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POST-BIFURCATION ANALYSIS OF SHALLOW SPHERICAL SHELLS UNDER UNIFORM PRESSURE

等分布荷重を受ける偏平球殻の分岐座屈解析

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Introduction

In shallow spherical shells subjected to uniform external pressure, it is often observed to buckle at a lower pressure load than the axi-symmetrical snap-through buckling load. At this lower buckling point on the equilibrium path, which is called a bifurcation point, the deflection of an asymmetrical mode is added suddenly to the axi-symmetrical one.

The determination of the critical pressure for bifurcation points has been a problem of practical importance and various numerical analyses[1-3] were presented in good agreement with each other.

However, due to the numerical instability in the neighborhood of the bifurcation point, there are few investigators[4] who pursued the complete post-bifurcation paths which may provide us the basis for not only judgement of the stability of post-buckling behaviours but also assessment of the influence of imperfection.

The second and third anthors presented the method of systematic approach to construct post-bifurcation paths by means of static perturbation technique in the previous paper[5].

In this article, the finite element in the form of conical frustum is adopted and high-order nonlinear equilibrium equations are derived for the case where an asymmetric mode of displacement is superposed on the axi-symmetric one.

Then the complete post-buckling paths including the asymmetrical deformation of a clamped shallow spherical shell under uniform pressure are constructed by applying the perturbation method cited in the above to the nonlinear equations.

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Equilibrium Equations

Let us consider a conical frustum element shown in Fig. 1. Nodal displacements of node $\varDelta$ in the global coordinate are defined as

$$D_{i\varDelta}=\begin{Bmatrix} \bar{u}_\varDelta \\ \bar{w}_\varDelta \\ \beta_\varDelta \\ v_\varDelta \end{Bmatrix}, \qquad i=1\sim4 \qquad (1)$$

If we assume that the load components are represented as

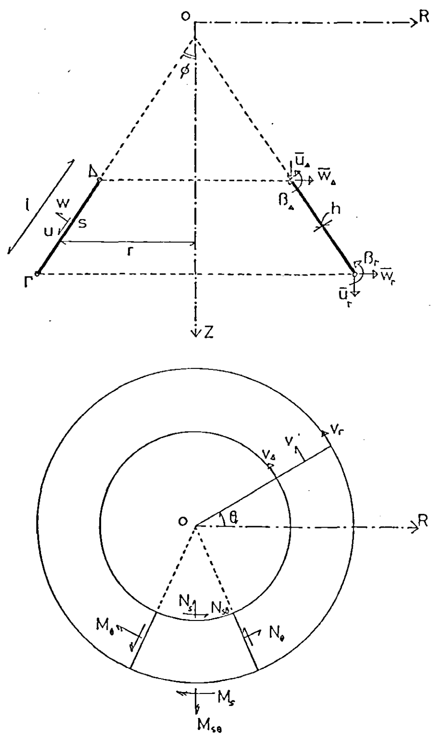

Fig. 1 Finite element model.

$$F_u=F_u^{(0)}(\xi)+F_u^{(n)}(\xi)\cdot\cos n\theta$$
$$F_w=F_w^{(0)}(\xi)+F_w^{(n)}(\xi)\cdot\cos n\theta \qquad (2)$$
$$F_v=F_v^{(n)}(\xi)\cdot\sin n\theta$$

in which $\xi=\frac{s}{l}$, the displacements may be written in the following form, corresponding to the above representation.

$$u=u^{(0)}(\xi)+u^{(n)}(\xi)\cdot\cos n\theta$$
$$w=w^{(0)}(\xi)+w^{(n)}(\xi)\cdot\cos n\theta \qquad (3)$$
$$v=v^{(n)}(\xi)\cdot\sin n\theta$$

where the upper indices (0) and (n) indicate the number of harmonics in the circumferential direction.

For the convenience of later formulation, we rewrite Eq. (3) in the following form.

$$u_i=u_i^{(0)}+u_i^{(n)}\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix}, \qquad i=1\sim3 \qquad (4)$$

where $\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix}$ means $\cos n\theta$ for both $i=1(u)$ and $i=2(w)$, and $\sin n\theta$ for $i=3(v)$.

The variation of the displacements in the direction of the generator is assumed according to the following shape functions

$$u^{(0)}=\alpha_1^{(0)}+\alpha_2^{(0)}\xi$$
$$w^{(0)}=\alpha_3^{(0)}+\alpha_4^{(0)}\xi+\alpha_5^{(0)}\xi^2+\alpha_6^{(0)}\xi^3 \qquad (5\text{–}1)$$
$$u^{(n)}=\alpha_1^{(n)}+\alpha_2^{(n)}\xi$$
$$w^{(n)}=\alpha_3^{(n)}+\alpha_4^{(n)}\xi+\alpha_5^{(n)}\xi^2+\alpha_6^{(n)}\xi^3 \qquad (5\text{–}2)$$
$$v^{(n)}=\alpha_7^{(n)}+\alpha_8^{(n)}\xi$$

The nodal displacements D_i are related to the coefficients α_i in the usual manner[6], leading to the following expression,

$$\{u_i\}=N_{ij}\cdot D_j=N_{ik}^{(0)}\cdot D_k^{(0)}+N_{il}^{(n)}\cdot D_l^{(n)}\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix},$$
$$i=1\sim3,\ j=1\sim7,\ k=1\sim3,\ l=1\sim4 \qquad (6)$$

The symbol $\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix}$ has the same meaning as in equation (4), and $D_k^{(0)}$, $D_l^{(n)}$ are generalized nodal displacements whose components are defined as follows.

$$D_{i\Delta}^{(0)}=\begin{Bmatrix}\bar{u}_\Delta^{(0)}\\ \bar{w}_\Delta^{(0)}\\ \beta_\Delta^{(0)}\end{Bmatrix},\quad i=1\sim3;\quad D_{j\Delta}^{(n)}=\begin{Bmatrix}\bar{u}_\Delta^{(n)}\\ \bar{w}_\Delta^{(n)}\\ \beta_\Delta^{(n)}\\ v_\Delta^{(n)}\end{Bmatrix},$$

$$j=1\sim4 \qquad (7\text{-}1,\ 7\text{-}2)$$

Strain-displacement relation where only the nonlinear terms due to the normal displacement w are retained is adopted according to Novozhilov[7].

$$\varepsilon_i=\begin{Bmatrix}\varepsilon_j\\ \kappa_k\end{Bmatrix}=\begin{Bmatrix}\varepsilon_{1j}\\ \kappa_k\end{Bmatrix}+\begin{Bmatrix}\varepsilon_{2j}\\ 0\end{Bmatrix}$$
$$i=1\sim6,\quad j=1\sim3,\quad k=1\sim3 \qquad (8)$$

Substituting Eq. (6) into Eq. (8), the strains are represented in terms of nodal displacements $\{D_i\}$ as

$$\varepsilon_i=B_{ij}\cdot D_j+A_{ijk}D_jD_k$$
$$=B_{il}^{(0)}\cdot D_l^{(0)}+B_{im}^{(n)}\cdot D_m^{(n)}\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix}$$
$$+A_{ils}^{(00)}D_l^{(0)}D_s^{(0)}+A_{ilm}^{(0n)}D_l^{(0)}D_m^{(n)}\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix}$$
$$+A_{imt}^{(nn)}D_m^{(n)}D_t^{(n)}\begin{pmatrix}\cos^2 n\theta\\ \sin^2 n\theta\\ \cos n\theta\cdot\sin n\theta\end{pmatrix},$$
$$i=1\sim3,\ j,k=1\sim7,\ l,s=1\sim3,\ m,t=1\sim4 \qquad (9)$$

where the symbol $(\cos^2 n\theta,\ \sin^2 n\theta,\ \cos n\theta\sin n\theta)$ can be interpreted according to the combination of $D_m^{(n)}$ and $D_t^{(n)}$, which correspond to the circumferential variation given by Eq. (3).

The stress resultants shown in Fig. 1 are related to the strains as follows,

$$\sigma_i=E_{ij}\cdot\varepsilon_j, \qquad i,j=1\sim6 \qquad (10)$$

Then the strain energy expression is given by

$$U=\frac{1}{2}\int\sigma_i\varepsilon_i dV$$
$$=\frac{1}{2}\int E_{kl}B_{ki}B_{lj}dVD_iD_j$$
$$+\int E_{lm}B_{li}A_{mjk}dVD_iD_jD_k$$
$$+\frac{1}{2}\int E_{mn}A_{mij}A_{nkl}dVD_iD_jD_kD_l$$
$$=U_{1ij}D_iD_j+U_{2ijk}D_iD_jD_k$$
$$+U_{3ijkl}D_iD_jD_kD_l$$
$$=U_1+U_2+U_3 \qquad (11)$$

where the convention implies that summation should be made over all the nodal freedom for all suffices.

Now, introducing Eq. (9) into Eq. (11) the component U_1 is led to

$$U_1=\frac{1}{2}\int_0^1\int_0^{2\pi}E_{kl}B_{ki}^{(0)}B_{lj}^{(0)}\cdot rld\theta d\xi D_i^{(0)}D_j^{(0)}$$
$$+\int_0^1\int_0^{2\pi}E_{kl}B_{ki}^{(0)}B_{lm}^{(n)}\begin{pmatrix}\cos n\theta\\ \sin n\theta\end{pmatrix}rld\theta d\xi D_i^{(0)}D_m^{(n)}$$
$$+\frac{1}{2}\int_0^1\int_0^{2\pi}E_{kl}B_{km}^{(n)}B_{ls}^{(n)}\begin{pmatrix}\cos^2 n\theta\\ \sin^2 n\theta\end{pmatrix}rld\theta d\xi D_m^{(n)}D_s^{(n)} \tag{12}$$

Integration with regard to θ with consideration of the orthogonality of trigonometrical functions, e. g.

$$\int_0^{2\pi}d\theta=2\pi,\qquad \int_0^{2\pi}\cos n\theta=\int_0^{2\pi}\sin n\theta d\theta=0,$$
$$\int_0^{2\pi}\cos^2 n\theta d\theta=\int_0^{2\pi}\sin^2 n\theta d\theta=\pi \tag{13}$$

reduces the above expression to

$$U_1=\pi\int E_{kl}B_{ki}^{(0)}B_{lj}^{(0)}rld\xi D_i^{(0)}D_j^{(0)}$$
$$+\frac{\pi}{2}\int E_{kl}B_{km}^{(n)}B_{ls}^{(n)}rld\xi D_m^{(n)}D_s^{(n)}$$
$$=U_{1ij}^{(00)}D_i^{(0)}D_j^{(0)}+U_{1ms}^{(nn)}D_m^{(n)}D_s^{(n)} \tag{14-1}$$

In a similar way, the remaining two components can be led to the following forms:

$$U_2=U_{2ijk}^{(0000)}D_i^{(0)}D_j^{(0)}D_k^{(0)}+U_{2ilm}^{(0nn)}D_i^{(0)}D_l^{(n)}D_m^{(n)} \tag{14-2}$$

$$U_3=U_{3ijkl}^{(0000)}D_i^{(0)}D_j^{(0)}D_k^{(0)}D_l^{(0)}$$
$$+U_{3ijms}^{(00nn)}D_i^{(0)}D_j^{(0)}D_m^{(n)}D_s^{(n)}$$
$$+U_{3mstp}^{(nnnn)}D_m^{(n)}D_s^{(n)}D_t^{(n)}D_p^{(n)} \tag{14-3}$$

By using the stationary condition of the total potential energy function, the nonlinear equations of equilibrium are obtained in the partitioned matrix form as,

$$\begin{bmatrix}K_{ij}^{(00)} & 0\\ 0 & K_{ij}^{(nn)}\end{bmatrix}\begin{Bmatrix}D_j^{(0)}\\ D_j^{(n)}\end{Bmatrix}$$
$$+\begin{bmatrix}K_{ijk}^{(000)} & 0 & K_{ijk}^{(0nn)}\\ 0 & 2K_{ijk}^{(n0n)} & 0\end{bmatrix}\begin{Bmatrix}D_j^{(0)}D_k^{(0)}\\ D_j^{(0)}D_k^{(n)}\\ D_j^{(n)}D_k^{(n)}\end{Bmatrix}$$
$$+\begin{bmatrix}K_{ijkl}^{(0000)} & 0 & K_{ijkl}^{(00nn)} & 0\\ 0 & K_{ijkl}^{(nn00)} & 0 & K_{ijkl}^{(nnnn)}\end{bmatrix}$$
$$\begin{Bmatrix}D_j^{(0)}D_k^{(0)}D_l^{(0)}\\ D_j^{(n)}D_k^{(0)}D_l^{(0)}\\ D_j^{(0)}D_k^{(n)}D_l^{(n)}\\ D_j^{(n)}D_k^{(n)}D_l^{(n)}\end{Bmatrix}=\begin{bmatrix}F_i^{(0)}\\ F_i^{(n)}\end{bmatrix} \tag{15}$$

The condensed expressin of the above equation is

$$K_{ij}D_j+K_{ijk}D_jD_k+K_{ijkl}D_jD_kD_l=\Lambda P_i \tag{16}$$

where D_i : i-th component of displacements

Λ : load parameter

P_i : i-th component of loading mode.

Solution of the Nonlinear Equations

To use the static perturbation method presented in reference[5] for solving the foregoing nonlinear equations, let us reduce Eq. (16) to the following incremental equilibrium equations of the progress $(d_i,\ \lambda)$ from a known equilibrium point $P(D_i,\ \Lambda)$ to a new equilibrium point $Q(D_i+d_i,\ \Lambda+\lambda)$:

$$\bar{K}_{ij}d_j+\bar{K}_{ijk}d_jd_k+\bar{K}_{ijkl}d_jd_kd_l=\lambda\bar{P}_i \tag{17}$$

where

$$\bar{K}_{ij}=K_{ij}+2K_{ijk}D_k+3K_{ijkl}D_kD_l$$
$$\bar{K}_{ijk}=K_{ijk}+3K_{ijkl}D_l$$
$$\bar{K}_{ijkl}=K_{ijkl} \tag{18}$$
$$\bar{P}_i=P_i$$

The coefficient matrix $[\bar{K}_{ij}]$ in Eq. (18) has usually the following form:

$$[\bar{K}_{ij}]=\begin{bmatrix}\bar{K}_{ij}^{(00)} & \bar{K}_{ij}^{(0n)}\\ \bar{K}_{ij}^{(n0)} & \bar{K}_{ij}^{(nn)}\end{bmatrix} \tag{19}$$

where

$$\bar{K}_{ij}^{(00)}=K_{ij}^{(00)}+2K_{ijk}^{(000)}D_k^{(0)}+3K_{ijkl}^{(0000)}D_k^{(0)}D_l^{(0)}$$
$$+K_{ijkl}^{(00nn)}D_k^{(n)}D_l^{(n)}$$
$$\bar{K}_{ij}^{(0n)}=2K_{ijk}^{(0nn)}D_k^{(n)}+2K_{ijkl}^{(00nn)}D_k^{(0)}D_l^{(n)}\equiv[\bar{K}_{ij}^{(n0)}]^T$$
$$\bar{K}_{ij}^{(nn)}=K_{ij}^{(nn)}+2K_{ijk}^{(nn0)}D_k^{(0)}+2K_{ijkl}^{(nn00)}D_k^{(0)}D_l^{(0)}$$
$$+3K_{ijkl}^{(nnnn)}D_k^{(n)}D_l^{(n)} \tag{20}$$

If the equilibrium point, P, that is the starting point of the incremental progress, is in the axi-symmetrical state $(D_k^{(n)}=0)$, the coefficient $[\bar{K}_{ij}]$ is expressed as a diagonally partitioned matrix having the form

$$[\bar{K}_{ij}]=\begin{bmatrix}\bar{K}_{ij}^{(00)} & 0\\ 0 & \bar{K}_{ij}^{(nn)}\end{bmatrix} \tag{21}$$

If we introduce a continuous parameter t, the progress $(d_i,\ \lambda)$ along the equilibrium path can be written in a parametric form by expanding both $d_i(t)$ and $\lambda(t)$ into Maclaurin series as

$$d_i(t)=\dot{d}_it+\frac{1}{2}\ddot{d}_it^2+\frac{1}{6}\dddot{d}_it^3+\cdots$$

$$\lambda(t)=\dot{\lambda}t+\frac{1}{2}\ddot{\lambda}t^2+\frac{1}{6}\dddot{\lambda}t^3+\cdots \qquad (22)$$

where $\cdot\equiv\frac{d}{dt}$ and $\dot{d}_i=\dot{d}_i(0)$, $\dot{\lambda}=\dot{\lambda}(0)$ etc. By introducing Eqs. (22) into Eq. (17) and then putting the coefficient of each power of t to zero, we can obtain the set of linear equations;

$$\begin{aligned}&\bar{K}_{ij}\dot{d}=\dot{\lambda}\bar{P}_i\\&\bar{K}_{ij}\ddot{d}+2(\bar{K}_{ijk}\dot{d}_j\dot{d}_k)=\ddot{\lambda}\bar{P}_i\\&\bar{K}_{ij}\dddot{d}+3(\bar{K}_{ijk}+\bar{K}_{ikj})\ddot{d}_j\dot{d}_k+6\bar{K}_{ijkl}\dot{d}_j\dot{d}_k\dot{d}_l\\&\quad=\dddot{\lambda}\bar{P}_i\end{aligned} \qquad (23)$$

In the case where the determinant of coefficient matrix $[\bar{K}_{ij}]$ is not vanished at a starting point of the step, the increments of displacement d_i for a given load level λ are easily calculated by successively solving the linearized equations (23) and then a new equilibrium state can be determined. When the coefficient matrix becomes singular, displacements for a given load increment λ becomes indefinite and this means the occurrence of a critical point.

In general, the case where the sub-matrix $[\bar{K}_{ij}^{(00)}]$ of Eq. (19) becomes singular corresponds to an axi-symmetrical snap-through buckling and another case in which $|\bar{K}_{ij}^{(nn)}|=0$ corresponds to a bifurcation buckling.

Construction of Post-Buckling Curve Beyond Bifurcation Point

Now let us consider the case of $|\bar{K}_{ij}^{(nn)}|=0$ where an asymmetrical mode of deformation starts to develop.

If we consider that the deflection remains axi-symmetrical and Eq. (21) holds at the very point of branching, we can reduce Eq. (23) to

$$\begin{cases}\bar{K}_{ij}^{(00)}\dot{d}_j^{(0)}=\dot{\lambda}P_i^{(0)} & (24\text{-}1)\\ \bar{K}_{ij}^{(nn)}\dot{d}_j^{(n)}=0 & (24\text{-}2)\end{cases}$$

$$\begin{cases}\bar{K}_{ij}^{(00)}\ddot{d}_j^{(0)}+2\bar{K}_{ijk}^{(000)}\dot{d}_j^{(0)}\dot{d}_k^{(0)}+2\bar{K}_{ijk}^{(0nn)}\dot{d}_j^{(n)}\dot{d}_k^{(n)}\\ \qquad=\ddot{\lambda}P_i^{(0)} & (25\text{-}1)\\ \bar{K}_{ij}^{(nn)}\ddot{d}_j^{(n)}+2\bar{K}_{ijk}^{(n0n)}\dot{d}_j^{(0)}\dot{d}_k^{(n)}=0 & (25\text{-}2)\end{cases}$$

$$\begin{cases}\bar{K}_{ij}^{(00)}\dddot{d}_j^{(0)}+3(\bar{K}_{ijk}^{(000)})\ddot{d}_j^{(0)}\dot{d}_k^{(0)}\\ \qquad+3(\bar{K}_{ijk}^{(0nn)}+\bar{K}_{ikj}^{(0nn)})\ddot{d}_j^{(n)}\dot{d}_k^{(n)}\\ \qquad+6K_{ijkl}^{(0000)}\dot{d}_j^{(0)}\dot{d}_k^{(0)}\dot{d}_l^{(0)}\\ \qquad+6K_{ijkl}^{(00nn)}\dot{d}_j^{(0)}\dot{d}_k^{(n)}\dot{d}_l^{(n)}=\dddot{\lambda}P_i^{(0)} & (26\text{-}1)\\ \bar{K}_{ij}^{(nn)}\dddot{d}_j^{(n)}+3\bar{K}_{ijk}^{(n0n)}\dot{d}_j^{(0)}\ddot{d}_k^{(n)}+3\bar{K}_{ikj}^{(n0n)}\ddot{d}_k^{(0)}\dot{d}_j^{(n)}\\ \qquad+6\bar{K}_{ijkl}^{(nn00)}\dot{d}_j^{(n)}\dot{d}_k^{(0)}\dot{d}_l^{(0)}\\ \qquad+6K_{ijkl}^{(nnnn)}\dot{d}_j^{(n)}\dot{d}_k^{(n)}\dot{d}_l^{(n)}=0 & (26\text{-}2)\end{cases}$$

Because of $|\bar{K}_{ij}^{(nn)}|=0$, we cannot determine $\dot{d}_j^{(n)}$ from Eq. (22-2). However, if we use the higher order perturbation equations (23) and (24), we can get the values of $\dot{d}_i$, $\dot{\lambda}$, $\ddot{d}_i$ and $\ddot{\lambda}$. How to determine these values from Eqs. (24)~(26) is the main interest of this section.

For this purpose, we adopt $d_r^{(n)}$ as the pertuabation parameter i. e., we have $t=d_r^{(n)}$, $\dot{d}_r^{(n)}=1$, $\ddot{d}_r^{(n)}=\dddot{d}_r^{(n)}=0$. Here r is to be so selected that determinant of minor matrix $[\varDelta\bar{K}_{ij}^{(nn)}]$, which is formed by deleting the r-th row and column from $[\bar{K}_{ij}^{(nn)}]$, have a non-zero value. Then we can represent $\dot{d}_j^{(n)}(j\neq r)$ from Eq. (24-2) as

$$\dot{d}_j^{(n)}=-[\varDelta\bar{K}_{ij}^{(nn)}]^{-1}\cdot\bar{K}_{ir}^{(nn)},\quad i,j\neq r \qquad (27)$$

Multiplying both sides of Eq. (23-2) by $\dot{d}_i^{(n)}$, we get the following equation.

$$\bar{K}_{ij}^{(nn)}\ddot{d}_j^{(n)}+2\bar{K}_{ijk}^{(n0n)}\dot{d}_i^{(n)}\dot{d}_i^{(0)}\dot{d}_k^{(n)}=0 \qquad (28)$$

With the aid of Eq. (24-2), the first term of the above equation vanishes. As $\bar{K}_{ijk}^{(n0n)}\dot{d}_i^{(n)}\dot{d}_k^{(n)}\neq 0$ generally in the second term, we obtain

$$\dot{d}_j^{(0)}=0 \qquad (29)$$

Introduction of Eq. (29) into Eq. (24-1) leads to

$$\dot{\lambda}=0 \qquad (30)$$

The values of $\ddot{d}_i^{(0)}$ can be obtained as a following functions of unknown $\ddot{\lambda}$ by substituting Eqs. (27) and (29) into (25-1).

$$\ddot{d}_k^{(0)}=-[\bar{K}_{kl}^{(00)}]^{-1}\cdot(\ddot{\lambda}\bar{P}_l^{(0)}+2\bar{K}_{lij}^{(0nn)}\dot{d}_i^{(n)}\dot{d}_j^{(n)}) \quad (31)$$

The value of $\ddot{\lambda}$ can be obtained from Eq. (26-2) by introducing foregoing values of $\dot{d}_i^{(0)}$, $\dot{d}_i^{(n)}$ and $\ddot{d}_i^{(0)}$ and multiplying $\dot{d}_i^{(n)}$ by the both sides of Eq. (26-2) with the aid of Eq. (24-2), as the similar way of getting the values of $\dot{d}_j^{(0)}$. Then $\ddot{d}_k^{(0)}$ is determined from Eq. (31) by substituting the obtained value of $\ddot{\lambda}$.

Also, putting $\dot{d}_j^{(0)}=0$ in Eq. (35-2) leads to the result that $\dot{d}_i^{(n)}=0$. Basing on these results, the first incremental progress can be stepped from the bifurcation point to find the first equilibrium point on the post-bifurcation curve.

Illustrative Examples and Discussions

A clamped shallow spherical shell subjected to axi-symmetric external pressure is considered.

At first, we constructed pressure-deflection curve under the axisymmetrical mode beyond the snap-buckling point, where we suppressed the singular point of bifurcation by the use of the conditions of symmetry. In the next place, the bifurcation point on the equilibrium path of axi-symmetrical mode was determined by judging whether the determinant of the matrix $[\bar{K}_{ij}^{(nn)}]$ vanishes or not. As the third step, the first equilibrium point in the neighborhood of the branching point was obtained by means of the method described in the previous section. Lastly, post-bifurcation path was traced. In this region the first order coefficient matrix $[\bar{K}_{ij}]$ in Eq. (17) becomes full matrix, because the non-diagonal elements, $\bar{K}_{ij}^{(0n)}$ and $\bar{K}_{ij}^{(n0)}$, take non-zero value by the influence of the displacement of asymmetrical mode occured, as shown by Eq. (20).

For this tracing of the post-bifurcation path, we used the standard solution technique of the static pertubation method[5], as in the pre-buckling region.

The illustrative example is a clamped shallow shell whose geometry is shown in Fig. 2 and the axi-symmetrical paths, the critical pressures and post-bifrcation paths for $n=2$ mode were calculated which were plotted in Fig. 3 in the Λ-Volume Change coordinate.

A post-bifurcation path for asymmetrical mode ($n=2$) is displayed in Fig. 3 as the curve $A \rightarrow E$.

At a bifurcation point A, the deformation of an asymmetrical mode starts. The asymmetrical displacements are increased quickly from A to B. On the path from B to C the displacement of axi-symmetrical mode is increases and asymmetrical mode decreases slowly. At the point C the

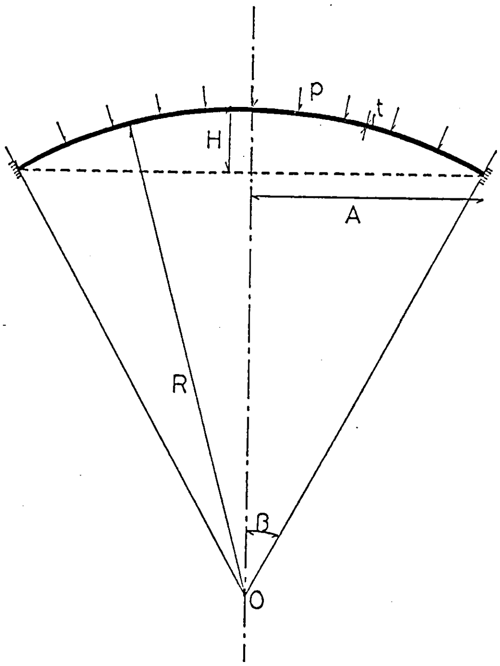

Fig. 2 Circular cap under uniform load.
$E=2.1\times10^6 \mathrm{kg/cm^2}$
$\nu=0.3$
$\lambda^2=\sqrt{12(1-\nu^2)}\cdot\beta^2\cdot R/t$
=(Geometric Parameter)

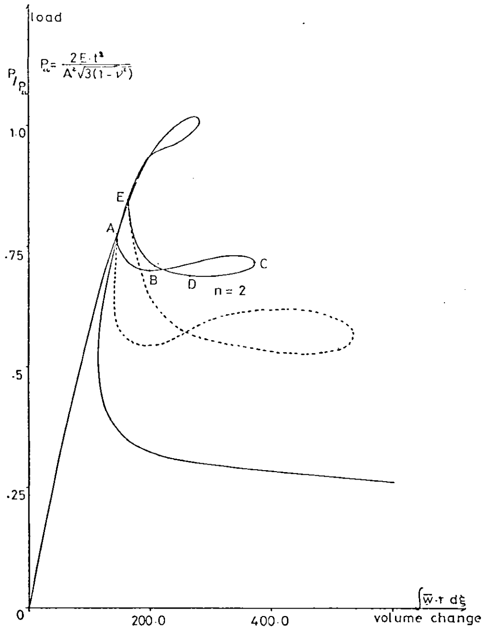

Fig. 3 Equilibrium path ($\lambda=6$)
……Ref. 4 ($\lambda \fallingdotseq 6.3$)

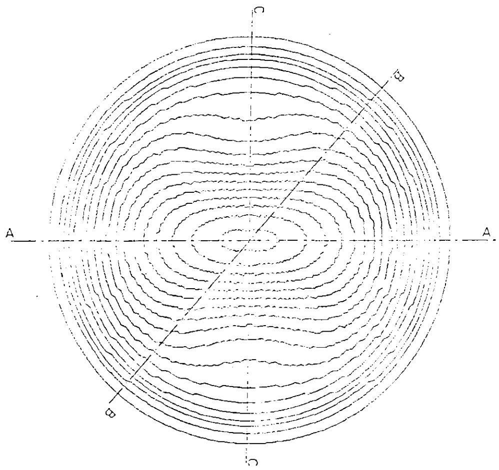

Fig. 4a Contours of deformation.

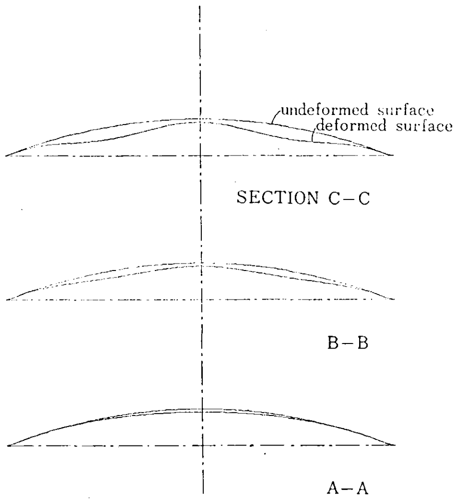

Fig. 4b Modes of deformation.

volume change is the largest.

The mode of deformation at point C is figured out in Fig. 4. From point C to D, the displacement decreases quickly and asymmetrical one disappeares completely at the point E.

For the purpose of comparison, similar result by another author[4] was indicated in Fig. 3.

In conclusion, it can be pointed out that, in spite of the numerical instability of the post-bifurcation behavior of the spherical cap, this example is indicating the effectiveness of the present method.

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