研　究　速　報

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ELASTIC-PLASTIC ANALYSIS OF BEAMS WITH UNIFORM CROSS-SECTION UNDER COMBINED LOADINGS

組合せ荷重を受ける梁の弾塑性解析

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1. Introduction

Intensive efforts have been devoted to the analysis of the plastic deformation of the beam under the combined loadings in 1950's[1),2)]. The difficulties inherent to the combined loading problems are attributed to the fact that the secondary stresses should appear to maintain the displacement-rates continuity along the elastic-plastic boundary. Thus even for the simple beam under pure bending, we must allow for the lateral and shear components of stress accompanying the longitudinal one which suffices in the elastic analysis of the beam. The major objective of the present paper is to demonstrate the power of the finite element method by applying the usual procedure of displacement formulation to the beam problem. The secondary or cross effects in the plastic range of deformation can be definitely seen through the non-zero components in the plastic stress-strain matrix $[D^p]$, which predicts the distortion under normal stress or inversely the normal strain under the shear loading.

For clarity of the presentation and as our first step, we consider here only the beam with uniform cross-section. Therefore, it is sufficient to take the element which has unit length in the axial direction z. The loadings cover the axial load, and the bending as well as twisting moments. In this respect, the present paper is an extension of those[3),4)] which treat the Saint-Venant torsion problem by the finite element displacement method.

Past analytical or numerical solutions[1),2))5)], if exist, mostly concern with the incompressible material whose Poisson's ratio is $\nu=1/2$ and/or the plastic rigid material. These solutions may be useful for the purpose of comparison with the finite element solutions. Unfortunately and as is well known[6)], however, the conventional finite element procedure fails for the material with $\nu=1/2$. Therefore we are forced to supplement our method of solution by applying the Herrmann's[7)] energy principle developed for the incompressible or nearly incompressible materials. A comment on our experiences with the Herrmann's method is given in the last section.

Numerical examples studied to the present are mainly on the combined loading of the axial load F_z and twisting moment M_z, but will be extended in the near future to the cases incorporating the bending moments.

2. Finite element solution procedure

(1) displacement function

We take the x and y axes in the plane of the cross-section and z axis to the axial direction of the beam. It has been shown by the semi-inverse method[2)] that the displacements u, v and w of the uniform beam under the combined axial, bending and twisting loadings are given by

$$\left.\begin{aligned} u&=f_1(x,y)-\theta yz+\frac{1}{2}\kappa_y z^2\\ v&=f_2(x,y)+\theta zx-\frac{1}{2}\kappa_x z^2\\ w&=f_3(x,y)-\kappa_y zx+\kappa_x yz+\varepsilon_0 z \end{aligned}\right\} \quad (1)$$

where ε_0, θ, κ_x and κ_y denote the longitudinal strain in the central layer, the angle of twist, the curvatures in the yz and zx planes respectively.

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The symmetry of the transverse cross-section is assumed and the bending moments M_x and M_y are applied in the planes of symmetry. The function $f_3(x, y)$ represents the warping of cross-section due to the twisting moment M_z. Note from Eq.(1) that we are concerned with the deformation which is not dependent on the axial coordinate z.

(2) triangular element

Fig. 1 shows the beam element with axial unit length which is divided into the triangular prisms.

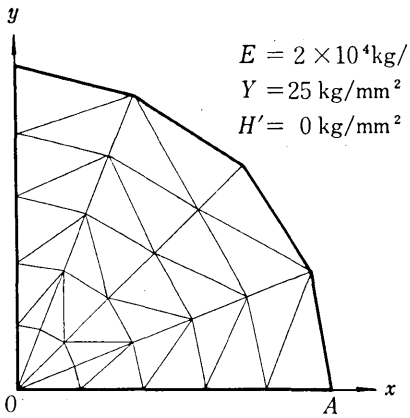

Fig. 1 Triangular element

We assume that the functions f_1, f_2 and f_3 of Eq. (1) are linear in x and y coordinates within each element. Then

$$\left.\begin{aligned} u&=\alpha_1+\alpha_2x+\alpha_3y-\theta yz+\frac{1}{2}\kappa_yz^2\\ v&=\alpha_4+\alpha_5x+\alpha_6y+\theta zx-\frac{1}{2}\kappa_xz^2\\ w&=\alpha_7+\alpha_8x+\alpha_9y+\varepsilon_0z+\kappa_xyz-\kappa_yzx \end{aligned}\right\} \quad (2)$$

The parameters or the generalized displacements $\lfloor\alpha_1, \alpha_2, \cdots, \alpha_9\rfloor$ can be expressed in terms of the displacement components at the nodes i. j and k as

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\\ \alpha_7\\ \alpha_8\\ \alpha_9\end{Bmatrix} = [\mathbf{A}] \begin{Bmatrix}\{u\}\\ \{v\}\\ \{w\}\\ \varepsilon_0\\ \theta\\ \kappa_x\\ \kappa_y\end{Bmatrix} \quad (3)$$

$$[\mathbf{A}]=\begin{bmatrix} [A] & [0] & [0] & \{0\} & \{a_y\} & \{0\} & -\{a_{\frac{1}{2}}\}\\ [0] & [A] & [0] & \{0\} & -\{a_x\} & \{a_{\frac{1}{2}}\} & \{0\}\\ [0] & [0] & [0] & -\{a_1\} & \{0\} & -\{a_y\} & \{a_x\} \end{bmatrix}$$

where

$$[A]=\begin{bmatrix}1 & x_i & y_i\\ 1 & x_j & y_j\\ 1 & x_k & y_k\end{bmatrix}^{-1}$$

$$\{u\}=\begin{Bmatrix}u_i\\ u_j\\ u_k\end{Bmatrix},\quad \{v\}=\begin{Bmatrix}v_i\\ v_j\\ v_k\end{Bmatrix},\quad \{w\}=\begin{Bmatrix}w_i\\ w_j\\ w_k\end{Bmatrix}$$

$$\{a_1\}=[A]\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix},\quad \{a_{\frac{1}{2}}\}=[A]\begin{Bmatrix}1/2\\ 1/2\\ 1/2\end{Bmatrix}$$

$$\{a_x\}=[A]\begin{Bmatrix}x_i\\ x_j\\ x_k\end{Bmatrix},\quad \{a_y\}=[A]\begin{Bmatrix}y_i\\ y_j\\ y_k\end{Bmatrix}$$

The suffixes denote quantities associated with the corresponding nodes i, j and k.

(3) stiffness equation

The strain components within each element are from Eq. (2)

$$\varepsilon_x=\frac{\partial u}{\partial x}=\alpha_2,\quad \varepsilon_y=\frac{\partial v}{\partial y}=\alpha_6$$

$$\varepsilon_z=\frac{\partial w}{\partial z}=\varepsilon_0+\kappa_xy-\kappa_yx$$

$$\gamma_{yz}=\left|\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right|_{z=1}=\alpha_9+\theta x$$

$$\gamma_{zx}=\left|\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right|_{z=1}=\alpha_8-\theta y$$

$$\gamma_{xy}=\left|\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right|_{z=1}=\alpha_3+\alpha_5$$

By substituting of $\alpha_1, \alpha_2, \cdots, \alpha_9$ from Eq. (3), we have

$$\{\varepsilon\}=[N]\{d\} \quad (4)$$

where

$$\{\varepsilon\}=\lfloor\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{yz}\ \gamma_{zx}\ \gamma_{xy}\rfloor^T,$$

$$\{d\}=\lfloor\lfloor u\rfloor\lfloor v\rfloor\lfloor w\rfloor\,\varepsilon_0\,\theta\,\kappa_x\,\kappa_y\rfloor^T$$

$$[N]=\begin{bmatrix} b_i & b_j & b_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_i & c_j & c_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & y & -x \\ 0 & 0 & 0 & 0 & 0 & 0 & c_i & c_j & c_k & 0 & x & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_i & b_j & b_k & 0 & -y & 0 & 1 \\ c_i & c_j & c_k & b_i & b_j & b_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $b_i=(y_j-y_k)/2\Delta$, $c_i=(x_k-x_j)/2\Delta$ and etc. in cyclic order i, j and k, and Δ is the area of the element triangle. Superfix T and symbol $\lfloor\ \rfloor$ indicate transpose and row vector, respectively. The associated stress field is

$$\{\sigma\}=[D]\{\varepsilon\},\ \text{or}\ \{d\sigma\}=[D]\{d\varepsilon\}$$

where $\{\sigma\}=\lfloor\sigma_x\,\sigma_y\,\sigma_z\,\tau_{yz}\,\tau_{zx}\,\tau_{xy}\rfloor^T$

We have to use the full 6×6 stress-strain matrix $[D]$ which is for the isotropic elastic deformation :

$$[D]=[D^e]=\frac{E}{1+\nu}\begin{bmatrix} \frac{1-\nu}{1-2\nu} & & & & & \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & & \text{SYM.} & & \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & & & \\ 0 & 0 & 0 & 1/2 & & \\ 0 & 0 & 0 & 0 & 1/2 & \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \qquad (5)$$

For the plastic deformation of the von Mises material[8),9)] :

$$[D]=[D^p]=[D^e]-\frac{2G}{S}\begin{bmatrix} \sigma_x'^2 & & & & & \\ \sigma_x'\sigma_y' & \sigma_y'^2 & & & & \\ \sigma_x'\sigma_z' & \sigma_y'\sigma_z' & \sigma_z'^2 & & \text{SYM.} & \\ \sigma_x'\tau_{yz} & \sigma_y'\tau_{yz} & \sigma_z'\tau_{yz} & \tau_{yz}^2 & & \\ \sigma_x'\tau_{zx} & \sigma_y'\tau_{zx} & \sigma_z'\tau_{zx} & \tau_{yz}\tau_{zx} & \tau_{zx}^2 & \\ \sigma_x'\tau_{xy} & \sigma_y'\tau_{xy} & \sigma_z'\tau_{xy} & \tau_{yz}\tau_{xy} & \tau_{zx}\tau_{xy} & \tau_{xy}^2 \end{bmatrix} \qquad (6)$$

$$S=\frac{2}{3}\bar{\sigma}^2\left(1+\frac{H'}{3G}\right)$$

The secondary stresses are due to the plastic stress-strain matrix $[D^p]$ which is fully populated and exhibits the cross effect between the normal and shear components.

The strain energy U stored in each element can be expressed by Eqs. (4) and (5) or (6) as

$$U=\frac{1}{2}\iint\lfloor\varepsilon\rfloor\{\sigma\}\,dxdy$$
$$=\frac{1}{2}\iint\lfloor\varepsilon\rfloor[D]\{\varepsilon\}\,dxdy$$
$$=\frac{1}{2}\lfloor d\rfloor\iint[N]^T[D][N]dxdy\{d\}$$
$$=\frac{1}{2}\lfloor d\rfloor[k]\{d\}$$

Then follows the stiffness matrix $[k]$ which is

$$[k]=\iint[N]^T[D][N]dxdy \qquad (7)$$

The stiffness equation of the relevant beam element connects the displacement vector $\{d\}$ with the force vector $\{F\}$ and can be expressed as

$$\{F\}=[k]\{d\} \qquad (8)$$

where

$$\{F\}=\lfloor 0\,0\,0\,\vdots\,0\,0\,0\,\vdots\,0\,0\,0\,\vdots\,F_z\,\vdots\,M_z\,\vdots\,M_x\,\vdots\,M_y\rfloor$$

The subsequent procedure follows the standard pattern of the finite element displacement method.

3. Numerical example

Our first numerical example concerns with the combination of axial load F_z and twisting moment M_z. For this combination, the theoretical solution of Gaydon[5)] is available for a circular bar of the non-hardening and incompressible material. In the computation, a quadrant of the cross-section is considered due to the symmetry ; numbers of the element division and nodes for the case of Fig. 2 being 36 and 26 respectively. The radius

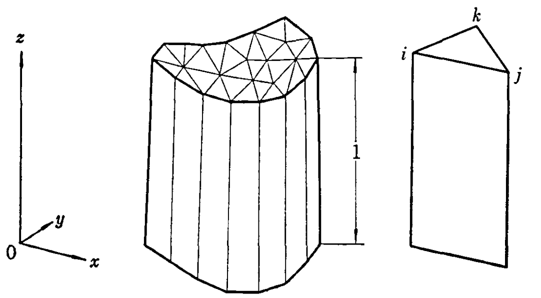

Fig. 2　Element division

of the bar is 10 mm. For simplicity, computation was carried out for the case where the value of the strain ratio $d\varepsilon_0/d\theta$ is kept constant during the whole process of loading.

Fig. 3 shows the calculated stress distributions at the elastic limit and the fully plastic state. As

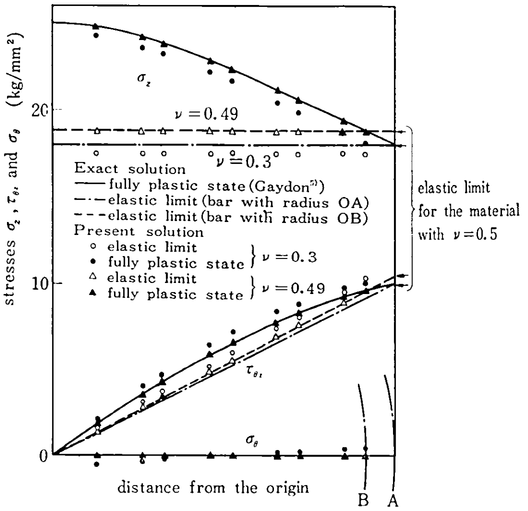

Fig. 3 Stress distributions ($d\varepsilon_0/d\theta=6$)

the stress in the present solution varies linearly in each element, the yielding of the element and especially the elastic limit of the bar are influenced by the choice of the reference point for the criterion of yielding. We used the centroid of the element for the criterion.

The limiting axial load $F_z{}^p$ and twisting moment $M_z{}^p$ for the fully plastic state are given in Table 1. The $F_z{}^p$ and $M_z{}^p$ for $\nu=0.49$ obtained by the present method are respectively around 2.3% and 3.5% less than the exact ones for $\nu=0.5$. These discrepancies seem to be attributable to the coarse element division, since the area of discretized cross-section of Fig. 2 is less than the actual circular one by about 2.5%.

4. Incompressible or nearly incompressible material

As indicated in the previous section, the usual finite element displacement method could be effective up to the case with the Poisson's ratio $\nu=0.49$. Moreover, the extrapolation is found to yield the solution for $\nu=0.5$ with suficient accuracy.

It may be useful, however, to apply the Herrmann's principle[7] to the present problem and assess the potential applicability and/or the drawbacks inherent to the principle. It should be emphasized that the Herrmann's formulation is most promising for the case where the mean normal stress or hydrostatic stress component is the major concern, in particular for the hydrodynamics of the incompressible fluid.

Under the assumption that the mean normal stress σ_m within each element is constant, the associated modifications pertaining to the present problem are as follows:

(1) The relation between the strain and displacement vector corresponding to Eq. (4) in the conventional displacement method is

$$\{\tilde{\varepsilon}\}=[\tilde{N}]\{\tilde{d}\}$$

where $\{\tilde{\varepsilon}\}=\lfloor \varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{yz}\ \gamma_{zx}\ \gamma_{xy}\ H\rfloor^T$,

$\{\tilde{d}\}=\lfloor \lfloor u\rfloor\ \lfloor v\rfloor\ \lfloor w\rfloor\ H\ \varepsilon_0\ \theta\ \kappa_x\ \kappa_y\rfloor^T$

$$[\tilde{N}]=\left[\begin{array}{ccc:ccc:ccc:ccccc}
b_i & b_j & b_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & c_i & c_j & c_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & y & -x\\
0 & 0 & 0 & 0 & 0 & 0 & c_i & c_j & c_k & 0 & 0 & x & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & b_i & b_j & b_k & 0 & 0 & -y & 0 & 1\\
c_i & c_j & c_k & b_i & b_j & b_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]$$

Table 1 The fully plastic loads $F_z{}^p$ and $M_z{}^p$

	calculated value					exact solution
	present formulation				Herrmann's principle	
ν	0.3	0.4	0.45	0.49	0.5	0.5
$F_z{}^p\times10^{-3}$ kg	6.24	6.34	6.40	6.43	6.42	6.58
$M_z{}^p\times10^{-4}$ kg-mm	1.78	1.72	1.69	1.67	1.66	1.73

Note that the generalized displacement $\{\tilde{d}\}$ as well as the generalized strain $\{\tilde{\varepsilon}\}$ have the additional component H which is defined by

$$H=3\varepsilon_m/(1-2\nu)=3\sigma_m/E$$

where ε_m and σ_m are mean normal strain and mean normal stress respectively. H is included in accordance with the Herrmann's formulation which incorporates H or the mean normal stress σ_m as the Lagrange multiplier.

(2) The stress-strain matrix $[\tilde{D}]$ corresponding to Eqs. (5) and (6) is for the isotropic elastic deformation:

$$[\tilde{D}]=[\tilde{D}^e]=2G\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & \text{SYM.} & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1/2 & & & \\ 0 & 0 & 0 & 0 & 1/2 & & \\ 0 & 0 & 0 & 0 & 0 & 1/2 & \\ \nu & \nu & \nu & 0 & 0 & 0 & -\nu(1-2\nu) \end{bmatrix} \quad (9)$$

For the plastic deformation of the Mises material:

$$[\tilde{D}]=[\tilde{D}^p]=[\tilde{D}^e]-\frac{2G}{S}\begin{bmatrix} \sigma_x'^2 & & & & & & \\ \sigma_x'\sigma_y' & \sigma_y'^2 & & & \text{SYM.} & & \\ \sigma_x'\sigma_z' & \sigma_y'\sigma_z' & \sigma_z'^2 & & & & \\ \sigma_x'\tau_{yz} & \sigma_y'\tau_{yz} & \sigma_z'\tau_{yz} & \tau_{yz}^2 & & & \\ \sigma_x'\tau_{zx} & \sigma_y'\tau_{zx} & \sigma_z'\tau_{zx} & \tau_{yz}\tau_{zx} & \tau_{zx}^2 & & \\ \sigma_x'\tau_{xy} & \sigma_y'\tau_{xy} & \sigma_z'\tau_{xy} & \tau_{yz}\tau_{xy} & \tau_{zx}\tau_{xy} & \tau_{xy}^2 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10)$$

The components of $[\tilde{D}]$ have been augmented corresponding to the inclusion of H in the displacement vector $\{\tilde{d}\}$. The last row of the $[\tilde{D}^p]-[\tilde{D}^e]$ is zero, since it is assumed that the permanent volume change does not take place in the von Mises material.

(3) The stiffness matrix $[\tilde{k}]$ is given in place of Eq. (7), by

$$[\tilde{k}]=\iint[\tilde{N}]^T[\tilde{D}][\tilde{N}]dxdy \quad (11)$$

The characteristic features to be notified is that the leading diagonal of the stress-strain matrix $[\tilde{D}]$ of Eqs. (9) and (10) contain negative or zero components, so that the stiffness matrix $[\tilde{k}]$ is ill-conditioned and can not be positive definite. These are embarassing and complicate the manipulation of the matrix, so the usual iterative solution procedure such as the conjugate gradient method, the Gauss-Seidel scheme and so forth can not be applied. We turned to the Cholesky method in our computation. As a further defect, we can point out that the dimension of the whole stiffness matrix becomes quite large.

The axial load F_z^p and the twisting moment M_z^p obtained by the Herrmann's formulation for the fully plastic state are included in Table 1. The solution by the conventional displacement method for $\nu=0.49$ correlates well with the exact solution, and it can be concluded that the Herrmann's method is not essential to the problem such as treated in the present paper.

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