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REEVALUATION OF RAYLEIGH-RITZ'S METHOD IN STRUCTURAL MECHANICS

—Some Improvement of Conventional Rayleigh-Ritz's Procedure and Finite Element Method—

構造力学における Rayleigh-Ritz の方法の再評価について
—Rayleigh-Ritz の方法および有限要素法の一修正案—

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1. Introduction

It is generally accepted that the finite element method is the most powerful and established method for analysis of the problems in linear structural mechanics to be encountered in all the fields of engineering and now very active research work having been made all over the world on the extended application to highly nonlinear problems where material as well as geometrical nonlinearities are coupled, and the load incremental procedure based on the displacement method is considered the most hopeful method for analysis of such problems.

Unfortunately, however, this method has a serious drawback from the practical point of view. That is, generally it requires a great amount of manpower and computer time and therefore so far only simple nonlinear problems have been solved even by using the large scale computing system and computer program. In order to reduce the computing time and cost of the analysis, many procedures have been proposed on the matrix condensation technique, and yet actual problem is far beyond the control of any of these methods. On the other hand classical Rayleigh-Ritz's or Galerkin's methods usually require solution of the linear equations with much smaller number of unknowns to compare with the finite element method for results of the same accuracy, but integration of the strain energy terms is formidable task.

Previously the author has proposed the combined use of the finite element method and Rayleigh-Ritz's procedure for analysis of complicated buckling and large deflection problems of elastic plates.[1)2)]

Recently T. Hori has suggested a method for transformation of the finite element method into Rayleigh-Ritz's procedure and demonstrated the merit of his method by solving some stress concentration problems.[3)]

However, his interest is concentrated in how to improve calculation of displacement as well as stress fields rather than matrix condensation technique. Extending idea of the present author conceived in his previous papers, attempt is being made on the improvement of conventional Rayleigh-Ritz's proceedure as well as the finite element method, the outline of which will be described briefely in this short note.

2. Reduction of the Finite Element Method to Rayleigh-Ritz's Procedure

For simplicity, consider the linear analysis of a three dimensional continuous body under the action of body forces and surface traction with prescribed geometrical boundary conditions.

Applying the principle of virtual work, the following variational equation is obtained:

$$\delta V-\delta W_{ex}=0 \qquad (1)$$

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δV is the strain energy variation and it is given by the following matrix equation:

$$\delta V=\sum_i \delta \boldsymbol{u}_i^T \boldsymbol{k}_i \boldsymbol{u}_i=\delta \boldsymbol{u}^T \boldsymbol{K}\boldsymbol{u} \qquad (2)$$

where $\boldsymbol{k}_i$ is the stiffnes matrix of the i th finite element, and $\boldsymbol{u}_i$ is the nodal displacement vector defined with respect to a given global coordinates, while $\boldsymbol{K}$, $\boldsymbol{u}$ are the overall stiffness matrix and displacement vector of agiven structure respectively. δW_{ex} is the virtual work due to external loads and it is expressed by the following equation:

$$\delta W_{ex}=\sum_i \delta \boldsymbol{u}_i^T \boldsymbol{f}_i=\delta \boldsymbol{u}^T \boldsymbol{F} \qquad (3)$$

where $\boldsymbol{F}$ is the corresponding equivalent external load vector.

From eqs. (1), (2) and (3) the following stiffness equation is obtained.

$$\boldsymbol{K}\boldsymbol{u}=\boldsymbol{F} \qquad (4)$$

This is only description of the standard finite element method.

Now assume that the same overall displacement field is also given by the following equation with respect to the same glabal coordinates:

$$\left.\begin{aligned} U(x,y,z)&=\boldsymbol{h}_1(x,y,z)\boldsymbol{a}\\ V(x,y,z)&=\boldsymbol{h}_2(x,y,z)\boldsymbol{b}\\ W(x,y,z)&=\boldsymbol{h}_3(x,y,z)\boldsymbol{c}\end{aligned}\right\} \qquad (5\text{-a})$$

or in matrix form

$$\underbrace{\begin{Bmatrix} U(x,y,z)\\ V(x,y,z)\\ W(x,y,z)\end{Bmatrix}}_{\boldsymbol{U}(x,y,z)} = \underbrace{\begin{bmatrix} \boldsymbol{h}_1(x,y,z) & 0 & 0\\ 0 & \boldsymbol{h}_2(x,y,z) & 0\\ 0 & 0 & \boldsymbol{h}_3(x,y,z)\end{bmatrix}}_{\boldsymbol{H}(x,y,z)} \underbrace{\begin{Bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\\ \boldsymbol{c}\end{Bmatrix}}_{\boldsymbol{a}} \qquad (5\text{-b})$$

$$\boldsymbol{U}(x,y,z)=\boldsymbol{H}(x,y,z)\boldsymbol{a} \qquad (5\text{-c})$$

Considering the nodal displacement vector $\boldsymbol{u}_i$ of the i th element as shown in the figure, the following expression is easily obtained:

$$\boldsymbol{u}_i=\boldsymbol{A}_i\boldsymbol{a} \qquad (6)$$

Where $\boldsymbol{A}_i$ is a transformation matrix between the nodal displacement vector $\boldsymbol{u}_i$ and the generalized coorinates vector $\boldsymbol{a}$, and it is generally a rectangular matrix.

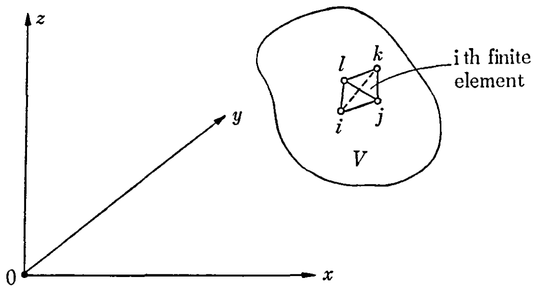

Fig. 1 Finite Element Idealization of a Structure by a Tetrahedron Element

Using eq. (6), the transformed formula of δV is obtained as follows:

$$\delta V=\sum_i \delta V_i=\sum_i \delta \boldsymbol{a}^T \boldsymbol{K}_i \boldsymbol{a}=\delta \boldsymbol{a}^T \boldsymbol{K}_R \boldsymbol{a} \qquad (7\text{-a})$$

where

$$\boldsymbol{K}_i=\boldsymbol{A}_i^T \boldsymbol{k}_i \boldsymbol{A}_i \qquad (7\text{-b})$$

$$\boldsymbol{K}_R=\sum_i \boldsymbol{K}_i \qquad (7\text{-c})$$

it should be noted that $\boldsymbol{K}_i$ is $n\times n$ square symmetric matrix where n is the number of components of column vector $\boldsymbol{a}$, and therefore $\boldsymbol{K}_R$ is obtained by simply adding $\boldsymbol{K}_i$ matrices.

In the same way δW_{ex} can be expressed as follows:

$$\delta W_{ex}=\sum_i \delta \boldsymbol{a}^T \boldsymbol{A}_i^T \boldsymbol{f}_i=\delta \boldsymbol{a}^T \boldsymbol{F}_R \qquad (8\text{-a})$$

$$\boldsymbol{F}_R=\sum_i \boldsymbol{F}_i\ ,\ \boldsymbol{F}_i=\boldsymbol{A}_i^T \boldsymbol{f}_i \qquad (8\text{-b})$$

Introducing eqs. (7-a) and (8-a) into eq. (1) the following variational equation is obtained:

$$\delta \boldsymbol{a}^T(\boldsymbol{K}_R\boldsymbol{a}-\boldsymbol{F}_R)=0$$

On account of independency of $\delta\boldsymbol{a}$, the following matrix equation can be deduced:

$$\boldsymbol{K}_R\boldsymbol{a}=\boldsymbol{F}_R \qquad (9)$$

This is the matrix equation which can be obtained by assuming the displacement function defined by eq. (5-a, b, c) and applying the conventional Rayleigh-Ritz's procedure.

In other word, the large scale system of the finite element stiffness equation (4) can be successfully transformed and reduced into the Rayleigh-Ritz's matrix equation (9) with the desired number of degree of freedom. It is obvious that this method is essentially different from the so-called "substructure method" which is a standard technique to solve the large scale matrix equation.[4] Because the former is a method of actually reducing the matrix size by imposing a finite number of constraints to the nodal displacement vector $\boldsymbol{u}$ of the total structure, while the latter is a method of solution of the large scale matrix equation by tempolarily hiding the slave nodal displacement and reducing the size of the mtraix equation to the number of master nodal displacements in the process of the structural analysis. This suggests a general method of matrix condensation in which the matrix $\boldsymbol{A}_i$ is called "the structural transducer", since this matrix has a function of transforming the stiffness equation and reducing its size, and the "structural transducer" may play a very important role in the structural analysis, because its selection gives subtantial influence on the convergency and accuracy of the said structural analysis.

From experience aquired by some numerical test analysis, the present author recommends to use the truncated power series of x, y and z for displacement functions to be assumed in Rayleigh-Ritz's method.

Perhaps selection of the terms requires analysists some skill or intuition, but the author believes that some guiding rule for selection can be established in the near future.

3. Conclusion

The pilot study on the feasibility of this method showed a satisfactory result and now extensive numerical study is underway.

For complete justification of this method, however, it should be necessary to establish an effective method of solution of fairly large matrix equations of full size.

It is believed that this method could be powerful in matrix condensation and solution of the large scale eigenvalue as well as stiffness equations, and with some appropriate modification of the procedure, it can be effectively applied to the solution of nonlinear problems in structural mechanics within reasonable computing time and labour. Extended application of this method to nonstructural problems is also now being explored.

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References

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