研 究 速 報

UDC 624.072.2.042

ELASTIC-PLASTIC ANALYSIS OF BEAMS WITH UNIFORM CROSS-SECTION UNDER COMBINED LOADINGS (2)

—Bending and Its Combination with Torsion—

組合せ荷重を受けるはりの弾塑性解析 (2)—純曲げおよび曲げとねじり—

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1. Introduction

The preceding work[1] in which we treated the combined loading of axial force and torsion of uniform beams is extended to include the pure bending and/or coexistence of the bending mode of deformation. When the bending predominates, it is appropriate to incorporate the quadratic shape function with respect to the coordinates x, y which are being taken in the cros-section of the beam or bar. Expository examples are concernd with the pure bending and its combination with torsion. The secondary stress inherent to the elastic-plastic deformation as well as the development of plastic enclave are studied in detail.

2. Finite element formulation

(1) displacement function

We take the x and y axes in the plane of the cross-section and z axis to the axial direction of the beam similarly as in the preceding paper (see Fig. 1). We base our formulation on the following expressions of displacements u, v, w which have been shown to be obtained by the semi-inverse method[2] for the uniform beam under the combined loadings of axial tensile (or compressive) force, bending and twisting moments:

$$\left.\begin{aligned} u&=f_1(x,y)+\frac{1}{2}\kappa_y z^2-\theta yz \\ v&=f_2(x,y)-\frac{1}{2}\kappa_x z^2+\theta zx \\ w&=f_3(x,y)-\kappa_y zx+\kappa_x yz+\varepsilon_0 z \end{aligned}\right\} \quad (1)$$

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where ε_0, θ, κ_x and κ_y denote the longitudinal strain in the central layer, the angle of twist, and the curvatures in the yz and zx planes respectively. The symmetry of the transverse cross-section is assumed and the bending moments M_x and M_y are applied respectively in the planes yz and zx of symmetry. It is known that the functions f_1 and f_2 of Eq. (1) are given for the elastic deformation

$$\left.\begin{aligned} f_1(x,y)&=-\nu\kappa_x xy+\frac{\nu}{2}\kappa_y(x^2-y^2)-\nu\varepsilon_0 x \\ f_2(x,y)&=-\frac{\nu}{2}\kappa_x(y^2-x^2)+\nu\kappa_y xy-\nu\varepsilon_0 y \end{aligned}\right\} \quad (2)$$

The function f_3 represents the warping of cross-section due to the twisting moment T. Note from Eq. (1) that we are concerned with the deformation which is not dependent on the axial coordinate z. Therefore, we can assume only unit length along z-axis.

(2) triangular element and shape function

Generally, the displacements u, v and w within each element shown in Fig. 1 are expressed as Eq. (1), and we assumed in the preceding paper

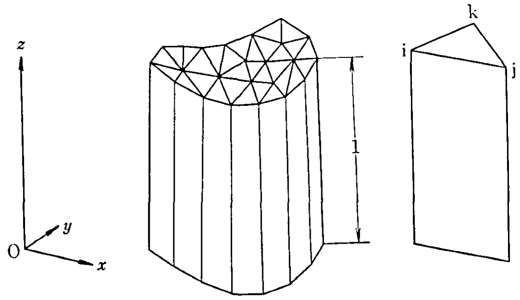


Fig. 1 Triangular element.

that functions f_1, f_2 and f_3 are all linear with respect to the x, y coordinates. It was found, however, that this assumption lead to the results seriously in error when applied to the problems where the bending deformations prevail. Further, Eq. (2) indicates that the quadratic expresions of f_1 and f_2 give the exact solution for the elastic problem which accompanies the bending modes of deformation. Thus the shape function adopted in the present formulation is, for the triangular element, in terms of the area coordinates ζ_i, ζ_j and ζ_k as follows:

$$\left.\begin{aligned} u=&\zeta_i(\zeta_i-\zeta_j-\zeta_k)u_i+\zeta_j(\zeta_j-\zeta_k-\zeta_i)u_j\\ &+\zeta_k(\zeta_k-\zeta_i-\zeta_j)u_k+4\zeta_j\zeta_k u_l+4\zeta_k\zeta_i u_m\\ &+4\zeta_i\zeta_j u_n+\theta y(1-z)-\frac{1}{2}\kappa_y(1-z^2)\\ v=&\zeta_i(\zeta_i-\zeta_j-\zeta_k)v_i+\zeta_j(\zeta_j-\zeta_k-\zeta_i)v_j\\ &+\zeta_k(\zeta_k-\zeta_i-\zeta_j)v_k+4\zeta_j\zeta_k v_l+4\zeta_k\zeta_i v_m\\ &+4\zeta_i\zeta_j v_n-\theta x(1-z)+\frac{1}{2}\kappa_x(1-z^2)\\ w=&\zeta_i w_i+\zeta_j w_j+\zeta_k w_k\\ &+\kappa_y x(1-z)-\kappa_x y(1-z)-\varepsilon_0(1-z) \end{aligned}\right\} \quad (3)$$

The suffixes denote quantities associated with the corresponing corner nodes i, j, k and/or mid-side nodes l, m, n.

(3) stiffness equation

The strain components within each element are derived from the displacement functions of Eq. (3) as follows

$$\{\varepsilon\}=[B]\{d\} \quad (4)$$

where $\lfloor\varepsilon\rfloor=\lfloor\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{yz}\ \gamma_{zx}\ \gamma_{xy}\rfloor$

$\lfloor d\rfloor=\lfloor u_i\ v_i\ w_i \vdots u_j\ v_j\ w_j \vdots u_k\ v_k\ w_k \vdots u_l\ v_l \vdots u_m\ v_m \vdots u_n\ v_n \vdots \varepsilon_0\ \theta\ \kappa_x\ \kappa_y\rfloor$

$[B]=\zeta_i[B_i]+\zeta_j[B_j]+\zeta_k[B_k]$

and $[B_i]$, $[B_j]$, $[B_k]$ are

$$[B_i]=\left[\begin{array}{ccc|ccc|ccc|cc|cc|cc|cccc} 3b_i & 0 & 0 & -b_j & 0 & 0 & -b_k & 0 & 0 & 0 & 0 & 4b_k & 0 & 4b_j & 0 & 0 & 0 & 0 & 0\\ 0 & 3c_i & 0 & 0 & -c_j & 0 & 0 & -c_k & 0 & 0 & 0 & 0 & 4c_k & 0 & 4c_j & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & y_i & -x_i\\ 0 & 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_i & -1 & 0\\ 0 & 0 & b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_i & 0 & 1\\ 3c_i & 3b_i & 0 & -c_j & -b_j & 0 & -c_k & -b_k & 0 & 0 & 0 & 4c_k & 4b_k & 4c_j & 4b_j & 0 & 0 & 0 & 0 \end{array}\right]$$

$$[B_j]=\left[\begin{array}{ccc|ccc|ccc|cc|cc|cc|cccc} -b_i & 0 & 0 & 3b_j & 0 & 0 & -b_k & 0 & 0 & 4b_k & 0 & 0 & 0 & 4b_i & 0 & 0 & 0 & 0 & 0\\ 0 & -c_i & 0 & 0 & 3c_j & 0 & 0 & -c_k & 0 & 0 & 4c_k & 0 & 0 & 0 & 4c_i & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & y_j & -x_j\\ 0 & 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_j & -1 & 0\\ 0 & 0 & b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_j & 0 & 1\\ -c_i & -b_i & 0 & 3c_j & 3b_j & 0 & -c_k & -b_k & 0 & 4c_k & 4b_k & 0 & 0 & 4c_i & 4b_i & 0 & 0 & 0 & 0 \end{array}\right]$$

$$[B_k]=\left[\begin{array}{ccc|ccc|ccc|cc|cc|cc|cccc} -b_i & 0 & 0 & -b_j & 0 & 0 & 3b_k & 0 & 0 & 4b_j & 0 & 4b_i & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -c_i & 0 & 0 & -c_j & 0 & 0 & 3c_k & 0 & 0 & 4c_j & 0 & 4c_i & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & y_k & -x_k\\ 0 & 0 & c_i & 0 & 0 & c_j & 0 & 0 & c_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_k & -1 & 0\\ 0 & 0 & b_i & 0 & 0 & b_j & 0 & 0 & b_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -y_k & 0 & 1\\ -c_i & -b_i & 0 & -c_j & -b_j & 0 & 3c_k & 3b_k & 0 & 4c_j & 4b_j & 4c_i & 4b_i & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

with

$$\begin{bmatrix} a_i & a_j & a_k\\ b_i & b_j & b_k\\ c_i & c_j & c_k \end{bmatrix}=\frac{1}{2A}\begin{bmatrix} x_jy_k-x_ky_j & x_ky_i-x_iy_k & x_iy_j-x_jy_i\\ y_j-y_k & y_k-y_i & y_i-y_j\\ x_k-x_j & x_i-x_k & x_j-x_i \end{bmatrix}$$

where $x_i, y_i, \cdots, y_k$ are the coordinates of corner nodes and A is the area of the element triangle.

By using the stress-strain matrix $[D]$[1),3)], the associated stress field is expressed as

$$\{\sigma\}=[D]\{\varepsilon\}=[D][B]\{d\} \quad (5)$$

where $\lfloor\sigma\rfloor=\lfloor\sigma_x\ \sigma_y\ \sigma_z\ \tau_{yz}\ \tau_{zx}\ \tau_{xy}\rfloor$

The strain energy U in each element can be expressed by Eqs. (4) and (5) as

$$U=\frac{1}{2}\iint\lfloor\sigma\rfloor\{\varepsilon\}\,dx\,dy$$

$$=\frac{1}{2}\lfloor d\rfloor\iint[B]^T[D][B]\,dx\,dy\{d\}=\frac{1}{2}\lfloor d\rfloor[k]\{d\}$$

Then follows the stiffness matrix $[k]$ which is expressed as, by integration over the area of element triangle

$$[k]=\iint[B]^T[D][B]\,dx\,dy$$

$$=\frac{A}{12}[[B_i]^T[D][B_i]+[B_j]^T[D][B_j]$$
$$+[B_k]^T[D][B_k]+[B_t]^T[D][B_t]]$$

where $\qquad [B_t]=[B_i]+[B_j]+[B_k]$

The stiffness equation of the relevant beam element connects the displacement vector $\{d\}$ with the force vector $\{F\}$ and can be expressed as

$$\{F\}=[k]\{d\}$$

where

$$\lfloor F\rfloor=\lfloor 0\;0\;0\,\vdots\,0\;0\;0\,\vdots\,0\;0\;0\,\vdots\,0\;0\,\vdots\,0\;0\,\vdots\,0\;0\,\vdots\,F_z\;TM_x\;M_y\rfloor$$

The subsequent procedure follows the standard pattern of the finite element displacement method.

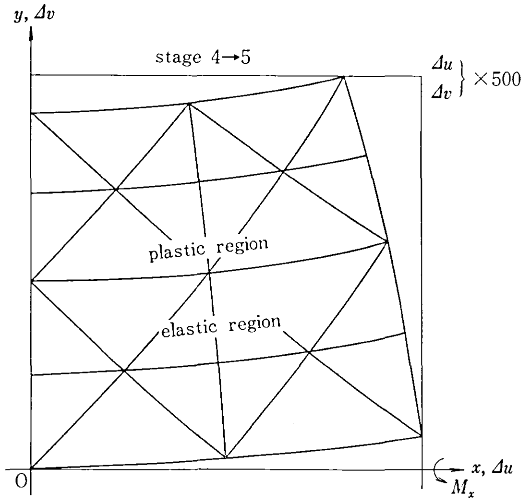
3. Numerical examples

As numerical examples, we solved the beams with square cross-section under pure bending and the combination of bending and torsion. In the computation, a quadrant of the cross-section is considered, by taking account of the symmetric nature of the problem. Material constants are $E=2\times10^4$ kg/mm², Y (yield stress) $=25$ kg/mm², H' (strain hardening modulus) $=0$ kg/mm² and ν (Poisson's ratio) $=0.3$. The side length of the square cross section is 20 mm.

(1) pure bending

The numerical results for the elastic range have been found naturally to correlate well with the exact solution, since the shape functions relevant to displacements u and v are quadratic in x and y and precise in elastic case. From this it is expected the reasonable accuracy of solution is maintained for the elastic-plastic realm of deformation.

Fig. 2 shows the displacement increments Δu and Δv between stages 4 and 5 in an example of elastic-plastic bending, and Fig. 3 shows the stress increment $\Delta\sigma_x$ from stage 1 to 2 and 4 to 5.



Fgi. 2 Displacements increments Δu and Δv. Distance measured from the surrounding square section indicates the displacements ×500.

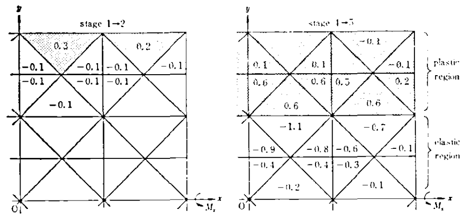


Fig. 3 Stress increment $\Delta\sigma_x$ (kg/mm²). Values in the elements which are left blank are less than 0.05 kg/mm².

The stress state of the element of nonhardening material treated here is not affected practically, once it becomes plastic, by the later displacement increment. Therefore, the deformation in the plastic region is considered to be nearly incompressible (i. e. $\nu=0.5$). On the other hand, the deformation of the elastic region is compressible ($\nu=0.3$). Noting these features, the displacement increments Δu and Δv in Fig. 2 agree well with the following expectation. In the vicinity of the elastic-plastic boundary, the plastic side is pulled in x direction by the elastic region. Conversely, the elastic side is contracted by the plastic region. In conformity with these, the stress increments $\Delta\sigma_x$ of Fig. 3 are positive in the plastic side of the elastic-plastic boundary and negative in the elastic side. The other secondary stress increments

$\Delta\sigma_y$ and $\Delta\tau_{xy}$ were found to exist, but they are considerably small, compared with $\Delta\sigma_x$.

The stress increment of the element generally reverses its sign as the elastic-plastic boundary traverses the element. Therefore, it is difficult to elucidate the general tendency of the secondary stress (not the stress increment at each stage of deformation), particularly by the rather coarse element division used in this article. So, it is only remarked here that the absolute value of secondary stresses σ_x, σ_y, and τ_{xy} is the order of 3 kg/mm^2, whereas the primary stress σ_z in axial direction is between 23. 1 and 25. 3 kg/mm^2 at the limit, i. e. fully plastic state of deformation.

The secondary stresses do not appear in the incompressible elastic-plastic material. Fig. 4 shows the secondary stress σ_x as well as its increment vanish, as the Poisson's ratio ν approaches to 0. 5.

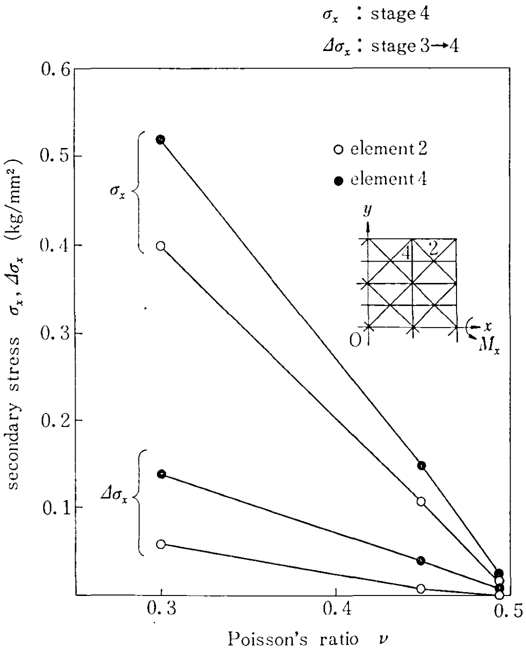


Fig. 4 Relation between secondary stress σ_x and Poisson's ratio ν.

(2) combined loading of bending and twist

For this combination, the numerical solution of Steele[4] is available for the square bar of the plastic-rigid material. In order to compare with Steele's solution, computation was carried out for two cases where the value of strain ratio $d\theta/d\kappa_x$ is kept constant ($d\theta/d\kappa_x=\sqrt{3}$ and $\sqrt{3}/2$ respectively) during the whole process of loading.

Fig. 5 depicts the development of plastic region for the case of strain ratio $d\theta/d\kappa_x=\sqrt{3}$. The plastic yielding starts at the extreme boundary

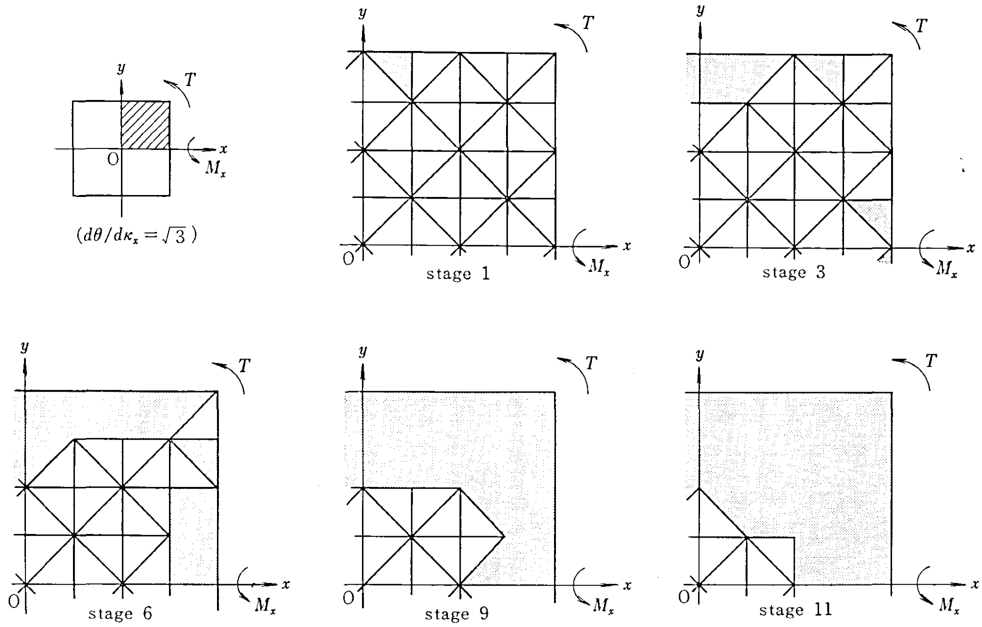


Fig. 5 Development of plastic region (combined loading of bending and twist)

element on y axis where both stresses due to bending and twist predominate. Subsequent yielding occurs in the element at the extreme of x, and then the plastic region expands to cover the whole cross-section as the loads increase.

Fig. 6 shows the axial stress distribution σ_z in the ratio to the yield stress σ_Y of the material at the fully plastic state of deformation. The stress at the centroid of the element is given. The relation between bending and twisting moments M, T at the fully plastic state is illustrated in Fig. 7 where a comparison with the interaction curves of plastic-rigid analysis is also made.

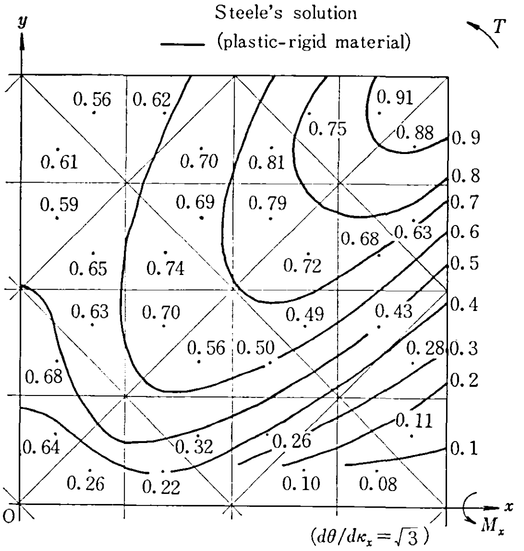


Fig. 6 Distribution of σ_z/σ_Y.

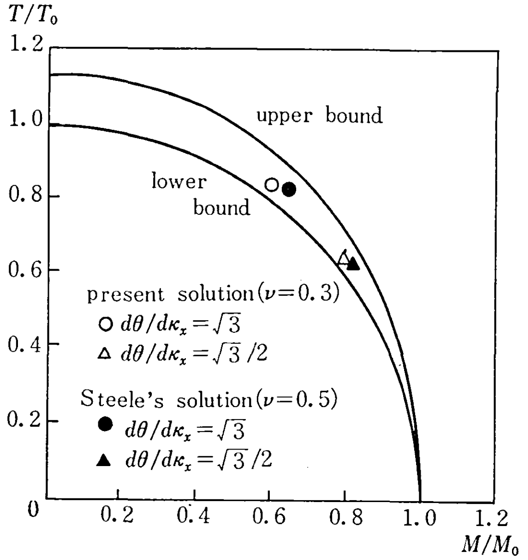


Fig. 7 Interaction curve (combined loading of bending and twist).

It should be emphasized that the present formulation facilitates the analysis at every stages of deformation, not merely at the limiting fully plastic stage. Moreover, it is of interest to note that the M-T relation at the limiting stage obtained for the compressible elastic-plastic material lies between the upper and lower bounds of interaction curve for the plastic-rigid material.

As for the features of the secondary stress, it can only be said that its maximum absolute value is 0.33 kg/mm^2 at the fully plastic state of the material whose yield stress is 25 kg/mm^2.

4. Conclusions

The finite element solution procedure is formulated for the elastic-plastic deformation of uniform beams subjected to combined loadings which consist of axial tensile (or compressive) force, bending and twisting moments. In the case where the bending deformation predominates, the use of the quadratic displacement function is imperative. Presesnt formulation facilitates the pursuit of the development of the plastic region. Further, it has become possible to make evaluation of the secondary stress which could not be assessed so far. An extension of the present formulation is attempted towards the problems which accompany the shear load and/or non-uniformity of cross-section, i. e. the cases where the deformation and stress are dependent on the axial coordinate z.

(Manuscript received March 6, 1973)

References

1) Y. Yamada and K. Takatsuka, "Elastsc-Plastic Analysis of Beams with Uniform Cross-Section under Combined Loadings", Seisan Kenkyu, **23**-12 (1971), pp. 531-535.
2) Y. Yamada, "Sosei-Rikigaku (Theory of Plasticity)", Nikkankogyo Shinbunsha (1965)
3) Y. Yamada, T. Kawai, N. Yoshimura and T. Sakurai, "Analysis of the Elastic-Plastic Problems by the Matrix Displacement Method", AFFDL-TR-68-150 (1968), pp. 1271-1299.
4) M. C. Steele, "The Plastic Bending and Twisting of Square Section Members", J. Mech. Phys. Solids, **3**-2 (1954), pp. 156-166.