研 究 速 報

UDC 624.042.7

FLUCTUATIONS OF RESPONSE SPECTRA —PART 4—

応答倍率の変動について —第4報—

——THEORETICAL EVALUATIONS IN CASE THAT RESPONSES AND EARTHQUAKES ARE STRONGLY CORRELATED——

——応答波と地震波の相関が強い場合の理論的考察——

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1. Introduction

In this paper, the fluctuations of response spectra are theoretically discussed. In a previous report[1], the fluctuations of response spectra were discussed under the assumption of independence between the average energy of output-and that of input-waves. But, when the output-and input-waves are strongly correlated, the results from expressions in the previous report are not consistent with the results from analog computation[2].

Here, the author has derived the probability distribution of response spectra including the correlation coefficient of average energy of output-and that of input-waves. By using this formula, the statistics of response spectra are given.

2. Fundamental Equations

In the previous report, the joint probability distribution of χ_i and χ_o (square root values of normalized average powers correspond input-and output-waves) was obtained under the assumption that χ_i and χ_0 are independent each other. When χ_i and χ_o are not independent, the determination of an exact expression on the joint probability distribution of χ_i and χ_o seems to be quite difficult. Here, an approximate expression on the joint probability distribution of χ_i and χ_o is derived by solving the following probabilistic model for χ^2-type-problem.

"Problem; Let $I_1, I_2, \ldots, I_m$ be the m real random variables belonging to $N(0, \sqrt{\phi_1})$, and $I_{m+1}, \ldots, I_{m+n}$ be the n real random variables belonging to $N(0, \sqrt{\phi_2})$.

What is the joint probability density $f_2(x_1, x_2)$ of the vector process $\boldsymbol{x}$, if $\overline{I_i I_{m+j}}=\overline{I_{m+j} I_i}=\phi_{12}=\rho, (i=1, 2, \ldots, m; j=1, 2, \ldots, n)$?
Where,

$$\boldsymbol{x}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} \tilde{\boldsymbol{I}}_1 \boldsymbol{J}_1 \boldsymbol{I}_1 \\ \tilde{\boldsymbol{I}}_2 \boldsymbol{J}_2 \boldsymbol{I}_2 \end{pmatrix} \quad (1)$$

and

$$\boldsymbol{I}_1=\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_m \end{pmatrix}, \quad \boldsymbol{I}_2=\begin{pmatrix} I_{m+1} \\ \vdots \\ I_{m+n} \end{pmatrix}, \quad \boldsymbol{I}=\begin{pmatrix} \boldsymbol{I}_1 \\ \boldsymbol{I}_2 \end{pmatrix}$$

and $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ are unit matrices of orders m and n and $\tilde{\boldsymbol{A}}$ denotes a transpose matrix (or vector) of $\boldsymbol{A}$."

We can solve the above problem as follows.

The characteristic function $F_2(i\xi_1, i\xi_2)_x$ of $f_2(x_1, x_2)$ is

$$F_2(i\xi_1, i\xi_2)_{\boldsymbol{x}}=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{m+n}(\boldsymbol{I})e^{i\tilde{\boldsymbol{\xi}}\boldsymbol{x}}d\boldsymbol{I} \quad (2)$$

from its definition, where $f_{m+n}(\boldsymbol{I})$ and $\boldsymbol{\xi}=\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ are the density of $\boldsymbol{I}$ and argument of a characteristic function. Also from its definition, the $f_{m+n}(\boldsymbol{I})$ is

$$f_{m+n}(\boldsymbol{I})=\frac{e^{-(1/2)\tilde{\boldsymbol{I}}\boldsymbol{K}^{-1}\boldsymbol{I}}}{(2\pi)^{(m+n)/2}(\det \boldsymbol{K})^{1/2}} \quad (3)$$

and covariance matrix $\boldsymbol{K}$ is

$$\boldsymbol{K}=\left(\begin{array}{ccc:ccc} \overline{I_1 I_1} & \cdots & \overline{I_1 I_m} & \overline{I_1 I_{m+1}} & \cdots & \overline{I_1 I_{m+n}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \overline{I_m I_1} & & \overline{I_m I_m} & \overline{I_m I_{m+1}} & \cdots & \overline{I_m I_{m+n}} \\ \hdashline \overline{I_{m+1} I_1} & \cdots & \overline{I_{m+1} I_m} & \overline{I_{m+1} I_{m+1}} & \cdots & \overline{I_{m+1} I_{m+n}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \overline{I_{m+n} I_1} & \cdots & \overline{I_{m+n} I_m} & \overline{I_{m+n} I_{m+1}} & \cdots & \overline{I_{m+n} I_{m+n}} \end{array}\right)\begin{array}{l} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} n \end{array} \quad (4)$$

and this becomes as

$$\boldsymbol{K}=\begin{pmatrix} \phi_1 & & 0 & \phi_{12} & \cdots & \phi_{12} \\ & \ddots & & \vdots & & \vdots \\ 0 & & \phi_1 & \phi_{12} & \cdots & \phi_{12} \\ \phi_{12} & \cdots & \phi_{12} & \phi_2 & & 0 \\ \vdots & & \vdots & & \ddots & \\ \phi_{12} & \cdots & \phi_{12} & 0 & & \phi_2 \end{pmatrix}. \quad (4)'$$

from the assumption of probabilistic model. As $\tilde{\boldsymbol{\xi}}\boldsymbol{x}$ is

$$\tilde{\boldsymbol{\xi}}\boldsymbol{x}=\tilde{\boldsymbol{I}}\boldsymbol{\Xi}\boldsymbol{I}, \quad \boldsymbol{\Xi}\equiv\left(\begin{array}{ccc:ccc} \xi_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & \xi_1 & & & \\ \hdashline & & & \xi_2 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \xi_2 \end{array}\right), \quad (5)$$

eq. (2) becomes as follows;

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$$F_{2x}=\int_{-\infty}^{\infty}\cdots\int\frac{e^{-(1/2)\tilde{I}(K^{-1}-2i\Xi)I}}{(2\pi)^{(m+n/2)}(\det \boldsymbol{K})^{1/2}}d\boldsymbol{I}\quad(6)$$
$$=[\det(\boldsymbol{J}-2i\boldsymbol{K\Xi})]^{-1/2},$$ [3]

where $\boldsymbol{J}$ is unit matrix of order $(m+n)$.

Introducing a new variable $P(\phi_1, \phi_2;\ \xi_1, \xi_2)$, that is,

$$P(\phi_1, \phi_2;\ \xi_1, \xi_2)\equiv(1-2i\phi_1\xi_1)^m(1-2i\phi_2\xi_2)^n,\quad(8)$$

F_{2x} is obtained from eq. (7) as,

$$F_{2x}=\left[\left(1-\phi_{12}{}^2\frac{\partial^2}{\partial\phi_1\partial\phi_2}\right)P\right]^{-1/2}.\quad(9)$$

Eq. (9) can be expanded to Taylor series if $\left|\phi_{12}{}^2\frac{\partial^2}{\partial\phi_1\partial\phi_2}\right|<1$, and we take the first 2 terms, then F_{2x} becomes as follows;

$$F_{2x}\cong\left(1+\frac{\phi_{12}{}^2}{2}\frac{\partial^2}{\partial\phi_1\partial\phi_2}\right)P^{-1/2}.\quad(10)$$

From eq. (10), $f_2(x_1, x_2)$ is obtained approximately as

$$f_2(x_1, x_2)\cong\left(1+\frac{\phi_{12}{}^2}{2}\frac{\partial^2}{\partial\phi_1\partial\phi_2}\right)f_1(x_1, \phi_1)f_1(x_2, \phi_2),\quad(11)$$

where

$$f_1(x_i, \phi_i)\equiv\frac{1}{\phi_i\cdot 2^{l/2}\Gamma(l/2)}\left(\frac{x_i}{\phi_i}\right)^{(l/2)-1}\cdot e^{-(1/2)(x_i/\phi_i)};\quad \begin{cases} l=m: \text{ if } i=1\\ l=n\ : \text{ if } i=2\end{cases}.\quad(12)$$

From eqs. (11) and (12), $f_2(u_1, u_2)$ is obtained as follows;

$$f_2(u_1, u_2)\cong\left[1+\frac{\rho^2}{2}\{u_1-(p_1+1)\}\{u_2-(p_2+1)\}\right]\times f_1(u_1)f_1(u_2),\quad(13)$$

where in this equation, we replaced

$$u_i\equiv\frac{1}{2}\frac{x_i}{\phi_i}(i=1,\ 2),\quad p_1\equiv\frac{m}{2}-1,\quad p_2\equiv\frac{n}{2}-1\quad(14)$$

and

$$f_1(u_i)\equiv\frac{u_i{}^{(l/2)-1}}{\Gamma(l/2)}e^{-u_i};\quad\begin{cases} l=m: \text{ if } i=1\\ l=n\ : \text{ if } i=2\end{cases}.\quad(15)$$

Now, the joint probability distribution $f_2(\chi_i, \chi_o)d\chi_i d\chi_o$ for the square root value of a normalized average power of input waves χ_i and that of output wave χ_o is

$$f(\chi_i\chi_o)d\chi_i d\chi_o=\left[1+\frac{\mu}{\sqrt{(n_i+1)(n_o+1)}}\times\{\chi_i{}^2-(n_i+1)\}\{\chi_o{}^2-(n_o+1)\}\right]$$
$$\times\frac{4\chi_i{}^{2n_i+1}\chi_o{}^{2n_o+1}}{\Gamma(n_i+1)\Gamma(n_o+1)}e^{-(\chi_i{}^2+\chi_o{}^2)}d\chi_i d\chi_o;\ \chi_i,\ \chi_o\geq 0$$
$$=0\quad;\ \chi_i,\ \chi_o \text{ otherwise}\quad(16)$$

from analogy of expression (5) in the previous report and eq. (14), where μ is a correlation coefficient of a normalized average power of input-and that of output-waves, n_i and n_o (correspond to input-and output-waves) are degrees of freedom of χ^2-type distribution in eq. (15).

The probability distribution of a normalized response spectra from the 2nd term in eq. (16) $f_2(\lambda^*)d\lambda^*$, $(\lambda^*\equiv\chi_o/\chi_i)$, is as follows (the 1st term was already discussed);

$$f_2(\lambda^*)d\lambda^*=\frac{2\mu}{\sqrt{(n_i+1)(n_o+1)}}\cdot\frac{1}{B(n_i+1, n_o+1)}$$
$$\cdot\frac{\lambda^{*2n_o+1}}{(1+\lambda^{*2})^{n_i+n_o+2}}\cdot\Big[(n_i+n_o+3)(n_i+n_o+2)\times\frac{\lambda^{*2}}{(1+\lambda^{*2})^2}-(n_o+1)(n_i+n_o+2)\frac{1}{(1+\lambda^{*2})}$$
$$-(n_i+1)(n_i+n_o+2)\frac{\lambda^{*2}}{(1+\lambda^{*2})}+(n_i+1)(n_o+1)\Big]$$
$$\times d\lambda^*;\ \lambda^*\geq 0$$
$$=0\quad;\ \lambda^*<0,\quad(17)$$

where $B(p, q)$ is Beta function of argument p and q.

3. Statistical Properties of Response Spectra

The moments of response spectra are obtained from eq. (17). Mean value $\bar{\lambda}$, standard deviation σ_λ and relative dispersion d_λ are as follows;

$$\bar{\lambda}=K\kappa_1\quad(18)$$
$$\sigma_\lambda=K\sqrt{\kappa_2-\kappa_1{}^2}\quad(19)$$
$$d_\lambda=\sqrt{\kappa_2-\kappa_1{}^2}/\kappa_1\quad(20)$$

Here, K is a constant, κ_1 and κ_2 are the first and second moments of λ^*, and these can be obtained from eq. (17), that is,

$$\kappa_1=\left[1-\frac{\mu}{4\sqrt{(n_i+1)(n_o+1)}}\right]\alpha_1,\quad(21)$$
$$\kappa_2=\left[1-\frac{\mu}{\sqrt{(n_i+1)(n_o+1)}}\right]\alpha_2,\quad(22)$$

where α_1 and α_2 are the 1st and 2nd moments of λ^* when $\mu=0$.[1]

From eqs. (21), (22), we see that the correlation coefficient μ gives an influence to κ_2 stronger than to κ_1, that is, to standard deviation than to mean value. And the larger μ becomes, the less κ_1 and κ_2 become.

4. Correlation Coefficient and Cross Covariance Function

Correlation coefficient μ is defined as

$$\mu\equiv\frac{\overline{\chi_i{}^2\chi_o{}^2}-\overline{\chi_i{}^2}\cdot\overline{\chi_o{}^2}}{\sqrt{\{\overline{(\chi_i{}^2)^2}-(\overline{\chi_i{}^2})^2\}\{\overline{(\chi_o{}^2)^2}-(\overline{\chi_o{}^2})^2\}}}$$
$$=\frac{\sigma^2[E_{io}]}{\sigma_{E_i}\cdot\sigma_{E_o}},\quad(23)$$

where

$$\sigma^2[E_{io}]=2\int_0^T(T-\tau)[\psi_{io}^2(\tau)+\psi_{oi}^2(\tau)]d\tau \tag{24}$$

$$\sigma[E_i]=4\int_0^T(T-\tau)\psi_i^2(\tau)d\tau \tag{25}$$

and

$$\sigma[E_o]=4\int_0^T(T-\tau)\psi_o^2(\tau)d\tau. \tag{26}$$

Here, $\psi_{oi}(\tau)$ is the cross covariance function of output-and input-waves, and $\psi_i(\tau)$ and $\psi_o(\tau)$ are auto covariance function of input-and output-waves respectively. Cross covariance function $\psi_{oi}(\tau)$ is obtained as Fourier Integral[4)] of cross spectral density $W_{oi}(\omega)$ for the output-and input-waves.

5. Numerical Computations and Results

The author comtutes $\bar{\lambda}$, d_λ in case of a single-degree-of-freedom dynamical system. The power spectrum of pseudo-earthquake is taken as,

$$A(\omega)=\frac{4\omega_g^2\zeta_g^2\omega^2+\omega_g^4}{\omega^4-2(1-2\zeta_g^2)\omega_g^2\omega^2+\omega_g^4}\cdot k \tag{27}$$

and transfer function for an absolute acceleration of the vibrating system is as,

$$H_s(i\omega)\equiv H_{\ddot{z}_b}(i\omega)=\frac{2\zeta_b\omega_b(i\omega)+\omega_b^2}{(i\omega)^2+2\zeta_b\omega_b(i\omega)+\omega_b^2}, \tag{28}$$

where ω_g, ζ_g are a dominant frequency and a corresponding damping ratio of the ground and ω_b, ζ_b are a natural frequency and a damping ratio of the system, and k is an intensity of gaussian white noise. From eqs. (27), (28), cross power spectrum is obtained as

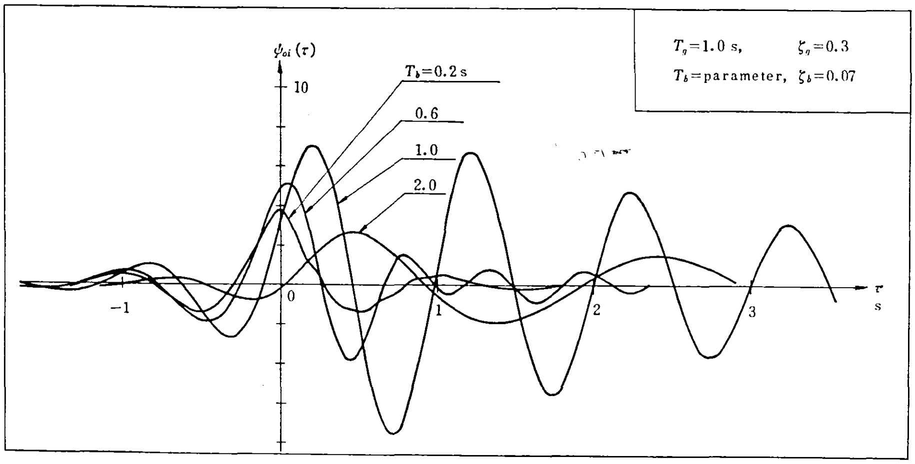

Fig. 1 CROSS-COVARIANCE FUNCTION OF $\ddot{z}_b(t+\tau)$ AND $\alpha(t)$
(1 freedom system, stationary-input)

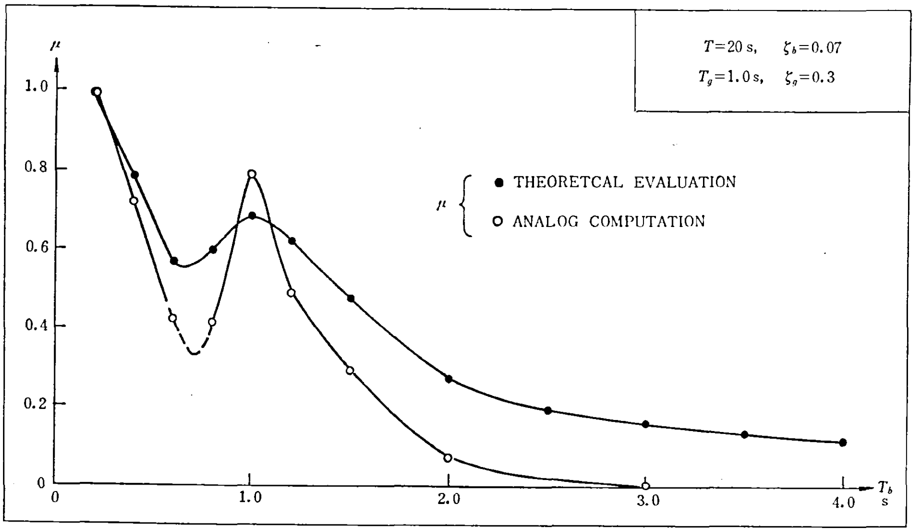

Fig. 2 CORRELATION COEFFICIENT OF E_i AND E_o
(1 freedom, stationary-input)

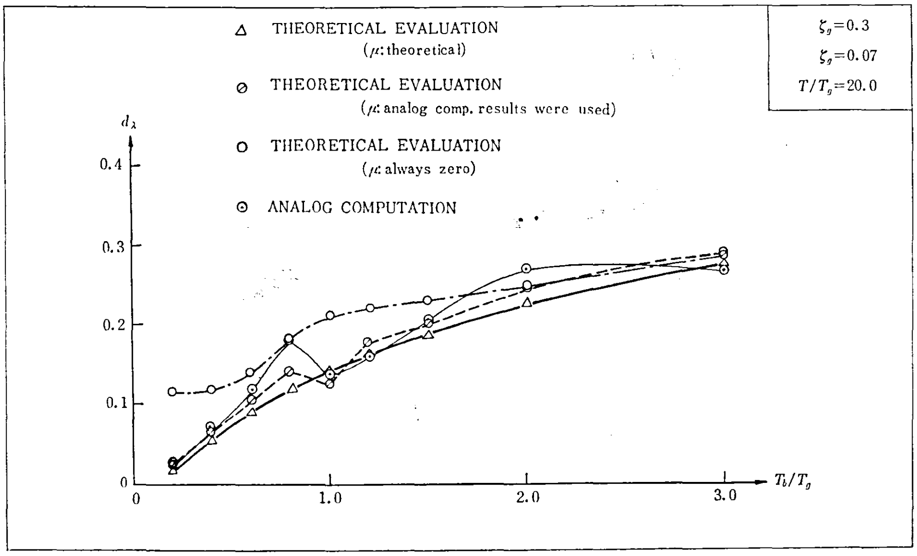

Fig. 3 RELATIVE DISPERSION OF RESPONSE SPECTRA
(1 freedom, absolute acc.)

$$W_{oi}(i\omega)=\frac{2\zeta_b\omega_b(i\omega)+\omega_b^2}{(i\omega)^2+2\zeta_b\omega_b(i\omega)+\omega_b^2}\cdot\left|\frac{2\zeta_g\omega_g(i\omega)^2+\omega_g^2}{(i\omega)^2+2\zeta_g\omega_g(i\omega)+\omega_g^2}\right|^2 k. \quad (29)$$

Fig. 1 shows the cross covariance function $\psi_{oi}(\tau)$ for the absolute acceleration of the system and pseudo-earthquake. Fig. 2 shows the correlation coefficient μ of the average energy of the absolute acceleration of the system and that of pseudo-earthquake. In this figure, the results of analog simulation are also involved. A tendency of the two curves is consistent, but the curve from analog simulation is turned more shaper. Quantitative defference between the two curves would be caused by the nostationary effect (i. e. initial condition), although input waves are stationary processes, for analog computation.

Fig. 3 shows the curves of the relative dispersion d_λ for T_b/T_g. We see that the results of theoretical evaluation from eq. (19) are better agreement to the results of analog simulation than those of theoretical evaluation as $\mu=0$. When μ is small, so also is d_λ. This fact is known directly from eq. (20). For $T_b/T_g \gtrsim 1.5$, the relative dispersion is considerably large and in this region, the effect of μ to the d_λ is almost none, that is, we can see that the assumption of the independency between the average power of the output-and that of input-waves is sufficiently satisfied. As for mean value $\bar{\lambda}$, μ scarecely gives an influence to it.

6. Conclusions and Acknowledgements

The author obtained the following conclusions from the above discussions;

(1) New formulae are given for the probability distribution of response spectra as eq. (17), and the joint probability distribution χ_i and and χ_o as eq. (16).

(2) Correlation coefficient of an average power of output-and that of input-waves is an important factor for the fluctuations of response spectra, but

(3) Mean value is little influenced.

(4) Expressions derived in Chap. 2 & 3 are valid for the estimation of response spectra, and

(5) These expressions can be used to a multi-degrees-of-freedom vibrating system because μ is included. This problem will be reported in the following papers.

The author expresses his great gratitudes to Professor Shibata for his valuable discussions.

(Manuscript received Mar. 24 1970.)

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