UDC 534.282
539.374

DYNAMIC RESPONSE OF VISCOELASTIC MATERIALS TO SINUSOIDALLY VARYING LOADINGS

有限要素法による粘弾性材料の周波数応答解析

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1. Introduction

The material damping can be easily incorporated into the finite element analysis of dynamic problems by representing the material behavior through appropriate mechanical models, i.e. generalized Maxwell or Voigt (Kelvin) models[1].

This procedure is featured by the fact that it does not necessarily involve the assumption of damping property which is proportionate to velocity and has been successfully used in the analysis of transient wave in the bar[2]. The present paper aims to extend the procedure to the frequency response of flexural vibration of beams under sinusoidally varying loadings. Generalized Maxwell model which includes simple Voigt material is assumed. The relevant stiffness matrix as well as force and displacement amplitudes are complex, and these are divided into real and imaginary parts in the process of solution. Frequency responses of simple Voigt, three and five element models of generalized Maxwell type have been studied.

The practical application of the method concerns with the reconsideration of the well-known Onogi's method[3] which has long been used for assessment of model constants principally of the simple Voigt material.

In the last part, a comparative study of consistent and lumped mass formulations will be made, as applied to frequency response analysis.

2. Basic Equation

In the analysis of frequency response of viscoelastic materials, the relationship between stress $\bar{\sigma}$ and strain $\bar{\varepsilon}$ can be written as

$$\bar{\sigma}=\bar{D}(j\omega)\bar{\varepsilon} \qquad (1)$$

Note $\bar{\sigma}$ and $\bar{\varepsilon}$ denote complex amplitudes and $\bar{D}(j\omega)$ is corresponding complex modulus. By applying the complex constitutive equation (1), the governing equation of lateral vibration of beam can be derived as

$$\bar{D}(j\omega)I\frac{\partial^4\bar{v}(x)}{\partial x^4}-\omega^2\rho A\bar{v}(x)=\bar{q}_y(x) \qquad (2)$$

where

ω = angular velocity
ρ = density of beam material
A = cross-sectional area
I = geometrical moment of inertia
$\bar{v}(x)$ = complex amplitude of lateral deflection
$\bar{q}_y(x)$ = complex distributed load.

In equation (2), the Bernoulli-Euler postulate of beam theory is adopted and so the shear deformation is neglected.

3. Finite Element Formulation

By following the standard procedure[4], we can obtain the finite element equivalent of (2) as

$$-\omega^2[M]\{\bar{V}_p\}+[\bar{K}]\{\bar{V}_p\}=\{\bar{Q}_p\} \qquad (3)$$

where

$\{\bar{V}_p\}$ = displacement vector whose components are lateral deflection and angle of inclination at nodes
$\{\bar{Q}_p\}$ = force vector whose components are shear force and moment at nodes
$[\bar{K}]$ = element stiffness matrix
$[M]$ = element mass matrix.

We write the complex stiffness matrix and force vectors as

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$$[\bar{K}]=[K_1]+j[K_2]$$
$$\{\bar{V}_p\}=\{V_{p1}\}+j\{V_{p2}\} \qquad (4)$$
$$\{\bar{Q}_p\}=\{Q_{p1}\}+j\{Q_{p2}\}$$

Then, by comparing the real and imaginary parts, we obtain from (3) the two sets of simultaneous linear equations as follows:

$$\begin{bmatrix} -\omega^2[M]+[K_1] & -[K_2] \\ [K_2] & -\omega^2[M]+[K_1] \end{bmatrix} \begin{Bmatrix} \{V_{p1}\} \\ \{V_{p2}\} \end{Bmatrix} = \begin{Bmatrix} \{Q_{p1}\} \\ \{Q_{p2}\} \end{Bmatrix} \qquad (5)$$

When cubic shape function is assumed to describe the lateral deflection of beam element of length L, the element stiffness matrix $[\bar{K}]$ is

$$[\bar{K}]=\bar{D}(j\omega)I \begin{bmatrix} 12/L^3 & & & \text{SYM.} \\ 6/L^2 & 4/L & & \\ -12/L^3 & -6/L^2 & 12/L^3 & \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix} \qquad (6)$$

The corresponding consistent mass matrix $[M_C]$ is given by

$$[M_C] = \rho AL \begin{bmatrix} 13/35 & & & \text{SYM.} \\ 11L/210 & L^2/105 & & \\ 9/70 & 13L/420 & 13/35 & \\ -13L/420 & -L^2/140 & -11L/210 & L^2/105 \end{bmatrix} \qquad (7)$$

Alternatively, the following lumped mass matrix $[M_L]$ can be used for the beam flexural deformation

$$[M_L]=\rho AL \begin{bmatrix} 1/2 & & & 0 \\ & \alpha L^2 & & \\ & & 1/2 & \\ 0 & & & \alpha L^2 \end{bmatrix} \qquad (8)$$

The coefficient α of gradient mass in equation (8) is dependent on assumption. Thus, $\alpha=0$ corresponds to the case when we neglect the rotatory inertia as in the $[M_C]$ of equation (7). $\alpha=1/96$, when we assume the centers of rotatory inertia at element nodes i and j of Fig. 1, and $\alpha=1/24$, when we devide the element into two parts and place the centers of rotation at mid points i′ and j′ of Fig. 1.

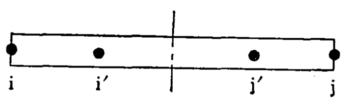


Fig. 1 Beam Element.

In passing, we refer here the resonant frequencies of cantilever beam which is excited by the constant amplitude external force or displacement at the fixed end. The frequencies are given in the ascending order as

$$\omega_1=(1.875/l)^2\sqrt{E\kappa^2/\rho}$$
$$\omega_2=(4.694/l)^2\sqrt{E\kappa^2/\rho}$$
$$\omega_3=(7.855/l)^2\sqrt{E\kappa^2/\rho}$$

or

$$\omega_2=6.267\,\omega_1, \quad \omega_3=17.551\omega_1, \quad \omega_4=34.390\,\omega_1 \qquad (9)$$

where E, κ, ρ and l are respectively Young's modulus, radius of gyration of area, density and length of beam. Note, however, that the results above neglect the effects of rotatory inertia and shear deformation.

4. Numerical Examples

Numerical examples principally concern with the cantilever beam of uniform cross-section which is excited by the input with constant displacement amplitude d_0 at the fixed end. The number of element divisions is 10, and the data in computation are

density of material: $\rho=0.0001$ (kg msec²/mm⁴)

length of beam: $l=100.0$ (mm)

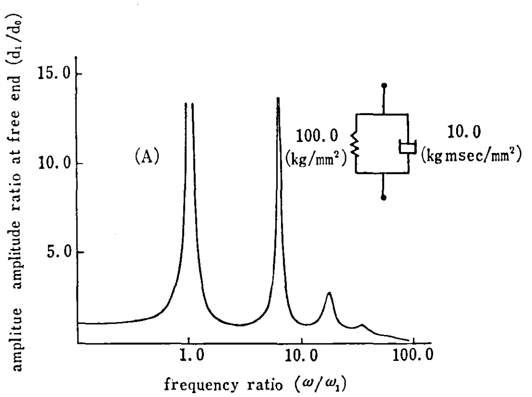
geometrical moment of inertia:

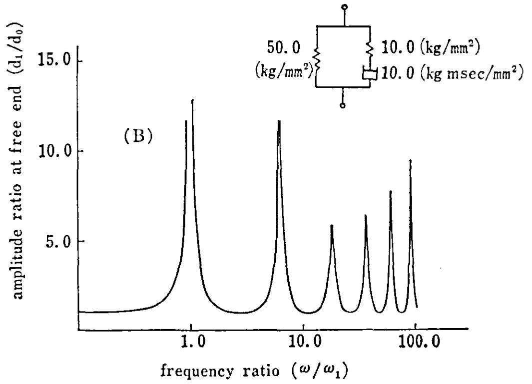
$I=5.0/6.0$ (mm⁴).

a) Frequency Response of Multi-Element Mechanical Models

Fig. 2 shows and compares frequency response curves obtained by the present procedure for three kinds of mechanical models; the ratio of free end amplitude d_1 to excitation amplitude d_0 at the fixed end being given in relation to frequency ratio ω/ω_1, where ω_1 denotes the first resonant frequency of elastic cantilever beam.

It is seen that the effect of material damping can be assessed reasonably by the present method. Further, it should be noted that the seperate





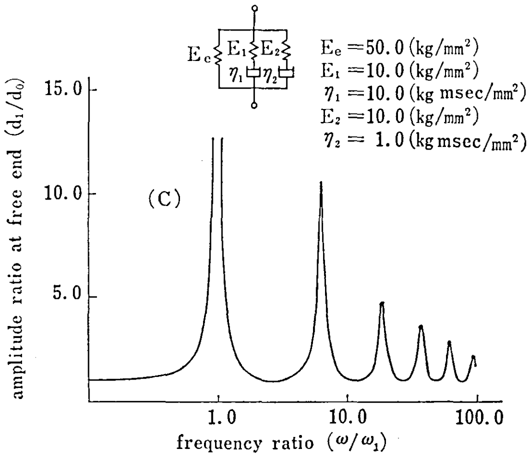


Fig. 2 Frequency response curve of viscoelastic beams.
(A) Voigt model
(B) Three-element model
(C) Five-element model

solution of eigen value problem which is rather complicated can be eliminated, at least, in the case of flexural vibration being studied.

b) Reconsideration of Onogi's Method

In Onogi's or so-called vibrating reed method[3], the material is implicitly assumed to be represented by simple Voigt model. The actual material behavior may not be of the Voigt type and be represented, for example, by the generalized multi-element Maxwell model. Therefore, we studied the validity of the Onogi's method by the following procedure.

(i) First, we compute the frequency response curve such as Fig. 2 (B) or solid curve (A) of Fig. 3 for the three element Maxwell model.

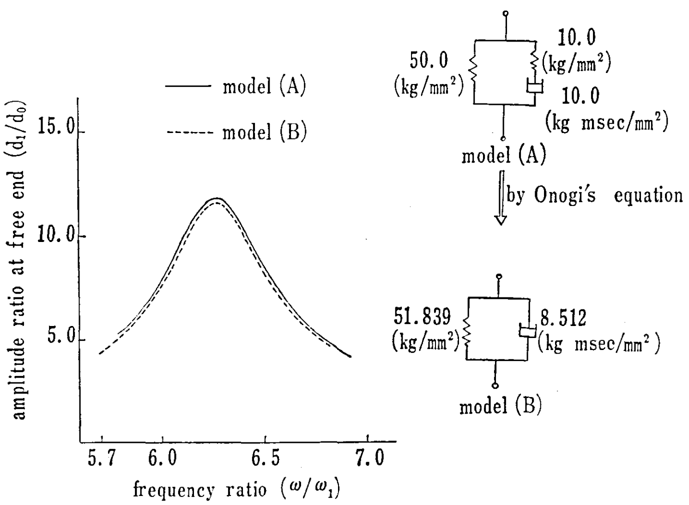


Fig. 3 Reconsideration of Onogi's method.

(ii) Then, Onogi's method is applied to the response curve and determine the element costants relevant to implicitly assumed simple Voigt model, i.e. model (B) of Fig. 3.

(iii) The same procedure as in (i) is applied to evaluate the frequency response for the Voigt model determined in step (ii).

The resultig response is shown by dotted curve in Fig. 3. The curve compares well with the original solid one and it is concluded that the Voigt model approximation or the Onogi's method is reasonable at least for the case studied.

c) Consistent vs. Lumped Mass Matrices

It has been frequently argued that the choice of mass matrices affects the numerical results as well as the stability of solution in the computation process[5]. The effects on resonant frequencies

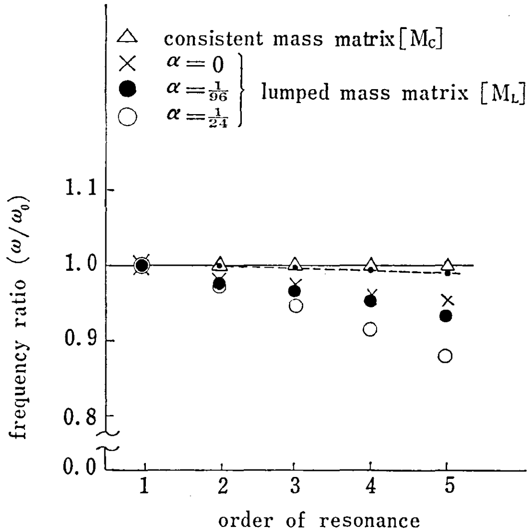


Fig. 4 A comparison of effects of consistent and lumped mass matrices upon computed resonant frequency.

have been studied in the present paper. The results are summarized in Fig. 4, where resonant frequencies ω for lumped mass matrices $[M_L]$ with $\alpha=0$, $\alpha=1/96$ and $\alpha=1/24$ are given in relation to ω_0's which correspond to the exact resonance frequencies at each mode, i.e. ω_1, ω_2, $\omega_3,\ldots$ in equation (9). Comment should be made that the dotted line of Fig. 4 represents the analytical solution which incorporates the translational as well as rotatory inertias.

5. Conclusions

Besides the reasonable numerical results, it should be emphasized that the analysis of frequency response can be a promising substitute for eigen solution which requires the rather complicated procedure and is different, in many respects, from the conventional finite element formulation.

(Manuscript received March 31, 1973)

References

1) Y. Yamada, "Plasticity and Viscoelasticity" (in Japanese) Baifukan, 1972.
2) Y. Yamada and Y. Nagai, "Finite Element Analysis of Load Cell Response in High Speed Tensile Testing", Seisan Kenkyu, **25**, 259–262, 1973.
3) M. Horio and S. Onogi, "Forced Vibration of Reed as a Method of Determining Viscoelasticity", J. Appl. Phys., **22**, 977–981, 1951.
4) O.C. Zienkiewicz, "Finite Element Method in Engineering Science", McGraw-Hill, 1971.
5) R.D. Krieg and S.W. Key, "Transient Shell Response by Numerical Time Integration", Advances in Computational Methods in Structural Mechanics and Design, Univ. of Alabama Press, 237–258, 1972.