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A METHOD OF BIFURCATION ANALYSIS IN PERTURBATION TECHNIQUE

摂動法による分岐座屈の解析

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1. Introduction

In the investigation of various types of elastic instability phenomena, it is an important and fundamental subject to construct both pre- and post-buckling equilibrium paths, which provide us the basis for assessing not only the buckling load but also the influence of imperfection.

In many cases the analysis of the critical points belongs to an eigen-value problem considering nonlinear pre-buckling deformation.

Especially in the construction of equilibrium paths through a bifurcation point, we face more difficult and complicated problems as this point does not satisfy the uniqueness condition of solution.

The principle for the post-buckling analysis has been presented, for example by Koiter[1] and Thompson[2], mainly by means of static perturbation method in which the diagonalization process of coefficient matrix of the equilibrium equations were introduced to make the discussion simple and clear.

However, as an actual process of numerical analysis of bifurcation problem, this diagonalization scheme becomes a serious drawback, because it requires the whole set of eigenvectors of the original coefficient matrix[3].

Thompson[2] proposed a method to avoid this difficulty by introducing "sliding coordinates" in which the analysis is fixed in a prescribed load level in the vicinity of the bifurcation point. This method has been applied by Mau and Gallagher[4] to the stability analysis in the form of finite element method.

The present paper gives a new approach of constructing post-buckling paths beyond bifurcation point without diagonalization technique for nonlinear structural systems represented by discrete variables. A characteristic of this approach is a direct derivation of branching paths depending exclusively basing on the condition of solution at the point of bifurcation.

2. Construction of Post-Buckling Curves

The fundamental equations to be solved are ones introduced by applying the perturbation technique to nonlinear equilibrium equations. The derivation is fully presented in reference[3] and hence let us begin with writing down the first and the second perturbation equations of the load-incremental type expressed in matrix form:

$$\begin{bmatrix} f^0{}_{1,1} & f^0{}_{1,2} & \cdots & f^0{}_{1,n} \\ f^0{}_{2,1} & f^0{}_{2,2} & \cdots & f^0{}_{2,n} \\ \vdots & \vdots & & \vdots \\ f^0{}_{n,1} & f^0{}_{n,2} & \cdots & f^0{}_{n,n} \end{bmatrix} \begin{Bmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \vdots \\ \dot{d}_n \end{Bmatrix} + \begin{Bmatrix} f^0{}_{1,\lambda} \\ f^0{}_{2,\lambda} \\ \vdots \\ f^0{}_{n,n} \end{Bmatrix} \dot{\lambda} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{1}$$

$$\begin{bmatrix} f^0{}_{1,1} & f^0{}_{1,2} & \cdots & f^0{}_{1,n} \\ f^0{}_{2,1} & f^0{}_{2,2} & \cdots & f^0{}_{2,n} \\ \vdots & \vdots & & \vdots \\ f^0{}_{n,1} & f^0{}_{n,2} & \cdots & f^0{}_{n,n} \end{bmatrix} \begin{Bmatrix} \ddot{d}_1 \\ \ddot{d}_2 \\ \vdots \\ \ddot{d}_n \end{Bmatrix} + \begin{Bmatrix} f^0{}_{1,\lambda} \\ f^0{}_{2,\lambda} \\ \vdots \\ f^0{}_{n,\lambda} \end{Bmatrix} \ddot{\lambda} + \begin{Bmatrix} f^0{}_{1,ij} \\ f^0{}_{2,ij} \\ \vdots \\ f^0{}_{n,ij} \end{Bmatrix} \dot{d}_i \dot{d}_j + 2 \begin{Bmatrix} f^0{}_{1,i\lambda} \\ f^0{}_{2,i\lambda} \\ \vdots \\ f^0{}_{n,i\lambda} \end{Bmatrix} \dot{d}_i \dot{\lambda} + \begin{Bmatrix} f^0{}_{1,\lambda\lambda} \\ f^0{}_{2,\lambda\lambda} \\ \vdots \\ f^0{}_{n,\lambda\lambda} \end{Bmatrix} \dot{\lambda}^2 = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{2}$$

where

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d_i : increment of nodal displacement

λ : increment of load parameter

t : perturbation parameter in Eqs. (3) and (4)

f_i : i-th equilibrium function

and

$$\cdot\equiv\frac{d}{dt},\quad f^0{}_{i,j}=\frac{\partial f_i}{\partial d_j}\bigg|_{t=0},\quad f^0{}_{i,\lambda}=\frac{\partial f_i}{\partial\lambda}\bigg|_{t=0}$$

Once the derivatives, $\dot{d}_i$, $\dot{\lambda}$, $\ddot{d}_i$, $\ddot{\lambda}$, ……, are obtained from the above equations, the increments of displacements and load parameter are represented in a parametric form as follows.

$$d_i(t)=\dot{d}_it+\frac{1}{2}\ddot{d}_it^2+\cdots\cdots \qquad (3)$$

$$\lambda(t)=\dot{\lambda}t+\frac{1}{2}\ddot{\lambda}t^2+\cdots\cdots \qquad (4)$$

The first equation of the displacement-incremental type which is obtained by the exchange of the locations of $\dot{d}_1$ and $\dot{\lambda}$ in Eq. (1) becomes

$$\begin{bmatrix} f^0{}_{1,\lambda} & f^0{}_{1,2}\cdots f^0{}_{1,n} \\ f^0{}_{2,\lambda} & f^0{}_{n,2}\cdots f^0{}_{2,n} \\ \vdots & \vdots \quad\quad \vdots \\ f^0{}_{n,\lambda} & f^0{}_{n,2}\cdots f^0{}_{n,n} \end{bmatrix}\begin{Bmatrix}\dot{\lambda}\\ \dot{d}_2\\ \vdots\\ \dot{d}_n\end{Bmatrix}+\begin{Bmatrix}f^0{}_{1,1}\\ f^0{}_{2,1}\\ \vdots\\ f^0{}_{n,1}\end{Bmatrix}\dot{d}_1=\begin{Bmatrix}0\\0\\ \vdots\\0\end{Bmatrix} \qquad (5)$$

At the bifurcation point, the determinants not only of the coefficient matrix $[f^0{}_{r,i}]$ for $\{\dot{d}_i\}$ in Eq. (1), but also of the matrix given in Eq. (5) become zero[3]. This means that the exchange of the column vectors $\{f^0{}_{i,1}\}$ and $\{f^0{}_{i,\lambda}\}$ does not recover the rank of the coefficient matrix. This is a difference of a bifurcation point from the case of limit point corresponding the non-uniqueness of solution at the bifurcation point.

Now let us consider the case where only the smallest eigenvalue of the matrix $[f^0{}_{r,i}]$ is zero, i.e. the rank is $n-1$. Then we have the following non-zero minor determinant of the matrix $[f^0{}_{r,i}]$.

$$|\triangle f^0{}_{1,1}|\neq 0 \qquad (6)$$

in which

$$[\triangle f^0{}_{1,1}]=\begin{bmatrix} f^0{}_{2,2}\cdots f^0{}_{2,n}\\ \vdots \quad\quad \vdots\\ f^0{}_{n,2}\cdots f^0{}_{n,n}\end{bmatrix} \qquad (7)$$

In the actual process, the row and the column to be deleted can be sought by Gauss-Jordan elimination procedure.

Though, taking d_1 as the parameter t, the displacement-incremental process is explained hereafter, the similar formulation is possible for the load-incremental process, because the matrix in Eq. (7) is common for both the cases of Eqs. (1) and (5).

By the definition, we have

$$t=d_1 : \dot{d}_1=1,\ \ddot{d}_1=0,\cdots\cdots \qquad (8)$$

Eqs. (3) and (4) take the following form in view of the above equations.

$$d_i(d_1)=\dot{d}_id_1+\frac{1}{2}\ddot{d}_id_1{}^2+\cdots\cdots \quad (i\neq 1) \qquad (9)$$

$$\lambda(d_1)=\dot{\lambda}d_1+\frac{1}{2}\ddot{\lambda}d_1{}^2+\cdots\cdots \qquad (10)$$

The final goal is to determine $\dot{d}_i$, $\dot{\lambda}$, $\ddot{d}_i$, $\ddot{\lambda}$, …… without diagonalization process.

Introduction of Eq. (8) into Eq. (5) leads to the following two equations:

$$f^0{}_{1,\lambda}\dot{\lambda}+(f^0{}_{1,2},\cdots f^0{}_{1,n})\begin{Bmatrix}\dot{d}_2\\ \vdots\\ \dot{d}_n\end{Bmatrix}+f^0{}_{1,1}=0 \qquad (11)$$

$$\begin{Bmatrix}f^0{}_{2,\lambda}\\ \vdots\\ f^0{}_{n,\lambda}\end{Bmatrix}\dot{\lambda}+\begin{bmatrix} f^0{}_{2,2}\cdots f^0{}_{2,n}\\ \vdots \quad\quad \vdots\\ f^0{}_{n,2}\cdots f^0{}_{n,n}\end{bmatrix}\begin{Bmatrix}\dot{d}_2\\ \vdots\\ \dot{d}_n\end{Bmatrix}+\begin{Bmatrix}f^0{}_{2,1}\\ \vdots\\ f^0{}_{n,1}\end{Bmatrix}=\begin{Bmatrix}0\\ \vdots\\ 0\end{Bmatrix} \qquad (12)$$

From the above equation, we have

$$\begin{Bmatrix}\dot{d}_2\\ \vdots\\ \dot{d}_n\end{Bmatrix}=-\begin{bmatrix} f^0{}_{2,2}\cdots f^0{}_{2,n}\\ \vdots \quad\quad \vdots\\ f^0{}_{n,2}\cdots f^0{}_{n,n}\end{bmatrix}^{-1}\left(\begin{Bmatrix}f^0{}_{2,\lambda}\\ \vdots\\ f^0{}_{n,\lambda}\end{Bmatrix}\dot{\lambda}+\begin{Bmatrix}f^0{}_{2,1}\\ \vdots\\ f^0{}_{n,1}\end{Bmatrix}\right) \qquad (13)$$

By substitution of Eq. (13) into Eq. (11), the following linear equation of $\dot{\lambda}$ is obtained.

$$\alpha\dot{\lambda}+\beta=0 \qquad (14)$$

where

$$\alpha=f^0{}_{1,\lambda}-(f^0{}_{1,2}\cdots f^0{}_{1,n})\begin{bmatrix} f^0{}_{2,2}\cdots f^0{}_{2,n}\\ \vdots \quad\quad \vdots\\ f^0{}_{n,2}\cdots f^0{}_{n,n}\end{bmatrix}^{-1}\begin{Bmatrix}f^0{}_{2,\lambda}\\ \vdots\\ f^0{}_{n,n}\end{Bmatrix} \qquad (15)$$

$$\beta=f^0_{1,1}-(f^0_{1,2}\cdots f^0_{1,n})\begin{bmatrix} f^0_{2,2} & \cdots & f^0_{2,n}\\ \vdots & & \vdots \\ f^0_{n,2} & \cdots & f^0_{n,n}\end{bmatrix}^{-1}\begin{Bmatrix} f^0_{2,1}\\ \vdots \\ f^0_{n,1}\end{Bmatrix} \quad (16)$$

As the determinant of the coefficient matrix in Eq. (5) is zero, $\dot\lambda$ is indefinite in Eq. (5) and accordingly in Eq. (14). Therefore, in order that the bifurcation point satisfies the first perturbation equations (11) and (12),

$$\alpha=\beta=0 \quad (17)$$

must hold in Eq. (14). A proof of Eq. (17) in mathematical form is presented in Appendix.

In other words, the first perturbation equation (5) does not yield an unique solution of $\dot\lambda$, giving only relations between coefficients expressed in Eq. (17). In order to obtain the values for $\dot\lambda$, it is necessary to make use of the second perturbation equation.

Introduction of Eq. (8) into Eq. (2) leads to

$$f^0_{1,\lambda}\ddot\lambda+(f^0_{1,2}\cdots f^0_{1,n})\begin{Bmatrix}\ddot d_2\\ \vdots\\ \ddot d_n\end{Bmatrix}+f^0_{1,ij}\dot d_i\dot d_j+2f^0_{1,i\lambda}\dot d_i\dot\lambda+f^0_{1,\lambda\lambda}\dot\lambda^2=0 \quad (18)$$

and

$$\begin{Bmatrix} f^0_{2,\lambda}\\ \vdots\\ f^0_{n,\lambda}\end{Bmatrix}\ddot\lambda+\begin{bmatrix} f^0_{2,2} & \cdots & f^0_{2,n}\\ \vdots & & \vdots \\ f^0_{n,2} & \cdots & f^0_{n,n}\end{bmatrix}\begin{Bmatrix}\ddot d_2\\ \vdots\\ \ddot d_n\end{Bmatrix}+\begin{Bmatrix} f^0_{2,ij}\\ \vdots\\ f^0_{n,ij}\end{Bmatrix}\dot d_i\dot d_j+2\begin{Bmatrix} f^0_{2,i\lambda}\\ \vdots\\ f^0_{n,i\lambda}\end{Bmatrix}\dot d_i\dot\lambda+\begin{Bmatrix} f^0_{2,\lambda\lambda}\\ \vdots\\ f^0_{n,\lambda\lambda}\end{Bmatrix}\dot\lambda^2=\begin{Bmatrix}0\\ \vdots\\ 0\end{Bmatrix} \quad (19)$$

From the above equation, we have

$$\begin{Bmatrix}\ddot d_2\\ \vdots\\ \ddot d_n\end{Bmatrix}=-\begin{bmatrix} f^0_{2,2} & \cdots & f^0_{2,n}\\ \vdots & & \vdots \\ f^0_{n,2} & \cdots & f^0_{n,n}\end{bmatrix}^{-1}\left(\begin{Bmatrix} f^0_{2,\lambda}\\ \vdots\\ f^0_{n,\lambda}\end{Bmatrix}\ddot\lambda+\begin{Bmatrix} f^0_{2,ij}\\ \vdots\\ f^0_{n,ij}\end{Bmatrix}\dot d_i\dot d_j+2\begin{Bmatrix} f^0_{2,i\lambda}\\ \vdots\\ f^0_{n,ij}\end{Bmatrix}\dot d_i\dot\lambda+\begin{Bmatrix} f^0_{n,\lambda\lambda}\\ \vdots\\ f^0_{n,\lambda\lambda}\end{Bmatrix}\dot\lambda^2\right) \quad (20)$$

Substitution of Eq. (20) into Eq. (18) in view of Eq. (9) gives

$$-(f^0_{1,2}\cdots f^0_{1,n})\begin{bmatrix} f^0_{2,2} & \cdots & f^0_{2,n}\\ \vdots & & \vdots \\ f^0_{n,2} & \cdots & f^0_{,ij}\end{bmatrix}^{-1}\left(\begin{Bmatrix} f^0_{2,ij}\\ \vdots\\ f^0_{n,ij}\end{Bmatrix}\dot d_i\dot d_j+2\begin{Bmatrix} f^0_{2,i\lambda}\\ \vdots\\ f^0_{n,i\lambda}\end{Bmatrix}\dot d_i\dot\lambda+\begin{Bmatrix} f^0_{2,\lambda\lambda}\\ \vdots\\ f^0_{n,\lambda\lambda}\end{Bmatrix}\dot\lambda^2\right)+f^0_{1,ij}\dot d_i\dot d_j+2f^0_{1,i\lambda}\dot d_i\dot\lambda+f^0_{1,\lambda\lambda}\dot\lambda^2=0 \quad (21)$$

It is to be noted that the term of $\ddot\lambda$ has been eliminated in Eq. (21).

Finally, the following quadratic equation of $\dot\lambda$ is obtained by substituting Eqs. (8) and (13) into Eq. (21).

$$a\dot\lambda^2+b\dot\lambda+c=0 \quad (22)$$

where

$$\begin{aligned}
a=&g^0_{1,k}f^0_{k,ij}g^0_{i,\lambda}g^0_{j,\lambda}+2g^0_{1,k}f^0_{k,i\lambda}g^0_{i,\lambda}\\
&+g^0_{i,k}f^0_{k,\lambda\lambda}+f^0_{1,\lambda\lambda}+f^0_{1,ij}g^0_{i,\lambda}g^0_{j,\lambda}\\
&+2f^0_{1,i\lambda}g^0_{i,\lambda}\\
b=&g^0_{1,k}(f^0_{k,1i}+f^0_{k,i1})g^0_{i,\lambda}\\
&+g^0_{1,k}f^0_{k,ij}(g^0_{i,\lambda}g^0_{1,j}+g^0_{1,j}g^0_{j,\lambda})\\
&+2g^0_{1,k}f^0_{k,1\lambda}+2g^0_{1,k}f^0_{k,i\lambda}g^0_{1,i}\\
&+(f^0_{1,i1}+f^0_{1,1i})g^0_{i,\lambda}+2f^0_{1,i\lambda}g^0_{1,i}\\
&+f^0_{1,ij}(g^0_{1,i}g^0_{j,\lambda}+g^0_{i,\lambda}g^0_{1,j})+2f^0_{1,1\lambda}\\
c=&g^0_{1,k}f^0_{k,11}+g^0_{1,k}(f^0_{k,1i}+f^0_{k,i1})g^0_{1,i}\\
&+g^0_{1,k}f^0_{k,ij}g^0_{1,i}g^0_{1,j}+f^0_{1,11}\\
&+(f^0_{1,i1}+f^0_{1,1i})g^0_{1,i}+f^0_{1,ij}g^0_{1,i}g^0_{1,j}\\
&(i,\ j,\ k=2\sim n)
\end{aligned} \quad (23)$$

in which

$$\begin{Bmatrix} g^0_{1,2}\\ \vdots\\ g^0_{1,k}\\ \vdots\\ g^0_{1,n}\end{Bmatrix}=-\begin{bmatrix} f^0_{2,2} & \cdots & f^0_{2,n}\\ \vdots & & \vdots \\ f^0_{k,2} & \cdots & f^0_{k,n}\\ \vdots & & \vdots \\ f^0_{n,2} & \cdots & f^0_{n,n}\end{bmatrix}^{-1}\begin{Bmatrix} f^0_{2,1}\\ \vdots\\ f^0_{k,1}\\ \vdots\\ f^0_{n,1}\end{Bmatrix} \quad (24)$$

$$\begin{Bmatrix} g^0_{2,\lambda}\\ \vdots\\ g^0_{i,\lambda}\\ \cdots\\ g^0_{n,\lambda}\end{Bmatrix}=-\begin{bmatrix} f^0_{2,2} & \cdots & f^0_{2,n}\\ \vdots & & \vdots \\ f^0_{i,2} & \cdots & f^0_{i,n}\\ \vdots & & \vdots \\ f^0_{n,2} & \cdots & f^0_{n,n}\end{bmatrix}^{-1}\begin{Bmatrix} f^0_{2,\lambda}\\ \vdots\\ f^0_{i,\lambda}\\ \vdots\\ f^0_{n,\lambda}\end{Bmatrix} \quad (25)$$

Eq. (22), obtained without diagonalization process, corresponds with Eq. (37) in reference [3].

It is apparent that the root of Eq. (22) is given in the form

$$\begin{Bmatrix}\dot\lambda^1\\ \dot\lambda_2\end{Bmatrix}=\frac{-b\pm\sqrt{b_2-4ac}}{2a} \quad (26)$$

By introduction of $\dot\lambda_1$ into $\dot\lambda_2$ into Eq. (13)

respectively, two sets of $(\dot{d}_2 \cdots \dot{d}_n)$ are obtained.

Further, by substituting $\dot{\lambda}$ and $\dot{d}_i$, obtained above, into higher order perturbation equations, $\ddot{\lambda}$ and $\ddot{d}_i$ can be determined, if necessary.

Upon using Eqs. (9), (10) and the above results, we can construct post-buckling curves beyond bifurcation point without diagonalization process.

3. Illustrative Example

In order to examine the validity of the present method, model calculation was carried out on the examples of a simple three-dimensional hinged truss of dome type configuration having 21 degrees of freedom. The stiffness matrices retaining the complete nonlinear terms up to the third order of displacements[5)] were used.

In Figs. 1 through 3, the bold lines are the load-displacement curves of the perfect system, where the post-buckling curves branching from the bifurcation point were calculated by the

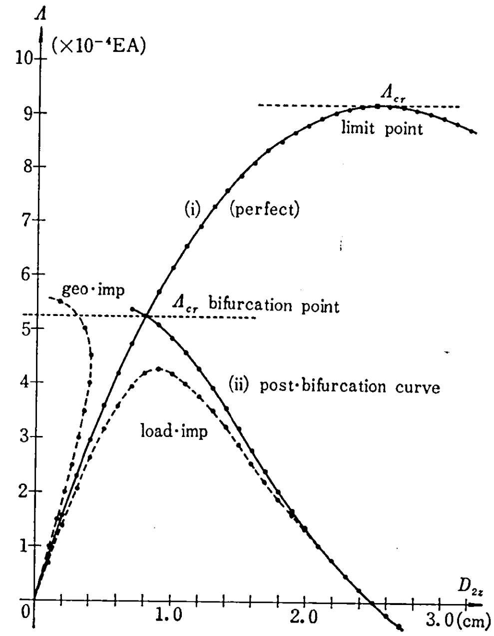


Fig. 1 Vertical displacement of the node 2.

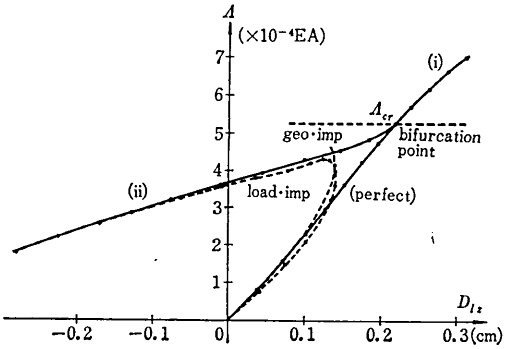


Fig. 2 Vertical displacement of the central node.

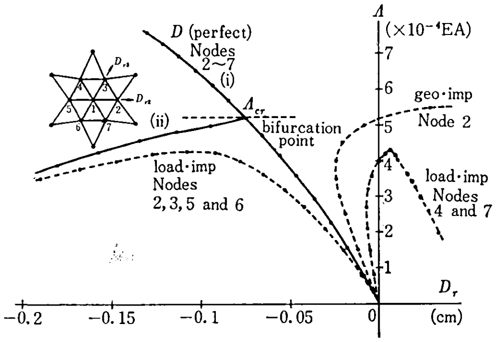
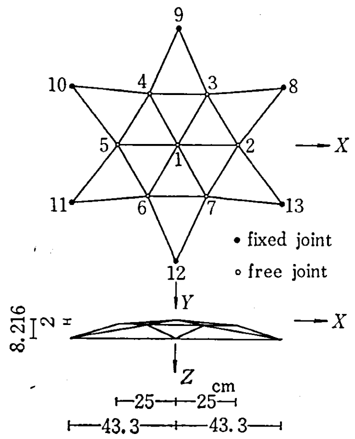


Fig. 3 Horizontal radial displacement of non-central nodes.

Table I

(a) Coordinates of the Joints, in cm

Joint	X (perfect system)	Y (perfect system)	Z (perfect system)	Z geo・imp (imperfect)
1	0.0	0.0	0.0	0.0
2	25.0	0.0	2.0	1.8
3	12.5	−21.65	2.0	2.0
4	−12.5	−21.65	2.0	2.0
5	−25.0	0.0	2.0	1.8
6	−12.5	21.5	2.0	2.0
7	12.5	21.5	2.0	2.0
8	43.30	−25.0	8.216	8.216
9	0.0	−50.0	8.216	8.216
10	−43.30	−25.0	8.216	8.216
11	−43.30	25.0	8.216	8.216
12	0.0	50.0	8.216	8.216
13	43.3)	25.0	8.216	8.216



Goemetry of the Trussed Dome Analyzed.

(b) Mode of Load Distribution

Z	Z (load・imp)
1.0	1.0
2.0	2.0
2.0	2.0
2.0	1.5
2.0	2.0
2.0	2.0
2.0	1.5

$X=Y=0$ for all types.

proposed method.

The response for two kinds of imperfect systems were analyzed. One is the imperfection of geometry shown in Table–I in the column "geo·imp" and another is the loading imperfection indicated by Table–I in the column "load·imp". The obtained results are shown by broken lines in the figures.

4. Conclusion

A new method of constructing equilibrium paths through a bifurcation point was presented. This method enables a direct and easy approach to the post-buckling branch analysis by avoiding the laborious process of diagonalization of coefficient matrix.

On the basis of the perturbation method of solving nonlinear equilibrium equations, the initial post-buckling paths branching from a bifurcation point are directly obtained by considering the condition of solution for the first and the second perturbation equations.

As an illustrative example, complete equilibrium paths for a bifurcation buckling of a three-dimensional truss were shown, where the branch analysis was carried out by the proposed method.

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Appendix: Proof of Eq. (17)

Let us represent the coefficient matrix $[f^0{}_{r,i}]$ in the form:

$$\boldsymbol{A}=\left[\begin{array}{c|c} a_{11} & \boldsymbol{a}_{12} \\ \hline \boldsymbol{a}_{21} & \boldsymbol{A}_{22} \end{array}\right], \quad |\boldsymbol{A}_{22}| \neq 0 \qquad \text{(a–1)}$$

Multiplication of the above by the regular matrix leads to

$$\left[\begin{array}{c|c} 1 & -a_{12}\boldsymbol{A}_{22}^{-1} \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right]\left[\begin{array}{c|c} a_{11} & \boldsymbol{a}_{12} \\ \hline \boldsymbol{a}_{21} & \boldsymbol{A}_{22} \end{array}\right] = \left[\begin{array}{c|c} a_{11}-\boldsymbol{a}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{a}_{21} & \boldsymbol{0} \\ \hline \boldsymbol{a}_{21} & \boldsymbol{A}_{22} \end{array}\right] \qquad \text{(a–2)}$$

By calculating the determinants of both sides of Eq. (a–2), we have

$$a_{11}-\boldsymbol{a}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{a}_{21}=0 \qquad \text{(a–3)}$$

The left term of the above equation corresponds with Eq. (16), i.e. β. Hence,

$$\beta=0 \qquad \text{(a–4)}$$

In a similar manner, we can prove $\alpha=0$.