e} : \mathbf{e} \in \tilde{E}^0, (i_1, i_2, \dots, i_n) \in \{0, 1, \dots, d\}^n \Big\}. & (3.10) \end{aligned}$$

Let $V^n = \{2^n \mathbf{v} : \mathbf{v} \in \tilde{V}^n\}$ and $E^n = \{2^n \mathbf{e} : \mathbf{e} \in \tilde{E}^n\}$. Here we write $2^n \mathbf{e} = \overline{2^n \mathbf{v} 2^n \mathbf{v}'}$ where $\mathbf{e} = \overline{\mathbf{v}\mathbf{v}'}$. We define the vertex set $V = \bigcup_{n=0}^{\infty} V^n$ and the edge set $E = \bigcup_{n=0}^{\infty} E^n$. We call the graph $G = (V, E)$ *d-dimensional pre-Sierpinski gasket.* Note that (i) all edges in E have length 1, (ii) all vertices except $\mathbf{O}$ have four adjacent edges and vertices. We denote $\mathbf{a}_i^n = 2^n \big((f_i \circ f_i \circ \cdots \circ f_i)\mathbf{a}_i \big) = 2^n \mathbf{a}_i$, and we see $|\mathbf{a}_i^n| = 2^n$. (See Figure 3.2.)

Now we define the probability space with density parameter $0 \leq p \leq 1$. We take configuration space $\Omega = \{0, 1\}^E$. For $\omega = \{\omega(\mathbf{e}) : \mathbf{e} \in E\} \in \Omega$, we call the edge $\mathbf{e}$ is *open* if $\omega(\mathbf{e}) = 1$ and $\mathbf{e}$ is *closed* if $\omega(\mathbf{e}) = 0$. Let $\mu = \mu_{\mathbf{e}}$ be marginal distribution on $\mathbf{e}$ such that

$$\mu(\omega(\mathbf{e}) = 1) = p, \quad \mu(\omega(\mathbf{e}) = 0) = 1 - p$$

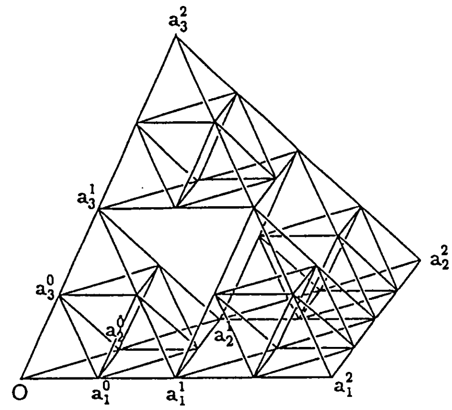

Figure 3.2: G^2 of 3-dimensional pre-Sierpinski gasket

independently of any other edges and identically distributed. We take the product probability measure on Ω such that $P_p = \prod_{\mathbf{e}\in E} \mu_{\mathbf{e}}$. We call $\mathbf{v}$ *is connected to* $\mathbf{v}'$ if there is a sequence of vertices $\mathbf{v}_0 = \mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n = \mathbf{v}'$ and sequence of open edges $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ such that $\mathbf{e}_i = \overline{\mathbf{v}_{i-1}\mathbf{v}_i}$ for $1 \le i \le n$. We denote this event by $\mathbf{v} \leftrightarrow \mathbf{v}'$ and the complement by $\mathbf{v} \not\leftrightarrow \mathbf{v}'$. We call $C(\mathbf{v}) = \{\mathbf{v}' \in V : \mathbf{v} \leftrightarrow \mathbf{v}'\}$ the *open cluster containing* $\mathbf{v}$. Especially we denote the open cluster containing $\mathbf{O}$ by C. Percolation probability and critical point are defined as (3.1), (3.2). We easily see $p_c = 1$ for all d.

The *correlation length* is defined, equivalent to (3.3), as follows:

$$\xi(p) = \lim_{n\to\infty} \left\{ -\frac{1}{2^n} \log P_p(\mathbf{O} \leftrightarrow \mathbf{a}_1^n) \right\}^{-1}. \tag{3.11}$$

We state the main theorem.

Theorem 3.2 (d-dimensional pre-Sierpinski gasket)

$$\xi(p) \approx \exp\left\{ \frac{\log 2}{2^d(d-1)} (1-p)^{-(d^2-d)} \right\} \quad as \quad p \to 1. \tag{3.12}$$

This theorem contains (3.6).

3.2.2 Existence of the correlation length

To simplify notations, we often denote $\mathbf{O}$ by $\mathbf{a}_0^n$. Let $A = \{A_\lambda\}$ be a partition of $\{0, 1, 2, \dots, d\}$ and $\mathcal{A}$ the set of all partitions. We define

$$Q_A^n = \{\mathbf{a}_i^n \leftrightarrow \mathbf{a}_j^n \text{ for } i \in A_\lambda, j \in A_{\lambda'} \text{ and } \lambda = \lambda', \mathbf{a}_i^n \not\leftrightarrow \mathbf{a}_j^n \text{ otherwise } \}. \tag{3.13}$$

Q_A^n *in* G^n denotes the event that Q_A^n occurs in G^n, where $G^n = (V^n, E^n)$ is the subgraph of G. We write Q_A^n *in* $G^n + \mathbf{a}_i^n$ for the event shifted to $G^n + \mathbf{a}_i^n$ for short. We define the *connectivity function*

$$\Phi_A^n(p) = P_p(Q_A^n \text{ in } G^n)$$

and

$$\Phi_{\mathcal{B}}^n(p) = \sum_{B\in\mathcal{B}} \Phi_B^n(p)$$

for $\mathcal{B} \subset \mathcal{A}$. It is clear that these probabilities are not changed by the shift of $\mathbf{a}_i^n$.

For the family of $\{\Phi_A^n(p)\}_{A\in\mathcal{A}}$, we give a numbering $\Phi_1^n(p), \Phi_2^n(p), \dots, \Phi_l^n(p)$ where l is the cardinality of $\mathcal{A}$. Note that $\sum_{k=1}^{l}\Phi_k^n(p) = \Phi_{\mathcal{A}}^n(p) = 1$. Set $D = \{(p_1, p_2, \dots, p_l) \in [0,1]^l : \sum_{k=1}^{l} p_k = 1\}$. It is clear $\mathbf{p}^n = \mathbf{p}^n(p) = (\Phi_1^n(p), \Phi_2^n(p), \dots, \Phi_l^n(p)) \in D$ for all n, p by the remark above.

Proposition 3.3 *There exist functions* $\{F_k\}_{1\le k\le l} : D \to D$ *such that*

$$F_k(\mathbf{p}^n) = \Phi_k^{n+1}(p)$$

for $1 \le k \le l$.

This proposition says that the probability $\Phi_k^{n+1}(p)$ is given as a function of $\Phi_1^n(p), \Phi_2^n(p), \dots, \Phi_l^n(p)$. Note that we know whether an event Q_A^{n+1} in G^{n+1} occurs or not whenever we know for all $0 \le i \le d$ which event of $\{Q_{A'}^n \text{ in } G^n + \mathbf{a}_i^n\}_{A'\in\mathcal{A}}$ occurs. This is because of the finitely ramification of Sierpinski gaskets.

Remark. We have the concrete expression of recursion functions for $d = 2$ ([1]). Let $A_1 = \{\{0,1,2\}\}$, $A_2 = \{\{0,1\},\{2\}\}$, $A_3 = \{\{0\},\{1\},\{2\}\}$. Set $\Phi_k^n(p) = \Phi_{A_k}^n(p)$. By symmetry, $\Phi_1^n(p)+3\Phi_2^n(p)+\Phi_3^n(p) = 1$. We have

$$\begin{aligned}
\Phi_1^{n+1}(p) &= (\Phi_1^n(p))^3 + 6(\Phi_1^n(p))^2\Phi_2^n(p) + 3\Phi_1^n(p)(\Phi_2^n(p))^2, \\
\Phi_2^{n+1}(p) &= (\Phi_1^n(p))^2 + 2\Phi_1^n(p)\Phi_2^n(p) + (\Phi_2^n(p))^2 - 4(\Phi_1^n(p))^2\Phi_2^n(p) \\
&\qquad -(\Phi_1^n(p))^3 + (\Phi_2^n(p))^3.
\end{aligned}$$

We define

$$R_A^n = \{\mathbf{a}_i^n \leftrightarrow \mathbf{a}_j^n \text{ in } G^n \text{ for } i \in A_\lambda, j \in A_{\lambda'} \text{ and } \lambda = \lambda'\}.$$

(Compare this definition with (3.13).) And the definitions of R_A^n in G^n, R_A^n in $G^n + \mathbf{a}_i^n$ follow above. Let $\Psi_A^n(p) = P_p(R_A^n \text{ in } G^n)$.

We confirm the existence of *correlation length.* We write $[0,1] = \{\{0,1\},\{2\},\{3\},\dots,\{d\}\}$.

Lemma 3.4 *Set* $\Psi^n(p) = \Psi_{[0,1]}^n(p)$, *that is the probability of the event* $\mathbf{O} \leftrightarrow \mathbf{a}_1^n$ *in* G^n. *The limit*

$$\xi(p) = \lim_{n\to\infty}\left\{-\frac{1}{2^n}\log\Psi^n(p)\right\}^{-1} \tag{3.14}$$

exists. We call $\xi(p)$ *the correlation length.*

Remark. We give some remarks about definitions of $\xi(p)$. (3.14) differs from (3.11), but there is no effection of restriction *in* G^n because $\lim_{n\to\infty}\left\{P_p(R_{[0,1]}^n \text{ in } G^n)/P_p(R_{[0,1]}^n)\right\} = 1$. (See Lemma 2.6 in [1].)

Set $A_{min} = \{\{0\},\{1\},\dots,\{d\}\}$. (The meaning of the minimum will be mentioned in the next section.) Set $\hat{\Psi}^n(p) = 1 - \Phi_{A_{min}}^n(p)$, the probability of the event that there exist i,j $(i \ne j)$ such that $\mathbf{a}_i^n \leftrightarrow \mathbf{a}_j^n$ in G^n. Then

$$\xi(p) = \lim_{n\to\infty}\left\{-\frac{1}{2^n}\log\hat{\Psi}^n(p)\right\}^{-1},$$

because $\Psi^n(p) \le \hat{\Psi}^n(p) \le c\Psi^n(p)$ for some constant c. This implies the equivalence between (3.4) and (3.5).

We prepare two propositions to prove Lemma 3.4.

Proposition 3.5 *There exists a constant c which depends only on d such that*

$$(\Psi^n(p))^2 \le \Psi^{n+1}(p) \le (\Psi^n(p))^2 + c(\Psi^n(p))^3. \tag{3.15}$$

Proof. The left-hand inequality is clear because $(R^n_{[0,1]}$ in $G^n) \cap (R^n_{[0,1]}$ in $G^n + \mathbf{a}^n_1) \subset (R^{n+1}_{[0,1]}$ in $G^{n+1})$. For the right-hand side, we consider the self-avoiding walks from $\mathbf{O}$ to $\mathbf{a}^1_1$ in G^1. There is only one walk with length 2. Another walks are with length more than 3, and the number of walks are finite. (The number is depend on dimension d.) □

The next proposition is a generalization of proposition 2.5 in [1].

Proposition 3.6 *Suppose that a strictly positive sequence $\{x_n\}_{n=0,1,\cdots}$ and a constant $\alpha > 1$ satisfy*

$$0 < c_1 = \liminf_{n\to\infty} \frac{x_{n+1}}{x_n^\alpha} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n^\alpha} = c_2 < \infty \tag{3.16}$$

for all n. Then there exists $\beta \ge 0$ such that

$$c_1^{\frac{1}{\alpha-1}} \le \liminf_{n\to\infty} \frac{e^{-\alpha^n\beta}}{x_n} \le \limsup_{n\to\infty} \frac{e^{-\alpha^n\beta}}{x_n} \le c_2^{\frac{1}{\alpha-1}}. \tag{3.17}$$

Proof. Set $y_n = x_{n+1}/x_n^\alpha$. We see

$$\begin{aligned} \frac{1}{\alpha^n}\log x_n &= \frac{1}{\alpha^n}\log\Big(x_0^{\alpha^n}\cdot\frac{x_1^{\alpha^{n-1}}}{x_0^{\alpha^n}}\cdot\frac{x_2^{\alpha^{n-2}}}{x_1^{\alpha^{n-1}}}\cdots\frac{x_n}{x_{n-1}^\alpha}\Big) \\ &= \log x_0 + \frac{1}{\alpha}\log y_0 + \frac{1}{\alpha^2}\log y_1 + \cdots + \frac{1}{\alpha^n}\log y_{n-1}. \end{aligned} \tag{3.18}$$

The right hand side of (3.18) converges as $n \to \infty$ since y_n is finite. Let $-\beta$ be the limit. From (3.18) we see $-\alpha^n\beta - \log x_n = \alpha^{-1}\log y_n + \alpha^{-2}\log y_{n+1} + \cdots$. So we have

$$\frac{1}{\alpha-1}\log\Big(\inf_{m\ge n} y_m\Big) \le -\alpha^n\beta - \log x_n \le \frac{1}{\alpha-1}\log\Big(\sup_{m\ge n} y_m\Big)$$

by assumption (3.16). This completes the proof. □

We see the justification of the definition (3.14) as a corollary of the above proposition. Set $x_n = \Psi^n(p)$ and $\alpha = 2$. By (3.15), we can take $c_1 = c_2 = 1$ for $p > 0$. (Note that $\lim_{n\to\infty}\Psi^n(p) = 0$ for $p < 1$.) We have $\lim_{n\to\infty}\{e^{-2^n/\xi(p)}/\Psi^n(p)\} = 1$ where $\xi(p) = \beta^{-1}$. We see that $\xi(p)$ is a continuous function by the proof above. Clearly $\xi(p)$ is increasing by definition.

3.3 Asymptotic behavior of the correlation length

3.3.1 Sufficient conditions to have the asymptotic behavior

We give some definitions first in this section. We introduce the partially order $\prec$ on $\mathcal{A}$ such that

$$A \prec A' \iff A \text{ is a subpartition of } A'.$$

That is $A_\lambda \subset A'_\eta$ if $A_\lambda \cap A'_\eta \ne \emptyset$. It is clear that $A_{max} = \{\{0,1,\dots,d\}\}$ is the maximal partition and $A_{min} = \{\{0\},\{1\},\dots,\{d\}\}$ is the minimal partition of $\mathcal{A}$. A subset $\mathcal{I} \subset \mathcal{A}$ is *increasing set* if and only if

$$I \prec I' \text{ and } I \in \mathcal{I} \implies I' \in \mathcal{I}$$

holds. $\Im$ denotes the set of all increasing sets.

An event $Q \subset \Omega$ is called *increasing event* if and only if

$$\omega \in Q \text{ and } \omega(\mathbf{e}) \leq \omega'(\mathbf{e}) \text{ for all } \mathbf{e} \in E \quad \Longrightarrow \quad \omega' \in Q$$

holds. For instance, Q^n_A is not an increasing event for $A \neq A_{max}$, and R^n_A is an increasing event for any A. We see $Q^n_{\mathcal{I}} = \bigcup_{I \in \mathcal{I}} Q^n_I$ is an increasing event if and only if $\mathcal{I} \in \Im$.

The next lemma is the key to prove the main theorem.

Lemma 3.7 *Suppose*

$$\Phi^n_{\mathcal{I}}(p) \leq \Phi^{n+1}_{\mathcal{I}}(p') \tag{3.19}$$

for all $\mathcal{I} \in \Im$. *Then*

$$\Phi^{n+1}_{\mathcal{I}'}(p) \leq \Phi^{n+2}_{\mathcal{I}'}(p') \tag{3.20}$$

holds for all $\mathcal{I}' \in \Im$.

We give a proof for a modified version of Lemma 3.7.

Lemma 3.8 *Let μ be the Lebesgue measure on $[0,1]$ and ν be a probability measure on $\mathcal{A}$. Let $F : [0,1] \to \mathcal{A}$ be a function that $F^{-1}(A)$ is μ-measurable for all $A \in \mathcal{A}$. Suppose*

$$\mu F^{-1}(\mathcal{I}) \leq \nu(\mathcal{I}) \tag{3.21}$$

for all $\mathcal{I} \in \Im$. Then there is a function $G : [0,1] \to \mathcal{A}$ such that

$$\mu G^{-1} = \nu \tag{3.22}$$

and

$$F(x) \prec G(x) \tag{3.23}$$

almost surely.

We prove Lemma 3.7 as a corollary of Lemma 3.8. There is a function F with $\mu F^{-1}(A) = \Phi^n_A(p)$, because $\mu([0,1]) = \Phi_{\mathcal{A}}(p) = 1$. Set $\nu(A) = \Phi^{n+1}_A(p')$, and (3.19) induce (3.21).

Suppose $G(x)$ with (3.22) and (3.23) is given. Let G_1 be a copy of G^{n+1} and G_2 a copy of G^{n+2}. μ_{d+1} denotes the $(d+1)$-dimensional Lebesgue measure on $[0,1]^{d+1}$. Regard μ_{d+1} as the probability measure which has the uniform distribution. Assume we pick a point $\mathbf{x} = (x_0, x_1, \cdots, x_d)$ with respect to μ_{d+1}. We determine what occur in G_1 and G_2 by the following rule. For each $0 \leq i \leq d$, we regard as

$$\text{the event } (Q^n_{F(x_i)} \text{ in } G^n + \mathbf{a}^n_i) \text{ occurs in } G^n + \mathbf{a}^n_i \tag{3.24}$$

for G_1, and

$$\text{the event } (Q^{n+1}_{G(x_i)} \text{ in } G^{n+1} + \mathbf{a}^{n+1}_i) \text{ occurs in } G^{n+1} + \mathbf{a}^{n+1}_i \tag{3.25}$$

for G_2. The events (Q^{n+1}_A in G_1) and (Q^{n+2}_A in G_2) are measurable by Proposition 2.1. We see

$$\Phi^{n+1}_A(p) = \mu_{d+1}(\mathbf{x} \in [0,1]^{d+1} : Q^{n+1}_A \text{ in } G_1 \text{ occurs by rule (3.24) })$$

and

$$\Phi_A^{n+2}(p') = \mu_{d+1}(\mathbf{x} \in [0,1]^{d+1} : Q_A^{n+2} \text{ in } G_2 \text{ occurs by rule (3.25) })$$

by construction. If $\mathcal{I}' \in \Im$ we have (3.20), because $F(x_i) \prec G(x_i)$ and $Q_{\mathcal{I}'}^n$ in G^n is increasing for all n.

Proof of Lemma 3.8. We write $(\nu - \mu F^{-1})(\cdot) = \nu(\cdot) - \mu F^{-1}(\cdot)$. Set $\mathcal{A}^+ = \{A \in \mathcal{A} : (\nu - \mu F^{-1})(A) \geq 0\}$, $\mathcal{A}^- = \{A \in \mathcal{A} : (\nu - \mu F^{-1})(A) < 0\}$. If $\mathcal{A}^- = \emptyset$, then take $G = F$ and the proof is finished.

Pick a maximal element K of $\mathcal{A}^-$. Set $\mathcal{U}_K = \{A \in \mathcal{A} : K \preceq A\}$ and $\mathcal{U}_K^+ = \mathcal{U}_K \cap \mathcal{A}^+$. Clearly $\mathcal{U}_K^+ = \mathcal{U}_K \setminus \{K\}$, which contains A_{max}. For $\mathcal{U}_K^+$, we give a numbering $U_1 = A_{max}, U_2, \dots, U_k$. Set

$$M_i = \max\{-(\nu - \mu F^{-1})(K) - \sum_{j=i+1}^{k} (\nu - \mu F^{-1})(U_j), 0\}.$$

Remark that M_i is non-decreasing with respect to i. Set $\Delta_{U_1} = M_1$ and $\Delta_{U_i} = M_i - M_{i-1}$ for $2 \leq i \leq k$. We see

$$\Delta_{U_i} \leq (\nu - \mu F^{-1})(U_i). \tag{3.26}$$

For $2 \leq i \leq k$, (3.26) is clear. For $i = 1$,

$$\begin{aligned} & (\nu - \mu F^{-1})(U_1) - M_1 \\ \geq \; & (\nu - \mu F^{-1})(U_1) + (\nu - \mu F^{-1})(K) + \sum_{j=2}^{k} (\nu - \mu F^{-1})(U_j) \\ = \; & (\nu - \mu F^{-1})(\mathcal{U}_K) \geq 0 \end{aligned} \tag{3.27}$$

since $\mathcal{U}_K \in \Im$.

We construct $\tilde{F} : [0,1] \to \mathcal{A}$ as follows.

(i) $\tilde{F}(x) = F(x)$ if $F(x) \neq K$.

(ii) Set $F^{-1}(K) = S$. Take $S_0 \subset S$ such that $\mu(S_0) = \nu(K)$, and define $\tilde{F}(x) = K$ for $x \in S_0$.

(iii) Take a sequence of subsets $\{S_i\}$ for $1 \leq i \leq k$ such that

(a) $S_i \cap S_{i'} = \emptyset$ if $i \neq i'$,

(b) $\bigcup_{i=1}^{k} S_i = S \setminus S_0$,

(c) $\mu(S_i) = \Delta_{U_i}$.

It is possible to satisfy (a), (b) and (c) because $\sum_{i=1}^{k} \Delta_{U_i} = M_k = -(\nu - \mu F^{-1})(K)$. We define $\tilde{F}(x) = U_i$ if $x \in S_i$.

Clearly this map satisfies $F(x) \preceq \tilde{F}(x)$ by construction. Moreover,

$$(\nu - \mu \tilde{F}^{-1})(\mathcal{I}) \geq 0$$

for any $\mathcal{I} \in \Im$. To prove this inequality, we may assume $K \notin \mathcal{I}$.

$$\begin{aligned} (\nu - \mu \tilde{F}^{-1})(\mathcal{I}) & \geq (\nu - \mu F^{-1})(\mathcal{I}) - \sum_{U_i \in \mathcal{U}_K^+ \cap \mathcal{I}} \Delta_{U_i} \\ & \geq (\nu - \mu F^{-1})(\mathcal{I}) + (\nu - \mu F^{-1})(K) - \sum_{U_i \in \mathcal{U}_K^+ \cap \mathcal{I}} (\nu - \mu F^{-1})(U_i) \\ & = (\nu - \mu F^{-1})(\mathcal{I}) + (\nu - \mu F^{-1})(K) - (\nu - \mu F^{-1})(\mathcal{U}_K^+ \cap \mathcal{I}) \\ & = (\nu - \mu F^{-1})(\mathcal{U}_K \cap \mathcal{I}) \geq 0. \end{aligned}$$

We use (3.26) and (3.27) for the second line.

Replace F by $\tilde{F}$ and repeat this procedure to be $\mathcal{A}^- = \emptyset$. □

3.3.2 Probabilities of increasing events on Sierpinski gaskets

We apply Lemma 3.7 to see the asymptotic behavior of the correlation length. Owing to the lemma, if we take p, p' and n which satisfy (3.19) for all $\mathcal{I} \in \Im$, then for any $m > n$ we have

$$\Phi^m_{\mathcal{I}'}(p) \leq \Phi^{m+1}_{\mathcal{I}'}(p') \tag{3.28}$$

for all $\mathcal{I}' \in \Im$. By definition (3.14), we obtain $\dfrac{\xi(p')}{\xi(p)} \geq 2$ from (3.28). Consider the converse. If we take p, p' and n which satisfies

$$\Phi^n_{\mathcal{I}}(p) \geq \Phi^{n+1}_{\mathcal{I}}(p') \tag{3.29}$$

for all $\mathcal{I} \in \Im$, we have $\dfrac{\xi(p')}{\xi(p)} \leq 2$. So it is the problem how to take p, p' and n to satisfy (3.19) or (3.29).

Lemma 3.9

$$\liminf_{p\to 1} \frac{\xi(p + k(1-p)^{d^2-d+1})}{\xi(p)} \geq 2 \quad \textit{if} \quad k > \frac{2^d}{d}, \tag{3.30}$$

and

$$\limsup_{p\to 1} \frac{\xi(p + k(1-p)^{d^2-d+1})}{\xi(p)} \leq 2 \quad \textit{if} \quad k < \frac{2^d}{d}.$$

Proof. We prove (3.30). It is sufficient to show (3.19) for some n. We want to have the expansion of $\Phi^n_{\mathcal{I}}(p)$ with respect to $(1-p)$ because we observe probabilities near the critical point. First, we consider the case $\mathcal{I} = \{A_{max}\}$.

Proposition 3.10 *There exists $N = N(d)$ such that for any $n \geq N$*

$$\begin{aligned} & \Phi^n_{A_{max}}(p) \\ = \ & 1 - (d+1)(1-p)^d + V(1-p) - 2^d(d+1)n(1-p)^{d^2} + W(n, 1-p) \end{aligned} \tag{3.31}$$

where V, W are polynomials of finite degree and $V(x) = o(x^d)$, $W(n,x) = o(x^{d^2})$ as $x \to 0$.

Proof. Observe when the event $Q^n_{A_{max}}$ in G^n does not occur. If it does not occur, at least one vertex of $\mathbf{a}^n_0, \mathbf{a}^n_1, \ldots, \mathbf{a}^n_d$ is not connected to any other d vertices in G^n. We say the vertex is *isolated.*

We consider two typical case for $\mathbf{O} = \mathbf{a}^n_0$ to be isolated. If all adjacent edges of $\mathbf{O}$ are closed, $\mathbf{O}$ cannot be connected to any other vertices. This probability is $(1-p)^d$. Consider the second case. For fixed k $(1 \leq k \leq n-1)$, let E^{k-}_i be set of adjacent edges of $\mathbf{a}^k_i$ contained in E^k and E^{k+}_i set of those not contained in E^k. If edges in E^{k-}_i or in E^{k+}_i are all closed, it cannot go through $\mathbf{a}^k_i$. This probability is approximately $2(1-p)^d$ if p is near to 1. So the probability that it cannot go through $\mathbf{a}^k_1$, $\mathbf{a}^k_2$, ... ,$\mathbf{a}^k_d$ is approximately $2^d(1-p)^{d^2}$, which is independent of k.

We see

$$\begin{aligned} & P_p(\mathbf{O} \text{ is isolated in } G^{n+1}) - P_p(\mathbf{O} \text{ is isolated in } G^n) \\ = \ & 2^d(1-p)^{d^2} + o((1-p)^{d^2}). \end{aligned} \tag{3.32}$$

The first term of the second line corresponds to the second typical case. Except the two typical cases mentioned above, it is necessary more than d^2 edges to be closed to make the event ($\mathbf{O}$ is isolated in G^{n+1}) $\setminus$ ($\mathbf{O}$ is isolated in G^n) occur when n is sufficiently large. So we obtain (3.32). Thus

$$P_p(\mathbf{O} \text{ is isolated in } G^n) = (1-p)^d + V(1-p) + 2^d n(1-p)^{d^2} + o((1-p)^{d^2}).$$

Since $\Phi^n_{A_{max}}(p) = \bigcap_{i=0}^{d}(\mathbf{a}^n_i \text{ is not isolated in } G^n)$, we have (3.31). □

Next, set $\mathcal{I} = \mathcal{A} \setminus \{A_{min}\}$, which is also an increasing set. Observe when the event $Q_{A_{min}}$ in G^n occur. (Recall (3.13), the definition of Q. The event $Q_{A_{min}}$ can be regarded as all $\mathbf{a}^n_0, \mathbf{a}^n_1, \dots, \mathbf{a}^n_d$ are isolated.) Suppose $\mathbf{a}^n_0, \mathbf{a}^n_1, \dots, \mathbf{a}^n_{d-1}$ are isolated. Then $\mathbf{a}^n_d$ is isolated automatically. First typical case is that all adjacent edges of d vertices are closed. This probability is $(1-p)^{d^2}$. Consider the second typical case. That is, it cannot go through $\mathbf{a}^k_1$, $\mathbf{a}^k_2$, ... ,$\mathbf{a}^k_d$ from $\mathbf{O}$ and all adjacent edges of $\mathbf{a}^n_1, \dots, \mathbf{a}^n_{d-1}$ are closed. This probability is approximately $2^d(1-p)^{d^2} \times (1-p)^{d(d-1)}$. Take note of the possibility of choice of the vertices, we have

$$\begin{aligned}\Phi^n_{\mathcal{I}}(p) &= 1 - \Phi^n_{A_{min}}(p)\\ &= 1-(d+1)(1-p)^{d^2} + V(1-p) - 2^d d(d+1)n(1-p)^{2d^2-d} + W(n, 1-p)\end{aligned}$$

where $V(x) = o(x^{d^2})$ and $W(n,x) = o(x^{2d^2-d})$ as $x \to 0$.

As a conclusion for $\mathcal{I} \in \Im$, the top terms of the expansion of $\Phi^n_{\mathcal{I}}(p)$ with respect to $(1-p)$ depend on the minimal number of isolated vertices to make the event $Q^n_{\mathcal{I}}$ in G^n does not occur and possibility of choices of vertices which attains the minimum. $m_1 = m_1(\mathcal{I})$ denotes the minimal number, and $m_2 = m_2(\mathcal{I})$ denotes the possibility of choices. As we see, $m_1 = 1$, $m_2 = d+1$ for $\mathcal{I} = \{A_{max}\}$, and $m_1 = d$, $m_2 = d+1$ for $\mathcal{I} = \mathcal{A} \setminus \{A_{min}\}$. We conclude

$$\begin{aligned}&\Phi^n_{\mathcal{I}}(p) \\ =\;& 1 - m_2(1-p)^{dm_1} + V(1-p) - 2^d m_1 m_2 n(1-p)^{d^2+d(m_1-1)} + W(n, 1-p)\end{aligned} \tag{3.33}$$

where V, W are polynomials of finite degree and $V(x) = o(x^{dm_1})$, $W(n,x) = o(x^{d^2+d(m_1-1)})$ as $x \to 0$.

Set $p' = p + k(1-p)^{d^2-d+1}$. All we have to do is confirm (3.19). By (3.33), we have

$$\Phi^{n+1}_{\mathcal{I}}(p') - \Phi^n_{\mathcal{I}}(p) = (kd - 2^d)m_1 m_2(1-p)^{d^2+d(m_1-1)} + o((1-p)^{d^2+d(m_1-1)}).$$

Since $\Phi^{n+1}_{\mathcal{I}}(p')$, $\Phi^n_{\mathcal{I}}(p)$ are of finite degree, we complete the proof. □

Proof of Theorem 3.2. Set $g(p) = \log \xi(p)$. Note that $g(p)$ is an increasing function. Assume

$$\limsup_{p \to 1} (1-p)^{d^2-d} g(p) < c < \frac{\log 2}{2^d(d-1)}, \tag{3.34}$$

and we lead a contradiction.

Set $h_c(x) = c\left\{\frac{1}{(x-kx^{s+1})^s} - \frac{1}{x^s}\right\}$ where $k = 2^d/d$, $s = d^2-d$. Applying the L'Hospital's theorem,

$$\lim_{x \to 0} h_c(x) = c \lim_{x \to 0} \frac{(1-(1-kx^s)^s)'}{((x-kx^{s+1})^s)'} = cks.$$

That is $\lim_{x \to 0} h_c(x) < \log 2$. Since $h_c(x)$ is continuous near to 0, we can pick p_0 such that

$$h_c(x) < \alpha < \log 2 \quad \text{for} \quad 0 < x < 1 - p_0. \tag{3.35}$$

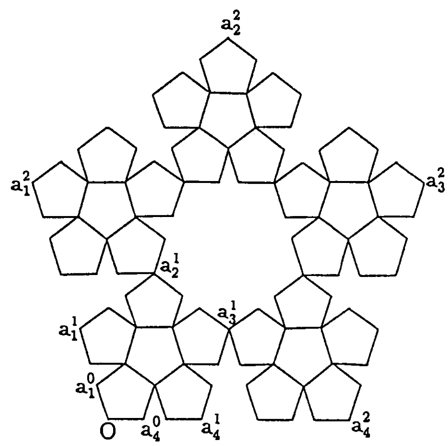

Figure 3.3: G^2 of the pentakun lattice

Set $\tau(p) = p + k(1-p)^{s+1}$. Define $p_{n+1} = \tau(p_n)$ inductively, and we see p_n is increasing with respect to n and $\lim_{n\to\infty} p_n = 1$. Let $N = N(p_0)$ be a large integer. There exists t such that $p_N < t < 1$ and $g(t) < c(1-t)^{-s}$ by (3.34). For this t, pick N' with $p_{N'} \le t < p_{N'+1}$. We see

$$\begin{aligned} g(t) &< \frac{c}{(1-t)^s} \\ &< \frac{c}{(1-p_{N'+1})^s} \\ &= h_c(1-p_{N'}) + h_c(1-p_{N'-1}) + \cdots + h_c(1-p_0) + \frac{c}{(1-p_0)^s} \\ &< (N'+1)\alpha + \frac{c}{(1-p_0)^s} \end{aligned} \tag{3.36}$$

by the definition of $h_c(x)$ and (3.34). On the other hand, we see

$$g(t) \ge g(p_{N'}) > g(p_0) + N' \log 2 \tag{3.37}$$

by (3.30). Combining (3.36) and (3.37), we have

$$N'(\log 2 - \alpha) < \alpha + \frac{c}{(1-p_0)^s} - g(p_0).$$

Take N sufficiently large, that leads to a contradiction. □

3.4 Some other examples; the pentakun lattice and the snowflake lattice

3.4.1 The pentakun lattice

In this section we study percolation on another fractal graphs. First, we define the *pentakun*. Recall equation (3.7). Let $\mathbf{a}_0 = \mathbf{O}$ be origin of $\mathbf{R}^2$, and let $\mathbf{a}_i$ $(i = 0,1,2,3,4)$ be vertices of the regular pentagon on $\mathbf{R}^2$ with $|\mathbf{a}_i - \mathbf{a}_{i+1}| = 1$. Here we define $\mathbf{a}_5 = \mathbf{a}_0$ for simplicity. Let $f_i : \mathbf{R}^2 \to \mathbf{R}^2$ $(i = 0,1,2,3,4)$ be contraction mappings

$$f_i(\mathbf{x}) = \frac{1}{\beta}(\mathbf{x} - \mathbf{a}_i) + \mathbf{a}_i \tag{3.38}$$

where $\beta = \dfrac{3+\sqrt{5}}{2}$. The solution of equation (3.7) for (3.38) is called the pentakun. Let $\tilde{V}^0 = \{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_4\}$ and $\tilde{E}^0 = \{\overline{\mathbf{a}_i\mathbf{a}_j} : 0 \le i < j \le 4\}$. We define $\tilde{V}^n$, $\tilde{E}^n$ the same as (3.9), (3.10).

Let $V^n = \{\beta^n \mathbf{v} : \mathbf{v} \in \tilde{V}^n\}$ and $E^n = \{\beta^n \mathbf{e} : \mathbf{e} \in \tilde{E}^n\}$. We define the vertex set $V = \bigcup_{n=0}^{\infty} V^n$ and the edge set $E = \bigcup_{n=0}^{\infty} E^n$. The accomplished graph $G = (V, E)$ is the *pentakun lattice*. (See Figure 3.3.) Notations follow those in Section 2 and 3. The pentakun lattice is not symmetric with respect to the change of $\mathbf{a}_1^n$ and $\mathbf{a}_2^n$, which differs from Sierpinski gaskets.

We consider bond percolation on this graph. P_p denotes its distribution. We define correlation length

$$\xi(p) = \lim_{n\to\infty} \left\{ -\frac{1}{\alpha^n} \log P_p(\mathbf{O} \leftrightarrow \mathbf{a}_1^n) \right\}^{-1} \tag{3.39}$$

where $\alpha = 1 + \sqrt{3}$. α means a scale factor. Remark that α does not coincide β, the ratio of contraction. This constant is determined by the length of the shortest path from $\mathbf{O}$ to $\mathbf{a}_1^n$. The exsistence of the limit in (3.39) will be mentioned below.

On this graph, we have a concrete expression of the recursion formulas. We concentrate three connective probabilities, $\Theta_I^n(p) = \Psi_{[0,1]}^n(p)$, $\Theta_{II}^n(p) = \Psi_{[0,2]}^n(p)$ and $\Theta_{III}^n(p) = \Psi_{[0,1,3]}^n(p)$. (Here $\Theta_{III}^n(p)$ is the probability of the event $\mathbf{O} \leftrightarrow \mathbf{a}_1^n$ and $\mathbf{O} \leftrightarrow \mathbf{a}_3^n$ in G^n.) We have

$$\begin{aligned}
\Theta_I^{n+1}(p) &= (\Theta_{II}^n(p))^2 + (\Theta_I^n(p))^3(\Theta_{II}^n(p))^2 - (\Theta_I^n(p))^3(\Theta_{III}^n(p))^2, &(3.40)\\
\Theta_{II}^{n+1}(p) &= \Theta_I^n(p)(\Theta_{II}^n(p))^2 + (\Theta_I^n(p))^2(\Theta_{II}^n(p))^2 - (\Theta_I^n(p))^3(\Theta_{III}^n(p))^2, &(3.41)\\
\Theta_{III}^{n+1}(p) &= 2\Theta_I^n(p)(\Theta_{II}^n(p))^2\Theta_{III}^n(p) + (\Theta_I^n(p))^2(\Theta_{II}^n(p))^2\Theta_{III}^n(p) \\
&\quad -2(\Theta_I^n(p))^2(\Theta_{III}^n(p))^3.
\end{aligned}$$

We see

$$\Theta_I^n(p)\Theta_{II}^n(p) \le \Theta_{III}^n(p) \le \Theta_{II}^n(p) \le \Theta_I^n(p). \tag{3.42}$$

We use FKG inequality for the first inequality. The second inequality is given by a symmetry of the graph. Combining (3.40) – (3.42), we have

$$\lim_{n\to\infty} \frac{\Theta_I^{n+1}(p)}{\{\Theta_{II}^n(p)\}^2} = \lim_{n\to\infty} \frac{\Theta_{II}^{n+1}(p)}{\Theta_I^n(p)\{\Theta_{II}^n(p)\}^2} = 1. \tag{3.43}$$

Then

$$\lim_{n\to\infty} \frac{\Theta_I^{n+1}(p)\{\Theta_{II}^{n+1}(p)\}^\alpha}{\left\{\Theta_I^n(p)\{\Theta_{II}^n(p)\}^\alpha\right\}^\alpha} = 1 \quad \text{where} \quad \alpha = 1 + \sqrt{3}$$

follows. There exists the limit

$$\hat{\xi}(p) = \lim_{n\to\infty} \left\{ -\frac{1}{\alpha^n} \log\Big(\Theta_I^n(p)\{\Theta_{II}^n(p)\}^\alpha\Big) \right\}^{-1}$$

by Proposition 3.3. Thus we obtain

$$\begin{aligned}
\xi(p) &= \lim_{n\to\infty} \left\{ -\frac{1}{\alpha^n} \log \Theta_I^n(p) \right\}^{-1} \\
&= (3 + \sqrt{3})\hat{\xi}(p).
\end{aligned}$$

Remark. We mention about the length of the shortest path. We denote the number of edges of the shortest path from $\mathbf{v}$ to $\mathbf{v}'$ by $d(\mathbf{v}, \mathbf{v}')$. Set $d_i^n = d(\mathbf{O}, \mathbf{a}_i^n)$. It is clear $d_1^{n+1} = 2d_2^n$ and $d_2^{n+1} = d_1^n + 2d_2^n$, which correspond to (3.43). Moreover,

$$\lim_{n\to\infty} \frac{d_1^n}{d_2^n} = \lim_{n\to\infty} \frac{\{-\alpha^{-n} \log \Theta_{II}^n(p)\}^{-1}}{\xi(p)} = (\sqrt{3} - 1).$$

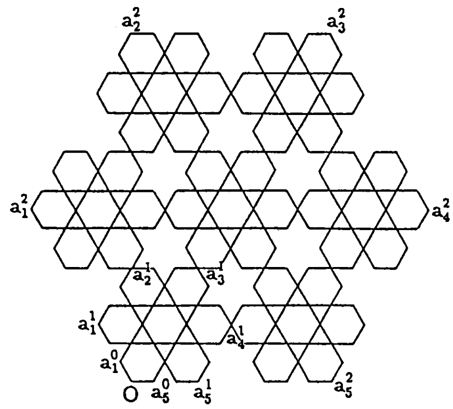

Figure 3.4: G^2 of the snowflake lattice

We proceed to have the expansion of $\Phi_{\mathcal{I}}^n(p)$. Here we consider two typical case for $\mathbf{O}$ to be isolated in G^n. Take note of E^0. We write $\mathbf{e}_i = \overline{\mathbf{a}_i^0 \mathbf{a}_{i+1}^0}$. If at least one pair of edges $(\mathbf{e}_0, \mathbf{e}_3)$, $(\mathbf{e}_0, \mathbf{e}_4)$, $(\mathbf{e}_1, \mathbf{e}_3)$ or $(\mathbf{e}_1, \mathbf{e}_4)$ are both closed, then $\mathbf{O}$ is isolated. This probability is approximately $4(1-p)^2$. Consider the second typical case. Set $\mathbf{b}_1^n = \beta^{n+1} f_2(\mathbf{a}_0)$, $\mathbf{b}_2^n = \beta^{n+1} f_1(\mathbf{a}_4)$, $\mathbf{b}_3^n = \beta^{n+1} f_0(\mathbf{a}_3)$ and $\mathbf{b}_4^n = \beta^{n+1} f_4(\mathbf{a}_2)$. (Here $\mathbf{b}_2^n = \mathbf{a}_2^n$, $\mathbf{b}_3^n = \mathbf{a}_3^n$.) For fixed k $(1 \le k \le n-1)$, if it cannot go through at least one pair of vertices $(\mathbf{b}_2^k, \mathbf{b}_3^k)$, $(\mathbf{b}_2^k, \mathbf{b}_4^k)$, $(\mathbf{b}_1^k, \mathbf{b}_3^k)$ or $(\mathbf{b}_1^k, \mathbf{b}_4^k)$, then $\mathbf{O}$ is isolated. This probability is approximately $4 \cdot 8^2 (1-p)^4$, which is independent of k.

Generally for $\mathcal{I} \in \Im$, we have

$$\Phi_{\mathcal{I}}^n(p) = 1 - 4^{m_1} m_2 (1-p)^{2m_1} + V(1-p) - 4^{m_1 - 1} m_1 m_2 \cdot 4 \cdot 8^2 n (1-p)^{2m_1+2} + W(n, 1-p)$$

where m_1, m_2 are defined as Section 3, and $V(x) = o(x^{2m_1})$, $W(n,x) = o(x^{2m_1+2})$ as $x \to 0$. We obtain the estimate of the correlation length

$$\liminf_{p \to 1} \frac{\xi(p + k(1-p)^3)}{\xi(p)} \ge \alpha \quad \text{if} \quad k > 32$$

and

$$\limsup_{p \to 1} \frac{\xi(p + k(1-p)^3)}{\xi(p)} \le \alpha \quad \text{if} \quad k < 32.$$

As a conclusion, we have the following theorem.

Theorem 3.11 (the pentakun lattice)

$$\xi(p) \approx \exp\left\{ \frac{\log \alpha}{64} (1-p)^{-2} \right\} \quad as \quad p \to 1.$$

3.4.2 The snowflake lattice

Next, we consider percolation on the *snowflake lattice.* We define *snowflake.* Let $\mathbf{a}_0 = \mathbf{O}$ be origin of $\mathbf{R}^2$, and let $\mathbf{a}_i$ $(0 \le i \le 5)$ be vertices of the regular hexagon on $\mathbf{R}^2$ with $|\mathbf{a}_i - \mathbf{a}_{i+1}| = 1$. Here we define $\mathbf{a}_6 = \mathbf{a}_0$ for simplicity. And let $\mathbf{a}_{-1}$ be the center of the hexagon. Let $f_i : \mathbf{R}^2 \to \mathbf{R}^2$ $(-1 \le i \le 5)$ be contraction mappings

$$f_i(\mathbf{x}) = \frac{1}{3}(\mathbf{x} - \mathbf{a}_i) + \mathbf{a}_i. \tag{3.44}$$

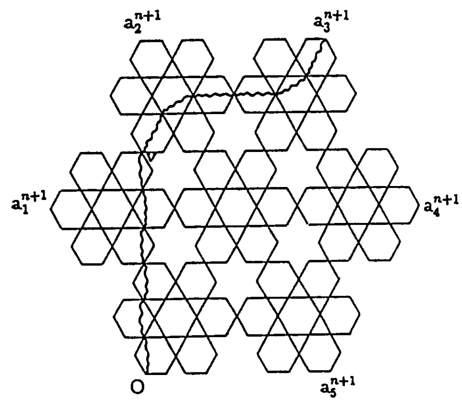

Figure 3.5: $\mathbf{O} \leftrightarrow \mathbf{v}$ and $\mathbf{v} \leftrightarrow \mathbf{a}_3{}^{n+1}$ induce $\mathbf{O} \leftrightarrow \mathbf{a}_3{}^{n+1}$. $P_p(\mathbf{O} \leftrightarrow \mathbf{v}) \geq P_p(\mathbf{O} \leftrightarrow \mathbf{a}_1{}^{n+1})$ by symmetry.

The solution of equation (3.7) for (3.44) is called the snowflake. Note that the number of contraction mappings is 7, which is not coincide the cardinality of $\tilde{V}^0$. This is the difference from the examples mentioned above. Let $\tilde{V}^0 = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_5\}$ and $\tilde{E}^0 = \{\overline{\mathbf{a}_i \mathbf{a}_{i+1}} : 0 \leq i \leq 5\}$. We define $\tilde{V}^n$, $\tilde{E}^n$, V^n, E^n, V and E in the same way as the previous sections. The accomplished graph $G = (V, E)$ is the *snowflake lattice.* (See Figure 3.4.)

Set $\Theta_I^n(p) = \Psi_{[0,1]}^n(p)$, $\Theta_{II}^n(p) = \Psi_{[0,2]}^n(p)$ and $\Theta_{III}^n(p) = \Psi_{[0,3]}^n(p)$. We have

$$\{\Theta_I^n(p)\}^2 \leq \Theta_{III}^n(p) \leq \Theta_{II}^n(p) \leq \Theta_I^n(p). \tag{3.45}$$

See Figure 3.5 to have the first inequality. We have

$$\lim_{n\to\infty} \frac{\Theta_I^{n+1}(p)}{\{\Theta_{II}^n(p)\}^2} = 1 \quad \text{and} \quad 1 \leq \frac{\Theta_{II}^{n+1}(p)}{\{\Theta_{II}^n(p)\}^3} \leq C \tag{3.46}$$

for some $C < \infty$. To see (3.46), estimate $\Theta_I^{n+1}(p)$, $\Theta_{II}^{n+1}(p)$ like (3.15) and use (3.45). Thus the limit

$$\begin{aligned} \xi(p) &= \lim_{n\to\infty} \left\{-\frac{1}{3^n} \log P_p(\mathbf{O} \leftrightarrow \mathbf{a}_1^n)\right\}^{-1} \\ &= \frac{2}{3}\left(\lim_{n\to\infty} \left\{-\frac{1}{3^n} \log \Theta_{II}^n(p)\right\}^{-1}\right) \end{aligned}$$

exists by (3.17).

For $\mathcal{I} \in \Im$, we have

$$\Phi_{\mathcal{I}}^n(p) = 1 - 4^{m_1} m_2 (1-p)^{2m_1} + V(1-p) - 4^{m_1-1} \cdot 8^3 m_1 m_2 n (1-p)^{2m_1+4} + W(n, 1-p).$$

We have the following theorem.

Theorem 3.12 (the snowflake lattice)

$$\xi(p) \approx \exp\left\{\frac{\log 3}{256}(1-p)^{-4}\right\} \quad as \quad p \to 1.$$

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第4章 Existence of phase transition of percolation on Sierpinski carpet lattices

4.1 Introduction

Percolation is studied as a very important subject in statistical mechanics because this is one of the simplest models which contains phase transitions of disordered media. Percolation has close relations to disordered electrical networks, ferromagnetism, epidemic models and so on. Percolation models were proposed by Broadbent-Hammersley [1], and have been well studied in the last thirty years. See Grimmett [7] to view the whole of this field.

In this paper we consider percolation on *fractal-like lattices*. A fractal-like lattice is a graph which corresponds to a fractal. It has a kind of self-similarity but it may not have translation invariance.

Now we explain two well-known examples, *the Sierpinski gasket* and *the Sierpinski carpet*. The former is a *finite* ramified fractal and the latter is an *infinite* ramified fractal. In a previous paper [14] we analyse percolation on the Sierpinski gasket lattice, which has no phase transition. The non-existence of the phase transition is induced by the character of finite ramified fractals. In this paper we treat *Sierpinski carpet lattices*. Sierpinski carpet lattices is the class of graphs which correspond to *generalized Sierpinski carpets* and it contains infinite ramified fractals. Kumagai [9] gave a sufficient condition for Sierpinski carpet lattices to have a phase transition. We will describe his results below.

We define the Sierpinski carpet and generalized Sierpinski carpets on $\mathbf{R}^2$ as follows. Set $L \geq 2$ to be an integer and set $\mathbf{T}_L = \{0, 1, \ldots, L-1\}^2$. For $(i,j) \in \mathbf{T}_L$, we set an affine map $\Psi_{(i,j)}$ from $[0,1]^2$ to $[i/L, (i+1)/L] \times [j/L, (j+1)/L]$ which preserves the directions. For a nonempty subset $T \subset \mathbf{T}_L$, it is well-known (see Falconer [6], for example) that there exists a unique nonempty compact set $K_T \subset [0,1]^2$ which satisfies the equation

$$K_T = \bigcup_{t \in T} \Psi_t(K_T).$$

We call these K_T's generalized Sierpinski carpets. The Sierpinski carpet is an element of generalized Sierpinski carpets.

Example 4.1 *Set $L = 3$ and $T = \mathbf{T}_3 \setminus \{(1,1)\}$. K_T is the Sierpinski carpet.*

We remark that the Sierpinski gasket is also an element of generalized Sierpinski carpets.

Example 4.2 *Set $L = 2$ and $T = \mathbf{T}_2 \setminus \{(1,1)\}$. K_T is the Sierpinski gasket.*

Let us define the graph corresponding to K_T. Set

$$F_T^n = \bigcup_{t_1, t_2, \cdots, t_n \in T} \Psi_{t_1} \circ \Psi_{t_2} \circ \cdots \circ \Psi_{t_n}([0,1]^2).$$

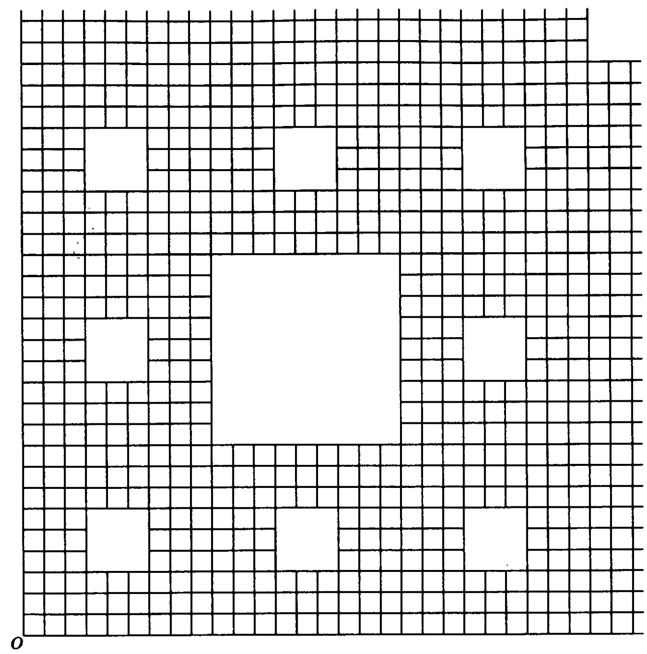

Figure 4.1: The Sierpinski carpet lattice, G_T for $= \mathbf{T}_3 \setminus \{(1,1)\}$

Note that K_T can be constructed as the limit of F_T^n. Set a graph $G_T^n = (V(G_T^n), E(G_T^n))$, where $V(G_T^n) = \mathbf{Z}^2 \cap L^n F_T^n$ and $E(G_T^n) = \{\langle u, v\rangle : u, v \in V(G_T^n),$
$|u - v| = 1\}$. Here we write $\langle u, v\rangle$ as a bond with endvertices u and v. From now on we assume through this paper that K_T is connected, and

$$(0,0) \in T. \tag{4.1}$$

Under these assumptions we set $G_T = \bigcup_{n=1}^{\infty} G_T^n$. That is, $V(G_T) = \bigcup_{n=1}^{\infty} V(G_T^n)$ and $E(G_T) = \bigcup_{n=1}^{\infty} E(G_T^n)$. Note that $V(G_T^n)$ and $E(G_T^n)$ are increasing sequences with respect to n under (4.1). We call the family of G_T corresponding K_T's *Sierpinski carpet lattices. The Sierpinski carpet lattice* given in Figure 4.1 is an example of Sierpinski carpet lattices.
We consider bond percolation on G_T. Set $0 \le p \le 1$. Each $e \in E_T$ is declared to be *open* with probability p and *closed* with probability $1 - p$ independently. We denote the product measure by P_p. We define $\theta(p) = P_p(|C(0)| = \infty)$ where $C(0)$ is the open cluster containing the origin and $|C(0)|$ is the number of vertices in $C(0)$. Let $p_c(G_T) = \inf\{p : \theta(p) > 0\}$. We study the problem of finding a necessary and sufficient condition for T to be $p_c(G_T) < 1$.
The difficulty of this problem is that we cannot apply Peierl's argument (see [7], for example) because the ratio of the holes of G_T tends to 1.
In the case of $L = 2$, we can completely answer the problem; $P_c(G_T) < 1$ if and only if $T = \mathbf{T}_2$. Hereafter we assume $L \ge 3$. In [9] Kumagai obtained a sufficient condition for this problem. Set $\partial_{int}\mathbf{T} = \{(0,j) : 0 \le j \le L-1\} \cup \{(L-1,j) : 0 \le j \le L-1\} \cup \{(i,0) : 0 \le i \le L-1\} \cup \{(i, L-1) : 0 \le i \le L-1\}$.

Theorem 4.1 (Kumagai [9]) $p_c(G_{\partial_{int}\mathbf{T}}) < 1$.

By the monotonicity, that is $T \supset T'$ implies $p_c(G_T) \le p_c(G_{T'})$, we see $p_c(G_T) < 1$ if

$$T \supset \partial_{int}\mathbf{T}. \tag{4.2}$$

In this paper we give a weaker condition than the theorem above. We write $T_l = \{j : (0,j) \in T\}$, $T_r = \{j : (L-1,j) \in T\}$, $T_d = \{i : (i,0) \in T\}$ and $T_u = \{i : (i, L-1) \in T\}$. We say T *is connected* if for any $t, t' \in T$ there exists a sequence $t_1 = t,\ t_2,\ \ldots,\ t_n = t'$ which satisfies $t_i \in T$ and $|t_i - t_{i+1}| = 1$ for $1 \leq i \leq n-1$.

Theorem 4.2 *Assume*

$$T \setminus \{t\} \text{ is connected for any } t \in T \tag{4.3}$$

and

$$|T_l \cap T_r| \geq 2 \text{ and } \ |T_d \cap T_u| \geq 2. \tag{4.4}$$

Then $p_c(G_T) < 1$.

$\partial_{int}\mathbf{T}$ satisfies (4.3) and (4.4), so this theorem contains Theorem 4.1. We will give examples to which Theorem 4.2 is applicable but Theorem 4.1 is not applicable. Moreover these conditions will be further relaxed as we explain in the following sections.

Remark. If $T \supset T'$ and T' satisfies (4.3) and (4.4), then $p_c(G_T) < 1$ by monotonicity. This is a trivial extension of Theorem 4.2.

We note the existence of phase transition on general graphs.
Häggström [8] showed that if the maximum degree of the vertices is finite then the existence of phase transitions of bond percolation, site percolation and the Ising model are equivalent. So we can consider the critical phenomena of bond, site percolation or the Ising model on Sierpinski carpet lattices.
For a general connected graph G, we define *isoperimetric dimension* $\mathrm{Dim}(G)$ by

$$\mathrm{Dim}(G) = \sup\Big\{D > 0 : \inf \frac{|\partial S|}{|S|^{\frac{D-1}{D}}} > 0\Big\}$$

where S is a finite connected subset of the bonds of G and ∂S is the outer boundary of S. We hope to clarify the relation between $\mathrm{Dim}(G)$ and $p_c(G)$. In [2], Benjamini and Schramm proposed the problem of whether $\mathrm{Dim}(G) > 1$ implies $p_c(G) < 1$.
We check this problem in the case of Sierpinski carpet lattices. Now we can say only that it seems that $\mathrm{Dim}(G_T) > 1$ implies $p_c(G_T) < 1$, but we do not yet have the proof, and $\mathrm{Dim}(G_T) = 1$ does not imply $p_c(G_T) = 1$. We will discuss about the problem above with giving some examples.
We treat Sierpinski carpet lattices which satisfy (4.3) and (4.4) in Section 2 and give a proof of Theorem 4.2. In Section 3 we discuss the crucial examples which do not satisfy (4.3) or (4.4).

4.2 Proof of Theorem 4.2

To prove Theorem 4.2, we use a fractal percolation technique. See Chayes-Chayes-Durrett [3], Dekking-Meester [5], Grimmett [7] for details on fractal percolation. To use the technique, we define box-percolation on G_T. Let

$$\mathcal{B}_T^n = \left\{ [iL^n, (i+1)L^n] \times [jL^n, (j+1)L^n] : \begin{array}{c} i, j \geq 0 \text{ and there exists } (k,l) \in T \\ \text{such that } i \equiv k, j \equiv l \mod L \end{array} \right\}$$

for $n \geq 0$. We say a box $b \in \mathcal{B}_T^0$ is *open* if the bottom edge and the left edge of b are open. The probability of the event that b is open is p^2, independent of any other boxes. Note that if there exists an infinite sequence of connected open boxes then there exists an infinite sequence of connected open bonds, which implies the existence of an infinite open cluster. We say a box $b \in \mathcal{B}_T^1$ is *good* if at most one subbox $b' \in b \cap \mathcal{B}_T^0$ is not open. (that is, at least $|T|-1$ subboxes of b are open.) For $n \geq 2$, we say a box $b \in \mathcal{B}_T^n$ is *(very)$^{n-1}$ good* if at most one subbox $b' \in b \cap \mathcal{B}_T^{n-1}$ is not (very)$^{n-2}$ good.

Lemma 4.3 *For a sufficiently large p, there exists* $0 < \theta < 1$ *such that*

$$P_p(b \in \mathcal{B}_T^n \text{ is (very)}^{n-1}\text{good}) \geq 1 - \theta^n \tag{4.1}$$

for any n.

Proof. We write $p^2 = \hat{p}$ to simplify the notation. Set $f_T(x) = x^{|T|} + |T|x^{|T|-1}(1-x)$. It is clear that $P_p(b \in \mathcal{B}_T^1 \text{ is good }) \geq f_T(\hat{p})$ by definition, and inductively we can see that

$$P_p(b \in \mathcal{B}_T^n \text{ is (very)}^{n-1} \text{ good}) \geq f_T^n(\hat{p})$$

where f_T^n is the nth iterate of f_T. Let α be the largest solution of $f_T(x) = x$ contained in $(0,1)$. Note that $x < f_T(x)$ and $f_T''(x) < 0$ in $(\alpha, 1)$. We observe that $(1 - f_T(x))/(1-x)$ is decreasing and smaller than 1 in this interval. If $\hat{p} > \alpha$, then there exists $0 < \theta < 1$ such that

$$1 - f_T^n(\hat{p}) \leq \theta(1 - f_T^{n-1}(\hat{p})) \leq \theta^2(1 - f_T^{n-2}(\hat{p})) \leq \cdots \leq \theta^n(1-\hat{p}). \quad \square$$

Proof of Theorem 4.2. At first we assume

$$|T_l \cap T_r| \geq 3 \text{ and } |T_d \cap T_u| \geq 3. \tag{4.2}$$

Suppose $b \in \mathcal{B}_T^1$ is good. Then $\{b' \in b \cap \mathcal{B}_T^0 : b' \text{ is open}\}$ is connected because of (4.3). Suppose $b \in \mathcal{B}_T^2$ is very good. Then $\{b'' \in \mathcal{B}_T^0 : b''$ is open and contained in a good
$b' \in b \cap \mathcal{B}_T^0\}$ is connected by (4.3) and (4.2). Here (4.2) assures that good boxes $b_1', b_2' \in \mathcal{B}_T^1$ are mutually connected when they are neighbors, even if each of them has one box which is not open. By the same observation, we can see that if $b \in \mathcal{B}_T^n$ is (very)$^{n-1}$ good then there exist some sequences of open boxes from the left (resp. bottom) side of b to the right (resp. top) side of b. We have

$$\{|C(0)| = \infty\} \subset \left\{ \begin{array}{l} b \in \mathcal{B}_T^0 \cap b^1(0) \text{ are all open },\\ b' \in \mathcal{B}_T^1 \cap b^2(0) \text{ are all good },\\ \cdots, \text{ and} \\ b^{(n)} \in \mathcal{B}_T^n \cap b^{n+1}(0) \text{ are all (very)}^{n-1} \text{ good },\\ \cdots \end{array} \right\}. \tag{4.3}$$

Here we denote by $b^n(0)$ the box such that $b^n(0) \in \mathcal{B}_T^n$ and $0 \in b^n(0)$. Then by (4.1) and FKG inequality, we have

$$\theta(p) \geq (1-\theta)^{|T|}(1-\theta^2)^{|T|}\cdots(1-\theta^n)^{|T|}\cdots > 0.$$

To complete the proof, we show the conditions (4.3) and (4.4) is enough to be $P_c(G_T) < 1$. Set

$$T^2 = \left\{ (i,j) : \begin{array}{l} 0 \leq i,j \leq L^2 - 1 \text{ and there exist } (k_1, l_1), (k_2, l_2) \in T \\ \text{such that } i = k_1 L + k_2 \text{ and } j = l_1 L + l_2 \end{array} \right\}.$$

That is, $T^2(\subset \mathbf{T}_{L^2})$ represents the second iteration of T. We can consider that G_T is generated by T^2 because $G_T = \bigcup_{n=1}^{\infty} G_T^n = \bigcup_{m=1}^{\infty} G_T^{2m}$. It is clear that T^2 satisfies (4.3) and (4.2) when T satisfies (4.3) and (4.4). $\square$

By the proof above, we have the following estimate.

Corollary 4.4 *Under the same assumptions of Theorem 4.2,*

$$p_c(G_T) \le \sqrt{\alpha},$$

where α is the largest solution of $f_T(x) = x$ in $(0,1)$.

Corollary 4.5

$$p_c(\text{the Sierpinski carpet lattice}) < 0.9224.$$

Remark. Under the same assumptions of Theorem 4.2 we can show that $p_c^{site}(G_T)$ < 1, that is, there exists the phase transition in the case of site percolation on G_T. We also can observe $p_c^{site}(G_T) \le \alpha$. Grimmett-Stacey [7] showed

$$\frac{1}{\Delta - 1} \le p_c^{bond}(G) \le p_c^{site}(G) \le 1 - (1 - p_c^{bond}(G))^{\Delta - 1}$$

where Δ is the maximum degree of the vertices of G. By using of this inequality, we have $p_c(G_T) \le \alpha$, which is an improvement of Corollary 4.5. In the case of the Sierpinski carpet lattice, $L = 3$, $T = \mathbf{T}_3 \setminus \{(1,1)\}$ and $\alpha = 0.9576\cdots$. We give a better upper bound in the following proof.

Proof of Corollary 4.5. In this proof we denote the Sierpinski carpet lattice by $G = (V, E)$. We change the definition of a box being good. For $b = [3i, 3(i+1)] \times [3j, 3(j+1)]$, set

$$V^b = \{(k,l) \in V : 3i \le k \le 3(i+1), 3j \le l \le 3(j+1)\},$$

$$\tilde{V}^b = \{(k,l) \in V : 3i \le k < 3(i+1), 3j \le l < 3(j+1)\},$$

$$V_l^b = \{(3i,l) \in V : 3j \le l < 3(j+1)\}, V_r^b = \{(3(i+1),l) \in V : 3j \le l < 3(j+1)\},$$

$$V_d^b = \{(k,3j) \in V : 3i \le k < 3(i+1)\}, V_u^b = \{(k,3(j+1)) \in V : 3i \le k < 3(i+1)\}$$

and

$$E^b = \{\langle v, v+(1,0)\rangle \in E : v \in \tilde{V}^b\} \cap \{\langle v, v+(0,1)\rangle \in E : v \in \tilde{V}^b\}.$$

Set $G^b = (V^b, E^b)$ and consider sub-percolation on G^b. We say b is good if there exists a open cluster C of G^b which satisfies $|C \cap V_\eta^b| \ge 2$ for $\eta = l, r, d$ and u. We have

$$\begin{aligned} & P_p(b \text{ is good}) \\ = \;& p^{18} + 18p^{17}(1-p) + 147p^{16}(1-p)^2 + 704p^{15}(1-p)^3 + 2129p^{14}(1-p)^4 \\ & + 4002p^{13}(1-p)^5 + 4165p^{12}(1-p)^6 + 1780p^{11}(1-p)^7 + 142p^{10}(1-p)^8 \end{aligned} \tag{4.4}$$

by direct calculation. We follow the definition of (very)$^{n-1}$ good for $n \geq 2$ in the proof of Theorem 4.2. It is sufficient to be $\theta(p) > 0$ that the probability (4.4) is greater than 0.9577. $\square$

At the end of this section, we comment on Theorem 4.2.

First, we emphasize that this method to show $p_c(G) < 1$ is a different approach from Peierl's argument. Peierl's argument is suitable for graphs which have translation invariance and our method is suitable for graphs which have self-similarity.

Second, we show that Theorem 4.2 is a true extension of Theorem 4.1.

Example 4.3 *Set $L \geq 3$ and $T = \{(i,j) \in \mathbf{T}_L : i \in \{0,1\}$ or $j \in \{0,1\}\}$. Then (4.2) does not hold, but we have $p_c(G_T) < 1$ because (4.3) and (4.4) hold.*

Last, we remark on the extension of this theorem to subgraphs in $\mathbf{R}^d$, $d \geq 3$. We can define d-dimensional Sierpinski carpet lattices. Set $\mathbf{T}_L = \{0, 1, \dots, L-1\}^d$, an affine map $\Psi_{(i_1, i_2, \dots, i_d)}$ from $[0,1]^d$ to $[i_1/L, (i_1+1)/L] \times [i_2/L, (i_2+1)/L] \times \cdots \times [i_d/L, (i_d+1)/L]$, and so on. We call the family of G_T corresponding K_T's *d-dimensional Sierpinski carpet lattices.*

Example 4.4 *Set $L = 3$ and $T = \mathbf{T}_3 \setminus \{(1, 1, \dots, 1)\}$. Then K_T is the d-dimensional Sierpinski carpet. We call G_T corresponding this K_T the d-dimensional Sierpinski carpet lattice.*

Example 4.5 *Set $L = 3$ and $T = \mathbf{T}_3 \setminus \{(i_1, i_2, \dots, i_d) : |\{l : i_l = 1\}| \leq 1\}$. Then K_T is the d-dimensional Menger Sponge. We call G_T corresponding this K_T the d-dimensional Menger sponge lattice.*

G_T in Example 4.4 contains the $\mathbf{Z}^2$ lattice and G_T in Example 4.5 contains the 2-dimensional Sierpinski carpet lattice as a subgraph, so $p_c(G_T) < 1$ is clear. Murai [13] studied an asymptotic behavior of $p_c(G_T)$ as $d \to \infty$. Generally, if a Sierpinski carpet lattice contains a 2-dimensional sub-Sierpinski carpet lattice which satisfies the assumptions of Theorem 4.2, then $p_c(G_T) < 1$ follows. Moreover, it is easy to modify Theorem 4.2, being suitable for d-dimensional Sierpinski carpet lattices, to make it is applicable to the following example.

Example 4.6 *Set $d = 3$ and $L = 3$. Let $T = \{(i,j,k) : (i,j) \in H$ and $k \in \{0,2\}\} \cup \{(0,0,1), (2,2,1)\}$ where $H = \{0,1,2\}^2 \setminus \{(1,0), (1,2)\}$.*

This G_T contains G_H (a 2-dimensional Sierpinski carpet lattice) as a subgraph but this does not imply $p_c(G_T) < 1$ because $p_c(G_H) = 1$, as we will show in the next section. But this T satisfies the modification of (4.3) and (4.4) and we can obtain $p_c(G_T) < 1$.

4.3 Remark on the isoperimetric dimension

In this section we discuss the case where T does not satisfy (4.3) or (4.4). To study the relation between the phase transition and the isoperimetric dimension of the graph, we give three examples; G_T with

I. $\mathrm{Dim}(G_T) = 1$ and $p_c(G_T) = 1$,

II. $\mathrm{Dim}(G_T) > 1$ and $p_c(G_T) < 1$,

III. $\mathrm{Dim}(G_T) = 1$ and $p_c(G_T) < 1$.

I. The following is a simple example of G_T on which $p_c(G_T) = 1$ is not so clear.

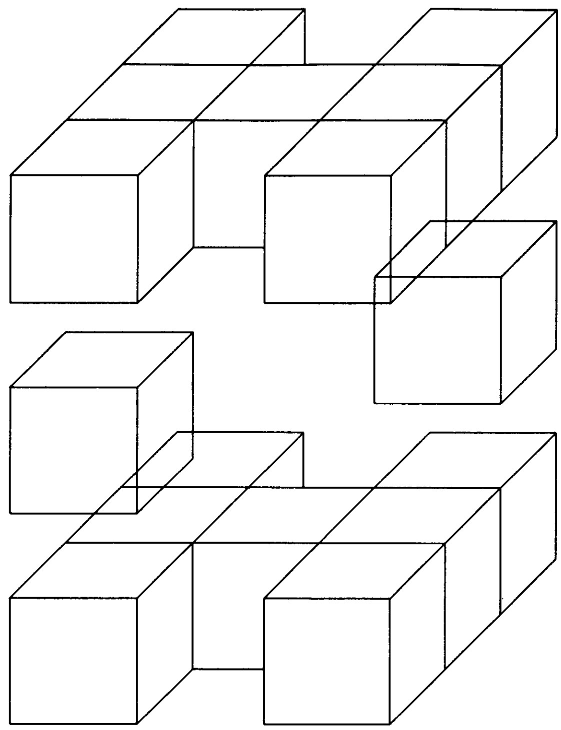

Figure 4.1: The image of T in Example 4.6.

Example 4.7 *Set* $L = 2k+1$ $(k \geq 1)$ *and* $T = \{(0,j) : 0 \leq j \leq L-1\} \cup \{(L-1,j) : 0 \leq j \leq L-1\} \cup \{(i,k) : 0 \leq i \leq L-1\}$. *Then* $\mathrm{Dim}(G_T) = 1$, *because we can take* $\{S_n\}$ *as Figure 4.3 to satisfy* $|\partial S_n| \leq 8$.

Proposition 4.6 *Let* T *be defined in Example 4.7. Then* $p_c(G_T) = 1$.

When $k \geq 2$, this proposition is shown in Kumagai [9]. Here we give a proof for the $k = 1$ case, that is $L = 3$. (This T corresponds to H in Example 4.6.) Recall the definition of G_T^n, that is $G_T \cap [0, L^n]^2$. We say *there exists an left-right* (resp. *top-bottom*) *open crossing of* G_T^n if there exist $u \in \{x = 0\} \cap V(G_T^n)$ and $v \in \{x = L^n\} \cap V(G_T^n)$ (resp. $u \in \{y = 0\} \cap V(G_T^n)$ and $v \in \{y = L^n\} \cap V(G_T^n)$) such that u and v are in the same open cluster of G_T^n. Set

$$\begin{aligned} x_n(p) &= P_p(\text{there exists a left-right open crossing of } G_T^n), \\ y_n(p) &= P_p(\text{there exists a top-bottom open crossing of } G_T^n). \end{aligned}$$

From now on, we assume $0 < p < 1$. For briefly we write x_n and y_n instead of $x_n(p)$ and $y_n(p)$.

Lemma 4.7

$$\lim_{n \to \infty} x_n = 0. \tag{4.1}$$

Proof. We observe

$$x_n \leq x_{n-1} \cdot \{1 - (1 - x_{n-2})^3\}^2 \leq \cdots \leq \{1 - (1 - x_1)^3\}^{2(n-1)}, \tag{4.2}$$

and $x_1 < 1$ induces (4.1). □

By the proof above, we can also see that x_n is strictly decreasing with respect to n.

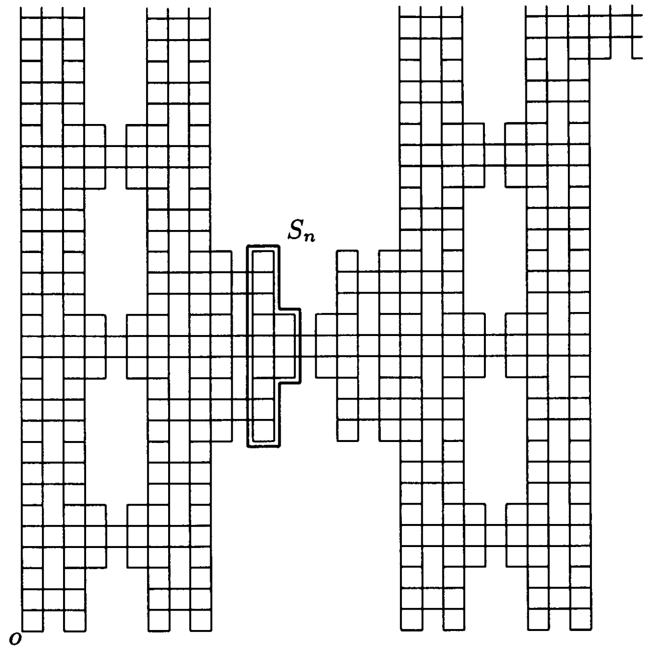

Figure 4.1: G_T for $T = \mathbf{T}_3 \setminus \{(1,0),(1,2)\}$

Lemma 4.8 *There exists $\beta > 2$ and $0 < \theta < 1$ such that*

$$x_n \leq \theta^{\beta^n}. \tag{4.3}$$

Proof. To observe the construction of T more precisely, we have better estimate of x_n than (4.2). That is

$$x_n \leq x_{n-1} \cdot \{1 - (1 - x_{n-2})^3\}^2 \{1 - (1 - x_{n-3})^9\}^4 \leq x_{n-1} \cdot (3x_{n-2})^2 \cdot (9x_{n-3})^4.$$

Set $z_n = \log x_n$ and we have $z_n \leq z_{n-1} + 2z_{n-2} + 4z_{n-3} + 10\log 3$. By (4.1), we can pick m for z_n being sufficiently small. Set $\{Z_n\}_{n \geq m}$ such that $Z_m = z_m$, $z_{m+1} = Z_{m+1}$, $z_{m+2} = Z_{m+2}$ and

$$Z_{n+3} = Z_{n+2} + 2Z_{n+1} + 4Z_n + 10\log 3 \tag{4.4}$$

for $n \geq m$. Then $z_n \leq Z_n$ for all $n \geq m$. By (4.4) Z_n can be written as

$$Z_n = c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n$$

with $c_1 < 0$ where λ_i are the eigenvalues of the matrix $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\lambda_1 > 2$ and $|\lambda_1| > |\lambda_2| > |\lambda_3|$. So we can choose $\lambda_1 > \beta > 2$ and $c > 0$ to be $Z_n < -c\beta^n$, that induces (4.3). □

Lemma 4.9

$$y_{m+n} \leq 2^m y_n^{2^m} + 7^{m-1} x_n. \tag{4.5}$$

Proof. If there is a top-bottom open crossing of G_T^{m+n}, either of the events (i),(ii) must occur.

(i) There is a top-bottom open crossing in at least one of the rectangles, $\{[iL^n, (i+1)L^n] \times [0, L^{m+n}]\}$ where $i = \sum_{l=0}^{m-1} i_l 3^l, i_l \in \{0, 2\}$.

(ii) There is a left-right open crossing in the center subbox (size L^n) of the boxes with size L^{n+1}.

As for the event (i) we can take 2^m boxes with size L^n disjointly from each rectangles. That is, $\{[iL^n,(i+1)L^n]\times[jL^n,(j+1)L^n]\}$ where $j=\sum_{l=0}^{m-1} j_l 3^l, j_l \in \{0,2\}$. In these boxes there must be top-bottom open crossings, and this probability is not greater than $2^m y_n^{2^m}$. As for the event (ii) we count the number of boxes with size L^{n+1}. We see the probability of this event is less than $7^{m-1}x_n$. □.

Proof of Proposition 4.6. Choose $2<\gamma<\beta$. Set an integer m to be $\gamma^n \le m < \gamma^n+1$, and we have

$$y_{n+m} \le 2^{\gamma^n+1} y_n^{2^{\gamma^n}} + 7^{\gamma^n}\theta^{\beta^n}$$

by (4.3) and (4.5). Clearly the second term of the right hand side goes to 0 as $n\to\infty$. For the first term, we note that $y_n \le 1-(1-p)^{2^{n+1}}$ by definition and we have only to prove

$$\lim_{n\to\infty} 2^{\gamma^n+1}\{1-(1-p)^{2^{n+1}}\}^{2^{\gamma^n}}=0.$$

This equation is true because $2<\gamma$. □

Example 4.8 *Set* $0\le I,J\le L-1$. *Set* $T=\mathbf{T}_L\setminus\{(I,j):j\ne J\}$. *This is a generalization of Example 4.7. Then* $\mathrm{Dim}(G_T)=1$ *and* $p_c(G_T)=1$.

By this example, we can see that for any $0\le j\le L-1$

$$|T\cap\{(i,j):0\le i\le L-1\}|\ge 2 \tag{4.6}$$

is a necessary condition of $p_c(G_T)<1$.

II. Generally speaking, it is difficult to determine the exact isoperimetoric dimension of Sierpinski carpet lattices. Osada [11] established the dimension of the d-dimensional Sierpinski carpet, that induces $\mathrm{Dim}(G_T)=\log(3^d-1)/(\log(3^d-1)-\log(3^{d-1}-1))$ for the d-dimensional Sierpinski carpet lattices mentioned in Example 4.4. But if G_T does not have good symmetries then it seems hard even to be sure that $\mathrm{Dim}(G_T)>1$. We believe the assumptions of Theorem 4.2 are sufficient for $\mathrm{Dim}(G_T)>1$, but we do not yet have the proof. Conversely, we think the assumptions of Theorem 4.2 are stronger than $\mathrm{Dim}(G_T)>1$, and we wonder that the proof of Theorem 4.2 is effective for all Sierpinski carpet lattices with $\mathrm{Dim}(G_T)<1$. In that proof, we need (4.3) and (4.2) only to assure (4.3). We can change the assumptions so long as they assure (4.3). If we can show $\mathrm{Dim}(G_T)>1$ implies (4.3), then $p_c(G_T)<1$ holds.

Remark. We can see that (4.4) is a necessary condition for $p_c(G_T)<1$, which is similar to (4.6). So we should change (4.3) to a suitable alternative.

Example 4.9 *Set* $L=7$ *and* $T=\mathbf{T}_7\setminus\{(2,4),(2,5),(2,6),(3,2),(3,4),(4,0),(4,1),(4,2)\}$. *In this case (4.3) does not hold but (4.3) holds, and* $p_c(G_T)<1$.

Example 4.10 *Set* $L\ge 3$ *and* $T=\{(i,j):i\in\{0,1\}$ *and* $0\le j\le L-1\}\cup\{(i,j):0\le i\le L-1$ *and* $j\in\{0,L-1\}\}$. *Then* $p_c(G_T)<1$ *holds.*

III. Here we show $\mathrm{Dim}(G_T)>1$ is *not* necessary for $p_c(G_T)<1$. We give an example.

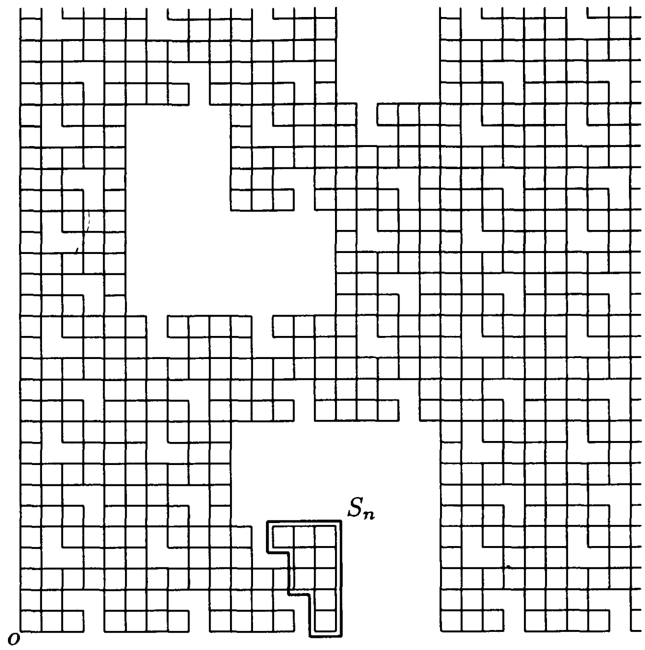

Figure 4.2: G_T for $T = \mathbf{T}_5 \setminus \{(1,3),(1,4),(2,1),(2,3),(3,0),(3,1)\}$

Example 4.11 *Set* $L = 5$ *and* $T = \mathbf{T}_5 \setminus \{(1,3),(1,4),(2,1),(2,3),(3,0),(3,1)\}$. *In this case* $\mathrm{Dim}(G_T) = 1$ *because we can take* $\{S_n\}$ *as in Figure 4.4 to satisfy* $|\partial S_n| = 2$. *On this graph* $p_c(G_T) < 1$.

In this example the component corresponding $(2,0)$ seems a dangling subgraph, but if we delete $(2,0)$ from T then $p_c(G_T) = 1$.

We conclude this paper with a further problem; Is $p_c(G_T) < 1$ eqivalent to the condition that G_T contains a subgraph G with $Dim(G) > 1$?

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第5章 Non-existence of phase transition of oriented percolation on Sierpinski carpet lattices

5.1 Introduction

Percolation is studied as an important subject in statistical mechanics because this is one of the simplest models which contains phase transitions of disordered media. Percolation has close relations to disordered electrical networks, ferromagnetism, epidemic models and so on. Percolation models were proposed by Broadbent and Hammersley [1], and have been well studied in the last thirty years. See Grimmett [7] to view the whole of this field.

Percolation problems had been studied mostly on $\mathbb{Z}^d$ lattice until recent years. We note that $\mathbb{Z}^d$ lattice has translation-invariances. In this paper we consider percolation on *fractal-like lattices*. Fractal-like lattices are graphs which correspond to fractals. All of them have a kind of self-similarity, but most of them have no translation invariances. *The Sierpinski gasket* and *the Sierpinski carpet* are well-known examples of fractals. The former is a *finite* ramified fractal (that is, it can be disconnected by removing a finite number of points) and the latter is an *infinite* ramified fractal. See Mandelbrot [12] for details of fractals. In a previous paper [14] we have analysed percolation on the Sierpinski gasket lattice, which has no phase transition. The non-existence of phase transition is induced by the character of finite ramified fractals. Now we focus on *the Sierpinski carpet lattice.* The Sierpinski carpet lattice is a graph which corresponds to the Sierpinski carpet.

Let us define the Sierpinski carpet on $\mathbb{R}^2$ as follows. For $(i,j) \in \{0,1,2\}^2$ we set an affine map $\Psi_{(i,j)}$ from $[0,1]^2$ to $[i/3,(i+1)/3]\times[j/3,(j+1)/3]$ which preserves the directions. Set $T = \{(i,j) \in \{0,1,2\}^2 \mid (i,j) \neq (1,1)\}$. It is well-known (see Falconer [6] for example) that there exists a unique nonempty compact set $K \subset [0,1]^2$ which satisfies the equation that $K = \bigcup_{t\in T} \Psi_t(K)$. We call this K the Sierpinski carpet. Let us define the graph corresponding to K. Set $F^n = \bigcup_{t_1,t_2,\cdots,t_n\in T} \Psi_{t_1}\circ\Psi_{t_2}\circ\cdots\circ\Psi_{t_n}([0,1]^2)$. We note that K can be constructed as the limit of F^n. We write $kA = \{ka \mid a \in A\}$. Set $V^n = \mathbb{Z}^2 \cap 3^n F^n$. We denote by $||x||$ the Euclidean norm of x. For a vertex set W we define a bond set $E(W) = \{\langle u,v\rangle \mid u,v \in W, ||u-v|| = 1\}$. Here we wrote $\langle u,v\rangle$ as a bond with endvertices u and v. Set a graph $G^n = (V^n, E(V^n))$. Note that V^n and $E(V^n)$ are increasing sequences with respect to n. Set $G = \bigcup_{n=1}^{\infty} G^n$, that is $G = (V,E)$ where $V = \bigcup_{n=1}^{\infty} V^n$ and $E = \bigcup_{n=1}^{\infty} E(V^n)$. We call this G the Sierpinski carpet lattice. We will define a family of Sierpinski carpet lattices in Section 3.

We consider bond percolation and oriented bond percolation on G. Let $0 \le p \le 1$. Each $e \in E$ is declared to be *open* with probability p and *closed* with probability $1-p$ independently. We denote by P_p the product measure. Next let us consider a sequence of vertices $\pi = (v_0, v_1, \cdots, v_m)$ where $v_i \in V$ for $0 \le i \le m$. We say π is a *path* when $\langle v_{i-1}, v_i\rangle \in E$ for $1 \le i \le m$ and $v_i \neq v_j$ for $i \neq j$. We give a partial order on $\mathbb{Z}^2$ such that $(x_1,x_2) \le (y_1,y_2)$ if and only if $x_1 \le y_1$ and $x_2 \le y_2$. We say π is an

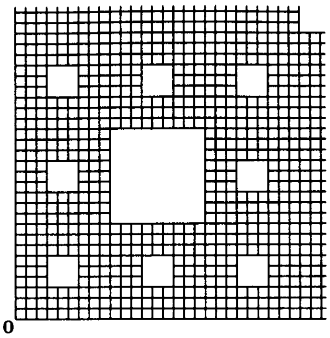

Figure 5.1: the Sierpinski carpet lattice

oriented path when π is a path and $v_{i-1} \leq v_i$ for $1 \leq i \leq m$. We write $u \leftrightarrow v$ if and only if there exists a path π with $v_0 = u$, $v_m = v$ and $\langle v_{i-1}, v_i \rangle$ are open for $1 \leq i \leq m$. We denote $C(v) = \{u \in V | \ v \leftrightarrow u\}$. We call $C(v)$ the *open cluster containing* v, and we denote by C the open cluster containing the origin. We define $\theta(p) = P_p(|C| = \infty)$ where $|C|$ means the number of vertices in C. Set $p_c = \inf\{p | \ \theta(p) > 0\}$. We write $u \to v$ if and only if there exists an oriented path π with $v_0 = u$, $v_m = v$ and $\langle v_{i-1}, v_i \rangle$ are open for $1 \leq i \leq m$. We define $\overrightarrow{C}(v) = \{u \in V | \ v \to u\}$, $\overrightarrow{C}$, $\overrightarrow{\theta}(p)$ and $\overrightarrow{p_c}$ in the same way as $C(v)$, C, $\theta(p)$ and p_c. We write $p_c(S.C.)$ and $\overrightarrow{p_c}(S.C.)$ for p_c and $\overrightarrow{p_c}$ respectively when we want to emphasize its dependence on the graph (the Sierpinski carpet lattice in this case).

We explain studies of percolation on Sierpinski carpet lattices. Kumagai [9] showed that $p_c < 1$ for a family of Sierpinski carpet lattices (which includes the Sierpinski carpet lattice) and studied under an assumption its critical phenomena and uniqueness of infinite cluster for $p > p_c$. Lü [11] gave an alternative proof of $p_c < 1$ using a Peierls argument. Shinoda [15] gave sufficient conditions and necessary conditions to have $p_c < 1$ for generalized Sierpinski carpet lattices. Murai [13] studied an asymptotic behavior as $d \to \infty$ of the critical probability of d-dimensional Sierpinski carpet lattices. Dekking and Meester [5] studied the fractal percolation process (Mandelbrot percolation) on the Sierpinski carpet.

In this paper we study oriented percolation on Sierpinski carpet lattices. Oriented percolation is significant as a model of disordered media because it has close relations to media of semiconductors, contact processes and so on. On $\mathbb{Z}^2$ we may regard this model as a one-dimensional contact process in discrete time. See Durrett [4] and [7] for details. On $\mathbb{Z}^d$ ($d \geq 2$), it is well-known that the critical probability $p_c(\mathbb{Z}^d)$ of percolation and that $\overrightarrow{p_c}(\mathbb{Z}^d)$ of oriented percolation are strictly less than 1. In particular, $p_c(\mathbb{Z}^2) = 1/2$ has been shown by Kesten [9] and $\overrightarrow{p_c}(\mathbb{Z}^2) \leq 2/3$ has been shown by Liggett [10]. We shall determine the critical probability $\overrightarrow{p_c}(S.C.)$ of oriented percolation on the Sierpinski carpet lattice. By definition $p_c(S.C.) \leq \overrightarrow{p_c}(S.C.)$ is clear. We obtain the following result.

Theorem 5.1 *The critical probability $\overrightarrow{p_c}(S.C.)$ of oriented percolation on the Sierpinski carpet lattice is equal to* 1.

This result is interesting because it shows a difference between the Sierpinski carpet lattice and $\mathbb{Z}^2$ lattice. Theorem 5.1 says that there exists no phase transition of oriented percolation on the Sierpinski carpet lattice, in spite of the existence of phase transition of percolation on it. This kind of extinction of phase transition had been shown by Chayes [2] and Chayes, Pemantle and Peres [3] in the case of the fractal percolation process on the unit square. Theorem 5.1 says also that the contact process will die out if $p < 1$ on the Sierpinski carpet lattice.

We give a proof of Theorem 5.1 in Section 2. In Section 3 we consider this problem on a family of Sierpinski carpet lattices, and give sufficient conditions for non-existence of phase transition.

5.2 Proof of main theorem

In this section we shall prove the main theorem. In this proof, events of a crossing in a rectangle play important roles. For a rectangle $R \subset \mathbb{R}^2$, we say *left-right crossing* (respectively *bottom-top crossing*) of R exists if $u \to v$ for some u on the left (respectively lower) side of R and some v on the right (respectively upper) side of R. We write $LR(R)$ (respectively $BT(R)$) for the event. This event depends on the configuration of $\{\langle u, v\rangle |\ u, v \in R\}$. For a positive integer k, we write $x_k^n(p) = P_p(LR([0, k\cdot 3^n] \times [0, 3^n]))$. Note that $x_k^n(p)$ is non-increasing with respect to k. In order to show Theorem 1.1 it is enough to prove

$$\lim_{n\to\infty} x_2^n(p) = 0, \tag{5.1}$$

because for any n

$$\{|\overrightarrow{C}| = \infty\} \subset LR([0, 2\cdot 3^n] \times [0, 3^n]) \cup BT([0, 3^n] \times [0, 2\cdot 3^n])$$

which implies $\overrightarrow{\theta}(p) \leq 2x_2^n(p)$ by symmetry. We will use the following lemmas.

Lemma 5.2 *Let $p < 1$. There exist $k_0 \geq 1$ and $\varphi > 0$ such that*

$$x_{k_0}^n(p) \leq \mathrm{e}^{-3^n\varphi} \tag{5.2}$$

for any n.

Lemma 5.3 *Let $k \geq 3$. For any n and p,*

$$x_k^{n+1}(p) \leq 2x_{k+1}^n(p). \tag{5.3}$$

Lemma 5.4 *For any n and p,*

$$x_2^{n+1}(p) \leq x_2^n(p)^2 + 2x_5^n(p). \tag{5.4}$$

Lemma 5.5 *For any n and p,*

$$x_2^n(p) \leq \left\{1 - (1-p)^{2^{n+1}}\right\}^2. \tag{5.5}$$

We give a proof of these lemmas one by one.

Proof of Lemma 5.2. For $m \geq 1$ we define a random variable

$$X_m^n = \inf\{j |\ \text{there exists } w \text{ such that } 0 \leq w \leq 3^n \text{ and } (0, w) \to (m, j)\}. \tag{5.6}$$

For convenience we set $X_0^n = 0$, and we set $X_m^n = \infty$ if the right-hand of (5.6) is empty. X_m^n is non-decreasing with respect to m. Set $V_m = ([0, m] \times [0, \infty)) \cap V$ and $E_m = E(V_m)$. Note that X_m^n is determined by the configuration of E_m. For any configuration ω_m of E_m we have

$$\begin{aligned} P_p(X_{m+1}^n = X_m^n |\ \omega_m) &\leq p, & (5.7)\\ P_p(X_{m+1}^n \geq X_m^n + 1 |\ \omega_m) &\geq 1 - p. & (5.8) \end{aligned}$$

It is clear that

$$x_k^n(p) = P_p(X_{k\cdot 3^n}^n \leq 3^n)$$

by definition. Let $\{Y_i\}_{i=1,2,\ldots}$ be the sequence of independent random variables with $P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - p$ for any i. Then we have

$$P_p(X_{k\cdot 3^n}^n \leq 3^n) \leq P\left(\sum_{i=1}^{k\cdot 3^n} Y_i \leq 3^n\right)$$

by (5.7) and (5.8). The event of right-hand side has been studied well as a sum of independent random variables, such as random walks (see Spitzer [16] for example). If $1/k < 1-p$ then the probability decays exponentially with respect to 3^n. $\square$

Remark. Lemma 5.2 is true also on $\mathbb{Z}^2$ lattice. In case of $\mathbb{Z}^2$ lattice the conditional probabilities in (5.7) and (5.8) are equal to p and $1-p$ respectively.

Proof of Lemma 5.3. We set $s = \lfloor (k-1)/2 \rfloor$ where $\lfloor x \rfloor$ means the greatest integer not greater than x. Note that $2s+1 \leq k$. We observe that

$$LR([0, k\cdot 3^{n+1}] \times [0, 3^{n+1}]) \subset A_1^n \cup A_2^n$$

where

$$\begin{aligned} A_1^n &= LR([0,(3s+2)3^n] \times [0,3^n]), \\ A_2^n &= LR([(3s+1)3^n, (2s+1)3^{n+1}] \times [2\cdot 3^n, 3^{n+1}]). \end{aligned}$$

Here we used the property of G that there exists a hole with size $3^n \times 3^n$ centered at $[(2s+1)3^{n+1}/2, 3^{n+1}/2]$. Thus $x_k^{n+1}(p) \leq 2x_{3s+2}^n(p)$ follows. We note that $k+1 \leq 3s+2$ when $k \geq 3$, and we have completed the proof. $\square$

Proof of Lemma 5.4. We observe that

$$LR([0, 2\cdot 3^{n+1}] \times [0, 3^{n+1}]) \subset (A_3^n \cap A_4^n) \cup A_5^n \cup A_6^n$$

where

$$\begin{aligned} A_3^n &= LR([0, 2\cdot 3^n] \times [0, 3^n]), \\ A_4^n &= LR([4\cdot 3^n, 2\cdot 3^{n+1}] \times [2\cdot 3^n, 3^{n+1}]), \\ A_5^n &= LR([0, 5\cdot 3^n] \times [0, 3^n]), \\ A_6^n &= LR([3^n, 2\cdot 3^{n+1}] \times [2\cdot 3^n, 3^{n+1}]). \end{aligned}$$

We have (5.4) immediately from this relation. $\square$

Proof of Lemma 5.5. Set $E_m^n = \{\langle (m,w),(m+1,w)\rangle \mid 0 \leq w \leq 3^n\} \cap E$. If $LR([0, 2\cdot 3^n] \times [0, 3^n])$ occurs, then at least one bond in $E_{(3^n-1)/2}^n$ must be open and so as in $E_{(3^{n+1}-1)/2}^n$. We obtain (5.5) immediately because $|E_{(3^n-1)/2}^n| = |E_{(3^{n+1}-1)/2}^n| = 2^{n+1}$. $\square$

We give a proof of Theorem 5.1 by using of these lemmas.

Proof of Theorem 5.1. For $p < 1$ we pick k_0 and $\varphi > 0$ which satisfy (5.2). By (5.3) we obtain

$$x_5^n(p) \leq 2x_6^{n-1}(p) \leq \cdots \leq 2^{k_0-5} x_{k_0}^{n-k_0+5}(p) \leq 2^{k_0-5} \mathrm{e}^{-3^{n-k_0+5}\varphi}$$

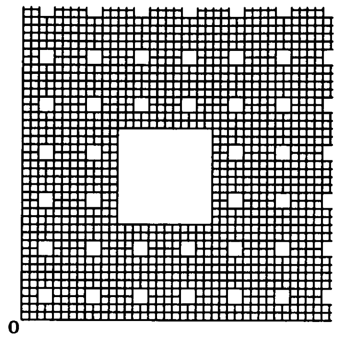

Figure 5.1: the graph of $G^2_{2,2}$

for $n \geq k_0 - 5$. By this inequality and (5.4) we have

$$x_2^{n+1}(p) \leq x_2^n(p)^2 + \mathrm{c}e^{-3^n \psi} \tag{5.9}$$

for some $c < \infty$ and $\psi > 0$. If $\liminf_{n\to\infty} x_2^n(p) < 1$ then (5.1) follows because $\lim_{n\to\infty} \mathrm{c}e^{-3^n\psi} = 0$. Suppose that $\lim_{n\to\infty} x_2^n(p) = 1$. Pick N such that $x_2^n(p) \geq 1/2$ for any $n > N$. By (5.9) and (5.5) we have

$$\begin{aligned} x_2^{n+1}(p) &\leq x_2^n(p)^{3/2}\{x_2^n(p)^{1/2} + 2^{3/2}\mathrm{c}e^{-3^n\psi}\} \\ &\leq x_2^n(p)^{3/2}\{1 - (1-p)^{2^{n+1}} + 2^{3/2}\mathrm{c}e^{-3^n\psi}\} \end{aligned}$$

for $n > N$. So we can pick N' such that $x_2^{n+1}(p) < x_2^n(p)^{3/2}$ for any $n > N'$. This contradicts to $\lim_{n\to\infty} x_2^n(p) = 1$. □

5.3 On generalized Sierpinski carpet lattices

In this section we consider oriented percolation on a family of Sierpinski carpet lattices in $\mathbb{Z}^d$, $d \geq 2$. Let a and b be positive integers. We write $L = 2a + b$. For $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \{0, 1, \ldots, L-1\}^d$ we set an affine map $\Psi_{\mathbf{i}}$ from $[0,1]^d$ to $[i_1/L, (i_1+1)/L] \times [i_2/L, (i_2+1)/L] \times \cdots \times [i_d/L, (i_d+1)/L]$ which preserves the directions. Set

$$T^d_{a,b} = \left\{(i_1, i_2, \ldots, i_d) \in \{0, 1, \ldots, L-1\}^d \;\middle|\; |\{j \mid a \leq i_j \leq a+b-1\}| \leq 1\right\}.$$

We take the unique nonempty compact set $K^d_{a,b} \subset [0,1]^d$ which satisfies the equation that $K^d_{a,b} = \bigcup_{\mathbf{i}\in T^d_{a,b}} \Psi_{\mathbf{i}}(K^d_{a,b})$. We note that $K^d_{1,1}$ is called d-dimensional *Menger sponge* (see [12] for example). Set $F^{d,n}_{a,b} = \bigcup_{\mathbf{i}_1,\mathbf{i}_2,\cdots,\mathbf{i}_n \in T^d_{a,b}} \Psi_{\mathbf{i}_1} \circ \Psi_{\mathbf{i}_2} \circ \cdots \circ \Psi_{\mathbf{i}_n}([0,1]^d)$. Set $V^{d,n}_{a,b} = \mathbb{Z}^d \cap L^n F^{d,n}_{a,b}$ and $G^{d,n}_{a,b} = (V^{d,n}_{a,b}, E(V^{d,n}_{a,b}))$. We define a graph $G^d_{a,b} = \bigcup_{n=1}^\infty G^{d,n}_{a,b}$, that is $G^d_{a,b} = (V^d_{a,b}, E^d_{a,b})$ where $V^d_{a,b} = \bigcup_{n=1}^\infty V^{d,n}_{a,b}$ and $E^d_{a,b} = \bigcup_{n=1}^\infty E(V^{d,n}_{a,b})$. As an example, the graph of $G^2_{2,2}$ is illustrated in Figure 5.2.
We consider bond percolation and oriented bond percolation on $G^d_{a,b}$. We give a partial order on $\mathbb{Z}^d$ such that $(x_1, x_2, \ldots, x_d) \leq (y_1, y_2, \ldots, y_d)$ if and only if $x_i \leq y_i$ for $1 \leq i \leq d$. We define $\theta^d_{a,b}(p)$, $p_c(G^d_{a,b})$, $\vec{\theta}^d_{a,b}(p)$ and $\vec{p_c}(G^d_{a,b})$ in a similar fashion as in Section 1. In case of percolation, $p_c(G^d_{a,b}) < 1$ has been shown for all a and b in [9]. In contrast we obtain two theorems in case of oriented percolation.

Theorem 5.6 *Let $d = 2$ and $a \leq b$. Then $\vec{p_c}(G^2_{a,b}) = 1$.*

Theorem 5.7 *Let $2 \le d \le b$. Then $\overrightarrow{p_c}(G^d_{1,b}) = 1$.*

Theorem 5.6 says that on two-dimensional Sierpinski carpet lattices if the ratio of its hole in $T^2_{a,b}$ is not smaller than $1/3^2$ then there is no phase transition. Theorem 5.7 says that for any $d \ge 2$ there exist d-dimensional Sierpinski carpet lattices on which there is no phase transition. We do not know whether $\overrightarrow{p_c}(G^d_{a,b}) = 1$ for *all* d,a and b or not.

Remark. We may define generalized Sierpinski carpet lattices in a different manner. Set $L = 3$ and $T^d_{sc} = \{0,1,2\}^d \setminus \{(1,1,\dots,1)\}$. Let K^d_{sc} be the unique nonempty compact set which satisfies the equation that $K^d_{sc} = \bigcup_{\mathrm{i}\in T^d_{sc}} \Psi_{\mathrm{i}}(K^d_{sc})$. K^d_{sc} is called *d-dimensional Sierpinski carpet.* Both $K^d_{1,1}$ and K^d_{sc} are a generalization of the Sierpinski carpet in d dimensions. Let G^d_{sc} be the graph corresponding to K^d_{sc}. We note that G^d_{sc} contains $\mathbb{Z}^{d-1}$ lattice as a subgraph, and we observe that $\overrightarrow{p_c}(G^d_{sc}) \le \overrightarrow{p_c}(\mathbb{Z}^{d-1}) < 1$ when $d \ge 3$.

For a rectangle $R = [s_1,t_1] \times [s_2,t_2] \times \cdots \times [s_n,t_n] \subset \mathbb{R}^d$ we denote by $LR(R)$ the event $\{u \to v \text{ for some } u,v \in R \text{ with } u_1 = s_1, v_1 = t_1\}$ where u_1 and v_1 mean the first coordinate of u and v respectively. Set $x^n_{k,l}(p) = P_p(LR([0,kL^n] \times [0,lL^n]^{d-1}))$. We notice that $x^n_{k,l}(p)$ depends on d, a and b but we omit to write them. Note that $x^n_{k,l}(p)$ is non-increasing with respect to k and non-decreasing with respect to l.

First we shall prove Theorem 5.6. Recall that $d = 2$ and $a \le b$ in this case. It is enough to show that

$$\lim_{n\to\infty} x^n_{a+b,a}(p) = 0. \tag{5.10}$$

We have already shown this theorem in case of $a = b = 1$ in Section 2. Also in case of $a = 1$ and $b \ge 2$ we can prove (5.10) in exactly the same way. Hereafter we assume that $2 \le a \le b$. We will use the following lemmas.

Lemma 5.8 *Let $p < 1$. There exist $k_0 \ge 1$ and $\varphi > 0$ such that*

$$x^n_{k_0,2a}(p) \le \mathrm{e}^{-L^n\varphi} \tag{5.11}$$

for any n.

Lemma 5.9 *(i) Let $k \ge 2a+3$. For any n and p,*

$$x^{n+1}_{k,a}(p) \le 2a x^n_{k+1,a}(p). \tag{5.12}$$

(ii) Let $k \ge 4a+2$. For any n and p,

$$x^{n+1}_{k,2a}(p) \le 2x^n_{3a+2b,a}(p) + (2a-1)x^n_{k+1,2a}(p). \tag{5.13}$$

Lemma 5.10 *For any n and p,*

$$x^{n+1}_{a+b,a}(p) \le x^n_{a+b,a}(p)^2 + 2\{x^n_{3a+2b,a}(p) + (a-1)x^n_{4a+3b,2a}(p)\}. \tag{5.14}$$

Lemma 5.11 *For any n and p,*

$$x^n_{a+b,a}(p) \le \{1-(1-p)^{(2a)^{n+1}}\}^{a+b}. \tag{5.15}$$

Lemma 5.8 and Lemma 5.11 are obtained in exactly the same way as Lemma 5.2 and Lemma 5.5 respectively. We give a proof of Lemma 5.9 and Lemma 5.10 briefly.

Proof of Lemma 5.9. Let $\alpha_1 = \lfloor\lfloor(k-1)/a\rfloor/2\rfloor$ and $\alpha_2 = \lfloor(k-1)/a\rfloor$. Note that $2\alpha_1 + (a-1)\alpha_2 + 1 \le k$. We have the following relation:

$$LR([0, kL^{n+1}] \times [0, aL^{n+1}]) \subset \bigcup_{j=0}^{a} B_j^n$$

where

$$\begin{aligned}
B_0^n &= LR([0, (\alpha_1 L + a + b)L^n] \times [0, aL^n]), \\
B_j^n &= LR([(\alpha_1 L + (j-1)\alpha_2 L + a)L^n, (\alpha_1 L + j\alpha_2 L + a + b)L^n] \\
&\qquad \times [(jL - a)L^n, (jL + a)L^n]) \quad \text{for } 1 \le j \le a-1, \\
B_a^n &= LR([(\alpha_1 L + (a-1)\alpha_2 L + a)L^n, (2\alpha_1 + (a-1)\alpha_2 + 1)L^{n+1}] \\
&\qquad \times [(aL - a)L^n, aL^{n+1}]).
\end{aligned}$$

Thus we obtain

$$x_{k,a}^{n+1}(p) \le 2x_{\alpha_1 L + a + b, a}^n(p) + (a-1)x_{\alpha_2 L + b, 2a}^n(p).$$

We note that $x_{k,2l}^n(p) \le 2x_{\lfloor k/2\rfloor, l}^n(p)$ holds for any k and l. So we have

$$x_{k,a}^{n+1}(p) \le 2x_{\alpha_1 L + a + b, a}^n(p) + 2(a-1)x_{\lfloor(\alpha_2 L + b)/2\rfloor, a}^n(p).$$

We note that $\alpha_1 L + a + b \ge \lfloor(\alpha_2 L + b)/2\rfloor$ holds for any k, a and b by the definition of α_1 and α_2. If $k \ge 2a + 3$ then $\lfloor(\alpha_2 L + b)/2\rfloor \ge k + 1$ because

$$\begin{aligned}
\left\lfloor \frac{\alpha_2 L + b}{2} \right\rfloor - (k+1) &\ge \frac{1}{2}\left\{ \left\lfloor \frac{k-1}{a} \right\rfloor L + b - 1 \right\} - (k+1) \\
&\ge \frac{k-a}{2a} L + \frac{b-1}{2} - (k+1) \\
&= \frac{bk - 2a^2 - 3a}{2a}
\end{aligned}$$

and $a \le b$. Thus we have proved (5.12). Let us prove (5.13) in a similar fashion. Let $\beta_1 = \lfloor\lfloor(k-1)/(2a)\rfloor/2\rfloor$ and $\beta_2 = \lfloor(k-1)/(2a)\rfloor$. Note that $2\beta_1 + (2a-1)\beta_2 + 1 \le k$. We obtain

$$x_{k,2a}^{n+1}(p) \le 2x_{\beta_1 L + a + b, a}^n(p) + (2a-1)x_{\beta_2 L + b, 2a}^n(p).$$

We observe that $k \ge 4a + 1$ implies $\beta_1 \ge 1$ and $bk \ge 4a^2 + 2a$ implies $\beta_2 L + b \ge k + 1$. Thus we have obtained (5.13). □

Proof of Lemma 5.10. Let $\gamma = \lfloor(a + b - 2)/(a-1)\rfloor$. We have

$$x_{a+b,a}^{n+1}(p) \le x_{a+b,a}^n(p)^2 + 2\{x_{3a+2b,a}^n(p) + (a-1)x_{\gamma L + b, 2a}^n(p)\}.$$

We observe that $a \le b$ implies $\gamma \ge 2$, and we have obtained (5.14) similarly to the proof of Lemma 5.4.
□

Proof of Theorem 5.6. Let $p < 1$. If we prove that there exist $c < \infty$ and $\psi > 0$ such that

$$x_{3a+2b,a}^n(p) + (a-1)x_{4a+3b,2a}^n(p) \le ce^{-L^n \psi}, \tag{5.16}$$

then by (5.14) and (5.15) we can prove (5.10) in the same way as the proof of Theorem 1.1 in Section 2. Let us prove (5.16). Let k_0 and $\varphi > 0$ satisfy (5.11). By using (5.12) repeatedly we have

$$x^n_{3a+2b,a}(p) \le (2a)^q x^{n-q}_{k_0,a}(p) \le (2a)^q x^{n-q}_{k_0,2a}(p) \le (2a)^q \mathrm{e}^{-L^{n-q}\varphi} \tag{5.17}$$

where $q = k_0 - 3a - 2b$. We have also $x^n_{4a+3b,2a}(p) \le c' \mathrm{e}^{-L^n \varphi'}$ for some $c' < \infty$ and $\varphi' > 0$ in the same way by (5.13),(5.11) and (5.17). □

We turn to prove Theorem 5.7. Recall that $d \ge 2$, $a = 1$ and $L = 2 + b \ge d + 2$ in this case. It is enough to show that $\lim_{n\to\infty} x^n_{1+b,1}(p) = 0$. Theorem 5.7 follows immediately from the following two lemmas.

Lemma 5.12 *Let $p < 1$. There exist $k_0 \ge 1$ and $\varphi > 0$ such that*

$$x^n_{k_0,1}(p) \le \mathrm{e}^{-L^n\varphi} \tag{5.18}$$

for any n.

Lemma 5.13 *Let $k \ge d+1$. For any n and p,*

$$x^{n+1}_{k,1}(p) \le d!\; x^n_{k+1,1}(p).$$

Proof of Lemma 5.12. Recall that $G^{d,n}_{1,b} = G^d_{1,b} \cap [0, L^n]^d$, and we regard $G^d_{1,b} \cap ([0, kL^n] \times [0, L^n]^{d-1})$ as a subset of $[0, kL^n] \times G^{d-1,n}_{1,b}$. We denote by Π the set of the oriented paths on $G^{d-1,n}_{1,b}$ starting at the origin. For $\pi \in \Pi$ we define $H(\pi) = \{v \in V^d_{a,b} |\ 0 \le v_1 \le kL^n$ and $(v_2, v_3, \dots, v_d)$ is a vertex of $\pi\}$. We have

$$\begin{aligned} & LR([0,kL^n] \times [0,L^n]^{d-1}) \\ = \ & \bigcup_{\pi\in\Pi} \{u \to v \text{ in } H(\pi) \text{ for some } u, v \text{ with } u_1 = 0, v_1 = kL^n\}. \end{aligned}$$

Note that the length of $\pi \in \Pi$ is not more than $(d-1)L^n$. The number of the paths in Π is not more than $d^{(d-1)L^n}$. We have

$$x^n_{k,1}(p) \le d^{(d-1)L^n} P\Big(\sum_{i=1}^{kL^n} Y_i \le (d-1)L^n\Big)$$

where Y_i is the random variable defined in the proof of Lemma 2.1. We can pick k_0 sufficiently large to satisfy $P(\sum_{i=1}^{k_0 L^n} Y_i \le (d-1)L^n) \le \mathrm{e}^{-L^n\varphi}$ and $\mathrm{e}^{-\varphi} < d^{-(d-1)}$. Then (5.18) follows. □

Proof of Lemma 5.13. Let Ξ be the set of the oriented paths from $(0,0,\dots,0)$ to $(1,1,\dots,1)$ on $\mathbb{Z}^{d-1}$: that is, $\xi = (\xi^1, \xi^2, \dots, \xi^d) \in \Xi$ if and only if $\xi^1 = (0,0,\dots,0)$, $\xi^d = (1,1,\dots,1)$ and $\xi^i \le \xi^{i+1}$ for $1 \le i \le d-1$ with respect to the partial order on $\mathbb{Z}^{d-1}$. We write $A + x = \{a + x |\ a \in A\}$. For $\xi \in \Xi$ we set $R_{\xi,i} = [0, L^n]^{d-1} + (L-1)L^n\xi^i$. Let $s = \lfloor (k-1)/d \rfloor$. Note that $s \ge 1$ and $ds + 1 \le k$. We observe that

$$LR([0, kL^{n+1}] \times [0, L^{n+1}]^{d-1}) \subset \bigcup_{\xi\in\Xi} \left(\bigcup_{i=1}^{d} A_{\xi,i} \right)$$

where $A_{\xi,i} = LR([((i-1)sL+1)L^n, (isL+1+b)L^n] \times R_{\xi,i})$. We have

$$x^{n+1}_{k,1}(p) \le (d-1)! \cdot d \cdot x^n_{sL+b,1}(p) = d! x^n_{sL+b,1}(p)$$

because the number of the paths in Ξ equals $(d-1)!$. Let us prove that $sL+b \geq k+1$ to complete this proof. If $k \geq 3d/2$ then

$$\begin{aligned}
sL+b-(k+1) &= \left\lfloor \frac{k-1}{d} \right\rfloor L+b-(k+1) \\
&\geq \frac{k-d}{d} L+b-(k+1) \\
&= \frac{(b+2-d)k-3d}{d} \\
&\geq \frac{2k-3d}{d} \\
&\geq 0.
\end{aligned}$$

Suppose that $d+1 \leq k < 3d/2$. Then $s=1$, and $sL+b=2b+2 \geq 2d+2 \geq k+1$. $\square$

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