Essays on Tolerant Economic Systems

OHASHI Yoshihiro

Preface

Our human beings cannot live without economic systems. Economic systems are to be characterized by some abstract form of *mechanisms*, e.g., trading rules of stock markets or social decision processes.

The economic system is said to be *tolerant* if it does not "collapse" against particular changes in its details. For example, consider a stock trading market. Assume that the prices are determined by some pricing rules and those pricing rules prevent any arbitrage by traders. However, it is another question whether such pricing rules remain to prevent arbitrages against a certain small change in the environment. If the small change allows arbitrages, the pricing rule cannot be tolerant.

The tolerant economic system is required to keep operating well against particular changes in the environment we take up. This thesis investigates tolerant economic systems in the market with middlemen and in the abstract mechanism design problem known as implementation theory.

In the literature of Financial economics, "arbitrage-free" prices are desired. In the competitive market model with symmetric information, it is well known that arbitrage-free prices are supported in equilibrium under mild assumption on preferences (e.g., see Mas-Colell, Whinston and Green (1995) or LeRoy and Werner (2001)). However, in other trading systems or in markets with asymmetric information, we have to reconsider what conditions support arbitrage-free prices in equilibrium. Kyle (1985) investigates equilibrium price formation in a market making trading system under asymmetric information about trading goods. He shows the existence of linear equilibrium price formation rules. Hubermann and Stanzl (2004) show that such linear pricing rules are arbitrage-free against "normal" behavior of participants. However it is unknown whether the results remain valid or not against abnormal behavior of participants. Chapter 1 and 2 in the thesis analyze this issue.

Chapter 1 investigates how market makers should set prices in order to prevent speculative price bubbles in sequential trading stock markets, i.e., market making systems. I analyze a situation in which a rational speculator can gain from a price bubble caused by his speculative strategy which exploits irrational *feedback traders*. Under the assumption that market makers set prices by a *linear pricing rule*, I characterize a class of "speculation-proof" linear pricing rules. The speculation-proof rule is the arbitrage-free pricing rule in the model. In an application to a simple three periods trading model, I show that competitive market makers can set prices in equilibrium which follow a speculation-proof linear pricing rule.

Chapter 2 expands the setting in Chapter 1 and argues *market control* in an infinite period market making systems. The market control means a market intervention for stabilization. I analyze a situation in which two-periods-lived rational speculators gains by speculations which exploit irrational *feedback traders* and cause price bubbles. Under the assumption that market prices follow a *linear pricing rule*, I characterize the rules which do not lead price bubbles. If there is no control, then we have to considerably restrict the pricing rules for no price bubbles. On the other hand, if there is a control, we can achieve no price bubbles in equilibrium without any substantial restriction on the pricing.

In the literature of Game theory, the "Wilson doctrine" is well known as an admonition to too much dependence on common knowledge (Wilson (1987)). Recent papers in Mechanism design and auction theory have been investigated along with the Wilson doctrine (e.g., Dasgupta and Maskin (2000), Matsushima (2005, 2008) Bergemann and Morris (2005).) This research direction is favorable toward the construction of tolerant economic systems. Chapter 3 and 4 in the thesis are related to this issue.

Chapter 3 takes up the *full* implementation problem under conditions of incomplete information. The solution concept I use is *ex post equilibrium*. I provide a necessary and an almost sufficient condition for ex post implementation. I show that the *ex post selective elimination* condition and ex post incentive compatibility are necessary conditions for which social choice set X is ex post implementable. Moreover, social choice set X is ex post implementable if both the conditions are satisfied in an economic environment.

Chapter 4 also investigates *full* implementation problem under conditions of incomplete information. This chapter, in particular, focuses on a robustness of mechanisms. I introduce a new concept to implementation problem — *belief-free implementation*. Social choice function x is said to be *quasi belief-free implementable* if there exists a mechanism which implements x for any full-support belief system of agents. The main solution concept is iterative deletion of ex post *weakly* dominated strategies. If social choice function x is implementable in iterative deletion of ex post weakly dominated strategies, then it implies that x is quasi belief-free implementable. I provide a sufficient condition, the *uniformly effective elimination condition*, for quasi belief-free implementation in an economic environment.

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Part I

Pricing Rules in Market Making Systems

Chapter 1

Speculative Bubbles Prevention by Market Makers

1.1 Introduction

This chapter focuses on a relation between momentum trading by irrational traders and price bubbles in a stock market with a *market making system*. The market is composed of discrete N trading periods. There are three kinds of market participants: a speculator, a (representative) market maker, and irrational *feedback traders*. The behavior of feedback traders is determined by price trends, i.e., they buy stocks today when they observed a price gain yesterday. The price gain leads to their purchase of stock; their purchase of stock implies a future price gain. This self-feeding behavior continuously raises market prices and may lead a price bubble.[1]

I consider a situation in which a rational speculator can get money through a speculation which exploits feedback traders. The speculation begins with purchase of stocks in the first period. The speculator's purchase raises the market price and then triggers feedback traders' purchase. If feedback traders monotonically raise the market price by their self-feeding behavior, the speculator can get money only by selling shares in the last period. This is the scenario of the speculator; all the speculator has to do is to wait till the last

[1] Shiller (2008) writes on price increases observed in recent subprime loan tragedy as follows:

> Feedback loop appear, as price increases encourage belief in "new era" stories,... and so lead to further price increases. The price-story-price loop repeats again and again during a speculative bubble. The feedback loops also take the form of price-economic activity-price loops.(pp45-46).

period arrives after his purchase.[2]

The main purpose of this article is to characterize the market maker's pricing behavior which gives the speculator at most nonpositive payoff from the speculation. Market making systems allow the market maker to offer a price after he observes an aggregate market order. Under the assumption that the market maker offers a price which is derived by a *linear pricing rule* (a linear mapping from an aggregate placed order into a nonnegative price), I characterize a class of "speculation-proof" pricing rules, i.e., under the rule of which, the speculator can get at most nonpositive payoff from the speculation. The set of speculation-proof pricing rules is nonempty but its measure is shrunk by a proportional to a parameter of momentum. (Section 1.3.)

Finally I apply the obtained result to a three periods trading model under asymmetric information about true stock value and investigate whether market makers set a speculation-proof pricing rule in equilibrium. A contribution of this article is to show that competition among market makers can prevent the speculation in equilibrium (hence it can prevent speculative bubbles) if market makers correctly estimate trading volume from feedback traders. It means that the equilibrium pricing behavior is not only a linear pricing rule, but consistent with setting a speculation-proof pricing rule. (Section 1.4.)

1.1.1 Related literature

The model in this article is based on Huberman and Stanzl (2004). Huberman and Stanzl assume a single arbitrageur, market makers, and normally-distributed noise traders. Market prices follow a (common) pricing rule of market makers. *Price manipulation* is an arbitrage behavior which includes the speculative strategy defined in this article. Whereas Huberman and Stanzl show that most linear pricing rules prevent price manipulation, I show that most linear pricing rules *cannot* prevent price manipulation once we assume feedback traders, which is independent whether there are normally-distributed noise traders or not.

The feedback trader is identical with the *positive feedback trader* introduced by De Long, Shleifer, Summers, Waldmann (1990). De Long, et al propose a model without market makers in which the positive feedback trader is

[2] Judging from descriptions in Soros (2003, pp49-72), his speculative strategy in stock markets exploits irrational traders. In fact it brought him large profits, while price was bubbling. Importantly, his strategy does not rely on fundamentals but on anticipation of behavior of irrational traders.

the source of price bubble; in competitive equilibrium, informed speculators' optimal decision triggers positive feedback traders' demand, which makes a market price surpass the fundamental value of stock. Such a bubble surely occurs in equilibrium whenever informed speculators exactly know the fundamental value is good. On the other hand, I propose a model based on Kyle (1985) such that competitive market makers set a speculation-proof linear pricing rule in equilibrium. Therefore the speculator does not have speculative incentive and so price bubbles never occur in the equilibrium no matter what the fundamental value is.

1.2 The Model

I analyze a single stock market with discrete finite N trading periods in the interval $(0,1] \subset \mathbb{R}_+$. A trading is taken place at each time $n\Delta_N$, where $1 \le n \le N$ and $\Delta_N := 1/N$. The market opens at $n = 1$ and closes at $n = N$. There are three kinds of market participants: a rational speculator, a rational market maker, and irrational feedback traders. In each trading period n, the speculator and feedback traders simultaneously place a "market order" (quantities they want to trade), x_n and ξ_n respectively, where $x_n, \xi_n \in \mathbb{R}$. Then the market maker offers a price $p_n \in \mathbb{R}_+$ for the aggregate market order, $q_n = x_n + \xi_n$, and clears the market. The market maker offers a trading price in period n by the following pricing rule:

$$\begin{aligned} p_n &= p_{n-1} + (U_{n-1}(q_{n-1}) - P_{n-1}(q_{n-1})) + P_n(q_n) \\ &= p_0 + \sum_{k=1}^{n-1} U_k(q_k) + P_n(q_n), \end{aligned} \tag{1.1}$$

where p_0 is an opening price in the market. This pricing rule follows the one in Huberman and Stanzl (2004). Function $P_n(q_n)$, say *price impact function*, captures immediate price reaction to a market order in current period. On the other hand, $U_n(q_n)$, say *price update function*, captures only the permanent price impact from trade. Here is an assumption for price functions.

Assumption 1 *Functions $U_n(q_n)$ and $P_n(q_n)$ are time-independent and linear, i.e.,*

$$U_n(q_n) = \lambda q_n, \qquad P_n(q_n) = \mu q_n,$$

where $(\lambda, \mu) \in \mathbb{R}^2_+$.

We call (λ, μ) a pair of *price coefficients*. A pricing rule is said to be *linear* when both $U_n(q_n)$ and $P_n(q_n)$ are linear for all n.

The decision of feedback traders depends on the latest price difference (trend): $\xi_n = \beta(p_{n-1} - p_{n-2})$ with a momentum parameter $\beta > 0$. This behavioral assumption is the same as the *positive feedback trader* introduced by De Long, Shleifer, Summers, and Waldmann (1990).

I consider a situation in which the speculator knows β and correctly expects (λ, μ) in period 0, and so the speculator tries to implement a speculation by using the knowledge about them. Let $\boldsymbol{x} = (x_1, \cdots, x_N)$ denote a *strategy* of the speculator. The speculator implements a *round-trip strategy* defined by

$$\sum_{n=1}^{N} x_n = 0. \tag{1.2}$$

A round-trip strategy is said to be *nonzero* if $x_n \neq 0$ for some n. The payoff function of the speculator is

$$\pi(\boldsymbol{x}; N) = -\sum_{n=1}^{N} p_n x_n,$$

where p_n is defined by (1.1) with Assumption 1. Thus the speculator maximizes $\pi(\boldsymbol{x}, N)$ subject to equality (1.2). A (risk-neutral) *price manipulation* is a round-trip strategy $\boldsymbol{x} = (x_1, \cdots, x_N)$ which makes $\pi(\boldsymbol{x}, N) > 0$. A *simple strategy* is a round-trip strategy such that $x_1 = x \in \mathbb{R}$, $x_N = -x$, and $x_n \equiv 0$ for all n but $n = 1, N$. The main analysis in this article is to characterize price coefficients which satisfy the "speculation-proof" conditions I define now.

Definition 1 *A pair of price coefficients (λ, μ) is **feasible** if, for all $N \in \mathbb{Z}_+$, for all $x \in \mathbb{R}$, we have $(p_N - p_1)x \leq 0$, where p_1 and p_N are defined by (1.1) with Assumption 1.*

The *feasible set* of price coefficients is defined by the collection of feasible pair of price coefficients. We call an element of the feasible set *feasible price coefficients*.

Definition 2 *A pair of price coefficients (λ, μ) is **semi-feasible** in N periods market if it makes $\pi(\boldsymbol{x}, N) < 0$ for any nonzero round-trip strategy $\boldsymbol{x}$ in N periods market.*

Note that these definitions implicitly assume p_0 to be sufficiently large.

1.3 Main Results

Theorem 1 characterizes the feasible set of price coefficients. The main point is that feasible linear pricing rules must sufficiently depress the momentum of feedback traders. As a result, few price coefficients are feasible. Note that we *only* take into account simple strategies. The set proves to be nonempty, but the area of feasible price coefficients is inversely proportional to β^2. Proposition 1 proposes a characterization of the semi-feasible set of price coefficients in the case of $\lambda = \mu$.

1.3.1 A characterization of feasible price coefficients

Theorem 1 *If (λ, μ) is feasible, then (λ, μ) satisfies the following inequalities.*

$$\beta\mu - 1 < \beta\lambda, \quad 0 \leq \beta\lambda < 1, \quad \frac{\lambda - 2\mu + 2\beta\lambda\mu}{1 - \beta\lambda} \leq 0. \tag{1.3}$$

In particular, if $(\lambda, \mu) \in \mathbb{R}^2_+$ satisfies (1.3) and $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda > 0$, then (λ, μ) is feasible.

Remark: It is difficult to characterize the necessary and sufficient feasible price coefficients since the payoff from a simple strategy does not monotonically increase with N.
Proof. See Appendix. ■

I demonstrate the case that $N = 2$. Let $x_1 = x \in \mathbb{R}$ and $x_2 = -x$. Then

$$\begin{aligned} p_1 &= p_0 + \mu x, \\ p_2 &= p_0 + \lambda x + \mu(-x + \beta\mu x). \end{aligned}$$

Therefore,

$$\begin{aligned} -x_1 p_1 - x_2 p_2 &= x(p_2 - p_1) \\ &= x^2(\beta\mu^2 - 2\mu + \lambda) \\ &\leq 0 \end{aligned} \tag{1.4}$$

is necessary and sufficient for preventing the speculation.

Figure 1.1 indicates a graphical image of Theorem 1. The finer shaded area in Figure 1.1, which is defined by inequalities (1.3) and $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda > 0$, indicates sufficient feasible price coefficients. The union of the finer and the coarser shaded area, which corresponds to the area defined by

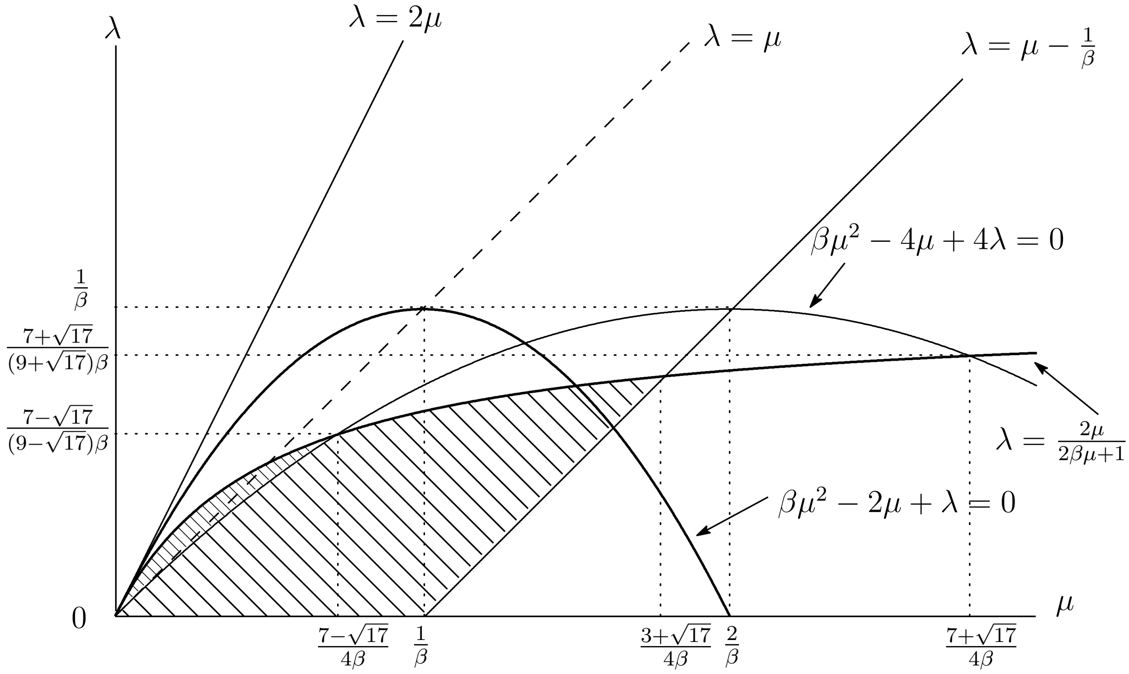

Figure 1.1: The finer shaded area indicates a feasible set. Union of the finer and the coarser shaded area indicates an upper bound of the feasible set.

inequalities (1.3), indicates an upper bound of the feasible set. Figure 1.1 also shows that the area defined by inequality (1.4) with $(\lambda, \mu) \in \mathbb{R}_+^2$ completely includes the one defined by inequalities (1.3) and $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda > 0$. Appendix shows that both the finer and the coarser shaded area have the strictly positive measure for any $\beta > 0$. As $\beta \to 0$, the feasible set becomes large, and in the limit, it is defined by $\{ (\lambda, \mu) \mid \lambda \geq 0,\ \mu \geq 0,\ \lambda \leq 2\mu \}$. Appendix also shows that prices are bounded in the limit $N \to \infty$ only if (λ, μ) is in

$$B = \left\{ (\lambda, \mu) \;\middle|\; 0 \leq \lambda \leq \frac{1}{\beta},\ 0 \leq \mu \leq \frac{2}{\beta},\ \lambda \leq 2\mu \right\}. \tag{1.5}$$

We call B the *bounded area* of price coefficients. If $(\lambda, \mu) \in B$ we obtain $|(p_N - p_1)x| < \infty$ when $N \to \infty$ for any x, and if $(\lambda, \mu) \notin B$, there exists N such that $(p_N - p_1)x > 0$ for all $x > 0$. I concern how much feasible price coefficients account for within B. Appendix shows that approximately only 2.6% of bounded area corresponds to the finer shaded area in Figure 1.1, and approximately 41.0% of bounded area corresponds to the union of the finer and the coarser shaded area. Note that we obtain these intolerant results in spite of putting tractable assumptions: only one-time purchase and sale by the speculator and only one-period trend chasing feedback traders.

1.3.2 An insight into the feasible pricing rules

Price coefficients which belong to the union of the finer and the coarser shaded area in Figure 1.1 are characterized by relatively smaller λ compared to μ. In other words, the price update coefficient should relatively be smaller than the price impact coefficient. It is the effective way to depress the momentum of feedback traders. To see why, we first look at the behavior of feedback traders. Feedback traders' demand ξ_n is defined by

$$\begin{aligned} \xi_n &= \beta(p_{n-1} - p_{n-2}) \\ &= \beta(\mu q_{n-1} + (\lambda - \mu) q_{n-2}) \end{aligned} \tag{1.6}$$

for $n \in \{2, \cdots, N\}$. ξ_n is equal to q_n when $n \in \{2, \cdots, N-1\}$ if the speculator implements a simple strategy. Suppose that the speculator implements the simple strategy with $x_1 > 0$ and that $\xi_n \geq 0$ for all $n \in \{2, \cdots, N\}$. Then, from equality (1.6), ξ_n increases with λ, which is favorable to the speculator because larger ξ_n pushes n period price further. Therefore decreasing λ is directly effective to discourage simple strategies. In fact, if

$(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda \geq 0$, ξ_n is positive for all $n \in \{2, \cdots, N\}$ in implementing simple strategies with $x_1 > 0$ (see Appendix). Since the price update function conveys price impact from current trade to future prices, letting λ be relatively small stands to reason. As a result, the vast area of price coefficients which satisfy $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda \geq 0$ in Figure 1.1 is excluded from the feasible point of view.

1.3.3 A characterization of semi-feasible price coefficients

In this subsection, I assume that $U_n(q_n) = P_n(q_n) = \lambda q_n$, namely,

$$p_n = p_{n-1} + \lambda q_n \quad (n = 1, 2, \cdots, N) \tag{1.7}$$

with $\lambda \in \mathbb{R}_+$. Consider the (N, N) matrix $\boldsymbol{D}$ such that

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1+\beta\lambda & 1 & 0 & & 0 \\ 1+\beta\lambda+(\beta\lambda)^2 & 1+\beta\lambda & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \sum_{i=1}^{N}(\beta\lambda)^{i-1} & \sum_{i=1}^{N-1}(\beta\lambda)^{i-1} & \sum_{i=1}^{N-2}(\beta\lambda)^{i-1} & \cdots & 1 \end{pmatrix}.$$

Let $\boldsymbol{x} = (x_1, \cdots, x_N)^\top$ be a round-trip strategy and $\boldsymbol{p} = (p_1, \cdots, p_N)^\top$ be a vector of market prices whose each element is defined by (1.7). Then the maximization problem for the speculator is written as follows:

$$\max -\boldsymbol{x} \cdot \boldsymbol{p} \quad \text{s.t.} \;\; \mathbf{1} \cdot \boldsymbol{x},$$

where $\mathbf{1}$ is a n-tuple of 1. This problem is equivalent to the maximization of $-\lambda \boldsymbol{x}^\top \boldsymbol{D} \boldsymbol{x}$ under the constraint $\mathbf{1} \cdot \boldsymbol{x}$. The objective function $-\lambda \boldsymbol{x}^\top \boldsymbol{D} \boldsymbol{x}$ is rewritten by matrix $\boldsymbol{A}$ such that

$$a_{ij} = \begin{cases} (d_{ij} + d_{ji})/2 & (i \neq j) \\ d_{ij} & (i = j), \end{cases}$$

where a_{ij} (d_{ij}) is (i, j)-element of $\boldsymbol{A}$ ($\boldsymbol{D}$), so that $\boldsymbol{x}^\top \boldsymbol{D} \boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x}$. Matrix $\boldsymbol{A}_l$ ($l = 1, \cdots, N$) is the (l, l) submatrix of $\boldsymbol{A}$ obtained by retaining only the first l rows and columns of $\boldsymbol{A}$. Let $\boldsymbol{C}_l$ be the $(l+1, l+1)$ matrix defined as follows:

$$\boldsymbol{C}_l = \begin{pmatrix} 0 & \mathbf{1}_l^\top \\ \mathbf{1}_l & \boldsymbol{A}_l \end{pmatrix},$$

where $\mathbf{1}_l$ is a l-tuple of 1. Here is a characterization result of semi-feasible price coefficients.

Proposition 1 *In N-periods market, a price coefficient $\lambda \in \mathbb{R}_+$ is semi-feasible if and only if λ satisfies $|\boldsymbol{C}_l| < 0$ for all $l = 2, 3, \cdots, N$.*

Proof. Since $\boldsymbol{x}^\top \boldsymbol{D}\boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}$, we want to be that $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$ with $\boldsymbol{1} \cdot \boldsymbol{x} = 0$. By Theorem 4 in Debreu (1952), it is equivalent that $|\boldsymbol{C}_l| < 0$ for all $l = 2, 3, \cdots, N$. ■

A feasible price coefficient can be semi-feasible in a market whose periods is lager than two. For example, in $N = 3$, $|\boldsymbol{C}_2| < 0$ is equivalent that $0 \leq \beta\lambda < 1$ and $|\boldsymbol{C}_3| < 0$ is equivalent that $0 \leq \beta\lambda < (-1 + \sqrt{5})/2$. From Theorem 1, a feasible price coefficient $\lambda = \mu$ satisfies $0 \leq \beta\lambda \leq 1/2$. From inequality (1.4), a semi-feasible price coefficient $\lambda = \mu$ in two periods market satisfies $0 \leq \beta\lambda < 1$. Therefore a feasible price coefficient can be semi-feasible in three periods market.

Table 1.1:

N	round-trip strategy	(semi-) feasible area
2	any	$0 \leq \beta\lambda < 1$
3	any	$0 \leq \beta\lambda < \frac{-1+\sqrt{5}}{2}$
any	simple	$0 \leq \beta\lambda \leq 1/2$

1.3.4 Why do speculative opportunities emerge?

This subsection gives an explanation by comparing Huberman and Stanzl (2004) why feedback traders generate a speculative opportunity. Huberman and Stanzl give a rationale of using linear pricing rule for a market maker; linear pricing rules assure no price manipulation without feedback traders. I briefly explain by an example that a nonlinear pricing rule brings a speculative incentive for the speculator, and then feedback traders generate a nonlinear effect for linear pricing rules from the speculator's point of view.

Consider eight periods model without feedback traders and normally-distributed noise traders. Participants are a speculator and a market maker. Then the speculator's capital gain $(p_N - p_1)x > 0$ implies the market maker's loss $(p_1 - p_N)x < 0$. Therefore the market maker seeks a pricing rule which brings no loss from trade. The result from Proposition 3 in Huberman and Stanzl (2004) implies that, when $U(q_n) = P(q_n)$, no price manipulation is the unique optimal strategy for the speculator if and only if $U(q_n) = \lambda q_n$ with $\lambda \geq 0$. Assume contrary that $U(q_n) = aq_n$ if $q_n \geq 0$ and $U(q_n) = bq_n$ if $q_n < 0$ with $a > b > 0$. Clearly this price function is not linear. Then the speculator can get a positive profit by a round-trip strategy $x_n = 1$ for $n = 1, 2, \cdots, 4$ and $x_n = -1$ for $n = 5, 6, \cdots, 8$. Figure 1.2 exhibits price paths from the round-trip strategy. A bold slope line with dots indicates the price path with the nonlinear pricing rule. Each dot indicates the price at each trading period. A dashed line indicates the price path with price function $U(q_n) = aq_n$ for $n = 5, 6, 7, 8$. Since $l_1 = |p_5 - p_4|$, $l_2 = |(p_7 - p_2) + (p_8 - p_1)|$ and $p_6 = p_3$, the speculator can get the positive payoff represented by $l_2 - l_1$, which implies the market maker's loss.

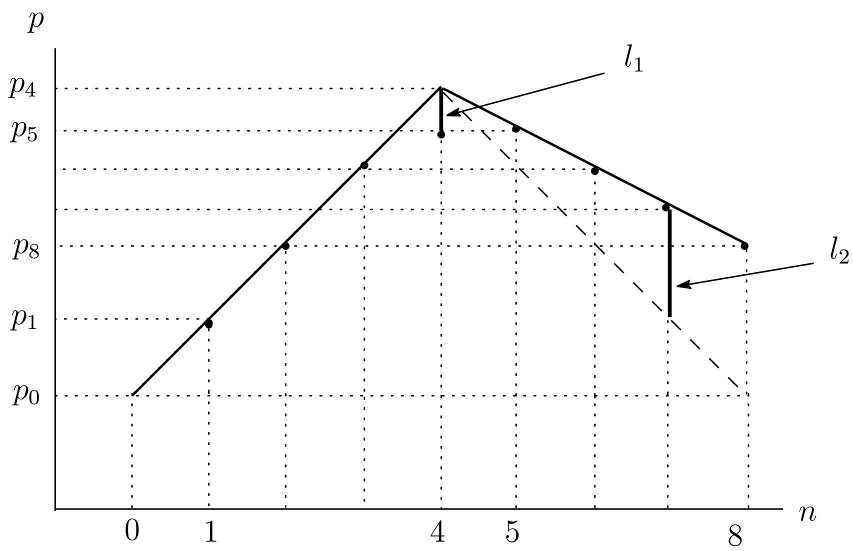

Figure 1.2:

Next I introduce feedback traders in the eight periods model with the assumption $U_n(q_n) = P_n(q_n) = \lambda q_n$ for all n. The goal is to see why feedback traders generate an speculative opportunity. Assume that $\beta = 3$ and $\lambda \in \{11/30, 1/4, 1/8\}$. Consider a simple strategy $x = 1$, which generates the demand function of feedback traders $\xi_n = (\beta\lambda)^{n-1} = (3\lambda)^{n-1}$ for $n \geq 2$. Figure 1.3 to 1.5 depict price paths in this example.

These figures tell us that feedback traders generate a *nonlinear* effect for the price function from the speculator's point of view. Indeed the price impact is λx when the speculator buys x but it looks for him as if $\lambda(-x) + \lambda\xi_8$ when he sells x. This is the source of speculative opportunity. In order to

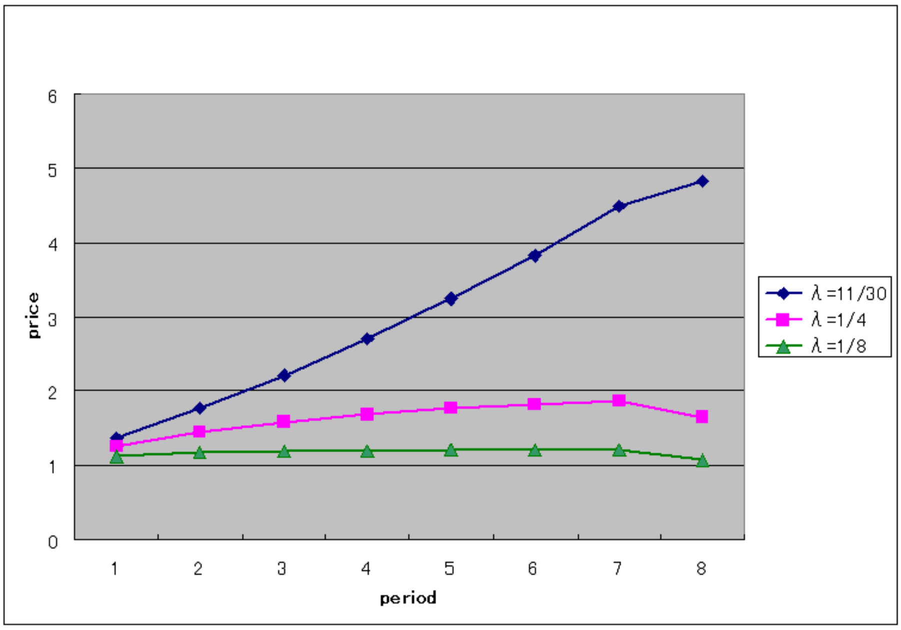

Figure 1.3: Top: $\lambda = 11/30$, middle: $\lambda = 1/4$, bottom: $\lambda = 1/8$. The speculator gets the payoff about 3.46 and 0.4 when $\lambda = 11/30$ and $\lambda = 1/4$ respectively. When $\lambda = 1/8$, the payoff is about -0.05.

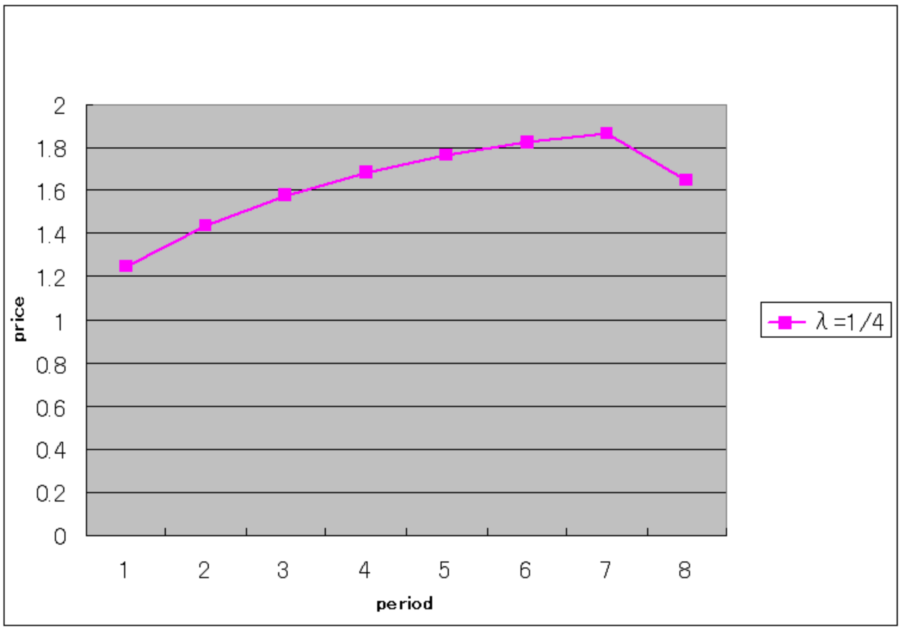

Figure 1.4: $\lambda = 1/4$.

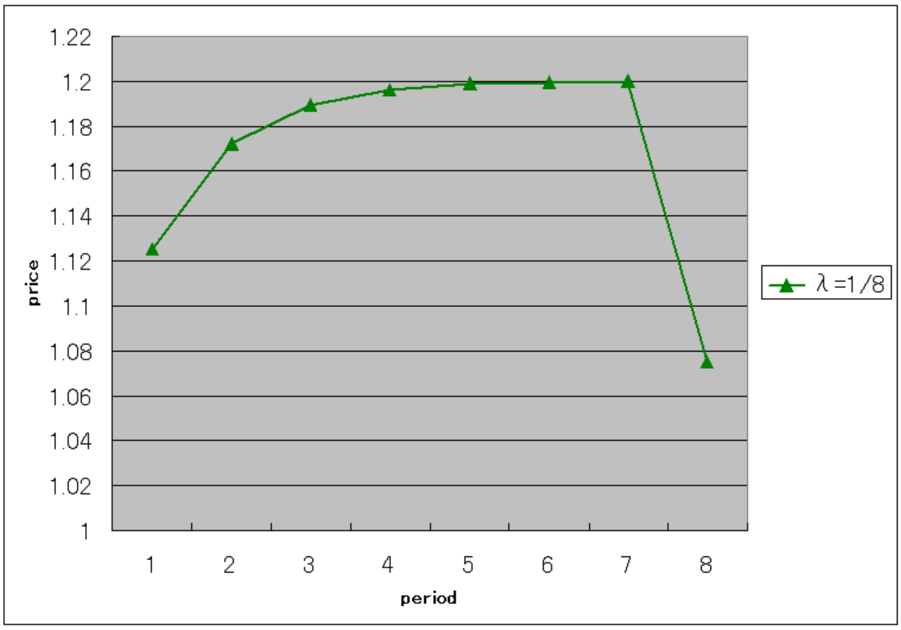

Figure 1.5: $\lambda = 1/8$.

make the speculations to be unprofitable, we must diminish the price impact from feedback traders. Therefore we require price coefficients to be small enough. When price coefficients are sufficiently small, the price impact from feedback traders becomes small in each trading period, so that no price manipulation is the unique optimum for the speculator because the situation is sufficiently close to Huberman and Stanzl' model. Theorem 1 or Proposition 1 proposes an answer how we should depress price coefficient(s) to achieve no price manipulation.

Finally, I point out that there is a big difference in market maker's incentive for preventing the speculation. If there is no feedback trader, a single market maker has incentive to set a feasible pricing rule against speculator's simple strategies because speculator's gain means market maker's loss. However, if there are feedback traders, a single risk-neutral market maker no longer sets any feasible pricing rule because he also can get a positive payoff from trading with feedback traders; he can choose a pair of non-feasible price coefficients which will make $\sum_{n=1}^{N} p_n q_n > 0$ by inducing simple strategies.[3] Therefore, we require market makers to be competitive to achieve no price

[3] Strictly speaking, there is no optimal solution for the speculator because he has no restriction on his position. We need an assumption to assure $\sum_{n=1}^{N} p_n q_n > 0$ such as boundary for order placement from a single participant in each period.

manipulation if there are feedback traders.

1.4 Application

I consider the following *three periods liquidation model.* The market is composed of period $n = 0, 1, 2, 3$. There is a stock whose liquidation value is v. The true value of v is not public until the beginning of period 3. Trade begins from $n = 1$. Let p_0 be the opening price. There are a risk-neutral speculator, risk-neutral market makers, feedback traders, and other noise traders in the market. Noise traders' aggregate order placement η_n in period n is a random variable with $\mathbf{E}[\eta_n] = 0$ for all n. I assume that the speculator privately knows v but other participants do not. In addition, the speculator exactly knows feedback traders' momentum β, and so it may be optimal for the speculator to implement a speculation which exploits feedback traders. The speculator's *speculative strategy* is a round-trip strategy denoted by $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 0$. Market makers offer a trading price in each period. We say that market makers *prevent speculations* if the speculator's ex-ante optimal strategy, if exists, is unique and it is $x_n = 0$ for all n. *Speculative bubbles prevention succeeds* if ex-ante expected price $\mathbf{E}[p_n]$ defined with ex-ante expectation $\mathbf{E}[x_n]$ satisfies $p_0 \le \mathbf{E}[p_n] \le v$ or $p_0 \ge \mathbf{E}[p_n] \ge v$, where x_n is an optimal strategy of the speculator for all n. (Note that optimal strategies may depend on a history of price paths.)

1.4.1 Exogenous speculative bubbles prevention

If someone wants to prevent speculative bubbles, price regulation may be the easiest way. Here I consider how we should regulate prices (i.e., pricing rules) for speculative bubbles prevention. I assume that market makers follow a pricing rule such that

$$p_n = p_{n-1} + (\lambda - \mu)q_{n-1} + \mu q_n \quad (n = 1, 2), \tag{1.8}$$

where $\lambda, \mu \in \mathbb{R}_+$.

First I consider the case that $v = p_0$, where p_0 is the commonly known opening price. Since market makers will set $p_3 = v$, the speculator plans a speculation between period 1 and 2 if it is profitable. The problem is how we regulate (λ, μ) in order to prevent speculations. Let $\boldsymbol{x} = (x_1, x_2)$ denote a round-trip strategy such that $x_1 = x \in \mathbb{R}$ and $x_2 = -x$. Then

$$\begin{aligned} p_1 &= p_0 + \mu(x_1 + \eta_1), \\ p_2 &= p_0 + \lambda(x_1 + \eta_1) + \mu(x_2 + \eta_2 + \beta\mu(x_1 + \eta_1)). \end{aligned}$$

The speculator wants to implement the speculation if and only if $\mathbf{E}[(p_2 - p_1)x] > 0$. From inequality (1.4) in the previous section, we requires price coefficients to satisfy that

$$\beta\mu^2 - 2\mu + \lambda \leq 0 \tag{1.9}$$

for $(\lambda, \mu) \in \mathbb{R}^2_+$. Otherwise, the speculator implements the speculation *even if he knows* $p_0 = v$.

Next I consider the case that $v \neq p_0$ with pricing rules being (1.8). In this case, we see that any pair of nonzero price coefficients cannot prevent speculations, but speculative bubbles prevention succeeds as long as inequality (1.9) holds for $(\lambda, \mu) \in \mathbb{R}^2_+$.

The speculator's maximization problem is

$$\max \mathbf{E}[-p_1 x_1 - p_2 x_2 - v x_3] \tag{1.10}$$

subject to $x_3 = -x_1 - x_2$. From (1.8), the first order condition of (1.10) is

$$\begin{pmatrix} 2\mu & \lambda + \beta\mu^2 \\ \lambda + \beta\mu^2 & 2\mu \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v - p_0 \\ v - p_0 \end{pmatrix}.$$

Therefore

$$x_1 = x_2 = \frac{v - p_0}{\beta\mu^2 + 2\mu + \lambda},$$

which is the optimal solution of (1.10) since the objective function is strictly concave. Then the expected prices are

$$p_1 = p_0 + \frac{\mu}{M}(v - p_0), \qquad p_2 = v - \frac{\mu}{M}(v - p_0),$$

where $M = \beta\mu^2 + 2\mu + \lambda$, and the expected payoff of the speculator is $(v - p_0)^2/M$. Since $p_2 - p_1 = (\beta\mu^2 + \lambda)(v - p_0)/M$, the expected price monotonically approaches to v. I summarize this argument as a corollary.

Corollary 1 *In the three periods liquidation model, if market prices are determined by (1.8) in period 1 and 2, then speculative bubbles prevention succeeds if and only if* $(\lambda, \mu) \in \mathbb{R}^2_+$ *satisfies* $\beta\mu^2 - 2\mu + \lambda \leq 0$.

1.4.2 Endogenous speculative bubbles prevention

This subsection considers whether market makers' optimal behavior can prevent speculations. I propose two main messages. One is that the principle of speculative bubbles prevention is essentially the same one as in the exogenous case; another is that the optimal behavior of the market makers succeeds in speculative bubbles prevention.

I assume that v and η_n are followed by Normal distributions: $v \sim \mathcal{N}(v_0, \Sigma_0)$ and $\eta_n \sim \mathcal{N}(0, \sigma_n^2)$, which are mutually and serially independent. Market makers are *competitive* and they know that the speculator privately knows v, whereas they only know its distribution, but they exactly know the true value of β as well as the speculator. Competitive market makers set market prices such that $\mathbf{E}[v|\{q_k\}_{k\leq n}] = p_n$. I assume common knowledge about the setting between the speculator and the market makers. Suppose that the market makers follow a pricing rule such that

$$p_n = p_{n-1} + \lambda_n q_n \quad (n = 1, 2) \tag{1.11}$$

with $p_0 = v_0$, where $\lambda_n \in \mathbb{R}_+$.

The situation is the same one as in the exogenous case except for strategic interaction between the informed speculator and the uninformed market makers. Therefore the result of Corollary 1 can apply to this case; since

$$\begin{aligned} p_1 &= p_0 + \lambda_1(x_1 + \eta_1), \\ p_2 &= p_1 + \lambda_2(x_2 + \eta_2 + \beta\lambda_1(x_1 + \eta_1)), \end{aligned}$$

then, it is necessary and sufficient for speculative bubbles prevention that

$$0 \leq \beta\lambda_1 \leq 1 \quad \text{or} \quad \lambda_2 = 0. \tag{1.12}$$

I show that pricing principle (1.12) is essentially achieved by the market makers' optimal pricing behavior. According to Kyle (1985), there exists the unique equilibrium in which market makers offer a price following from (1.11) in the case of no feedback traders. In Kyle's equilibrium, λ_1, λ_2 are uniquely determined by σ_n^2 and Σ_0. We see now that if the market makers know β, their optimal behavior discourages the speculator from implementing any simple strategy.

Let $\boldsymbol{x} = (x_1, x_2)$ denote a simple strategy such that $x_1 = x \in \mathbb{R}$ and $x_2 = -x$. Since feedback traders' order placement contains no information about v, market makers discount the effect from feedback traders and set prices optimally as follows:

$$
\begin{aligned}
p_1 &= p_0 + \lambda_1(x_1 + \eta_1), \\
p_2 &= p_1 + \lambda_2(x_2 + \eta_2 + \beta\lambda_1(x_1 + \eta_1) \underbrace{-\beta\lambda_1(x_1 + \eta_1)}_{discounting}).
\end{aligned}
$$

The speculator's expected payoff from the speculative strategy is $-x^2\lambda_2$. Thus the speculator gives up the speculation by simple strategies if the market makers know the true value of β.

Proposition 2 *In the three periods liquidation model, if market makers correctly estimate the order placement from feedback traders, then speculative bubbles prevention succeeds by competition among market makers.*

Remark: This proposition is a natural result from assumption $\mathbf{E}[v|\{q_k\}_{k\leq n}] = p_n$.

Proof. We only have to check that Kyle's equilibrium order placement strategy $\boldsymbol{x} = (x_1^*, x_2^*)$ do not make $\mathbf{E}[p_n]$ be over (or under) v. In Kyle's equilibrium, $\mathbf{E}[v] = v_0 = p_0$, and $x_n^* = \gamma_n(v - p_{n-1})$, where γ_n is uniquely determined by σ_n^2 and Σ_0. Taking ex-ante expectation about x_n^* yields $\mathbf{E}[x_1^*] = \mathbf{E}[x_2^*] = 0$. Since $q_1 = x_1^* + \eta_1$ and $q_2 = x_2^* + \eta_2 + \xi_2$, ex-ante expected prices are always $\mathbf{E}[p_n] = p_0$. ■

The optimal behavior of the market makers essentially follows the principle in (1.12). To see it, suppose that the market makers estimate the true value of the momentum parameter at $\widehat{\beta}$. Then $\mathbf{E}[(p_2 - p_1)x] \leq 0$ is equivalent that

$$
(\beta - \widehat{\beta})\lambda_1 \leq 1 \quad \text{or} \quad \lambda_2 = 0, \tag{1.13}
$$

which is essentially the same principle as in (1.12). Principle (1.13) holds for any λ_1 if $\widehat{\beta}$ can be arbitrarily close to β. Indeed, principle (1.13) holds in equilibrium because we assume that $\beta = \widehat{\beta}$.

Note that Kyle's equilibrium pricing rule itself does not assure speculative bubbles prevention. Suppose that market makers are unaware of feedback traders, then Kyle's equilibrium pricing rule may cause a speculative bubble. This is true when periods are arbitrary.

Corollary 2 *Suppose that market makers are unaware of feedback traders. Then some Kyle's equilibrium price coefficients $(\lambda_1, \cdots, \lambda_N)$ cannot prevent the speculator's simple strategies.*

Remark: Suppose that the market is composed of $N+1$ periods. It is trivial in the case that any of Kyle's equilibrium pricing coefficients $\{\lambda_1, \cdots, \lambda_N\}$ does not belong to the feasible area. The proof takes up the case.
Proof. See Appendix. ■

De Long et al (1990) analyze a three periods competitive market model without market makers and show that price bubbles due to feedback traders occur in equilibrium. Proposition 2 suggests that the choice of trading system or market structure is important for price bubbles prevention.

1.5 Conclusion

This chapter analyzes a trading model with market makers and proposes two new messages. One is a message about pricing rule of market makers. If we want to prevent speculative bubbles, we restrict pricing rules even if we focus on linear ones. Proposition 3 in Huberman and Stanzl (2004) says that, for any round-trip strategy, any linear pricing rule satisfying $\lambda \leq 2\mu$ is sufficient to achieve no price manipulation hence to prevent speculative bubbles if the noise is represented by a Normal distribution. On the other hand, I show that, if the noise is (or contains) a positive feedback, a heavy restriction on linear pricing rules is required for no price manipulation even if we consider simple strategies. In other words, the result of Huberman and Stanzl (2004) is not robust for noise structures.

The other message tells that speculative bubbles prevention by market makers can succeed. The result of Theorem 1, or Corollary 1 tells how we should set linear price functions in order to prevent the speculation. Proposition 2 shows that competitive market makers succeeds in speculative bubbles prevention in equilibrium if they are aware of feedback traders. These results shed new light on the relation between price bubbles emergence/prevention and trading structures. Empirical tests for these results remain as a feature work.

1.6 Appendix

1.6.1 Proof of Theorem 1

We prove the theorem for the case of simple strategy $x > 0$. (The argument is symmetric for the case $x < 0$ as long as p_0 is sufficiently large.) The simple strategy x generates feedback traders' demands as follows:

$$\xi_n = \beta\mu q_{n-1} + \beta(\lambda - \mu)q_{n-2}, \tag{1.14}$$

which is equal to q_n when $n \in \{2, \cdots, N-1\}$. Let $K = \beta\mu$ and $J = \beta(\lambda-\mu)$ for notational simplicity. Then, the equation (1.14) represents a second order linear homogeneous difference equation when $n \in \{2, \cdots, N-1\}$. Let $D = K^2 + 4J$. A theory of difference equation gives

$$q_n = \frac{x}{\sqrt{D}}\left\{\left(\frac{K+\sqrt{D}}{2}\right)^n - \left(\frac{K-\sqrt{D}}{2}\right)^n\right\}$$

when $D > 0$ for all $n \in \{1, \cdots, N-1\}$.[4]

Sufficiency part

First I show that the payoff increases monotonically with N. Define

$$\varphi_1 = \frac{K+\sqrt{D}}{2}, \quad \varphi_2 = \frac{K-\sqrt{D}}{2}$$

with $D = K^2 + 4J > 0$, and

$$\begin{aligned} f_n &:= \varphi_1^n - \varphi_2^n \\ &= \left(\frac{K+\sqrt{D}}{2}\right)^n - \left(\frac{K-\sqrt{D}}{2}\right)^n. \end{aligned}$$

I claim $f_n \geq 0$ for all $n \in \mathbb{Z}_+$ when $D > 0$. Since $|\varphi_1|^2 - |\varphi_2|^2 = K\sqrt{D}$, we can say $|\varphi_1| \geq |\varphi_2|$. This result and the fact $\varphi_1 > 0$ mean $f_n \geq 0$ no matter what the sign of φ_2. In particular, $f_n > 0$ for all $n \in \mathbb{Z}_{++}$ when $D > 0$ and $K \neq 0$. The payoff function is

$$\begin{aligned} x(p_N - p_1) &= x^2\left\{\frac{1}{\sqrt{D}}\left(\lambda\sum_{n=1}^{N-1} f_n + \mu f_N\right) - 2\mu\right\} \\ &= x^2\left\{\frac{1}{\sqrt{D}}\left(\lambda\sum_{n=1}^{N-1}(\varphi_1^n - \varphi_2^n) + \mu(\varphi_1^N - \varphi_2^N)\right) - 2\mu\right\}. \end{aligned} \tag{1.15}$$

Since $f_n > 0$, (1.15) increases monotonically with N.

[4] A reader who wishes to know more detail on difference equation is to refer Elaydi (2005), for example.

Next I derive the necessary and sufficient condition for feasible price coefficients when $D > 0$. We require $\lim_{N\to\infty} \pi(x; N) \leq 0$, which implies the requirement that $f_n \to 0$ as $n \to \infty$. If $\varphi_1 = 1$, then $\sum_{n=1}^{\infty} f_n$ does not converge since $1 > |\varphi_2|$. If $\varphi_1 > 1$, then $\varphi_2 > 1$ is required for convergence. However, $\{f_n\}_{n=1}^{\infty}$ does not converge since

$$\begin{aligned}\varphi_1^n - \varphi_2^n &= (\varphi_1 - \varphi_2)(\varphi_1^{n-1} + \varphi_1^{n-2}\varphi_2 + \cdots + \varphi_2^{n-1}) \\ &> n\sqrt{D}.\end{aligned}$$

Therefore, the sequence converges only if $\varphi_1 < 1$, which is equivalent that

$$\begin{aligned}&\frac{K + \sqrt{D}}{2} < 1 \\ &\Leftrightarrow \sqrt{D} < 2 - K \\ &\Leftrightarrow D < 4 - 4K + K^2 \\ &\Leftrightarrow \beta\lambda < 1.\end{aligned} \tag{1.16}$$

The second equation in (1.16) requires $\beta\mu < 2$ since $D > 0$. By the way, $\varphi_1 < 1$ is the necessary and sufficient condition for holding $\sum_{n=1}^{\infty} \varphi_1^n < \infty$, which implies absolute convergence of $\sum_{n=1}^{\infty} f_n$ since $|\varphi_1| > |\varphi_2|$. Thus the limit of (1.15) is

$$\begin{aligned}&x^2 \left\{ \frac{\lambda}{\sqrt{D}} \left(\frac{\varphi_1}{1 - \varphi_1} - \frac{\varphi_2}{1 - \varphi_2} \right) - 2\mu \right\} \\ &\Leftrightarrow x^2 \left\{ \frac{\lambda - 2\mu + 2\beta\lambda\mu}{1 - \beta\lambda} \right\}.\end{aligned}$$

The requirement $\lim_{N\to\infty} \pi(x; N) \leq 0$ implies the following when $D > 0$:

$$0 \leq \beta\mu < 2, \quad 0 \leq \beta\lambda < 1, \quad \frac{\lambda - 2\mu + 2\beta\lambda\mu}{1 - \beta\lambda} \leq 0. \tag{1.17}$$

The set of (λ, μ) satisfying (1.17) and $D > 0$ has the positive measure in $\mathbb{R}_+^2$ for all $\beta \in \mathbb{R}_{++}$: The measure of the finer shaded area in Figure 1.1 is

$$\begin{aligned}\int_0^{\frac{7-\sqrt{17}}{4\beta}} \frac{2\mu}{2\beta\mu + 1} - \frac{4\mu - \beta\mu^2}{4} d\mu &= \int_0^{\frac{7-\sqrt{17}}{4\beta}} \frac{1}{4}\beta\mu^2 - \mu + \frac{1}{\beta} - \frac{1}{\beta}\left(\frac{1}{2\beta\mu + 1}\right) d\mu \\ &= \frac{1}{192\beta^2}\left(115 - 5\sqrt{17} - 96 \log\left(\frac{9 - \sqrt{17}}{2}\right)\right) \\ &> \frac{1}{192\beta^2} \cdot \frac{88}{10} \, \left(= \frac{11}{240\beta^2}\right).\end{aligned} \tag{1.18}$$

Because we consider the case that $D > 0$, it is no harm to replace the condition $0 \leq \beta\mu < 2$ in (1.17) with $\beta\mu - 1 < \beta\lambda$ since $(7-\sqrt{17})/4\beta < 1/\beta < (3+\sqrt{17})/4\beta$. (See also Figure 1.1.) Therefore, the set of price coefficients defined by inequalities (1.3) and $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda > 0$ is sufficient to be the feasible set.

Necessity part

We consider the case that $D < 0$. Suppose that the speculator implements a simple strategy $x > 0$ and market makers choose a pair (λ, μ) which satisfies $D = K^2 + 4J < 0$. Then we obtain

$$q_n = \frac{x}{i\sqrt{D'}} \left\{ \left(\frac{K + i\sqrt{D'}}{2} \right)^n - \left(\frac{K - i\sqrt{D'}}{2} \right)^n \right\}, \tag{1.19}$$

where $D' = -D > 0$ and $i = \sqrt{-1}$.[5] Note that $D < 0$ implies $J < 0$. Let $J' = -J > 0$. In polar form, using Euler's Formula, $\varphi_1 = (K + i\sqrt{D'})/2 = \sqrt{J'}(\cos\theta + i\sin\theta) = \sqrt{J'}e^{i\theta}$ with some θ. Note that $\varphi_2 = \overline{\varphi_1} = (K - i\sqrt{D'})/2 = \sqrt{J'}e^{-i\theta}$. By using De Moivre's Theorem, equation (1.19) is equivalent that

$$q_n = \frac{2x}{\sqrt{D'}} \left(\sqrt{J'}\right)^n \sin(n\theta) \tag{1.20}$$

when the initial value is $(q_0, q_1) = (0, x)$. Since $\cos\theta = K/(2\sqrt{J'})$ and $\sin\theta = \sqrt{D'}/(2\sqrt{J'})$, the θ is determined by (λ, μ) and these values imply $\theta \in (0, \pi/2)$ when $\mu \neq 0$. The payoff of the speculator from the simple strategy is

$$\pi(x; N) = x^2 \left(\lambda \sum_{n=1}^{N-1} q_n + \mu q_N - 2\mu \right).$$

That $\lim_{N\to\infty} \pi(x; N) \leq 0$ requires to hold $\sqrt{J'} = \sqrt{\beta(\mu - \lambda)} < 1$, which means absolute convergence of $\sum_{n=1}^{\infty} q_n$. That $D < 0$ implies $\beta\lambda < 1$, so does $\beta\mu < 2$. We get from equation (1.19) that

$$\begin{aligned}\sum_{n=1}^{\infty} q_n &= \frac{x}{i\sqrt{D}} \left\{ \frac{\varphi_1}{1 - \varphi_1} - \frac{\varphi_2}{1 - \varphi_2} \right\} \\ &= \frac{x}{1 - \beta\lambda}.\end{aligned}$$

[5] See, for example, Elaydi (2005), pp75-76.

Thus,

$$\lim_{N\to\infty} \pi(x;N) = x^2\left(\frac{1-2\mu+2\beta\lambda\mu}{1-\beta\lambda}\right).$$

A necessary condition for feasible price coefficients when $D \neq 0$ is summarized as follows:

$$\beta\mu - 1 < \beta\lambda,\quad 0 \le \beta\lambda < 1,\quad \frac{\lambda-2\mu+2\beta\lambda\mu}{1-\beta\lambda} \le 0, \tag{1.21}$$

Next section proves that $0 \le \beta\mu < 2$ is required for feasible price coefficients. Since $D = 0$ is equivalent that $\beta\mu = 2(1 \pm \sqrt{1-\beta\lambda})$, we require $1-\beta\lambda > 0$. Thus a necessary condition for feasible price coefficients when $D = 0$ is characterized by $0 \le \beta\mu < 2$ and $0 \le \beta\lambda < 1$.

Payoff monotonicity

When $D = K^2 + 4J > 0$, the payoff from a simple strategy $x > 0$ is monotonically increasing with N, but this payoff monotonicity fails when $D < 0$ as we see in Figure 1.6. Here, I formally show the payoff monotonicity to fail necessarily for some price coefficients within $D < 0$.

Lemma 1 *Suppose that p_n is defined by (1.1) with Assumption 1. Pick a pair of price coefficients (λ, μ) arbitrarily which satisfies*

$$\beta\mu - 1 < \beta\lambda,\quad 0 \le \beta\lambda < 1,\ \textit{and}\ \ (\beta\mu)^2 - 4\beta\mu + 4\beta\lambda < 0. \tag{1.22}$$

Then, there are infinitely many N such that $p_N > \lim_{n\to\infty} p_n$.

Proof. Put $p^* = \lim_{n\to\infty} p_n$. We can check easily that $p^* = p_0 + x\lambda/(1-\beta\lambda)$, so $p^* > 0$ if $x > 0$. By using the equation (1.20), we get the following:

$$\begin{aligned}
& p_N > p^* \\
\Leftrightarrow\ & \frac{\mu}{\lambda} q_N > \sum_{n=N}^{\infty} q_n \\
\Leftrightarrow\ & \frac{\mu}{\lambda} q_N > \frac{2x}{\sqrt{D'}}\left(\sqrt{J'}\right)^N \frac{\sin(N\theta) - \sqrt{J'}\sin\bigl((N-1)\theta\bigr)}{1-\beta\lambda} \\
\Leftrightarrow\ & \{\mu(1-\beta\lambda) - \lambda\}\sin(N\theta) > -\lambda\sqrt{J'}\sin\bigl((N-1)\theta\bigr).
\end{aligned} \tag{1.23}$$

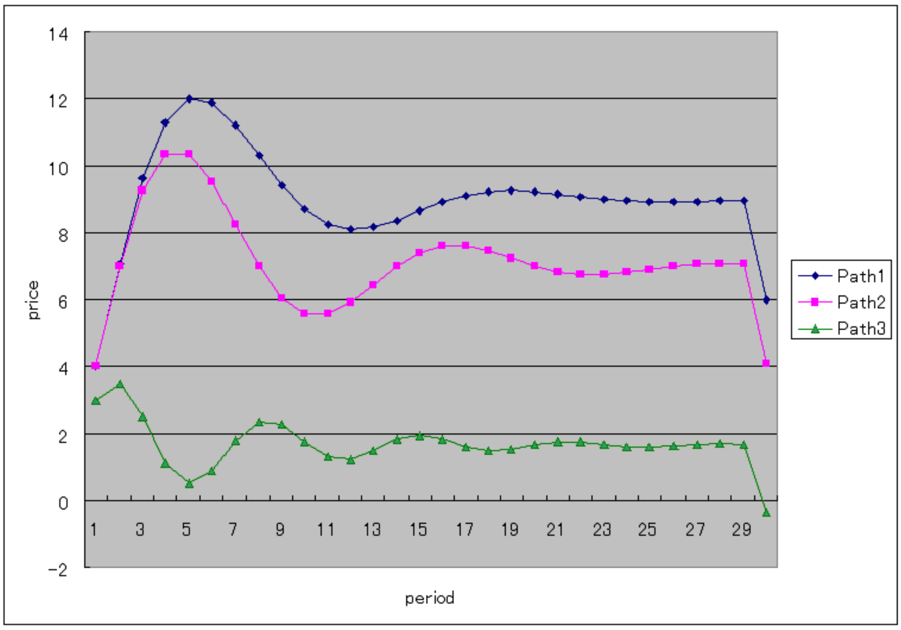

Figure 1.6: Common parameters:$(N, p_0, \beta) = (30, 1, 0.5)$. For (λ, μ), Path1 is $(1.6, 3)$, Path2 is $(1.5, 3)$, and Path3 is $(0.5, 2)$. The speculator implements the simple strategy $x = 1$. Some parameters within $D < 0$ give the positive payoff to the speculator, but some do not.

The RHS of the third inequality in (1.23) uses the fact of absolute convergence of $\sum_{n=1}^{N} q_n$ followed by (1.22).

$$\begin{aligned}\sum_{n=N}^{\infty} q_n &= \frac{\varphi_1^N - \varphi_2^N - \varphi_1^N\varphi_2 + \varphi_1\varphi_2^N}{(1-\varphi_1)(1-\varphi_2)} \\ &= \frac{2x}{\sqrt{D'}}\left(\sqrt{J'}\right)^N \frac{\sin(N\theta) - \sqrt{J'}\sin\big((N-1)\theta\big)}{1-\beta\lambda}\end{aligned}$$

by using Euler's Formula, De Moivre's Theorem, and the fact that $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$. Since $\theta \in (0, \pi/2)$, there are infinitely many N satisfying $p_N > p^*$ no matter what a pair we pick first. ■

Lemma 1 implies that the payoff monotonicity fails.

When $D = 0$, the payoff monotonicity holds. Corollary 2.24 in Elaydi (2005) tells us that $q_n = xn(K/2)^{n-1}$ for $n \in \{1, 2, \cdots, N-1\}$ when the speculator implements a simple strategy, which implies the payoff monotonicity. Since series $\sum_{n=1}^{\infty} xn(K/2)^{n-1}$ converges if and only if $K/2 < 1$. Therefore $0 \leq \beta\mu < 2$ is necessary for feasible price coefficients.

1.6.2 Relations between the bounded and the feasible area

Let R_a is the ratio of the assured feasible area defined by (1.17) and $D > 0$ to the bounded area defined by (1.5). The area of B is $7/(4\beta^2)$. From (1.18), we obtain

$$\frac{1}{192\beta^2}\cdot\frac{88}{10} < \frac{1}{192\beta^2}\left(115 - 5\sqrt{17} - 96\log\left(\frac{9-\sqrt{17}}{2}\right)\right) < \frac{1}{192\beta^2}\cdot\frac{89}{10}.$$

Dividing each side by $7/(4\beta^2)$ gives

$$0.0261 < R_a < 0.0264.$$

Similarly, let R_u is the ratio of the upper bound of feasible area defined by (1.21) to the bounded area defined by (1.5). Since $1 < (3+\sqrt{17})/4 < 2$, the upper bound area is

$$\begin{aligned}\int_0^{\frac{3+\sqrt{17}}{4\beta}} \frac{2\mu}{2\beta\mu+1}d\mu - \frac{(-1+\sqrt{17})^2}{32\beta^2} &= -\frac{1}{\beta}\left[\frac{1}{2\beta}\log(2\beta\mu+1) - \mu\right]_0^{\frac{3+\sqrt{17}}{4\beta}} - \frac{9-\sqrt{17}}{16\beta^2} \\ &= \frac{1}{16\beta^2}\left\{3 + 5\sqrt{17} - 8\log\left(\frac{5+\sqrt{17}}{2}\right)\right\}.\end{aligned}$$

Then we obtain

$$\frac{1}{16\beta^2}\cdot\frac{1147}{100} < \frac{1}{16\beta^2}\left\{3+5\sqrt{17}-8\log\left(\frac{5+\sqrt{17}}{2}\right)\right\} < \frac{1}{16\beta^2}\cdot\frac{1148}{100}.$$

Dividing each side by $7/(4\beta^2)$ gives

$$0.4096 < R_u < 0.4100.$$

1.6.3 Proof of Corollary 2

We prove the case of simple strategy $x > 0$ under the assumption that the true value of v is public in $N+1$ period. (The argument is symmetric for the case $x < 0$ as long as p_0 is sufficiently large.) When the pricing rule is defined by (1.7), the feasible set is

$$\left\{\lambda \in \mathbb{R}_+ \;\middle|\; 0 \le \beta\lambda < 1,\ \ \frac{2\beta\lambda - 1}{1-\beta\lambda} \le 0\right\}. \tag{1.24}$$

Assume that each of Kyle's equilibrium price coefficients $\{\lambda_1, \cdots, \lambda_N\}$ is not in (1.24), i.e., $\lambda_n > 1/(2\beta)$ for all n. Note that Kyle's equilibrium price coefficients satisfies $\lambda_n \le \lambda_{n-1}$ for all n. Then we have

$$\begin{aligned}\mathbf{E}[(p_N - p_1)x] &= x(\beta\lambda_2\lambda_1 + \cdots + \beta^{N-1}\lambda_N\cdots\lambda_1 - \lambda_N)\\ &> x\lambda_N\{\beta\lambda_N + \cdots + (\beta\lambda_N)^{N-1} - 1\}.\end{aligned}$$

Since $\lambda_N > 1/(2\beta)$, the speculator gets positive payoff if the period N is sufficiently large. In particular, if $\beta\lambda_N > 1$, the speculator gets a positive payoff for all markets with $N \ge 2$.

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Chapter 2

Market Control in Market Making Systems

2.1 Introduction

This chapter investigates a trading model in which market makers conduct pricing and dealing. The purpose of this chapter is to show that possibility of market intervention by a rational player not only discourages rational speculators to implement speculative strategies but enable market makers to price stocks almost freely. In equilibrium, neither the market intervention nor speculations occur, while market makers can act in the market without care for speculations.

The market intervention plainly means price control of the market by a third party, e.g., the government or the central bank. The article calls it *market control*. The market control succeeds if a rational player, say *controller*, can set both market prices and trading volumes to an arbitrary given target only by placing market orders without fluctuating prices and volumes.

I analyze an overlapping generation stock market model with an infinite discrete trading period. I explicitly assume three kinds of participants: an infinite number of two-periods-lived speculators, infinitely lived *feedback traders*, and an infinitely lived controller. Each speculator sequentially enters the market one by one. In each period, active speculators, feedback traders, and the controller simultaneously place a market order, then the market price is determined by a *pricing rule* — a mapping from an aggregate order to a nonnegative price — and each participant trades each amount of order at the price.

I consider a situation in which each speculator can get money through a speculation which exploits feedback traders. Feedback traders buy stocks today when they observed price gains yesterday. Price gains lead to their purchase of stock; their purchase of stock imply further price gains. This self-feeding behavior continuously raises market prices and may lead a price bubble. Speculators use this property of feedback traders and can gain a nonnegative profit through buying a unit of stock in the first period and selling it in the second.

The model assumes market prices to follow a pricing rule. Furthermore, I focus on *linear* pricing rule. A pricing rule is said to be *feasible* if it discourages the speculators from implementing any speculation, hence it prevents the speculative price bubble explained above. First I consider the case in which the controller does not exist and characterize the set of feasible linear pricing rules. We will find it very restrictive compared to the case of no feedback traders. Next I consider the case of control. In this case almost all linear pricing rules are feasible compared to the case of no feedback traders.

This permissive result owes to the linearity of trading mechanism. We can interpret the trading model I investigate as a market making system in which market makers set a linear pricing rule. Therefore the result of this chapter suggests an importance of market structure for speculative bubbles prevention.

2.1.1 Related literature

The model of this article is based on Ohashi (2008, Chapter 1 in this thesis), which analyzes speculative bubbles prevention in finite periods without control. New features are that (1) it is an infinite period model and (2) there is an infinite number of finitely lived speculators and their active periods are partially overlapping. The result changes drastically. The most different point is that some pricing rules which prevent a speculation in Ohashi (2008) never prevent the speculation in an infinite period OLG model without control. In particular, "Kyle type" pricing rule, i.e., $p_n = p_{n-1} + \lambda_n q_n$ (Kyle (1985)), never prevents the speculation. I explain why it does not work in Section 2.3.

A model of feedback trader follows from De Long, Shleifer, Summers, Waldmann (1990). De Long, et al shows by competitive market model without market makers that the feedback trader (they call it *positive feedback trader*) is a source of price bubbles. Recently, the issue of price bubbles and feedback traders is revisited (e.g., Shiller (2003, 2008)). While it is important to recognize that feedback trading have often caused price bubbles from historical and modeling point of view, my works (the present article and Ohashi (2008)) explain that we can prevent speculative bubbles due to feedback traders in market making systems.

The trading mechanism in this article can be interpreted that market makers set a linear pricing rule in advance and each market price is followed by the rule. A rationale for using linear pricing rules is exhibited in Huberman and Stanzl (2004).

This article is only concerned about prevention of speculative bubbles in an OLG model, hence no argument about efficiency appears, which is different from Tirole (1985).

2.2 The Model

A single stock market is composed of an infinite trading period $n = 0, 1, 2, \cdots$. There are three kinds of market participants: an infinite number of two-periods-lived (risk-neutral) speculators, a cluster of infinitely lived feedback traders, and a single infinitely lived market controller. Trading takes place in $n \geq 1$. Each speculator sequentially enters the market one by one: speculator n enters the market in period n and exits in $n + 1$. Let x_n^i denote the speculator n's market order in his ith period. The aggregate market order from active speculators is defined by $y_n := x_{n-1}^2 + x_n^1$ for $n \geq 2$. In each trading period $n \geq 2$, speculator $n - 1$ and n, feedback traders, and the controller simultaneously place a market order (quantities they want to trade), x_{n-1}^2, x_n^1, ξ_n, u_n respectively, where $x_{n-1}^2, x_n^1, \xi_n, u_n \in \mathbb{R}$. Then, a trading price $p_n \in \mathbb{R}_+$ is offered for the aggregate market order, $q_n := y_n + \xi_n + u_n$, and each participant trade at the price. A price is determined by a pricing rule such that

$$\begin{aligned} p_n &= p_{n-1} + (U_{n-1}(q_{n-1}) - P_{n-1}(q_{n-1})) + P_n(q_n) \\ &= p_0 + \sum_{k=1}^{n-1} U_k(q_k) + P_n(q_n), \end{aligned}$$

where p_0 is an opening price in the market. This pricing rule follows the one in Huberman and Stanzl (2004). *Price impact function* $P_n(q_n)$ captures immediate price reaction to a market order in period n. *Price update function* $U_n(q_n)$ captures only the permanent price impact from trade. A *price bubble* (or *speculative bubble*) is said to occur if price p_n departs from p_0.

Here I put assumptions on the model.

Assumption 1

A1. $x_n^i \in \{-1, 0, 1\}$ *and* $x_n^1 + x_n^2 = 0$.

A2. $\xi_n = \beta(p_{n-1} - p_{n-2})$ *with* $\beta \in \mathbb{R}_+$ *if* $n \geq 2$ *and* $\xi_1 \equiv 0$.

A3. $U_n(q_n) = \lambda q_n$ *and* $P_n(q_n) = \mu q_n$ *with* $\lambda, \mu \in \mathbb{R}_+$.

Assumption A1 means a consumption constraint; speculator's order placement is bounded and each speculator exits the market with null position. We will find it possible but substantially invariant for the result that we assume $x_n^i \in \{-x, 0, x\}$ for $x \in \mathbb{R}$. Assumption A2 is behavioral assumption for feedback traders. This behavioral assumption is the same one as in De Long, Shleifer, Summers, Waldmann (1990). Assumption A3 says both price impact and update functions are *linear* and *time independent*. We call $(\lambda, \mu) \in \mathbb{R}_+^2$ a pair of *price coefficients*. If both function $U_n(q_n)$ and $P_n(q_n)$ are linear for all n, we say that the pricing rule is *linear*. Assume further that all speculators can observe behavior of past speculators.

I define a game among speculators. Suppose that $(\lambda, \mu) \in \mathbb{R}_+^2$, $\beta \in \mathbb{R}_+$, and $p_0 \in \mathbb{R}_+$ are arbitrary given. Now assume that $u_n \equiv 0$ for all n. The set of pure strategies of speculator n is defined as follows:

$$X = X_n = \{\ (x_n^1, x_n^2) \mid x_n^i \in \{-1, 0, 1\},\ x_n^1 + x_n^2 = 0\}.$$

An element of X_n is denoted by $\boldsymbol{x}_n$. A mixed strategy of the speculator n is denoted by $\widetilde{\boldsymbol{x}}_n \in \Delta(X_n)$. A strategy is said to be *nonzero* if $\widetilde{\boldsymbol{x}}_n \neq \mathbf{0}$. From A1, the payoff of speculator n is described as $\pi_n(\widetilde{\boldsymbol{x}}_n, \widetilde{\boldsymbol{x}}_{n+1}; (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{n-1}))$. If all speculators take a pure strategy, then it is described by $(p_{n+1} - p_n)x_n^1$. Given a history $h_n := (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{n-1})$, a strategy $\boldsymbol{x}_n' \in X_n$ is *strictly dominated* if there is another strategy $\widetilde{\boldsymbol{x}}_n \in \Delta(X_n)$ such that

$$\pi_n(\widetilde{\boldsymbol{x}}_n, \boldsymbol{x}_{n+1}; h_n) > \pi_n(\boldsymbol{x}_n', \boldsymbol{x}_{n+1}; h_n)$$

for all $\boldsymbol{x}_{n+1} \in X_{n+1}$. Define X_n^t $(t = 0, 1, \cdots)$ recursively by

$$X_n^t = \Big\{ \boldsymbol{x}_n \in X_n^{t-1} \mid \ \nexists \widetilde{\boldsymbol{x}}_n \in \Delta(X_n^{t-1}),\ \text{s.t.}\ \forall \boldsymbol{x}_{n+1} \in X_{n+1}^{t-1},$$
$$\pi_n(\widetilde{\boldsymbol{x}}_n, \boldsymbol{x}_{n+1}; h_n) > \pi_n(\boldsymbol{x}_n, \boldsymbol{x}_{n+1}; h_n) \Big\},$$

where $\Delta(X_n^t)$ is the set of mixed strategies defined on X_n^t with $X_n^0 = X_n$ and $\Delta(X_n^0) = \Delta(X_n)$. A strategy $\boldsymbol{x}_n$ is *iteratively undominated* (under h_n) if $\boldsymbol{x}_n \in \bigcap_{t=1}^{\infty} X_n^t$. A pair of price coefficients (λ, μ) is said to be *feasible* if it makes the optimal strategy of any speculator be $(0, 0)$. The associated pricing rule is said to be *feasible pricing rule*.

2.3 The Results without Control

The purpose in this section is to characterize the class of (λ, μ) which leads speculator's optimum to be $(0,0)$ in the case that $u \equiv 0$. I show the optimum to be obtained by uniquely iteratively undominated strategy.

Suppose that all speculators implement $\boldsymbol{x} = (1, -1)$. Then $y_n = 0$ for any $n \geq 2$ and $y_1 = 1$. Therefore, from A2 and A3, sequence $\{y_n\}_{n=1}^{\infty}$ generates sequence $\{q_n\}_{n=1}^{\infty}$ such that

$$\begin{aligned} q_n = \xi_n &:= \beta(p_{n-1} - p_{n-2}) \\ &= \beta((\lambda - \mu)q_{n-2} + \mu q_{n-1}). \end{aligned}$$

for $n \geq 2$ and $q_1 = y_1 = 1$. Then the payoff of the speculator n is

$$\begin{aligned} p_{n+1} - p_n &= \mu q_{n+1} + (\lambda - \mu)q_n \\ &= \frac{q_{n+2}}{\beta}. \end{aligned}$$

Appendix shows that $q_n > 0$ for all n if $(\beta\mu)^2 - 4\beta\mu + 4\beta\lambda \geq 0$. In this case, the market price will grow monotonically, so that a price bubble occurs. Therefore it is necessary for preventing the bubble that $D < 0$. Indeed, there exist infinitely many n such that $q_n < 0$ if $D < 0$ (see Appendix). I characterize the set of (λ, μ) within $D < 0$ which makes the speculator's optimum be $(0,0)$.

Proposition 1 *If $\boldsymbol{x}_n = (0,0)$ is the optimum for some speculator n, then (λ, μ) satisfies $\beta\mu^2 - 4\mu + 4\lambda < 0$. If (λ, μ) further satisfies $\beta\mu^2 - 2\mu + 2\lambda < 0$, then $\boldsymbol{x}_n = (0,0)$ is the unique iteratively undominated strategy hence it is the unique optimum for any speculator n.*

Proof. The first statement is proved in Appendix. We show the second statement here. Suppose that speculator 1 implements a strategy $\boldsymbol{x}_1 = (1, -1)$. Then the payoff of speculator 2 from $\boldsymbol{x}_2$ is

$$\begin{aligned} \pi_2(\boldsymbol{x}_2, \boldsymbol{x}_3) &= \mu(\xi_3 + y_3) + (\lambda - \mu)(\xi_2 + y_2) \\ &= \mu\xi_3 + (\lambda - \mu)\xi_2 + \mu y_3 + (\lambda - \mu)y_2, \end{aligned}$$

where $\xi_2 = \beta\mu$ and $\xi_3 = (\beta\mu)^2 + (y_2 - 1)\beta\mu + \beta\lambda$. Suppose that $\mu\xi_3 + (\lambda - \mu)\xi_2 < 0$. It is equivalent that $\beta\mu^2 + (x_2^1 - 3)\mu + 2\lambda < 0$, which is maximized at $x_2^1 = 1$. Thus

$$\beta\mu^2 - 2\mu + 2\lambda < 0 \tag{2.1}$$

is sufficient for being $\mu\xi_3 + (\lambda - \mu)\xi_2 < 0$. When inequality (2.1) is satisfied, the action $x_2^1 = 1$ gives $\mu y_3 + (\lambda - \mu)y_2 = -\mu + \mu x_3^1$, so that the speculator 2 gets a negative payoff no matter what x_3^1 is. Therefore if (λ, μ) satisfies inequality (2.1), strategy $\boldsymbol{x}_2 = (1, -1)$ is strictly dominated under the history $\boldsymbol{x}_1 = (1, -1)$. When speculator 2 does not play $\boldsymbol{x}_2 = (1, -1)$, the payoff of the speculator 1 implementing $\boldsymbol{x}_1 = (1, -1)$ is $\mu(\beta\mu + x_2^1 - 1) + (\lambda - \mu)$ with $x_2^1 \in \{-1, 0\}$, which is maximized at $x_2^1 = 0$. Thus $\beta\mu^2 - 2\mu + \lambda < 0$ is sufficient for the speculator 1 to discourage $\boldsymbol{x}_1 = (1, -1)$, which is automatically satisfied when inequality (2.1) is satisfied. The argument is symmetric if we first take up the case that $-\boldsymbol{x}_1 = (-1, 1)$ as long as p_0 is sufficiently large. Thus, if a pair (λ, μ) satisfies $\beta\mu^2 - 2\mu + 2\lambda < 0$, $\boldsymbol{x}_n = (0, 0)$ is the unique iteratively undominated strategy for all speculators. ∎

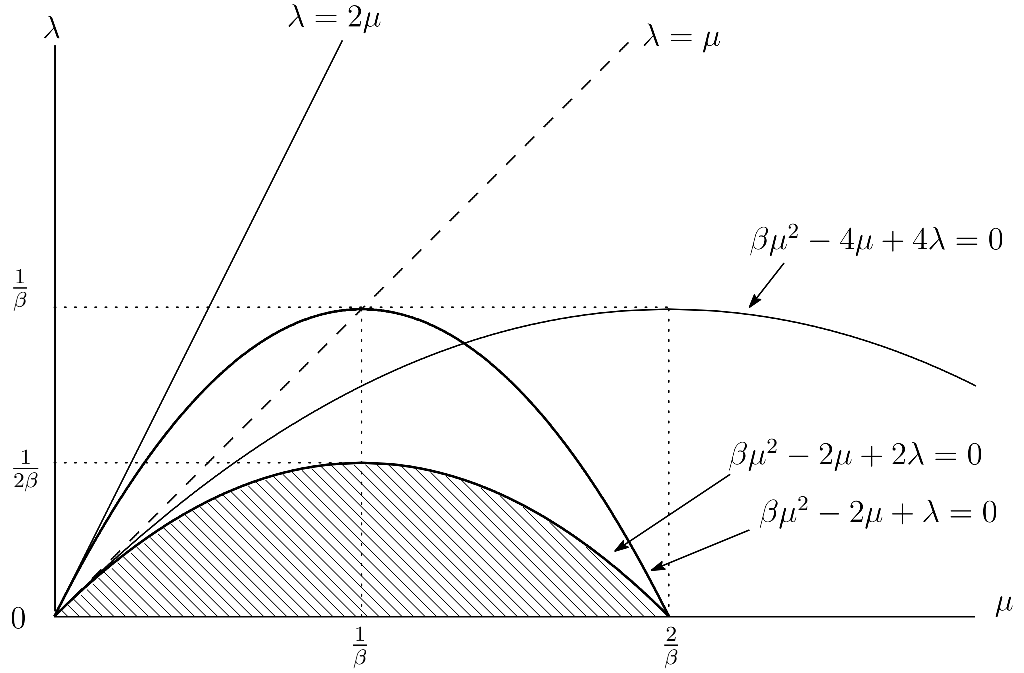

Figure 2.1: The shaded area is the feasible set of price coefficients.

The shaded area in Figure 2.1 indicates the "feasible area" — the area whose element (λ, μ) is feasible, i.e., (λ, μ) satisfies inequality (2.1) in Proposition 1 within $(\lambda, \mu) \in \mathbb{R}_+^2$. If (λ, μ) is in the area, the speculator 1 will not trade. No speculator wants to be the first participant in trade. Therefore, under the game among speculators, the speculative bubbles never occur in equilibrium. When $\beta = 0$, the feasible pair (λ, μ) necessarily and sufficiently satisfy $\lambda \leq \mu$ within $(\lambda, \mu) \in \mathbb{R}^2_+$.

The proof of Proposition 1 implies that the area defined by $\beta\mu^2 - 2\mu + \lambda < 0$ is the necessary and sufficient for the feasible area if we consider a single two-periods-lived speculator model without overlapping generation. Once generation is overlapped, the next entering speculator's purchase cancels out the negative impact to price caused by current speculator's sale. This "support buying" opportunity is favorable to speculators, so that the feasible area becomes restrictive compared to the case of a single speculator. Note also that, without overlapping generation and $\beta = 0$, it is necessary and sufficient for the feasible (λ, μ) to satisfy $\lambda \leq 2\mu$ within $(\lambda, \mu) \in \mathbb{R}^2_+$.

For speculative bubbles prevention, the price update coefficient λ, which weights on past trading volumes, should be relatively smaller than the price impact coefficient μ, which weights on a current trading volume. A reason of this restriction lies in the overlapping generation. Thanks to the support buying behavior, the speculators can cancel out negative current price impact, while feedback traders monotonically push prices up. This "pushing up" effect is proportional to the price impact *from past trading volumes*: the price update coefficient λ. That is why the price update coefficient should be relatively smaller than the price impact coefficient in order to prevent speculations.

2.4 State Control and Speculative Bubbles Prevention

The previous section shows speculative bubbles prevention to be possible but it restricts linear pricing rules compared to the case of a single two-periods-lived speculator model without overlapping generation. This section allows the controller to participate in trade.

We have already seen that the gain from trade of the speculators stems from that feedback traders push prices up. However, one may occur that the gain disappears if the controller succeeds in pulling back the price rising. I show that this conjecture is right when market prices follow a linear pricing rules which satisfies $\lambda < 2\mu$ with $(\lambda, \mu) \in \mathbb{R}^2_+$. Note that the area is almost identical to the feasible area in the model of a single two-periods-lived speculator without overlapping generation. Therefore, we can say that the market control succeeds without any substantial restriction for linear pricing rules.

2.4.1 The model and the results with control

The controller places market order u_n in period n. Sequence $\{u_n\}_{n=k}^{\infty}$ is called a *control*, which means the controller first intervenes in period k. We require that the control $\{u_n\}_{n=k}^{\infty}$ gets (p_n, q_n) to converge to (p_{k-1}, q_{k-1}), i.e., the market intervention must revert the market price and the trading volume to the origin. Here is the assumptions on the model.

Assumption 2

A4. There is common knowledge among the speculators and the controller about λ, μ, β, and X.

Assumption A4 is assumed for tractability. For simplicity, we identify with $p_0 = 0$. I define a game among the speculators and the controller. The strategy of the controller is to chose entering period k and construct sequence $\{u_n\}_{n=k}^{\infty}$ such that $u_{n+1} : (q_1, p_1, \cdots, q_n, p_n) \mapsto u \in \mathbb{R}$. The payoff of the controller is normalized to 1 if $\{u_n\}_{n=k}^{\infty}$ makes $(p_n, q_n) \to (p_{k-1}, q_{k-1})$ for some k, otherwise 0. The strategy and the payoff of speculators are the same one as in the previous section except for participation of the controller.

Suppose that each speculator implements the strategy $\boldsymbol{x} = (1, -1)$. The equations of a market price and a market order are described as follows:

$$\begin{aligned} p_{n+1} &= p_n + (\lambda - \mu)q_n + \mu q_{n+1} \\ q_{n+1} &= \beta\mu q_n + \beta(\lambda - \mu)q_{n-1} + u_{n+1}, \end{aligned}$$

which is equivalent that

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ q_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} p_n \\ q_n \\ q_{n-1} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} u_{n+1} \tag{2.2}$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & \lambda - \mu + \beta\mu^2 & \beta(\lambda - \mu)\mu \\ 0 & \beta\mu & \beta(\lambda - \mu) \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \mu \\ 1 \\ 0 \end{pmatrix}.$$

Let $\boldsymbol{z}_n := (p_n, q_n, q_{n-1})^\top$ be called a *state* in period n. Then system (2.2) is written by

$$\boldsymbol{z}_{n+1} = A\boldsymbol{z}_n + B\boldsymbol{z}_n, \tag{2.3}$$

where $\boldsymbol{u}_n := u_{n+1}$.

Definition 1 *System (2.3) is said to be* ***controllable*** *if for any* $k \in \mathbb{Z}_+$*, any initial state* $\boldsymbol{z}_{k-1}$*, and any given final state* $\boldsymbol{z}^*$*, there exists a finite number* $N > k-1$ *and a control* $\{\boldsymbol{u}_n\}$ $(k-1 < n \leq N)$*, such that* $\boldsymbol{z}_N = \boldsymbol{z}^*$*.*

If system (2.2) is controllable, the controller can set a market state to any given final state within finite periods only by placing the market orders $\{\boldsymbol{u}_n\}$. We say that *the market control succeeds* if and only if system (2.2) is controllable. Thanks to discrete control theory,[1] we know that system (2.2) is controllable if and only if matrix $W = [B,\ AB,\ A^2B]$ has full row rank. Then

$$W = \begin{pmatrix} \mu & \mu(\beta\mu)+\lambda & \mu(\beta\mu)^2-(\mu-2\lambda)\beta\mu+\lambda \\ 1 & \beta\mu & (\beta\mu)^2-\beta\mu+\beta\lambda \\ 0 & 1 & \beta\mu \end{pmatrix},$$

a calculation shows that $|W| = \lambda$. I summarize this result as a proposition.

Proposition 2 *The market control succeeds if and only if* $\lambda > 0$*.*

Recall that this market control in itself is required not to destabilize the market; the control $\{u_n\}_{n=k}^{\infty}$ must satisfy $(p_n, q_n) \to (p_{k-1}, q_{k-1})$. I will constitute such a sequence. Consider system (2.3) with the control defined by $\boldsymbol{u}_n = -S\boldsymbol{z}_n$, where $S = (\sigma_1, \sigma_2, \sigma_3)$ is a real (1×3) matrix. Then,

$$\begin{aligned} \boldsymbol{z}_{n+1} &= A\boldsymbol{z}_n + B\boldsymbol{u}_n \\ &= (A - BS)\boldsymbol{z}_n. \end{aligned}$$

System (2.3) is *stabilizable* if one can find a control $\boldsymbol{u}_n = -S\boldsymbol{z}_n$ with some matrix S and it achieves $\lim_{n\to\infty} \boldsymbol{z}_n = \boldsymbol{z}_{k-1}$. We say that *the market stabilization succeeds* if and only if system (2.3) is stabilizable. The following useful proposition is known.

Proposition 3 *(Elaydi (2005), Theorem 10.19.) Let* $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ *be an arbitrary set of complex numbers such that* $\overline{\Phi} = \{\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3\} = \Phi$*. Then the system (2.3) is controllable if and only if there exists a matrix* S *such that the eigenvalues of* $A - BS$ *are the set* Φ*.*

Applying this proposition to our system (2.2), we obtain the following proposition.

Proposition 4 *The market stabilization succeeds when* $\lambda > 0$*.*

[1] See, for example, Elaydi (2005), p433.

Proof. Without loss of generality, we identify with $\boldsymbol{z}_{k-1} = 0$. Let $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$ and $\delta_1 = -(\varphi_1 + \varphi_2 + \varphi_3)$, $\delta_2 = \varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_1$, and $\delta_3 = -\varphi_1\varphi_2\varphi_3$. The characteristic polynomial of $A - BS$ is

$$|A - BS - \varphi I| = \begin{vmatrix} 1-\sigma_1-\varphi & \lambda-\mu+\beta\mu^2-\mu\sigma_2 & \beta(\lambda-\mu)\mu-\mu\sigma_3 \\ -\sigma_1 & \beta\mu-\sigma_2-\varphi & \beta(\lambda-\mu)-\sigma_3 \\ 0 & 1 & -\varphi \end{vmatrix} = 0,$$

which is equivalent that

$$\varphi^3+(\mu\sigma_1+\sigma_2-\beta\mu-1)\varphi^2+((\lambda-\mu)\sigma_1-\sigma_2+\sigma_3-\beta(\lambda-\mu)+\beta\mu)\varphi-\sigma_3+\beta(\lambda-\mu) = 0.$$

Comparing the coefficients with the roots, we obtain

$$\begin{aligned} \mu\sigma_1 + \sigma_2 - \beta\mu - 1 &= \delta_1, \\ (\lambda-\mu)\sigma_1 - \sigma_2 + \sigma_3 - \beta(\lambda-\mu) + \beta\mu &= \delta_2, \\ -\sigma_3 + \beta(\lambda-\mu) &= \delta_3. \end{aligned}$$

It gives

$$\begin{aligned} S &= (\sigma_1, \sigma_2, \sigma_3) \\ &= \left(\frac{1+\delta_1+\delta_2+\delta_3}{\lambda},\ \frac{-\mu(1+\delta_1+\delta_2+\delta_3)}{\lambda} + \delta_1 + \beta\mu,\ -\delta_3 + \beta(\lambda-\mu) \right). \end{aligned}$$

By letting $|\varphi_i| < 1$ for each $i = 1, 2, 3$, a state $(p_n, q_n, q_{n-1})^\top$ converges to $(0, 0, 0)$ when $n \to \infty$. ■

The controller who participate in trade from period k places the order $u_k = 0$ and $u_{n+1} = -\sigma_1 p_n - \sigma_2 q_n - \sigma_3 q_{n-1}$ for $n \geq k$. Note that the controller enters the market only if the speculator trades; otherwise the controller never enters the market, i.e., no intervention.

I get the next theorem from Proposition 2 and 4.

Theorem 1 *If (λ, μ) is in the following set (2.4), then $\boldsymbol{x}_n = (0, 0)$ is the unique iteratively undominated strategy hence it is the unique optimum for any speculator n. Conversely, almost all feasible price coefficients must be in (2.4).*

$$\{(\lambda, \mu) \mid (\lambda, \mu) \in \mathbb{R}^2_+,\ 0 < \lambda < 2\mu\}. \tag{2.4}$$

Proof. Suppose that (λ, μ) is in (2.4). Consider the controller's strategy such that $u_1 = 0$ and, if the controller observe $q_1 \neq 0$, $u_n = -\sigma_1 p_{n-1} -$

$\sigma_2 q_{n-1} - \sigma_3 q_{n-2}$ for $n \geq 2$, where $(\sigma_1, \sigma_2, \sigma_3)$ is determined in Proposition 4, otherwise $u_n = 0$ for $n \geq 2$. Let $\Phi = \{1/3, 1/3, 1/3\}$, then

$$u_{n+1} = -\frac{1}{\lambda}\frac{8}{27}p_n + \frac{\mu}{\lambda}\left(\frac{8}{27} - \beta\lambda\right) q_n - \left(\frac{1}{27} + \beta(\lambda - \mu)\right) q_{n-1}.$$

Put $\boldsymbol{x}^* = (1, -1)$. Suppose that speculator 1 implements $\boldsymbol{x}_1 = \boldsymbol{x}^*$ but other speculators implement $\boldsymbol{x}_n \in X_n$. Then $u_2 = -\beta\mu$ and so $q_2 = -1 + x_2^1$, which leads to

$$\begin{aligned} p_2 &= \lambda - \mu + \mu x_2^1 \\ \xi_3 &= \lambda - 2\mu + \mu x_2^1 \\ u_3 &= -\frac{1}{3} - \xi_3 \\ p_3 &= \mu\left(x_3^1 - \frac{1}{3}\right) + (\lambda - \mu) x_2^1. \end{aligned}$$

Therefore $\pi_2(\boldsymbol{x}_2, \boldsymbol{x}_3, \cdots | \boldsymbol{x}^*) = \mu x_3^1 - \lambda - 2\mu/3 + (\lambda - 2\mu)x_2^1$ if $x_2^1 \neq 0$. Since $\pi_2(\boldsymbol{x}^*, \boldsymbol{x}_3, \cdots | \boldsymbol{x}^*) = \mu(x_3^1 - 4/3)$, $\boldsymbol{x}_2 = \boldsymbol{x}^*$ is strictly dominated by $(0, 0)$ for speculator 2. Then $\pi_1(\boldsymbol{x}^*, \boldsymbol{x}_2, \cdots) = \lambda - 2\mu + \mu x_2^1$ with $x_2^1 \in \{-1, 0\}$, so $\boldsymbol{x}_1 = \boldsymbol{x}^*$ is strictly dominated by $(0, 0)$ for speculator 1. The argument is symmetric when $\boldsymbol{x}_1 = -\boldsymbol{x}^*$. Thus $\boldsymbol{x}_n = (0, 0)$ is the unique iteratively undominated strategy for all speculators, so speculative bubbles do not occur when (λ, μ) is in (2.4).

All price coefficients (λ, μ) satisfying $\lambda > 2\mu$ are never feasible since speculator 1 always implement $\boldsymbol{x}_1 = (1, -1)$ in that case. If (λ, μ) satisfies $\lambda = 2\mu > 0$, no speculator can get a positive payoff by implementing $\boldsymbol{x}^*$ or $-\boldsymbol{x}^*$, to say nothing of the case $(\lambda, \mu) = (0, 0)$. Therefore almost all feasible price coefficients belong to (2.4). ■

In the previous section, we have seen that feasible price coefficients in a single two-periods-lived speculator model without control necessarily and sufficiently satisfy $0 \leq \lambda \leq 2\mu$ when $\beta = 0$. This model is very primitive one in the sense that it only assumes a single two-periods-lived speculator without feedback traders. Theorem 1 states that, if the market control succeeds, almost all feasible price coefficients in the primitive model are also feasible in our model containing infinitely many OLG two-lived speculators and infinitely lived feedback traders.

We can see Theorem 1 as a permissive result on speculative bubbles prevention. The result stems from the fact the market state can be described by a linear system, in particular, the linear price functions are used. Thanks to Huberman and Stanzl (2004), we have a rationale of using linear pricing functions in market making systems.

Huberman and Stanzl (2004) analyze a model of market making system without feedback traders. In their model, there is a single N-periods-lived speculator and his strategy is assumed that $(x_1, \cdots, x_N)$ with $\sum_{n=1}^{N} x_n = 0$. They show that if linear price functions satisfy $0 \leq \lambda \leq 2\mu$, the strategy $x_n = 0$ for all n is optimal for the speculator. They also show that, under some assumptions, it is necessary for x_n being the unique optimum of the speculator that the time-independent price update function $U(q_n)$ is to be linear. Therefore, any rational market maker does not set price functions such that $0 \leq 2\mu < \lambda$.

We can easily check this fact by considering the situation in which market participants are only a speculator and a market maker. If the market maker sets price coefficients $2\mu < \lambda$, the speculator can gain from trade by implementing the strategy $\boldsymbol{x} = (1, -1)$, which means a loss of the market maker. Therefore any rational market maker whose pricing rule follows (time-independent) linear price functions chooses (λ, μ) such that $0 \leq \lambda \leq 2\mu$, where $U(q_n) = \lambda q_n$ and $P(q_n) = \mu q_n$. Thus, if the market control succeeds, the market making system succeeds in speculative bubbles prevention without any substantial restriction on market maker's linear price functions.

2.5 Conclusion

This chapter proposes a simple trading model which describes rational speculators can gain from trade by exploiting irrational feedback traders with or without market control. The main statement is almost all pricing rules prove to be feasible in the market control. In particular, the control considerably improves the area of feasible price functions. The result shows the market making system to have very good property for speculative bubbles prevention because of its linearity.

I have simply assumed the control to be implemented by a third party. Whether the control can endogenously be implemented by market makers' optimal behavior remains unknown, so it is a future work.

2.6 Appendix

2.6.1 Proof of Proposition 1

Suppose that all speculators implement $\boldsymbol{x} = (1, -1)$. (The argument is symmetric for the case $-\boldsymbol{x}$ if p_0 is sufficiently large.) Then the market order is written by

$$q_n = \beta\mu q_{n-1} + \beta(\lambda - \mu)q_{n-2} \tag{2.5}$$

for all $n \geq 2$. Let $K = \beta\mu$ and $J = \beta(\lambda - \mu)$ for notational simplicity. Then, the equation (2.5) represents a second order linear homogeneous difference equation.[2] Put $D = K^2 + 4J$ and assume first that $D > 0$. By using a theory of difference equation, we obtain

$$q_n = \frac{x}{\sqrt{D}}\left\{\left(\frac{K+\sqrt{D}}{2}\right)^n - \left(\frac{K-\sqrt{D}}{2}\right)^n\right\}.$$

Define

$$\varphi_1 = \frac{K+\sqrt{D}}{2}, \quad \varphi_2 = \frac{K-\sqrt{D}}{2}$$

and $f_n := \varphi_1^n - \varphi_2^n$. Since $|\varphi_1|^2 - |\varphi_2|^2 = K\sqrt{D} > 0$, we can say $|\varphi_1| \geq |\varphi_2|$. This result and the fact $\varphi_1 > 0$ mean $f_n \geq 0$. Therefore $q_n > 0$ for all $n \geq 1$ when $K \neq 0$. Next we assume that $D < 0$. Then we obtain

$$q_n = \frac{x}{i\sqrt{D'}}\left\{\left(\frac{K+i\sqrt{D'}}{2}\right)^n - \left(\frac{K-i\sqrt{D'}}{2}\right)^n\right\}, \tag{2.6}$$

[2] A reader who wishes to know more detail on difference equation is to refer Elaydi (2005), for example.

where $D' = -D > 0$ and $i = \sqrt{-1}$.[3] Note that $D < 0$ implies $J < 0$, and we put $J' = -J > 0$. In polar form, using Euler's Formula, we obtain $\varphi_1 = (K + i\sqrt{D'})/2 = \sqrt{J'}(\cos\theta + i\sin\theta) = \sqrt{J'}e^{i\theta}$ with some θ. Note that $\varphi_2 = \overline{\varphi}_1 = (K - i\sqrt{D'})/2 = \sqrt{J'}e^{-i\theta}$. By using De Moivre's Theorem, equation (2.6) is equivalent that

$$q_n = \frac{2x}{\sqrt{D'}}\left(\sqrt{J'}\right)^n \sin(n\theta).$$

Therefore $q_n < 0$ for infinitely many n. Finally we assume $D = 0$. Corollary 2.24 in Elaydi (2005) tells us that $q_n = xn(K/2)^{n-1}$ for $n \geq 1$, which is always positive as long as $K \neq 0$.

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Part II

Implementation Theory

Chapter 3

Implementation in Ex Post Equilibrium

3.1 Introduction

The purpose of this chapter is to investigate the full implementation of a *social choice set* in ex post equilibrium under conditions of incomplete information and general interdependent values. The theory of implementation has been investigated under several environments, such as implementation in Nash equilibrium in a complete information environment (e.g., Maskin(1999)), and implementation in Bayesian equilibrium in an incomplete information (e.g., Mookherjee and Reichelstein (1990), Jackson(1991)). The typical solution concept of implementation in incomplete information environments is a Bayesian equilibrium. We should note, however, that the theory of Bayesian implementation, or more generally, the implementation problem under incomplete information, has assumed explicitly (or implicitly) that a planer (i.e., mechanism designer) knows the belief distribution of agents and he can use this information to design the mechanism which implements a social choice function (or social choice set). This assumption may be sometimes unrealistic.

In this article I consider the implementation problem without the assumption that a planner has full knowledge about prior distribution of types of agents on their belief systems. The solution concept I use in this article is *ex post equilibrium*, which can be seen as a Bayesian equilibrium, subject to a "no regret" condition — a formal definition of ex post equilibrium will be given in Section 3.2. This solution concept is stronger than that proposed by a Bayesian equilibrium, but the planner who knows nothing about the belief distribution of types of agents may implement a social choice function if the planner uses ex post equilibrium as the solution concept.

The main results of this article provide the necessary and almost sufficient condition for implementation of social choice set in ex post equilibrium, which we will refer to as *ex post implementation.* The literature of implementation has shown that it is necessary to satisfy an *incentive compatibility* condition in order for a social choice function to be implemented. This result is well known as the *revelation principle.* It is also well known, however, that the revelation principle itself cannot guarantee the full implementability of a social choice function. I show that the *ex post incentive compatibility* and the *ex post selective elimination* condition are necessary for ex post implementation of a social choice set. Moreover, both conditions are sufficient to implement a social choice set if the environment allows agents to transfer private goods. Ex post incentive compatibility is the version of incentive compatibility when we use ex post equilibrium as a solution concept, which requires truth-telling profile is an ex post equilibrium. Similarly the ex post selective elimination condition is the analogous version of *selective elimination* defined in Mookherjee and Reichelstein (1990).

3.1.1 Related literature

My approach to the ex post implementation follows Mookherjee and Reichelstein (1990) and the results of this article are analogous to their results. Mookherjee and Reichelstein introduce the idea of *augmented revelation* and show that (Bayesian) incentive compatibility and selective elimination (SE) conditions are necessarily held when a social choice *correspondence* is *Bayesian implementable.*[1] They also show that the converse is true when one considers the implementation of a social choice *function* and the environment is *economic.* The difference from their results is to show a mechanism which ex post *fully* implements social choice *set* rather than social choice *correspondence.*[2] I also show by example that the selective elimination condition

[1] They refer that their result of necessary condition can be expanded to the case of implementation of a social choice set.

[2] A social choice set X is a collection of social choice function, while a social choice correspondence is a collection of desired outcomes in each state. Implementing the former is more demanding than the latter. If social choice set can be implemented, it is said to succeed in full implementation. Mookherjee and Reichelstein (1990) write whether the constructions employed in their sufficiency results (i.e., their mechanism implements social choice correspondence) can be extended to obtain full implementation remains to be seen (p475).

neither imply the selective elimination nor *Maskin monotonicity*, which is a necessary and almost sufficient condition for Nash implementation in complete information environments (Maskin (1999)).

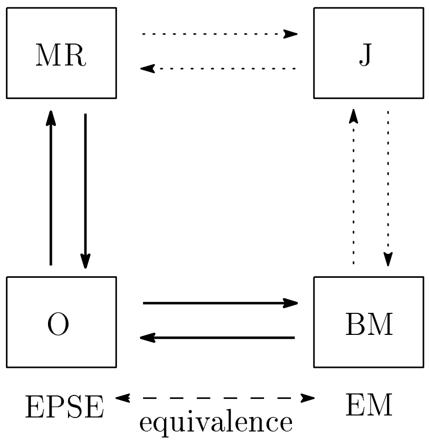

Figure 3.1: This article "O" provides the ex post selective elimination condition (EPSE) and shows that it is equivalent to the ex post monotonicity condition (EM) provided by Bergemann and Morris ("BM")(2008). My results extend the work of Mookherjee and Reichelstein ("MR")(1990), while the results of BM extend the work of Jackson ("J")(1991) for example. The bold line means the investigations are done in the present article.

Bergemann and Morris (2008) show that *ex post monotonicity* is a necessary and almost sufficient condition for ex post implementation of social choice set. I show that ex post monotonicity is equivalent to the ex post selective elimination condition. (See also Figure 3.1.) Therefore any result proved by using ex post monotonicity can be gotten by the ex post selective elimination condition. The main difference from their results is that I construct a mechanism which ex post implements a social choice set even if there are *two agents*. Bergemann and Morris assume the case of more than two agents and show that ex post monotonicity and ex post incentive compatibility is sufficient for ex post implementation in the economic environment. Since the environment I assume in the present article is included in Bergemann and Morris' economic environment, my result can be interpreted as an example in which ex post implementation succeeds by two agents. Whereas their constructions of the mechanisms which succeed in ex post implementation depend on the assumption of more than two agents, my result implies that there is no technical difficulty to expand their result to two agents case if we assume the economic environment because ex post monotonicity is equivalent to ex post selective elimination. Only the conceptual difference exists in constructing mechanisms; the idea of augmented revelation enables us to succeed in ex post implementation by two agents.

This chapter is organized as follows. Section 3.2 introduces the model and definitions. Section 3.3 provides a necessary condition for ex post implementation. Section 3.4 provides a sufficient condition for ex post implementation when private transfers are permitted. In this section we discusses the relationship between the ex post selective elimination condition and (1) ex post monotonicity of Bergemann and Morris (2008) and (2) the (original) selective elimination condition provided by Mookherjee and Reichelstein (1990).

3.2 The Model

I first describe the general environment $< A, N, \Theta >$ that will be taken into account. A is the set of alternatives or outcomes, $N = \{1, \cdots, n\}$ is the finite set of agents, and Θ is the set of possible states of the world, with a typical state denote as $\theta \in \Theta$. I describe each agent as $i \in N$ and assume that $n \geq 2$. Θ_i is the set of payoff relevant types for agent $i \in N$, and describe the typical type of i by $\theta_i \in \Theta_i$. I assume that each Θ_i is a finite set. The set of states Θ is defined by $\Theta := \Theta_1 \times \cdots \times \Theta_n$. The preference of each agent is represented by von Neumann-Morgenstern utility function $u_i : A \times \Theta \to \mathbb{R}$.

Mechanism $\Gamma = ((M_i)_{i=1}^n, g)$ is defined in the environment, which is composed of message space M_i of each i and outcome function $g : M_1 \times \cdots \times M_n \to A$. I denote a profile of message by $m = (m_i, m_{-i}) = (m_1, \cdots, m_n) \in M = M_1 \times \cdots \times M_n$. Given mechanism $\Gamma = ((M_i)_{i=1}^n, g)$, I define *game* $G = (\Gamma, (u_i)_{i=1}^n)$ with the environment.[3] Given an arbitrary game G, I assume that common knowledge among agents about the structure of G. Let $\alpha_i : \Theta_i \to M_i$ denote a *pure strategy* for $i \in N$ in G. I describe a typical strategy profile by $\alpha = (\alpha_i, \alpha_{-i}) = (\alpha_1, \cdots, \alpha_n)$.

Definition 1 *Given game $G = (\Gamma, (u_i)_{i=1}^n)$, pure strategy profile α is an* ***ex post equilibrium*** *in G if for all $i \in N$, for all $\theta \in \Theta$ and for all $m_i \in M_i$,*

$$u_i(g(\alpha(\theta)), \theta) \geq u_i(g(m_i, \alpha_{-i}(\theta_{-i})), \theta).$$

[3] Note that the term "game" is different from "game form", which is often used as the same meaning of mechanism. The reason why I use the term "game" is that I will compare two games: one of which is game $G = (\Gamma, (u_i)_{i=1}^n)$, the other is the game on which I assume common prior about type distributions. See Section 3.4.4.

For notational convenience, I describe $g \circ \alpha(\theta) := g(\alpha(\theta))$.

Social choice function x is a mapping such that $x : \Theta \to A$. *Social choice set (SCS)* X is a collection of social choice functions.

Definition 2 *Mechanism* $\Gamma = ((M_i)_{i=1}^n, g)$ ***ex post implements*** *SCS* X *by* $G = (\Gamma, (u_i)_{i=1}^n)$ *if the following both statements are satisfied:*

1. *For every pure ex post equilibrium* α *in* G*, it is true that* $g \circ \alpha = x$ *for some* $x \in X$*.*

2. *For any* $x \in X$*, there exists an pure ex post equilibrium* α *in* G *such that* $g \circ \alpha = x$*.*

Then X *is* ***ex post implementable****.*

Throughout this article, we only consider the *pure* strategy equilibrium.

Mookherjee and Reichelstein (1990) provide the mechanism whose message space is defined by union of the payoff relevant type space and an arbitrary set.

Definition 3 *An* ***augmented revelation mechanism*** *is a mechanism such that for all* $i \in N$ *:*

$$M_i = \Theta_i \cup T_i,$$

where T_i *is an arbitrary set.*

Mookherjee and Reichelstein (1990) introduce the *augmented revelation principle*, which states that if social choice set X is (Bayesian) implementable via some arbitrary mechanism, then it is also (Bayesian) implementable via augmented revelation mechanism. This statement is applicable to our solution concept even though their implementation concept is *weaker* than mine.[4]

Proposition 1 *If SCS* X *is ex post implementable, then* X *can be ex post implemented by an augmented revelation mechanism, in which truthful reporting is an ex post equilibrium.*

[4]According to the definition of implementation in Mookherjee and Reichelstein (1990), X is ex post implementable if there exists an pure ex post equilibrium α in G such that $g \circ \alpha \in X$ and for every *pure* ex post equilibrium α in G, it is true that $g \circ \alpha \in X$. Therefore, their implementation concept is weaker than mine because they do not require *every* social choice function x to be implemented.

Proof. Let α be an ex post equilibrium in $G = ((M_i)_i^n, g), \{u_i\}_{i=1}^n)$. Define $M_i^\alpha := \Theta_i \cup T_i$ for all i such that

$$T_i = \{m_i \in M_i : m_i \neq \alpha_i(\theta_i), \quad \forall \theta_i\}$$

We define outcome function $g^\alpha : M^\alpha \to A$ as follows:

$$g^\alpha(m) = \begin{cases} g(\alpha(\theta')) & \text{if } m = \theta' \in \Theta \\ g(m) & \text{otherwise.} \end{cases}$$

Then truth-telling profile is an ex post equilibrium in game $G^\alpha := ((M_i^\alpha)_{i=1}^n, g^\alpha), \{u_i\}_{i=1}^n)$. By assumption, mechanism $\Gamma^\alpha := (M_i^\alpha)_{i=1}^n, g^\alpha)$ implements some $x \in X$ in truth-telling ex post equilibrium in G^α. Mechanism Γ^α does not generate any ex post equilibrium α' in G^α such that $g^\alpha(\alpha'(\theta)) \notin X$. To see it, we consider mapping $\phi_i : M_i^\alpha \to M_i$ such that

$$\phi_i(m_i) = \begin{cases} \alpha_i(\theta_i') & \text{if } m_i = \theta_i' \in \Theta_i \\ m_i & \text{if } m_i \in T_i\ . \end{cases}$$

Let α_i^* be a strategy in G such that $\alpha_i^* = \phi_i \circ \alpha_i'$. If $\alpha'(\theta) = \theta' \in \Theta$, then $\phi(\alpha'(\theta)) = \alpha(\theta')$, where $\phi = (\phi_1, \cdots, \phi_n)$. Otherwise $\phi(\alpha'(\theta)) = m$ with $\alpha_i'(\theta_i) = m_i \neq \alpha_i(\theta_i)$ for some i. Thus $g \circ \alpha^*(\theta) = g^\alpha \circ \alpha'(\theta)$ for all $\theta \in \Theta$. Since α' is an ex post equilibrium in G^α, we have for any $i \in N$ and $\theta_i \in \Theta_i$:

$$\begin{aligned} u_i(g(\alpha^*(\theta)), \theta) &= u_i(g^\alpha(\alpha'(\theta)), \theta) \\ &\geq u_i(g^\alpha(m_i, \alpha_{-i}'(\theta_{-i})), \theta) \\ &= u_i(g(\phi_i(m_i), \alpha_{-i}^*(\theta_{-i})), \theta) \quad \forall m_i \in M_i^\alpha. \end{aligned}$$

Since ϕ_i is onto mapping to M_i, α^* is an ex post equilibrium in G. Applying this statement to every $x \in X$ completes the proof. ■

3.3 Necessary Condition for Ex Post Implementation

Definition 4 *Direct revelation mechanism $\Gamma_x^d = ((\Theta_i)_{i=1}^n, x)$ is said to be* ***ex post incentive compatible*** *(EPIC) if for all $i \in N$, for all $\theta \in \Theta$ and for all $\theta_i' \in \Theta_i$,*

$$u_i(x(\theta), \theta) \geq u_i(x(\theta_i', \theta_{-i}), \theta).$$

Let $G_x^d = (\Gamma_x^d, (u_i)_{i=1}^n)$ be a game with mechanism Γ_x^d.

Definition 5 *Ex post equilibrium α in G_x^d can be **ex post selectively eliminated**, if there exists $i^* \in N$ and social choice function $y : \Theta_{-i^*} \to A$ such that:*

$$\exists \theta \in \Theta, \quad u_{i^*}(y(\alpha_{-i^*}(\theta_{-i^*})), \theta) > u_{i^*}(x(\alpha(\theta)), \theta) \tag{3.1}$$

and

$$\forall \theta \in \Theta, \quad u_{i^*}(x(\theta), \theta) \geq u_{i^*}(y(\theta_{-i^*}), \theta). \tag{3.2}$$

Agent i^* in inequities (3.1) and (3.2) is labeled as *whistle-blower* at profile α in G_x^d.

Definition 6 *Direct revelation mechanism Γ_x^d satisfies the **ex post selective elimination** condition (**EPSE**) relative to X if $x \in X$ and if any ex post equilibrium α in G_x^d which satisfies $x \circ \alpha \notin X$ can be ex post selectively eliminated.*

*Furthermore, X **satisfies EPSE** if every $x \in X$ and the associated direct revelation mechanism satisfies EPSE relative to X.*

Proposition 2 *If SCS X is ex post implementable, then X satisfies EPSE.*

Proof. Since X is ex post implementable, and by the Proposition 1, we only focus on the augmented revelation mechanism which ex post implements X without loss of generality. I denote the game with this augmented revelation mechanism by $\widehat{G} = (\Gamma, (u_i)_{i=1}^n)$. Let $g|_\Theta$ be an outcome function of Γ whose domain is restricted to Θ. Take an arbitrary $x \in X$ and let $x = g|_\Theta$. From Proposition 1, truthful type reporting profile is an ex post equilibrium in $\widehat{G}$, so it also an ex post equilibrium in $G_x^d = (\Gamma_x^d, \{u_i\}_{i=1}^n)$.

Suppose that there exists ex post equilibrium α in G_x^d such that $x \circ \alpha \notin X$. Since Γ ex post implements X, we must have

$$\neg \left[\, \forall i \in N, \forall \theta \in \Theta, \forall m_i \in M_i, \quad u_i(g(\alpha(\theta)), \theta) \geq u_i(g(m_i, \alpha_{-i}(\theta_{-i})), \theta) \,\right],$$

which is equivalent that

$$\exists i \in N, \exists \theta \in \Theta, \exists \widehat{m}_i \in M_i, \quad u_i(g(\alpha(\theta)), \theta) < u_i(g(\widehat{m}_i, \alpha_{-i}(\theta_{-i})), \theta).$$

Since truth-telling profile is an ex post equilibrium in $\widehat{G}$, we must have

$$u_i(g(\theta'), \theta') \geq u_i(g(\widehat{m}_i, \theta'_{-i}), \theta').$$

for all $\theta' \in \Theta$. Define $y(\theta_{-i}) := g(\widehat{m}_i, \theta_{-i})$, so we have function $y : \Theta_{-i} \to A$. ■

3.4 Sufficient Condition for Ex Post Implementation

3.4.1 The economic environment

I consider the environment in which each agent can trade private goods, e.g., money. I define the following economic environment: the set of socially feasible alternatives is $A = \bar{A} \times \mathbb{R}^n$. Any social choice function x is denoted by $x = (\bar{x}, (t_i)_{i=1}^n)$, where $\bar{x} : \Theta \to \bar{A}$ is said to be a *public decision rule* and $t_i : \Theta \to \mathbb{R}$ to be a *transfer function* for i. I denote $t = (t_i)_{i=1}^n \in \mathbb{R}^n$. In addition I assume for simplicity that each agent has a quasi-linear utility function: $u_i((a,t),\theta) := v_i(a,\theta) + t_i$ for every $a \in \bar{A}$, $t \in \mathbb{R}^n$ and $\theta \in \Theta$, and also assume that $v_i(a,\theta)$ is a bounded function for every $i \in N$.[5]

In our economic environment, the outcome function of a mechanism can be denoted by $g(m) = (g_{\bar{x}}(m), (g_{t_i}(m))_{i=1}^n)$. I also use the notation such that $g(m) = ((g_i(m))_{i=1}^n)$ where $g_i(m) = (g_{\bar{x}}(m), g_{t_i}(m))$ for every $m \in M$ and $i \in N$.

3.4.2 Results

Our goal is to show that X is ex post implementable if X satisfies both EPIC and EPSE under the economic environment. Before prove it, I construct an augmented revelation mechanism which ex post implement X.

Let N^x be a set of whistle-blowers in G_x^d with $x \in X$. We denote $N^* = \cup_{x\in X} N^x$. Fix $x \in X$ arbitrarily. Let $\Phi[G_x^d]$ denote a set of ex post equilibrium in game G_x^d such that $x \circ \alpha \notin X$. Without loss of generality, we can say that G_x^d has the amount of K_x ex post equilibria which satisfy $x \circ \alpha \notin X$, i.e., $\#|\Phi[G_x^d]| = K_x$, where $\#|Y|$ represents the cardinality of set Y.

We inductively conduct an augmentation of message space of agents.

Step 1. For every $i \in N$ we set

$$M_i^{(1,x)} = (\Theta_i \times X) \cup N.$$

If $K_x = 0$, the augmentation ends for the game G_x^d. Otherwise, go to Step 2.

Step 2. Let α^k be an k-th ex post equilibrium in $\Phi[G_x^d]$. Then there exists a whistle-blower $i \in N^x$ at α^k in G_x^d. Pick one whistle-blower i^* at α^k in G_x^d

[5] Our result remains valid when we do not assume the quasi-linear utility. Along with our proof of Proposition 3, one can easily extend our result to the non-quasi-linear cases provided the private good is desirable.

arbitrarily and set
$$M_{i^*}^{(k+1,x)} = M_{i^*}^{(k,x)} \cup \{w^{(k,x)}\}$$
for the chosen whistle-blower i^*, where $w^{(k,x)}$ is an arbitrary message which satisfies $M_{i^*}^{(k,x)} \cap \{w^{(k,x)}\} = \emptyset$ and $\cap_{s \leq k}\{w^{(s,x)}\} = \emptyset$. For all i such that $i \neq i^*$, we set $M_i^{(k+1,x)} = M_i^{(k,x)}$. We conduct this inductive augmentation of message space from $k = 1$ to $k = K_x$. If it is finished, put $M_i^x := M_i^{(K_x+1,x)}$ for each $i \in N$.[6]

We apply this augmentation for each $x \in X$ through Step 1 and Step 2 keeping that
$$\bigcap_{x \in X} \bigcup_{k \in \{k : \alpha^k \in \Phi[G_x^d]\}} \{w^{(k,x)}\} = \emptyset.$$
Then, put $M_i := \cup_{x \in X} M_i^x$ for all $i \in N$, $\mathcal{W}^x := \{w^{(1,x)}, \cdots, w^{(K_x,x)}\}$, and $\mathcal{W} = \cup_{x \in X} \mathcal{W}^x$.

Next we define the outcome function in our message space. For notational ease, if $m \in \Theta \times X^n$, then we write $m_i = \alpha_i(\theta_i) = (\alpha_i^1(\theta_i), \alpha_i^2(\theta_i)) = (m_i^1, m_i^2)$ for every $i \in N$. Let $a = (a_i, (b)^{n-1})$ denote an element of $\mathbb{R}^n$ if $a_j = b$ for all $j \neq i$.

Rule 1. If $m \in \Theta \times X^n$ and $m_i^1 \in \Theta_i$, $m_i^2 = x$ for all i, then
$$g(m) = g(m_1^1, \cdots, m_n^1) = x(\widehat{\theta}).$$
Rule 2. If $m_i = w^{(k,x)}$ for some (k, x) and $m_{-i} \in \Theta_{-i} \times X^{n-1}$ such that $m_{-i}^1 = \widehat{\theta}_{-i} \in \Theta_{-i}$, $m_j^2 = x$ for all $j \in N \backslash \{i\}$, then
$$g(m) = y_i^{(\alpha^k,x)}(\widehat{\theta}_{-i}).$$
Rule 3. If $m_i \in N$ and $m_{-i} \in \Theta_{-i} \times X^{n-1}$, then
$$g(m) = (g_i(m), g_{-i}(m)) = \left(\left(z, -\frac{\delta}{2}\right), \left(z, \frac{\delta}{2(n-1)}\right)^{n-1}\right), \tag{3.3}$$
where $z \in \bar{A}$ is an arbitrary alternative and we take $\delta > 0$ such that for all $x \in X$, for all $\alpha \in \Phi_i[G_x^d]$ and for all $y_i^{(\alpha,x)}$ with $m_i = w^{(k,x)}$, where $\alpha = \alpha^k \in \Phi_i[G_x^d]$,
$$\forall j \in N,\ \forall \theta, \theta',\ \ u_j(y_i^{(\alpha,x)}(\theta'_{-i}), \theta) - v_j(z, \theta) < \delta. \tag{3.4}$$

[6]Under the game G_x^d with $\Phi[G_x^d] = K_x$, the augmentation of message space from k to $k+1$ implies that there exists only one agent $i \in N^x$ whose message space changes to $\#|M_i^{(k+1,x)}| = \#|M_i^{(k,x)}| + 1$, whereas the message spaces of the others are $M_j^{(k+1,x)} = M_j^{(k,x)}$ $(j \neq i)$. The message spaces especially to the agents $j \in N \backslash N^x$ are $M_j^{(k,x)} = M_j^{(1,x)}$ for all $k = 1, \cdots, K_x$.

Rule 4. If there are at least two agents i, j $(i \neq j)$ such that $m_i \notin \Theta_i \times X$ and $m_j \notin \Theta_j \times X$, then

$$g(m) = (g_h(m), g_{-h}(m)) = \left((z, \delta), \left(z, \frac{-\delta}{n-1} \right)^{n-1} \right), \tag{3.5}$$

where z and δ in (3.5) are the same ones in (3.3). In equation (3.5), h is the agent who matches up $h = \sum_{l=1}^{n} r_l(m_l) \mod n$, where $r_l : M_l \to \{0, 1, \cdots, n\}$ with its value

$$r_l(m_l) = \begin{cases} k & \text{if } m_l = k \in \{1, \cdots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

The agent h is called the *MG winner* (which means modulo game winner). By modulo operation, we identify $n = 0$ in determining who the MG winner is. We can check easily that a MG winner is always determined uniquely for any profile $m \in M$.[7]

Rule 5. If $m \in \Theta \times X^n$ and $m_i^2 \neq m_j^2$ for some i, j $(i \neq j)$, then

$$g_i(m) = (z, -\delta)$$

for all $i \in N$.

Consider game $G = (\Gamma, (u_i)_{i=1}^n)$ with $\Gamma = ((M_i)_{i=1}^n, g)$ where M_i $(i = 1, \cdots, n)$ and g follow the augmented revelation mechanism defined above.

Proposition 3 *In the economic environment, social choice set X is ex post implementable if X satisfies both EPIC and EPSE.*

Proof.

Our proof is complete if we show the following four claims;

(1) *There is no pure ex post equilibrium profile α in which any agent reports a type and function message and which reports the same function but their type reporting brings an undesired outcome.*

[7] Let i,j be MG winners at message profile m. Denote $r_l = r_l(m_l)$. Then there exists $K \in \mathbb{N}$ such that $\sum_{l=1}^{n} r_l = nK + i = nK + j$. Thus $i = j$.

Assume that α is an ex post equilibrium such that $\alpha_i^2 = x \in X$ for all $i \in N$ and $\alpha \in \Theta \times X^n$ with $g \circ \alpha \neq x$. Then Rule 1 applies, but there is the whistle-blower j against α since EPSE relative to X. Hence he wants to change his message $\alpha_j(\theta_j) \in \Theta_j \times X$ into $\alpha_j(\theta_j) = w^{(k,x)}$ with $\alpha^k = \alpha$ in some (θ_j, θ_{-j}). Therefore the assumption that α is an ex post equilibrium is wrong. $\|$

(2) *There is no pure ex post equilibrium profile α in which any agents reports a type and function message but some agents i, j $(i \neq j)$ report different functions.*

Suppose that α is a strategy profile such that $\alpha \in \Theta \times X^n$ but $\alpha_i^2(\theta_i) \neq \alpha_j^2(\theta_j)$ for some i, j $(i \neq j)$. Then Rule 5 applies, so the outcome for each agent is $(z, -\delta)$. Any agent $l \in N$ wants to change his message from $\alpha_l(\theta_l) \in \Theta_l \times X$ into $\alpha_l(\theta_l) \in N$ in state $(\theta_i, \theta_j, \theta_{-i,j})$ because he can get higher payoff by Rule 3. Thus this strategy profile cannot be an ex post equilibrium. $\|$

(3) *There is no pure ex post equilibrium profile α in which agent $i \in N$ reports non-type message and a message profile of the others is in $\Theta_{-i} \times X^{n-1}$.*

Assume that α is an ex post equilibrium such that $\alpha_i \notin \Theta_i \times X$ while $\alpha_{-i} \in \Theta_{-i} \times X^{n-1}$. Suppose that $\alpha_i(\theta_i) \in \mathcal{W}$. Then Rule 2 applies, so agent $j \neq i$ wants to change his message from $\alpha_j(\theta_j) \in \Theta_j \times X$ into $\alpha_j(\theta_j) \in N$ in some state $(\theta_i, \theta_j, \theta_{-i,j})$ because he can get higher payoff by Rule 4. Next suppose that $\alpha_i(\theta_i) \in N$. Without loss of generality, the agent i chooses the number which is congruent to i modulo n. Then Rule 3 applies, so agent $j \neq i$ wants to change his message from $\alpha_j(\theta_j) \in \Theta_j \times X$ into $\alpha_j(\theta_j) \in N$ in some state $(\theta_i, \theta_j, \theta_{-i,j})$ because he has a chance to get a higher payoff by being the MG winner. Therefore assumption that α is an ex post equilibrium is wrong $\|$

(4) *There is no pure ex post equilibrium profile such that at least two agents report a non-type message.*

Let α be a strategy profile such that $\alpha_i \notin \Theta_i \times X$ and $\alpha_j \notin \Theta_j \times X$ $(i \neq j)$. Suppose that $(\alpha_i(\theta_i), \alpha_j(\theta_j)) \in \mathcal{W} \times \mathcal{W}$. Then Rule 4 applies, so the MG winner is uniquely determined in state $(\theta_i, \theta_j, \theta_{-i,j})$ no matter what $\alpha_{-i,j}(\theta_{-i,j})$ is. Any agent l except for the MG winner wants to change unilaterally from his message $\alpha_l(\theta_l)$ into $\alpha_l(\theta_l) \in N$ which makes him the new MG winner. The same reasoning applies to the case of $(\alpha_i(\theta_i), \alpha_j(\theta_j)) \in \mathcal{W} \times N$ or $(\alpha_i(\theta_i), \alpha_j(\theta_j)) \in N \times N$. ■

3.4.3 Relationship between EPSE and EM

Equivalence

Bergemann and Morris (2008) define ex post monotonicity for ex post implementation. Ex post monotonicity is an equivalent condition with ex post selective elimination.

Definition 7 *Social choice set X satisfies **ex post monotonicity** (EM) if for every $x \in X$ and α with $x \circ \alpha \notin X$, there exists i, θ, $y \in X$ such that*

$$u_i(y, \theta) > u_i(x(\alpha(\theta)), \theta),$$

while

$$u_i(x(\theta_i', \alpha_{-i}(\theta_{-i})), (\theta_i', \alpha_{-i}(\theta_{-i}))) \geq u_i(y, (\theta_i', \alpha_{-i}(\theta_{-i}))), \forall \theta_i' \in \Theta_i.$$

Let $A_i^x(\theta_{-i})$ denote a subset of A such that

$$A_i^x(\theta_{-i}) = \{a : u_i(x(\theta_i', \theta_{-i}), (\theta_i', \theta_{-i})) \geq u_i(a, (\theta_i', \theta_{-i})), \forall \theta_i' \in \Theta_i\}.$$

Proposition 4 *Social choice set X satisfies EM if and only if X satisfies EPSE.*

Proof.

Suppose that X satisfies EM. Let α be an ex post equilibrium in G_x^d such that $x \circ \alpha \notin X$ for $x \in X$. Then there exist i, θ, $y \in X$ such that

$$u_i(y, \theta) > u_i(x(\alpha(\theta)), \theta),$$

and

$$u_i(x(\theta_i', \alpha_{-i}(\theta_{-i})), (\theta_i', \alpha_{-i}(\theta_{-i}))) \geq u_i(y, (\theta_i', \alpha_{-i}(\theta_{-i})))$$

for all $\theta_i' \in \Theta_i$. Define a function for i such that $y_i : \theta_{-i} \mapsto y' \in A_i^x(\theta_{-i})$. Since $y \in A_i^x(\alpha_{-i}(\theta_{-i}))$, function $y_i(\theta_{-i})$ satisfies the following:

$$u_i(y_i(\alpha_{-i}(\theta_{-i})), \theta) > u_i(x(\alpha(\theta)), \theta),$$

and for each $\theta_{-i} \in \Theta_{-i}$,

$$u_i(x(\theta_i', \theta_{-i}), (\theta_i', \theta_{-i})) \geq u_i(y_i(\theta_{-i}), (\theta_i', \theta_{-i})), \forall \theta_i' \in \Theta_i.$$

It means α can be ex post selectively eliminated.

Suppose that X satisfies EPSE. Let α be an ex post equilibrium such that $x \circ \alpha \notin X$ for $x \in X$. Then there exist i and $y_i : \Theta_{-i} \to A$ such that

$$\exists \theta \in \Theta, \quad u_i(y(\alpha_{-i}(\theta_{-i})), \theta) > u_i(x(\alpha(\theta)), \theta)$$

and

$$\forall \theta \in \Theta, \quad u_i(x(\theta), \theta) \geq u_i(y(\theta_{-i}), \theta).$$

Let y be an alternative such that $y = y_i(\alpha_{-i}(\theta_{-i}))$. Then for any ex post equilibrium profile α such that $x \circ \alpha \notin X$ with $x \in X$, we obtain $u_i(y, \theta) > u_i(x(\alpha(\theta)), \theta)$ for some i and θ. In the case that α is not ex post equilibrium, we obtain $u_i(x(\theta'_i, \theta_{-i}), \theta) > u_i(x(\alpha(\theta)), \theta)$ for some i and θ'_i by definition. In this case, we put $y = x(\theta'_i, \theta_{-i})$. ■

Implementation with two agents

Theorem 2 in Bergemann and Morris (2008) gives a sufficient condition for ex post implementation in the following *economic environment*.

Definition 8 *An environment is economic if, for every state $\theta \in \Theta$, allocation $a \in A$, there exist $i \neq j$ and allocation a_i and a_j, such that*

$$u_i(a_i, \theta) > u_i(a, \theta)$$

and

$$u_j(a_j, \theta) > u_j(a, \theta)$$

.

Here is the theorem:

Theorem 1 *(Bergemann and Morris (2008)). If $n \geq 3$, the environment is economic, and X satisfies (EPIC) and (EM), then X is ex post implementable.*

Not only this positive result but also their constructing mechanism depend on assumption $n \geq 3$. To see it, let me review their mechanism. The message space of agent i is defined by

$$M_i = \Theta_i \times X \times N \times A.$$

A message of agent i is denoted by $m_i = (\theta_i, x_i, z_i, a_i)$. The mechanism is described by three rules:

Rule 1'. If $x_i = x$ for all i, then $g(m) = x(\theta)$.

Rule 2'. If there exists j and x such that $x_i = x$ for all $i \neq j$ while $x_j \neq x$, then outcome a_j is chosen if $a_j \in A_j^x(\theta_{-j})$; otherwise outcome $x(\theta)$ is chosen.

Rule 3'. In all other cases, $a_{j(z)}$ is chosen, where $j(z)$ is the agent determined by the modulo game:

$$j(z) = \sum_{i=1}^{n} z_i \mod n$$

This is their mechanism, called *BM-mechanism.*

Rule 2' works only if there are at least three agents. If there are two agents and $x_1 = x' \neq x'' = x_2$, the rule cannot determine which function to be adapted as a social decision. Therefore, we should do away with Rule 2'. Then BM-mechanism cannot select a desired outcome, say $x(\theta)$, in ex post equilibria in the economic environment; agent i wants to unilaterally deviate, by utilizing Rule 3', to $m_i = (\theta_i, x_i, z_i, a_i)$, with $i = z_1 + z_2$, $x_i \neq x$, and a_i such that $u_i(a_i, \theta) > u_i(x(\theta), \theta)$.

BM-mechanism does not operate well if we assume that the social choice set is singleton, $X = \{x\}$, and that agents are prohibited to report any social choice function which is different from x. Suppose that $x \circ \alpha \neq x$. By ex post monotonicity, there exists i, θ and $y \in A_i^x(\alpha_j(\theta_j))$ such that $u_i(y, \theta) > u_i(x(\alpha(\theta)), \theta)$, but the rules do not realize y, so that the undesired ex post equilibrium cannot be eliminated. One reason why the mechanism I construct operates well in two persons case is to independently use non-type messages to eliminate undesired ex post equilibria.

3.4.4 Relationship between EPSE and SE

In this subsection, I discuss the relationship between the EPSE and SE (i.e., Selective Elimination) condition by example.

Let $BG = (\Gamma, (u_i)_{i=1}^n, p)$ denote a Bayesian game with mechanism Γ and common prior $p : \Theta \to [0, 1]$. Assume that a collection of θ such that $p(\theta) > 0$ is Θ, i.e., any state is possible.

Definition 9 *Mechanism* $\Gamma = ((M_i^n)_{i=1}^n, g)$ ***Bayesian implements*** *SCS* X *by* $BG = (\Gamma, (u_i)_{i=1}^n, p)$ *if the following both statements are satisfied:*

1. *For every pure Bayesian equilibrium α in BG, it is true that $g \circ \alpha = x$ for some $x \in X$.*

2. *For any $x \in X$, there exists an pure Bayesian equilibrium α in BG such that $g \circ \alpha = x$.*

*Then X is **Bayesian implementable.***

Definition 10 *Bayesian equilibrium α in game $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^n, p)$ can be **selectively eliminated**, if there exists $i \in N$ and $y : \Theta_{-i} \to A$ such that: for some $\theta_i \in \Theta_i$,*

$$\sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i}|\theta_i)\big[u_i(y(\alpha_{-i}(\theta_{-i})), \theta) - u_i(x(\alpha(\theta)), \theta)\big] > 0$$

and for all $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i}|\theta_i)\big[u_i(x(\theta), \theta) - u_i(y(\theta_{-i}), \theta)\big] \geq 0.$$

Definition 11 *Direct revelation mechanism $\Gamma_x^d = ((\Theta_i)_{i=1}^n, x)$ satisfies the **selective elimination** condition (**SE**) relative to X if $x \in X$ and if any Bayesian equilibrium α in $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^n, p)$ such that $x \circ \alpha \notin X$ can be selectively eliminated.*

*Furthermore, X **satisfies SE** if every $x \in X$ and the associated direct revelation mechanism satisfies SE relative to X.*

Mookherjee and Reichelstein (1990) show that SE is necessary for Bayesian implementation and it will be sufficient in economic environment. If X is Bayesian implementable, then Γ_x^d must satisfy SE for any $x \in X$. We are interested in the relation between SE and EPSE; which implies another or there is no inclusive relation. It is trivial to show that SE defined in game BG_x^d implies EPSE defined in G_x^d.[8] We will see that EPSE defined in game G_x^d does not imply SE defined in BG_x^d

Example 1 *X satisfies EPSE, but does not SE.*

[8] It is possible that there does not exist any ex post equilibria in $G = (\Gamma_x^d, (u_i)_{i=1}^n)$ even though there exists many Bayesian equilibria in $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^n, p)$ which is defined with some common prior p. In such a case, we should interpret that the EPSE condition for Γ_x^d is vacuously satisfied.

Consider the environment $N = \{1,2\}$, $A = \{a,b,c,d,e\}$ and $\Theta = \Theta_1 \times \Theta_2$ with $\Theta_i = \{\theta_i, \theta_i'\}$ for each $i = 1,2$. Their ex post payoffs (von Neumann-Morgenstern utilities) are described in Table 3.1. We want to implement the social choice function x as follows.

x	θ_2	θ_2'
θ_1	a	b
θ_1'	c	d

Direct mechanism Γ_x^d induces a truthful ex post equilibrium in G_x^d but also induces a suboptimal post equilibrium which brings undesired outcome. The suboptimal ex post equilibrium profile (α_1, α_2) is such that $\alpha_2(\theta_2) = \alpha_2(\theta_2') = \theta_2$ while α_1 is the truth-telling strategy. In addition, $\{x\}$ satisfies EPSE; define the social choice function such that $y(\theta_2) = b$ and $y(\theta_2') = d$.

Table 3.1: $D > B > M > L > 0$ and $B > D - M$.

a	θ_2	θ_2'
θ_1	B, B	$0, -L$
θ_1'	$-L, 0$	$0, -L$

b	θ_2	θ_2'
θ_1	$0, -L$	$0, -L$
θ_1'	$B, -L$	$-L, 0$

c	θ_2	θ_2'
θ_1	$-L, 0$	$0, -L$
θ_1'	B, B	$0, -L$

d	θ_2	θ_2'
θ_1	$B, -L$	$-L, 0$
θ_1'	$0, -L$	$0, -L$

e	θ_2	θ_2'
θ_1	$-M, 0$	$D, 0$
θ_1'	$0, 0$	$0, 0$

On the other hand, $\{x\}$ does not satisfy SE under some belief system. To see it, assume the common prior environment that $p(\theta) = 1/4$ with its marginal distribution $p(\theta_i) = p(\theta_i') = 1/2$ for $i = 1,2$. Hence the belief of type of agents is $p(\theta_j|\theta_i) = p(\theta_j'|\theta_i) = \frac{1}{2}$ for $i \neq j$ with $i,j = 1,2$. In this environment, Γ_x^d does not satisfy SE when we consider game $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^2, p)$. Suppose that Γ_x^d satisfy SE. Let agent 1 be a whistle-blower, then his payoff satisfies the following equation:

$$\frac{1}{2}(u_1(y_1(\theta_2), \theta_1, \theta_2) - B) + \frac{1}{2}u_1(y_1(\theta_2'), \theta_1, \theta_2') > 0$$

or

$$\frac{1}{2}\left(u_1(y_1(\theta_2),\theta_1',\theta_2)-B\right)+\frac{1}{2}\left(u_1(y_1(\theta_2')),\theta_1',\theta_2'\right)>0,$$

where $y:\Theta_2\to A$ is a social choice function. However, one can easily check that it is impossible to define such y that gives those payoffs for agent 1 in each state. By the same reasoning, we can say that agent 2 cannot be a whistle-blower.

This example also tells us that *Maskin monotonicity*, which is the necessary and almost sufficient condition for Nash implementation in complete information environment (Maskin (1999)), does not hold even if EPSE holds. Maskin monotonicity says that for every $x\in X$, α and θ with $x\circ\alpha(\theta)\neq x'$ for all $x'\in X$, there exists i and $\widetilde{a}\in A$ such that $u_i(\widetilde{a},\theta)>u_i(x(\alpha(\theta)),\theta)$, while $u_i(x(\alpha(\theta)),\alpha(\theta))\geq u_i(\widetilde{a},\alpha(\theta))$. In our example, we cannot find such $\widetilde{a}$ in state $\theta=(\theta_1',\theta_2')$ against $\alpha_1(\theta)=\alpha_2(\theta)=(\theta_1',\theta_2)$.

Indeed, Bergemann and Morris (2008) explain that ex post monotonicity does not imply Maskin monotonicity, and that Maskin monotonicity does not imply ex post monotonicity. Hence, from our equivalence result in Section 3.4.3, the ex post selective elimination condition neither implies nor is implied by Maskin monotonicity.

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Chapter 4

Implementation via Ex Post Dominance Solvable Mechanisms

4.1 Introduction

In this chapter, I consider the implementation problem under incomplete information. Implementation under incomplete information have been investigated in the framework of Bayesian implementation, e.g. Palfrey and Srivastava(1989), Mookherjee and Reichelstein(1990), and Jackson(1991).[1]

Bayesian implementation, or more generally, the implementation problem under incomplete information have often assumed that beliefs of agents are given. Moreover a planner knows beliefs of agents and is allowed to use them to implement a social choice function. Thus a mechanism might have to be redesigned when beliefs of some agents change. Needless to say, the assumption that the planner knows beliefs of agents is generally too strong.

This article considers implementation problem without the assumption that a planner has full knowledge about beliefs of agents. I introduce a new concept in the literature of implementation: the *belief-free implementation*. Social choice function x is belief-free implementable if there exist a mechanism which is constructed without precisely knowledge about belief systems and implements x under any belief system. I consider the implementation of social choice functions in *iterative deletion of ex post weakly dominated strategies* under the assumption of *full-support* beliefs of agents. If it succeeds, we obtain belief-free implementation under any full-support belief system. This approach for belief-free implementation is to be labeled as *quasi* belief-free

[1] There are many surveys for implementation including the case of complete information, e.g. Maskin and Sjöström (2002). Palfrey (1992) focus on Bayesian implementation.

implementation because of full-support assumption.

I show that social choice function x is *implementable in iterative deletion of ex post weakly dominated strategies* if x satisfies the *uniformly effective elimination condition (UEE)*, hence UEE is a sufficient condition for quasi belief-free implementation. This result is obtained by a *finite* mechanism with twice iteration. Moreover, it is independent from order of deletion. The condition assures any undesired type-reporting strategy to be selectively eliminated.

4.1.1 Related literature

Chung and Ely (2001) show that, in an auction model with interdependent values, an efficient allocation is *uniquely* achievable by iterative deletion of ex post weakly dominated strategies. Their model focuses on the generalized VCG mechanism. Therefore they investigate a direct mechanism in an auction environment with quasi-linear utilities and show that an *efficient* social choice function can be implementable in iterative deletion of ex post weakly dominated strategies. But it has been little known whether an arbitrary given social choice function is implementable in iterative deletion of ex post (weakly) dominated strategies when the mechanism is not direct one and the environment is more general.[2] Chung and Ely (2001) provide an example that an efficient social choice function is implementable in iterative deletion of ex post weakly dominated strategies only if the mechanism is an *indirect* one. Thus it is possible to see that this article generalizes the example provided in Chung and Ely (2001).

Bergemann and Morris (2003, 2008b) focus on belief-free implementation. (They call it *robust implementation.*) They show that social choice function x is belief-free implementable if and only if x is implementable in rationalizable strategies in the game played with some fixed mechanism. It is equivalent to iterative deletion of strictly dominated strategies on any type space if the message spaces are finite (or compact). They construct a mechanism which belief-free implements a social choice function via an *infinite* mechanism which permits lottery as an outcome. Their approach to belief-free implementation appeals to rationalizability. On the other hand, I construct a *finite* mechanism which quasi belief-free implements a social choice func-

[2] Recently, Bergemann and Morris (2008a) report that, in the auction environment investigated Chung and Ely (2001), strict ex post incentive compatibility and the *contraction property*("one's payoff does not too depend on other agents' type.") are necessary for belief-free implementation.

tion via *twice* iteration of deletion of ex post weakly dominated strategies. Furthermore, I use only deterministic outcomes. These acceptable results owe to quasi belief-free criterion and UEE. Implementation becomes easier if belief-free criterion weakens to quasi belief-free because we can employ iterative deletion of *weakly* dominated strategies, which is milder condition than iterative deletion of strictly dominated strategies.[3]

This chapter is organized as follows; Section 4.2 develops notations and definitions, Section 4.3 develops an example of task allocation problem, Section 4.4 is the main section in this chapter, which presents the theorem. Section 4.5 discusses my result and relations on the work of Bergemann and Morris (2008b). A minor proof is in Appendix.

4.2 Preliminaries

4.2.1 Set up

I first describe the general *environment* $< A, N, \Omega >$ that we take into account. A is the set of *alternatives* or *outcomes*, $N = \{1, \cdots, n\}$ is the finite set of *agents* with $n \geq 2$, and Ω is the set of possible states of the world, with a typical state is denoted $\omega \in \Omega$. A function $x : \Omega \to A$ is a *social choice function*. Each agent is denoted by $i \in N$. Ω_i is the set of *types* for agent $i \in N$, and typical type of i is denoted by $\omega_i \in \Omega_i$. I assume that each Ω_i is finite. The set of states is defined by $\Omega := \Omega_1 \times \cdots \times \Omega_n$. The preference of each agent is represented by von-Neumann and Morgenstern utility function $u_i : A \times \Omega \to \mathbb{R}$.

Mechanism $\Gamma = ((M_i)_{i=1}^n, g)$ is defined in the environment, which is composed of the message space M_i of each i and outcome function $g : M_1 \times \cdots \times M_n \to A$. I denote a profile of message by $m = (m_i, m_{-i}) = (m_1, \cdots, m_n) \in M = M_1 \times \cdots \times M_n$.

I define game $G = (\Gamma, \{u_i\}_{i=1}^n)$ with mechanism Γ on the environment. Given G, a pure strategy for $i \in N$ is denoted by $\alpha_i : \Omega_i \to M_i$. I describe a typical strategy profile by $\alpha = (\alpha_i, \alpha_{-i}) = (\alpha_1, \cdots, \alpha_n)$. A set of possible

[3]This argument reminds us Abreu and Matsushima (1992, 1994). When we employ iterative deletion of strictly dominated strategies as a solution concept, any social choice function is implementable in the sense of *virtual* (Abreu and Matsushima (1992)). On the other hand, when we relax the solution concept to iterative deletion of weakly dominated strategies, any social choice function is *exactly* implementable (Abreu and Matsushima (1994)). However, their mechanism utilizes lotteries as alternatives.

strategies for $i \in N$ is denoted by Φ_i and direct product of them by $\Phi = \Phi_1 \times \cdots \times \Phi_n$.

4.2.2 Solution concepts

The main solution concept in this article is the iterative deletion of ex post weakly dominated strategies.

Let $Q_i \subset \Phi_i$ be an arbitrary subset of possible strategies for $i \in N$. Put $Q = Q_1 \times \cdots \times Q_n$. $Q_{-i} \subset \Phi_{-i}$ is the set of strategies other than i.

Definition 1 *Strategy α_i is* ***ex post weakly dominated*** *against $Q_{-i} \subset \Phi_{-i}$ if there exist some $\widehat{\alpha}_i \in \Phi_i$ such that for all $\omega \in \Omega$,*

$$\forall \alpha_{-i} \in Q_{-i},\ u_i(g(\widehat{\alpha}_i(\omega_i), \alpha_{-i}(\omega_{-i})), \omega) \geq u_i(g(\alpha_i(\omega_i), \alpha_{-i}(\omega_{-i})), \omega) \quad (4.1)$$

with strict inequalities for some $\alpha_{-i} \in Q_{-i}$ and $\omega \in \Omega$.

I say that *action $\alpha_i(\omega_i)$ is ex post weakly dominated* if there exist some $m_i \in M_i$ such that for all α_{-i} and for all ω_{-i}, we have

$$u_i(g(m_i, \alpha_{-i}(\omega_{-i})), \omega) \geq u_i(g(\alpha_i(\omega_i), \alpha_{-i}(\omega_{-i})), \omega)$$

with strict inequalities for some α_{-i} and for some ω_{-i}. Note that strategy α_i is ex post weakly dominated if and only if there exist some $\omega_i \in \Omega_i$ and action $\alpha_i(\omega_i)$ is ex post weakly dominated.

Set of strategies Q_i is said to be *undominated* against Q_{-i} if for any pure strategies $\alpha_i \in Q_i$, there does not exist $\widehat{\alpha}_i \in Q_i$ which ex post dominates α_i against Q_{-i}. Q is said to be *internally undominated* if Q_i is undominated against Q_{-i} for each $i \in N$.

Let $(D^k)_{k=0}^{\infty}$ be a sequence of an arbitrary set, where $D^k := D_1^k \times \cdots \times D_n^k$ $(k = 0, 1, \cdots)$. We call it *deletion sequence* in G if it satisfies the following properties; $D_i^0 = \Phi_i$ for all i, $D^k \subset D^{k-1}$ for all $k \geq 1$, $\alpha_i \in D_i^{k-1} \backslash D_i^k$ only if α_i is ex post weakly dominated against D_{-i}^{k-1} in G, and $\cap_{k=1}^{\infty} D^k = D^*$, where D^* is internally undominated. Strategy profile α is said to be *ex post weakly iteratively undominated* if $\alpha \in D^*$.

Definition 2 *Social choice function x is* ***implementable in ex post weakly iteratively undominated strategies,*** *if there exist mechanism $\Gamma = ((M_i)_{i=1}^n, g)$ and game $G = (\Gamma, \{u_i\}_{i=1}^n)$ such that*

1. *For any deletion sequence $(D^k)_{k=0}^{\infty}$ in G, we have $\alpha \in D^*$ such that $x(\omega) = g(\alpha(\omega))$ for all $\omega \in \Omega$.*

2. *For any $\widehat{\boldsymbol{\alpha}} \in D^*$ coming from some deletion sequence $(\widehat{D}^k)_{k=0}^{\infty}$ in G, we must have $x(\omega) = g(\widehat{\boldsymbol{\alpha}}(\omega))$ for all $\omega \in \Omega$.*

In general the result in iterative deletion of weakly dominated strategies may depends on the order of deletion. Our definition of solution concept requires *any* sequence to have the same limit D^*.

4.2.3 Belief-free implementation

Let p_i denote a *belief* of agent i about the other agents' types which is defined by

$$p_i(\cdot|\omega_i) : \Omega_{-i} \to [0,1] \quad \text{and} \quad \sum_{\omega'_{-i} \in \Omega_{-i}} p_i(\omega'_{-i}|\omega_i) = 1$$

for each $\omega_i \in \Omega_i$. Incomplete information game G' is defined by $G' = (\Gamma, \{u_i\}_{i=1}^n, \{p_i\}_{i=1}^n)$, where $\{p_i\}_{i=1}^n$ indicates *belief system* in G'. Belief system $\{p_i\}_{i=1}^n$ is said to have *full-support* if $p_i(\omega'_{-i}|\omega_i) > 0$ for every $i \in N$, every $\omega_i \in \Omega_i$ and every $\omega'_{-i} \in \Omega_{-i}$.

Definition 3 *Fix game G'. Strategy α_i is* ***interim weakly dominated*** *against Φ_{-i} if there exist some $\widehat{\alpha}_i \in \Phi_i$ such that for all $\omega_i \in \Omega_i$ and for all $\alpha_{-i} \in \Phi_{-i}$,*

$$\begin{aligned} &\sum_{\omega_{-i} \in \Omega_{-i}} p_i(\omega_{-i}|\omega_i) u_i(g(\widehat{\alpha}_i(\omega_i), \alpha_{-i}(\omega_{-i})), \omega) \\ &\geq \sum_{\omega_{-i} \in \Omega_{-i}} p_i(\omega_{-i}|\omega_i) u_i(g(\alpha_i(\omega_i), \alpha_{-i}(\omega_{-i})), \omega) \end{aligned}$$

with strict inequalities for some $\omega'_i \in \Omega_i$ and $\alpha_{-i} \in \Phi_{-i}$.

We can check easily that if strategy α_i is ex post weakly dominated in G, then α_i is also interim weakly dominated in $G' = (G, \{p_i\}_{i=1}^n)$ with arbitrary belief system $\{p_i\}_{i=1}^n$ as long as it has *full-support.* The full-support condition is crucial if we presume weak dominance.[4]

Definition 4 *Strategy profile α is* ***interim equilibrium*** *in $G' = (G, \{p_i\}_{i=1}^n)$ if $\forall i, \omega_i, \alpha'_i$,*

$$\sum_{\omega_{-i} \in \Omega_{-i}} p_i(\omega_{-i}|\omega_i) u_i(g(\alpha(\omega), \omega) \geq \sum_{\omega_{-i} \in \Omega_{-i}} p_i(\omega_{-i}|\omega_i) u_i(g(\alpha'_i(\omega_i), \alpha_{-i}(\omega_{-i})), \omega).$$

[4] Suppose that $\alpha_i(\omega_i)$ is strictly preferred to $\alpha'_i(\omega_i)$ for all α_{-i} in state (ω_i, ω'_{-i}) but both actions are indifferent for all α_{-i} in any other state. If $p_i(\omega'_{-i}|\omega_i) = 0$, $\alpha'_i(\omega_i)$ is no longer interim dominated by $\alpha_i(\omega_i)$.

Definition 5 *Social choice function x is **interim implementable** by $G' = (G, \{p_i\}_{i=1}^n)$ if any interim equilibrium α in G' realizes $x(\omega) = g(\alpha(\omega))$ for all $\omega \in \Omega$.*

Definition 6 *Social choice function x is **(quasi) belief-free implementable** if there exists mechanism Γ such that for any (full-support) belief system $\{p_i\}_{i=1}^n$, x is interim implementable by $G' = (\Gamma, \{u_i\}_{i=1}^n, \{p_i\}_{i=1}^n)$.*

By definition, we get the following.

Lemma 1 *Social choice function x is quasi belief-free implementable if x is implementable in ex post weakly iteratively undominated strategies.*

4.3 Example: Task Allocation Problem

In this section, I provide an example of task allocation problem to illustrate the essence of implementation in ex post weakly iteratively undominated strategies.

The environmet $< A, N, \Omega >$ of this example is denoted as follows; $A := \widehat{A} \times \mathbb{R}^2$ where $\widehat{A} = \{a, b, c, d, e\}$, $N := \{1, 2\}$, $\Omega := \Omega_1 \times \Omega_2$ where $\Omega_i = \{\omega_i, \omega_i'\}$ for $i = 1, 2$. I assume that there exists a principal who assigns the optimal project. The optimality of these projects are depend on the states. I consider the case where agents collaborate the same project. The results of each project take the two values: "success" or "fail" . The probability of success is different on each state; it depends on the combination of each agent's type. The principal wants to assign the project whose probability of success is maximum on each state. Preference of each agent is denoted by $u_i(x, t, \omega) = v(x, \omega) + t_i$, where $v(\cdot, \omega) : \widehat{A} \to [0, 1]$ is the probability of success of projects in each state ω; the payoff which is gained through project is common value for each agent[5]. The project $e \in \widehat{A}$ represents the alternative that means "we abandon any projects". I normalize $v(e, \omega) = 0$ for every $\omega \in \Omega$.

A social choice function is denoted by $\widehat{x} : \Omega \to A$, where $\widehat{x}(\cdot) = (x(\cdot), t(\cdot))$, $t(\cdot) = (t_1(\cdot), t_2(\cdot))$, $x : \Omega \to \widehat{A}$ is called public decision function and $t_i : \Omega \to \mathbb{R}$ is monetary transfer function for i. The probabilities of each project in each state are denoted in Table 4.1. The principal wants to implement the

[5]This assumption is used when researcher focus on "team", e.g., Marschak and Radner (1972).

Table 4.1:

a	ω_2	ω_2'
ω_1	1	0
ω_1'	0	0.2

b	ω_2	ω_2'
ω_1	0	1
ω_1'	0.5	0

c	ω_2	ω_2'
ω_1	0	0.5
ω_1'	1	0

d	ω_2	ω_2'
ω_1	0.8	0
ω_1'	0	1

social choice function $\widehat{x}^* = (x^*, t^*)$, where $x^*(\omega_1, \omega_2) = a$, $x^*(\omega_1, \omega_2') = b$, $x^*(\omega_1', \omega_2) = c$, $x^*(\omega_1', \omega_2') = d$, and $t^*(\omega) = 0$ for every $\omega \in \Omega$.

In game $G = (\Gamma, u_1, u_2)$ with $\Gamma = ((\Omega_i)_{i=1,2}, \widehat{x}^*)$, agents' payoffs are in the Table 4.2. The true state is described on the upper-left corner in each table. We can check easily that $\widehat{x}^*$ cannot be implementable in ex post iteratively weakly undominated strategies in G.

Table 4.2:

(ω_1, ω_2)	ω_2	ω_2'
ω_1	$1, 1$	$0, 0$
ω_1'	$0, 0$	$0.8, 0.8$

(ω_1, ω_2')	ω_2	ω_2'
ω_1	$0, 0$	$1, 1$
ω_1'	$0.5, 0.5$	$0, 0$

(ω_1', ω_2)	ω_2	ω_2'
ω_1	$0, 0$	$0.5, 0.5$
ω_1'	$1, 1$	$0, 0$

(ω_1', ω_2')	ω_2	ω_2'
ω_1	$0.2, 0.2$	$0, 0$
ω_1'	$0, 0$	$1, 1$

Next I construct mechanism $\Gamma^* = ((M_i^*)_{i=1}^n, g^*)$ as follows; Each agent's message space is $M_i^* = \Omega_i \cup \{r_i\}$, and set the following rules as the outcome function g^*:

1. If both agents report $\omega_i \in \Omega_i$, then the public decision follows $\widehat{x}^*(\cdot)$.

2. If only one agent reports r_i, then i takes transfer 0.9 and $j \neq i$ takes transfer 0, with public decision e.

3. If both agents report r_i, then they take transfer 0, with public decision e.

The outcomes and ex post payoffs in mechanism Γ^* are described in Table 4.3 and Table 4.4 respectively. Then $\widehat{x}^*$ is implementable in ex post weakly iteratively undominated strategies. To see this we first check the fact that each agent' untruthful-type-reporting strategy is ex post dominated by the non-type-reporting strategy. Next, we check that the non-type-reporting strategy is ex post weakly dominated by truthful-type-reporting strategy against the set of strategy after deleting untruthful-type-reporting strategy for each i.

Table 4.3:

g^*	ω_2	ω_2'	r_2
ω_1	$(a,\ 0,\ 0)$	$(b,\ 0,\ 0)$	$(e,\ 0, 0.9)$
ω_1'	$(c,\ 0,\ 0)$	$(d,\ 0,\ 0)$	$(e,\ 0, 0.9)$
r_1	$(e, 0.9,\ 0)$	$(e, 0.9,\ 0)$	$(e,\ 0,\ 0)$

Table 4.4:

(ω_1, ω_2)	ω_2	ω_2'	r_2
ω_1	$1, 1$	$0, 0$	$0, 0.9$
ω_1'	$0, 0$	$0.8, 0.8$	$0, 0.9$
r_1	$0.9, 0$	$0.9, 0$	$0, 0$

(ω_1, ω_2')	ω_2	ω_2'	r_2
ω_1	$0, 0$	$1, 1$	$0, 0.9$
ω_1'	$0.5, 0.5$	$0, 0$	$0, 0.9$
r_1	$0.9, 0$	$0.9, 0$	$0, 0$

(ω_1', ω_2)	ω_2	ω_2'	r_2
ω_1	$0, 0$	$0.5, 0.5$	$0, 0.9$
ω_1'	$1, 1$	$0, 0$	$0, 0.9$
r_1	$0.9, 0$	$0.9, 0$	$0, 0$

(ω_1', ω_2')	ω_2	ω_2'	r_2
ω_1	$0.2, 0.2$	$0, 0$	$0, 0.9$
ω_1'	$0, 0$	$1, 1$	$0, 0.9$
r_1	$0.9, 0$	$0.9, 0$	$0, 0$

4.4 Main Results

4.4.1 Conditions

A *direct revelation mechanism relative to* x is the mechanism such that $M_i := \Omega_i$ for all i and $g := x$. Let Γ_x^d denote the direct revelation mechanism relative to x, i.e., $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$.

Definition 7 *Direct revelation mechanism $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$ satisfies the **effective elimination** condition (**EE**), if there exist $i \in N$ and $y_i : \Omega_{-i} \to A$ such that Γ_x^d satisfies the following inequalities;*

EE(1): *For every ω_i' and ω_{-i},*

$$\forall \alpha_{-i}(\omega_{-i}) \neq \omega_{-i},\ u_i(y_i(\alpha_{-i}(\omega_{-i})), (\omega_i', \omega_{-i})) \geq u_i(x(\omega_i'', \alpha_{-i}(\omega_{-i})), (\omega_i', \omega_{-i})),$$

for every $\omega_i'' \in \Omega_i$ with strict inequalities for some $\alpha_{-i}(\omega_{-i})$ in (ω_i', ω_{-i}).

EE(2): *For every $\omega \in \Omega$,*

$$u_i(x(\omega), \omega) \geq u_i(y_i(\omega_{-i}), \omega) \geq u_i(x(\tilde{\omega}_i, \omega_{-i}), \omega)$$

for every $\tilde{\omega}_i \in \Omega_i \backslash \{\omega_i\}$, and

$$\exists \omega_{-i} \in \Omega_{-i},\ u_i(x(\omega), \omega) > u_i(y_i(\omega_{-i}), \omega).$$

*Direct revelation mechanism Γ_x^d is said to satisfy the **uniformly effective elimination** condition (**UEE**) if for every $i \in N$, there exists social choice function $y_i : \Omega_{-i} \to A$ which satisfies both EE(1) and EE(2).*

Definition 8 *Direct revelation mechanism $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$ is said to be **ex post incentive compatible** if for every $i \in N$, for every $\omega \in \Omega$,*

$$u_i(x(\omega), \omega) \geq u_i(x(\omega_i', \omega_{-i}), \omega)$$

for every $\omega_i' \in \Omega_i$.

Definition 9 *Direct mechanism $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$ satisfies the **regularity condition** if Γ_x^d is ex post incentive compatible and it also satisfies that for all $i \in N$, for all $\omega_i \in \Omega_i$, there exist some $\omega_{-i} \in \Omega_{-i}$ such that*

$$u_i(x(\omega), \omega) > u_i(x(\omega_i', \omega_{-i}), \omega)$$

for all $\omega_i' \in \Omega_i \backslash \{\omega_i\}$.

In words, the regularity condition requires that there are some states in which truth-telling is the strict best reply if the others are reporting truthfully for any $i \in N$ in addition to ex post incentive compatibility. Note that UEE implies the regularity condition.

4.4.2 The environment

I give some restriction on the environment. Let $\widehat{A}$ denote the set of possible alternatives for each agent. I assume that $A := \widehat{A}^n = \widehat{A} \times \cdots \times \widehat{A}$ is the set of alternative. $F \subseteq A$ represents the set of socially feasible alternatives. This environment abstracts a pure exchange economy ($\widehat{A} := \mathbb{R}_+$ for example) or a single-unit demand auction ($\widehat{A} := \{\{0,1\} \times \mathbb{R}\}$). For convenience, I define description rules as follows: $a = (a^i, a^{-i}) \in A$ and $a = (a)^n = (a, \cdots, a)$ if $a^i = a^j$ for all $i, j \in N$. The utility of each agent is defined over $\widehat{A} \times \Omega$; $u_i : \widehat{A} \times \Omega \to \mathbb{R}$.

Assumption 1

(A1) There exists an alternative $e \in \widehat{A}$ such that $u_i(e, \omega) < u_i(a, \omega)$ for every $i \in N$, $\omega \in \Omega$, and $a \in \widehat{A} \backslash \{e\}$.

(A2) For any $a \in F$, an alternative $b \in A$ is such that $b^i = e$ and $b^{-i} = a^{-i}$ for some arbitrary $i \in N$, then $b \in F$.

Assumption (A1) implies that every agent $i \in N$ has the worst common alternative. Assumption (A2) can be seen the free-disposal condition. It implies that the worst alternative profile is always feasible, i.e.,$(e, \cdots, e) \in F$. In addition, it also implies that exclusive consumption is possible; we exclude *pure* public goods provision problem because pure public goods generally entail non-exclusiveness of consumption.

Social choice function $x : \Omega \to F$ is a collection $x = (x^i)_{i=1}^n$, where $x^i : \Omega \to \widehat{A}$. Direct revelation mechanism $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$ satisfies the UEE (or EE) condition if for every (resp. some) $i \in N$, there exists social choice function $y_i : \Omega_{-i} \to F$ which satisfies both EE(1) and EE(2), where $y_i = (y_i^j)_{j=1}^n$ and $y_i^j : \Omega_{-i} \to \widehat{A}$.

4.4.3 Sufficient condition for the implementation

Suppose that direct revelation mechanism Γ_x^d satisfies UEE. I consider the following *augmented revelation mechanism* [6]. The message space for each agent $i \in N$ is defined such that

$$M_i := \Omega_i \cup \{r_i\}.$$

[6] Mookherjee and Reichelstein (1990) first introduce the idea of augmented revelation to investigate Bayesian implementation.

For convenience, I define description rules as follows: $g(m) = (g^i, g^{-i})(m) = (g^i(m))_{i=1}^n$ and

$$u_i(g(\alpha), \omega) = u_i(g(\alpha_i, \alpha_{-i}), (\omega_i, \omega_{-i})) := u_i(g(\alpha(\omega)), \omega) = u_i(g^i(\alpha(\omega)), \omega).$$

The outcome function g is defined as follows ;
Rule 1: If $m = \widehat{\omega} \in \Omega$, then

$$\left(g^i(m)\right)_{i=1}^n = \left(x^i(\widehat{\omega})\right)_{i=1}^n .$$

Rule 2: If $m = (r_i, \widehat{\omega}_{-i})$, then

$$g(m) = (g^i, g^{-i})(m) = \left(y_i^i(\widehat{\omega}_{-i}), (e)^{n-1}\right)$$

where $y_i^i(\widehat{\omega}_{-i})$ is the social choice function for i which satisfies both EE(1) and EE(2) relative to x.
Rule 3: Otherwise.

$$g(m) = (e, \cdots, e) .$$

Note that the outcome function g satisfies feasibility.

Theorem 1 *Assume that (A-1) and (A-2). If direct mechanism $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$ satisfies UEE, then x is implementable in ex post weakly iteratively undominated strategies.*

Remark. If Γ_x^d satisfies UEE, we can implement x with *twice* iteration.
Proof. We define the game $G = (\Gamma, (u_i)_{i=1}^n)$ with the augmented revelation mechanism Γ define by Rule 1 to Rule 3 in the above. We show that x is implementable in ex post weakly iteratively undominated strategies in G Let $\alpha^* = (\alpha_i^*, \alpha_{-i}^*)$ denote the truthful-type-reporting strategy profile in G. Let D denote a deletion sequence in G.

Claim 1. *There exists D such that $D^0 = \Phi$, D^1 contains no untruthful-type-reporting strategies for each i, and $D^2 = \{\alpha^*\}$.*

The first round starts from putting $D^0 := \Phi$. We choose an agent $i \in N$, his type $\omega_i \in \Omega_i$, and strategy profile $\alpha_{-i} \in D_{-i}^0$ arbitrarily and fix them. Let $\alpha_i(\omega_i)$ denote an agent i's untruthful-type-reporting action in G. Thus $\alpha_i(\omega_i) \in \Omega_i \backslash \{\omega_i\}$. Pick an arbitrary profile $\omega_{-i} \in \Omega_{-i}$, then we have action profile $\alpha(\omega)$. For any action profile $\alpha_{-i}(\omega_{-i})$, if $\alpha_{-i}(\omega_{-i}) \neq \omega_{-i}$, we apply Rule 1 and Rule 2 and must have;

$$u_i(g(r_i, \alpha_{-i}(\omega_{-i})), \omega) \geq u_i(g(\alpha), \omega) \tag{4.2}$$

for all $\alpha_i(\omega_i) \in \Omega_i \backslash \{\omega_i\}$ (and it is strictly held for some α_{-i} and ω_{-i}) because of EE(1). Moreover equation (4.2) must be held when $\alpha_{-i}(\omega_{-i}) = \omega_{-i}$ because of EE(2) held for each i. Otherwise, Rule 3 applies, so we must have

$$u_i(g(r_i, \alpha_{-i}(\omega_{-i})), \omega) = u_i(g(\alpha), \omega) \tag{4.3}$$

for all $\alpha_i(\omega_i) \in \Omega_i \backslash \{\omega_i\}$. Since we choose $i \in N$ and $\omega_i \in \Omega_i$ arbitrarily, we can say that any untruthful-type-reporting strategies are ex post weakly dominated by non-type-message-reporting strategy against D^0_{-i} for every $i \in N$. Therefore we can delete them all and define D^1_i for each $i \in N$ which never contains untruthful-type-reporting strategies. Put $D^1 = D^1_1 \times \cdots \times D^1_n$ and go to the second round.

The second round shows that any strategies which contain the non-type-message reporting action are ex post weakly dominated by truth-telling strategy in G against D^1_{-i} for each agent $i \in N$. Let $\widehat{\alpha}(\omega) = (\alpha^*_i(\omega_i), \alpha_{-i}(\omega_{-i}))$ denote an action profile at any $\omega \in \Omega$. Since D^1_i does not contain untruthful-type-reporting strategies, the possible values of the outcome function for i are simply

$$g^i(\widehat{\alpha}(\omega)) = \begin{cases} x^i(\omega) & \text{if } \alpha_{-i}(\omega_{-i}) = \alpha^*_{-i}(\omega_{-i}). \\ e & \text{Otherwise.} \end{cases}$$

Therefore we must have that

$$u_i(g(r_i, \alpha_{-i}(\omega_{-i})), \omega) \leq u_i(g(\widehat{\alpha}(\omega)), \omega) \tag{4.4}$$

for every $\alpha_{-i} \in D^1_{-i}$ and every $\omega_{-i} \in \Omega_{-i}$ and must be held strictly in some case because of EE(2). $\|$

We have shown that *some* deletion sequence realize $x(\omega) = g(\alpha^*(\omega))$ for all $\omega \in \Omega$. Next we have to show that *any* deletion sequence in the G realizes the same outcome α^*. For this purpose, it is sufficient to show that it is impossible that some deletion sequence brings about;

(A) the single profile that $\alpha \neq \alpha^*$,
or

(B) the set of strategy profiles which contains at least two different profiles.

It is obvious to show that the case of (A) is impossible because of EE(2) and the definition of outcome function. Now we assume that some deletion sequence $(\tilde{D}^k)_{k=1}^{\infty}$ bring about the set of strategy profiles $D^* = \cap_k \tilde{D}^k$, which contains several strategies. Since (A) is impossible, we must have $\alpha^* \in D^*$.

We show that the case (B) is impossible by proving the following claim.

Claim 2. *Suppose that $D^k = D_1^k \times \cdots \times D_n^k$ is an element of the deletion sequence. If D_i^k contains the strategies that $\alpha_i(\omega_i) \in \Omega_i \backslash \{\omega_i\}$ for some ω_i, then it also contains the strategies that $\alpha_i(\omega_i) = r_i$.*

Proof. See Appendix.‖

Case (B) implies that there exists $\alpha \in D^*$ with $\alpha \neq \alpha^*$. We pick the strategies $\alpha_i^* \in D_i^*$ and $\alpha_i \in D_i^* \backslash \{\alpha_i^*\}$ for arbitrary $i \in N$. Without loss of generality, we assume that $\alpha(\bar{\omega}_i) \neq \bar{\omega}_i$ for some $\bar{\omega}_i \in \Omega_i$. Since $\alpha_{-i}^* \in D_{-i}^*$ and EE(2), there exist some $\omega_{-i} \in \Omega_{-i}$ such that

$$u_i\left(g(\alpha_i, \alpha_{-i}^*), (\bar{\omega}_i, \omega_{-i})\right) < u_i\left(g(\alpha^*), (\bar{\omega}_i, \omega_{-i})\right). \tag{4.5}$$

Fix $\bar{\omega}_i$. Since $\alpha_i \in D_i^*$, there exist some $\alpha'_{-i} \in D_{-i}^*$ and some $\omega'_{-i} \in \Omega_{-i}$, we must have

$$u_i(g(\alpha_i, \alpha'_{-i}), (\bar{\omega}_i, \omega'_{-i})) > u_i(g(\alpha_i^*, \alpha'_{-i}), (\bar{\omega}_i, \omega'_{-i})) \tag{4.6}$$

Because of Claim 2, we assume that $\alpha_i(\bar{\omega}_i) = r_i$ without loss of generality, so it is impossible to hold equation (4.6) for any $\omega'_{-i} \in \Omega_{-i}$ and any $\alpha'_{-i} \in D_{-i}^*$ since EE(2) and the definition of outcome function. Thus we conclude that x is implementable in ex post weakly iteratively undominated strategies. ■

Theorem 1 leads to the following statements.

Corollary 1 *Assume that (A-1) and (A-2). If direct mechanism $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$ satisfies UEE, then x is quasi belief-free implementable.*

UEE condition is demanding but it enables us *twice* iterative dominance. Furthermore, the weak dominance (or quasi belief-free) criterion allow the mechanism to offer indifferent outcomes for agent i against some action $\alpha_{-i}(\omega_{-i})$. Permitting indifferent outcomes makes it possible to construct *finite* mechanisms which are used in the implementation. It remains unknown whether it is possible or not to belief-free implement x via finite mechanisms. See also subsection 4.5.2.

4.5 Discussion

4.5.1 On effective elimination condition

We have investigated a sufficient condition for quasi belief-free implementation. However, x is implementable in ex post weakly iteratively undominated strategies even though EE does not hold. We will show it by an example.

Suppose that $A = \{a, b, c, d, s, t, w, z, e\}$, $N = \{1, 2\}$, and $\Omega_i = \{\omega_i, \omega_i'\}$ for each $i \in N$. Social choice function $x = (x^1, x^2)$ and mechanism $\Gamma = ((\Omega_i \cup \{r_i\})_{i=1,2}, (g^1, g^2))$ are defined in Table 4.5. Payoffs in the mechanism is described in Table 4.6. True state is denoted on the upper-left corner in each table in 4.6.

Table 4.5:

(x^1, x^2)	ω_2	ω_2'
ω_1	a, a	b, b
ω_1'	c, c	d, d

(g^1, g^2)	ω_2	ω_2'	r_2
ω_1	a, a	b, b	e, w
ω_1'	c, c	d, d	e, z
r_1	s, e	t, e	e, e

Table 4.6:

(ω_1, ω_2)	ω_2	ω_2'	r_2
ω_1	6, 6	2, 2	0, 4
ω_1'	2, 2	4, 4	0, 1
r_1	4, 0	6, 0	0, 0

(ω_1, ω_2')	ω_2	ω_2'	r_2
ω_1	2, 2	6, 6	0, 2
ω_1'	4, 4	2, 2	0, 4
r_1	6, 0	4, 0	0, 0

(ω_1', ω_2)	ω_2	ω_2'	r_2
ω_1	2, 2	4, 4	0, 6
ω_1'	6, 6	2, 2	0, 4
r_1	4, 0	1, 0	0, 0

(ω_1', ω_2')	ω_2	ω_2'	r_2
ω_1	4, 4	2, 2	0, 1
ω_1'	2, 2	6, 6	0, 4
r_1	6, 0	4, 0	0, 0

Note that direct mechanism $\Gamma_x^d = ((\Omega_1, \Omega_2), x)$ satisfies the regularity condition, but does not satisfy EE. However, x is implementable in ex post weakly iteratively undominated strategies. In game $G = (\Gamma, (u_i)_{i=1,2})$, we can delete iteratively the ex post weakly dominated actions in the following manner;

Phase 1. Since action $\alpha_1(\omega_1) = \omega_1'$ is ex post weakly dominated by $\alpha_1(\omega_1) = r_1$, we delete it.

Phase 2. After Phase 1, since action $\alpha_2(\omega_2) = \omega_2'$ is ex post weakly dominated by $\alpha_2(\omega_2) = r_2$, we delete it.

Phase 3. After Phase 2, since action $\alpha_1(\omega_1') = \omega_1$ is ex post weakly dominated by $\alpha_1(\omega_1') = r_1$, we delete it.

Phase 4. After Phase 3, since action $\alpha_2(\omega_2') = \omega_2$ is ex post weakly dominated by $\alpha_1(\omega_2') = r_2$, we delete it.

Phase 5. After Phase 4, since the strategies that contains action r_i are ex post weakly dominated by truth-telling strategy for each $i = 1, 2$, we delete them.

After Phase 5, remaining strategy is only the truth-telling strategy, x is implementable in ex post weakly iteratively undominated strategies. This example suggests us that a sufficient condition for implementation in ex post weakly iteratively undominated strategies is described as follows.

Definition 10 *Direct revelation mechanism $\Gamma_x^d = ((\Omega_i)_{i=1}^n, x)$ satisfies the* ***weak effective elimination*** *condition (**WEE**), if there exist $i \in N$, ω_i', and $y_i : \Omega_{-i} \to A$ such that Γ_x^d satisfies the following inequalities:*

WEE(1): *For every $\omega_{-i} \in \Omega_{-i}$ and $\omega_i'' \in \Omega_i$,*

$$\forall \alpha_{-i}(\omega_{-i}) \neq \omega_{-i},\ u_i(y_i(\alpha_{-i}(\omega_{-i})), (\omega_i', \omega_{-i})) \geq u_i(x(\omega_i'', \alpha_{-i}(\omega_{-i})), (\omega_i', \omega_{-i})),$$

with strict inequalities for some $\alpha_{-i}(\omega_{-i})$ in (ω_i', ω_{-i}).

EE(2): *For every $\omega \in \Omega$,*

$$u_i(x(\omega), \omega) \geq u_i(y_i(\omega_{-i}), \omega) \geq u_i(x(\tilde{\omega}_i, \omega_{-i}), \omega)$$

for every $\tilde{\omega}_i \in \Omega_i \backslash \{\omega_i\}$, and

$$\exists \omega_{-i} \in \Omega_{-i},\ u_i(x(\omega), \omega) > u_i(y_i(\omega_{-i}), \omega).$$

Direct revelation mechanism Γ_x^d is said to satisfy the ***uniformly weak effective elimination*** *condition (**UWEE**) if for every $i \in N$, there exists social choice function $y_i : \Omega_{-i} \to A$ which satisfies both WEE(1) and EE(2).*

WEE(1) requires that for some agent i the payoffs from the outcomes of x obtained from untruthful action $\alpha_{-i}(\omega'_{-i})$ can be weakly dominated in ex post when his type is ω'_i. EE(1) further requires that WEE(1) holds for any type $\omega'_i \in \Omega_i$. One may implement a social choice function if the associated direct revelation mechanism satisfies WEE or UWEE, but if does so, suggested in the example, twice iteration property generally no longer holds.

Moreover EE and WEE focus on untruthful strategies. It implicitly implies we consider full implementation only by the truth-telling strategy; we presume untruthful strategies never lead to desired outcomes.

4.5.2 Robust monotonicity and effective elimination

Bergemann and Morris (2008b) provide the following condition for belief-free implementation: *robust monotonicity.* If strategy profile α satisfies $x(\omega) \neq x(\alpha(\omega))$ for some $\omega \in \Omega$, then α is said to be *unacceptable deception.* Robust monotonicity requires any unacceptable deception to be unprofitable for some agent in the end.[7] For notational convenience, put $x \circ \alpha(\omega) := x(\alpha(\omega))$.

Definition 11 *Social choice function x satisfies* ***robust monotonicity*** *if, for every α with $x \neq x \circ \alpha$, there exist i, ω'_i such that, for all $\omega'_{-i} \in \Omega_{-i}$ there exists y such that for all $\omega_{-i} \in \alpha_{-i}^{-1}(\omega'_{-i})$:*

$$u_i(y, (\omega'_i, \omega_{-i})) > u_i(x(\alpha_i(\omega'_i), \omega'_{-i})), (\omega'_i, \omega_{-i}))$$

and for all $\omega''_i \in \Omega_i$:

$$u_i(x(\omega''_i, \omega'_{-i}), (\omega''_i, \omega'_{-i})) \geq u_i(y, (\omega''_i, \omega'_{-i})). \tag{4.7}$$

Social choice function x satisfies ***strict robust monotonicity*** *if for all ω''_i with $x(\omega''_i, \omega'_{-i}) \neq y$ the inequality (4.7) is strict.*

We can prove the following:

Lemma 2 *Social choice function x satisfies robust monotonicity if and only if for every α with $x \neq x \circ \alpha$, there exist i, ω'_i, and social choice function $y_i : \Omega_{-i} \to A$ such that for all $\omega_{-i} \in \Omega_{-i}$:*

$$u_i(y_i(\alpha_{-i}(\omega_{-i})), (\omega'_i, \omega_{-i})) > u_i(x(\alpha_i(\omega'_i)), \alpha_{-i}(\omega_{-i})), (\omega'_i, \omega_{-i})) \tag{4.8}$$

and for all $\omega''_i \in \Omega_i$:

$$u_i(x(\omega''_i, \omega_{-i}), (\omega''_i, \omega_{-i})) \geq u_i(y_i(\omega_{-i}), (\omega''_i, \omega_{-i})). \tag{4.9}$$

[7] Bergemann and Morris (2008b) define a deception to be a set-valued profile $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $\alpha_i : \Omega_i \to \mathcal{P}(\Omega_i) \backslash \{\emptyset\}$ where $\mathcal{P}(\Omega_i)$ represents the power set of Ω_i.

Proof. The "if part" is obvious, so that we omit the proof. Let $A_i^x(\omega_{-i})$ denote a subset of A such that

$$A_i^x(\omega_{-i}) = \{a : u_i(x(\omega_i'', \omega_{-i}), (\omega_i'', \omega_{-i})) \geq u_i(a, (\omega_i'', \omega_{-i})), \forall \omega_i'' \in \Omega_i\}.$$

By robust monotonicity, there exist a function $\phi : \Omega_{-i} \to A$ such that $y = \phi(\omega_{-i}')$ for all $\omega_{-i}' \in \Omega_{-i}$ with

$$u_i(y, (\omega_i', \omega_{-i})) > u_i(x(\alpha_i(\omega_i'), \omega_{-i}')), (\omega_i', \omega_{-i}))$$

for all $\omega_{-i} \in \alpha_{-i}^{-1}(\omega_{-i}')$. Robust monotonicity requires that $\phi(\omega_{-i}) \in A_i^x(\omega_{-i})$ for all $\omega_{-i} \in \Omega_{-i}$ because of inequality (4.7). Therefore, define a function $y_i : \omega_{-i} \mapsto y' \in A_i^x(\omega_{-i})$ such that

$$y_i(\omega_{-i}) = \begin{cases} \phi(\omega_{-i}') & \text{if } \exists \omega_{-i}' \; s.t. \; \omega_{-i}' = \alpha_{-i}(\omega_{-i}) \\ y' \in A^x(\omega_{-i}) & \text{if } \nexists \omega_{-i}' \; s.t. \; \omega_{-i}' = \alpha_{-i}(\omega_{-i}). \end{cases}$$

■

Bergemann and Morris (2008b) show that ex post incentive compatibility and robust monotonicity for social choice function x are sufficient for belief-free implementation with infinite mechanism under mild assumption for the environment (Theorem 1 in Bergemann and Morris (2008b)). They also show that if x is belief-free implementable by a game with a *finite* mechanism, then x satisfies strict robust monotonicity (Proposition 3 in Bergemann and Morris (2008b)). However they do not investigate quasi belief-free implementation. It remains as a future research to investigate a necessary and sufficient condition for quasi belief-free implementation or implementation in ex post weakly iteratively undominated strategies.

4.6 Appendix

Proof of Claim 2.

Proof. Suppose that D_i^k consists of the strategies that $\alpha_i(\omega_i) \neq r_i$ for all $\omega_i \in \Omega_i$. Therefore, by definition of deletion sequence, for all strategies $\alpha_i \in D_i^k$, we must have

$$\forall \alpha_{-i} \in D_{-i}^k, \; \forall \omega \in \Omega, \quad u_i(g(\alpha), \omega) \geq u_i(g(r_i, \alpha_{-i}(\omega_{-i})), \omega). \tag{4.10}$$

with strict inequality for some $\alpha_{-i} \in D_{-i}^k$ and $\omega_{-i} \in \Omega_{-i}$. If $\alpha_{-i} \neq \alpha_{-i}^*$, then equation (4.10) is contradiction against UEE and the definition of the outcome function, so it is necessary that $\alpha_{-i} = \alpha^*_{-i}$. Therefore we must have $\alpha_i = \alpha^*_i$. ∎

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