研　究　速　報
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A NOTE ON THE FINITE ELEMENT ANALYSIS OF TRANSIENT HEAT CONDUCTION WITH SPECIAL REFERENCE TO THE TIME INCREMENT

非定常熱伝導問題の有限要素解析で用いられる時間増分について

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1 Introduction

As a problem of the thermoelasticity, it is necessary to elucidate the temperature field of the body under interest. The heat conduction problem concerned has been well formulated in the finite element method, and discussion has been made on the short time accuracy, especially with respect to the transient heat flow[1),2),3)]. The governing equation used in the finite element formulation usually is expressed as follows,

$$[K]\{\theta\}+[C]\{\dot{\theta}\}+\{Q\}=\{0\} \qquad (1)$$

where [K] denotes the thermal conductivity matrix, [C] the heat capacity matrix, {Q} the thermal load vector, θ the temperature, and $\dot{\theta}$ the rate of the temperature change with time. Under the assumption that the temperature changes linearly within the time increment Δt, the equation (1) is converted into the following recursion form[1)].

$$([K]+2[C]/\Delta t)\{\theta\}_t=(2[C]/\Delta t-[K])\{\theta\}_{t-\Delta t}+\{Q\}_t+\{Q\}_{t-\Delta t} \qquad (2)$$

According to the formulation by Comini et al[2)], the left hand side matrix is given as $[K]+3[C]/2\Delta t$. From the point of view of the computer economy, Δt is desirable as large as possible. On the other hand, the recurrence formula (2) is proved stable unconditionally[3)], but the optimal value of Δt has not been discussed yet. This paper deals with the evaluation of Δt used in the finite element analysis of the transient heat conduction with the aid of the condition number determined from the relevant matrices.

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2 Condition number

According to the step-by-step procedure, the current temperature field is calculated by the equation (2), and the property of the solution depends on that of $[K]+2[C]/\Delta t$. The change of the temperature field propagates through the element, and the property of the element matrix is therefore essential in order to trace the propagation.

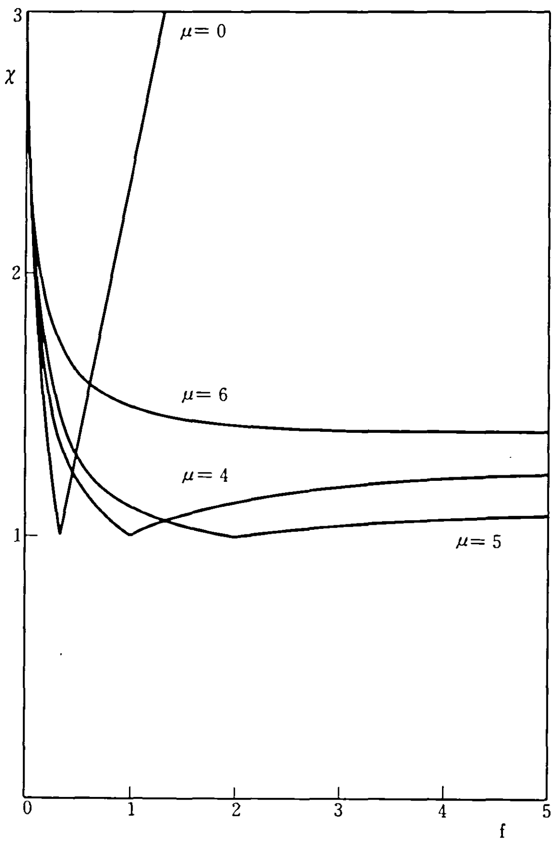

Fig. 1 Condition number vs. f

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As an example, one-dimensional bar element of A in the cross-sectional area, S in the peripheral length, and Δl in the longitudinal length is considered hereafter. The material constants are assumed uniform in the element, and the temperature distribution in the element is assumed linear for simplicity. Then the element matrices [k] and [c] are given as follows,

$$[k]=\frac{\lambda A}{\Delta l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{S\alpha\Delta l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$
$$[c]=\frac{\rho\tau A\Delta l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \quad (3)$$

where λ is the thermal conductivity, ρ the density, τ the heat capacity per volume, and α the heat transfer coefficient of the element flank surface. As for the element, we have

$$[k]+\frac{2}{\Delta t}[c]=\frac{\lambda A}{6\Delta l}\begin{bmatrix}6+2\mu+4/f & -6+\mu+2/f\\ \text{SYM.} & 6+2\mu+4/f\end{bmatrix} \quad (4)$$

where $\mu=S\alpha\Delta l^2/\lambda A$, and $f=\kappa\Delta t/\Delta l^2$ defined in the same way as the Fourier number concerned with the element by the use of the temperature conductivity $\kappa=\lambda/\rho\tau$.

A definition of the condition number of a matrix, χ, is the ratio of the maximum eigenvalue of it to the minimum one. Fig. 1 shows the relationship between the condition number χ thus defined and f of the matrix (4). It is seen that χ takes the minimum value of unity at $f_{cr}=2/(6-\mu)$, if μ is smaller than 6. If μ is equal to or larger than 6, the curve of χ has not a cusp but asymptote that lies over unity. The dependency of χ on f, say, the time increment Δt, is steep at the most in the case of $\mu=0$, and is moderate even for $f>f_{cr}$ in the case of $\mu>0$, in other words, when the heat transfer is taken into account. The off-diagonal part of the matrix (4) is positive, nil and negative, when f is smaller than, equal to and larger than f_{cr} rerspectively.

This property holds after merging the whole element matrices in the form of the equation (2). When f is taken equal to f_{cr}, the temperature at individual node is independent each other and determined only by the thermal load at the node, because that all the off-diagonal parts are nil. This is the best condition for solving the equation (2). For the other value of f, the condition number is larger than unity, and the equation (2) becomes not so well conditioned. Thus, it is recommendable to choose the value of $f=f_{cr}$ or $\Delta t=f_{cr}\Delta l^2/\kappa$ so that the condition number is minimized. When the heat transfer at the body surface is negligible, say, $\mu=0$, f_{cr} is 1/3, or 1/4 for the formulation by Comini et al[2].This means that the formulation by Wilson et al. allows Δt 4/3 times as large as that by Comini, resulting in computer time saving.

3 Numerical example

A transient heat conduction problem in one-dimension is solved herein by the finite element method[1] and compared with the relevant exact solution. The problem considered is the temperature field in a semi-infinite body, the initial temperature of which is uniform at θ_0, and the temperature at the boundary surface suddenly drops to zero at t=0[4]. This is simulated by the assembly of bar elements for which the heat transfer coefficient at the surface is nil.

Fig. 2 shows the effect of the value of f on the temperature field calculated by the use of the matrix (4) . In this calculation, 40 elements of $\Delta l=5$ mm are used, and the temperature conductivity κ is taken as 0.2779 mm²/sec, the material being assumed steel. The finite element solution obtained by taking f=1/30 (Δt=2.999 sec) or f=1/3 (Δt=29.99 sec) agrees well with the exact solution, and the result for f=1/30 is slightly larger than that

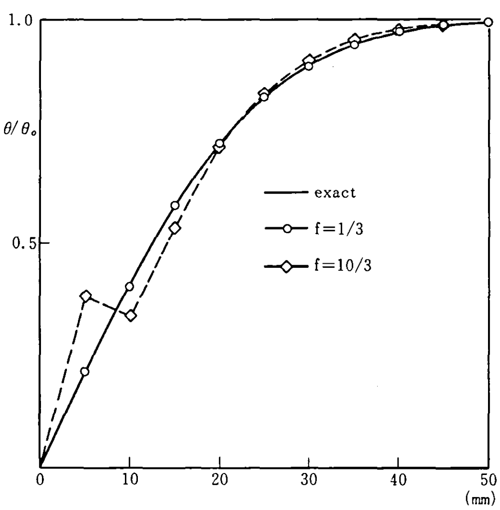

Fig. 2 Temperature distribution at t=599.7sec

for $f=1/3$ when compared strictly in numerals. In the case of $f=10/3$, the finite element solution meanders along the theoretical curve, largely in the region of small distance from the end boundary and slightly for large distance. The condition number in this case is 7, and the corresponding equation (2) is ill-conditioned in so far.

The temperature fall at the point $l=5$ mm is shown in Fig. 3. Also in such checking of the cooling rate, the finite element solution lies on the theoretical curve, if f is chosen smaller than $f_{cr}=1/3$. In the case of $f=10/3$, it is so oscillatory that there appears false temperature below zero and the convergence to the theoretical curve takes place only after some time elapses. Namely, the accuracy in short time region is inferior, when f is taken larger than f_{cr}. The oscillation can be expected by that the off-diagonal part of the matrix (4) is negative, and can be reduced by taken rather smaller value of f. Thus, the value of f_{cr} determined

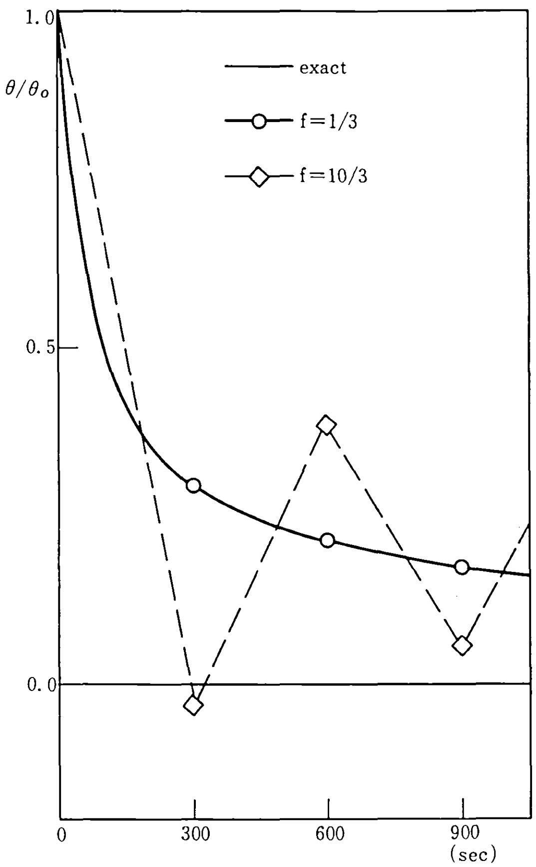

Fig. 3 Temperature change at $l=5$mm

so as to give the minimum condition number is the upper bound that the convergence of the finite element solution to the exact one is monotonic, as far as the problem of one-dimensional heat conduction is concerned.

4 Conclusions

The condition number of the matrix (4) consisting of the element thermal conductivity and heat capacity matrix is a function of the Fourier number defined as $f=\varkappa\Delta t/\Delta l^2$, and there is the minimum condition number at a certain value of f. It is possible to evaluate the optimal value of Δt by judging from the magnitude of the condition number.

Through a numerical example of one-dimensional transient heat conduction problem, it is shown that the finite element solution by the use of the step-by-step procedure and Δt corresponding to the minimum condition number agrees well with the exact solution and becomes oscillatory when larger value of Δt is taken.

It is expected that such evaluation of Δt is available for different interpolation function of the temperature in multidimensional element and different step-by-step procedure, because that the condition number can easily be calculated when the relevant matrices are given.

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