8793

Doctral Thesis

Inverse Spectral Problem for Systems of Ordinary Differential Equations

Masahiro YAMAMOTO

University of Tokyo

Contents

Chapter 1

Introduction

1. In this paper, we consider a boundary value problem for a system of ordinary differential equations of first order in the interval $(0, 1)$:

$$(1)\quad \begin{cases} \dfrac{du_2(x)}{dx} + p_{11}(x)u_1(x) + p_{12}(x)u_2(x) = \lambda u_1(x) \\ \dfrac{du_1(x)}{dx} + p_{21}(x)u_1(x) + p_{22}(x)u_2(x) = \lambda u_2(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le x \le 1) \\ u_2(0) + hu_1(0) = 0 \\ u_2(1) + Hu_1(1) = 0 \end{cases}$$

Here p_{ij} $(1 \le i,j \le 2)$ are real-valued functions on $[0, 1]$ with appropriate regularity, and h, H are real constants. Moreover the parameter λ corresponds to the eigenvalue.

The boundary value problem (1) describes vibrations for various phenomena such as an electric oscillation in a transmission line, a vibration of a string with viscous drag, a longitudinal vibration in media with discontinuities.

Let us assume that p_{ij} $(1 \le i,j \le 2)$, h and H are given. Then a problem of finding eigenvalues of (1) is one of what are called **forward problems**, and such a problem is closely related to determination of eigenfrequencies. Thus the forward problem for (1) is important from both mathematical and practical points of view.

On the other hand, the following two problems are significant also from the practical viewpoint :

1. *the uniqueness of physical systems which realize the specified characteristics of vibrations (eigenfrequencies, for example).*
2. *the construction of an algorithm of such a physical system.*

The former is an **identification problem** and the latter is a **synthesis problem.** In the identification problem and the synthesis problem for (1), assuming that the eigenvalues are given, we determine the coefficients $p_{ij}(x)$ $(1 \le i,j \le 2)$ and the real constants h, H. Thus in contrast with the forward problem, the identification problem and the synthesis problem are ones of **inverse problems**, or in particular, **inverse spectral problems** in our case.

The purpose of this paper is to study the identification problem and the **continuous dependence problem** for (1). Here by the continuous dependence problem we understand a problem of discussing in what sense do coefficients of the equations in (1) determined from the eigenvalues, depend continuously upon the eigenvalues. We note that the continuous dependence problem is associated with the synthesis problem.

As the theoretical features of the inverse spectral problem, we can point out the **nonlinearity** and the **ill-posedness**. That is, let an operation of determining the coefficients $p_{ij}(x)$ $(1 \le i,j \le 2)$, etc. of (1) from the specified sets of eigenvalues can be defined. Then the operation is nonlinear, although the original problem (1) is linear in (u_1, u_2) . Furthermore it is not certain whether this operation is continuous with respect to the metric introduced naturally in terms of the asymptotic behavior of eigenvalues, and this fact suggests that our inverse problem is not well-posed in the sense of Hadamard.

These nonlinearity and ill-posedness make the analysis of the problem difficult and interesting.

Thus the mathematical analysis of the inverse spectral problem for the boundary value problem (1) is significant from the practical viewpoint as well as the theoretical one.

On the other hand, as for the identification problem in terms of evolutionary systems, we can refer to Kitamura and Nakagiri [24], Nakagiri [37], Nakagiri, Kitamura and Murakami [38], and, Nakagiri and Yamamoto [39], for instance. In these papers, they consider an abstract evolution equation

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = 0 \qquad (t \geq 0) \\ u(0) = u_0 \end{cases}$$

in a Banach space and study an identification problem on the unique determination of the operator A from observations over a time interval of the solution $u(t)$. Therefore the identification problem in [24], [37], [38] and [39] differs from the one for an eigenvalue problem as is considered in the present paper.

As for the synthesis problem on design of distributed parameter circuits, we refer to Heim and Sharpe [11].

2. As for the Sturm - Liouville problem

$$(2) \qquad \begin{cases} -\dfrac{d^2u(x)}{dx^2} + p(x)u(x) = \lambda u(x) \qquad (0 \leq x \leq 1) \\ \dfrac{du(0)}{dx} - hu(0) = 0 \\ \dfrac{du(1)}{dx} + Hu(1) = 0 \quad , \end{cases}$$

the inverse spectral problem has been considered in detail. For instance, we can refer to Ambarzumian [1], Borg [2], Gel'fand and

Levitan [5], Hald [7, 8, 9, 10], Hochstadt [12, 13, 14, 15, 16], Hochstadt and Lieberman [17], Isaacson, McKean and Trubowitz [18], Isaacson and Trubowitz [19], Iwasaki [20, 21], Levinson [27], Levitan and Gasymov [28], Marchenko [30], McLaughlin [33], Mizutani [34], Murayama [35], Suzuki [46, 47, 48, 49, 50, 51], Willis [55], and Žikov [59].

On the other hand, as for the inverse spectral problem for equations of higher order such as $(-1)^m \frac{d^{2m}u(x)}{dx^{2m}} + p(x)u(x) = \lambda u(x)$ $(0 \leq x \leq 1 ; m = 1,2,3,\ldots)$, we refer to Leĭbenzon [26], McLaughlin [31, 32], Sahnovič [44, 45], and Uchiyama [54].

However, in spite of practical importance and the interesting features because of the fact that multiple coefficients in the system (1) have to be considered, as for the inverse spectral problem for systems of ordinary differential equations such as (1), there has been little work. This situation is the author's motivation for this research.

Of course, by eliminating u_1 or u_2 in the system (1) so as to get a single equation of second order, it is impossible to reduce our problem (1) to the Sturm - Liouville problem, except for particular cases such as $p_{11}(x) = p_{21}(x) = p_{22}(x) = 0$ $(0 \leq x \leq 1)$ $h = H = 0$. Moreover we note that (1) is a generalization of the Sturm - Liouville problem (2). That is, as is seen in §5 in Chapter 3, the problem (2) can be transformed to (1) in an appropriate manner, in the cases where the boundary conditions at $x = 0$, 1 are the homogeneous Dirichlet condition or the homogeneous Neumann condition.

3. This paper consists of four chapters including the present chapter.

In Chapter 2, we define the sets of eigenvalues which are given in order to determine the coefficients, etc. in (1) and we formulate our identification problem in the most general form (Definition 2 in $1). That is, let us introduce another boundary value problem (1)', where the boundary condition $u_2(1) + Hu_1(1) = 0$ in (1) is replaced by $u_2(1) + H^* u_1(1) = 0$ $(H \neq H^*)$:

$$
(1)' \quad \begin{cases} \dfrac{du_2(x)}{dx} + p_{11}(x)u_1(x) + p_{12}(x)u_2(x) = \lambda u_1(x) \\[2ex] \dfrac{du_1(x)}{dx} + p_{21}(x)u_1(x) + p_{22}(x)u_2(x) = \lambda u_2(x) \\[2ex] \qquad\qquad\qquad\qquad\qquad\qquad (0 \le x \le 1) \\[2ex] u_2(0) + hu_1(0) = 0 \\[2ex] u_2(1) + H^* u_1(1) = 0 \qquad\qquad . \end{cases}
$$

Throughout this paper, we adopt the spectra of both (1) and (1)' as the specified sets of eigenvalues given for the determination of the coefficients, etc.

The reason why we have to introduce another boundary value problem (1)' is because the spectrum of a single boundary value problem is not sufficient for the determinaion of the coefficients. Originally this formulation of the inverse spectral problem has been considered for the Sturm - Liouville problem since Borg [2]. On the other hand, it is natural to consider two kinds of boundary conditions at $x = 1$ from the realistic point of view, because such replacement is often easy, especially for our vibrating systems. In fact, in order to realize replacement of boundary conditions for electric oscillations in a transmission line, we have only to insulate or earth the ends of the line, for example. (See Remark 6 in $6 of Chapter 3.)

Furthermore in the same chapter, we characterize the boundary value problems which possess the same specified sets of eigenvalues (Theorem in $1 in Chapter 2). Actually we consider two pairs such that each pair is composed of two boundary value problems in the forms (1) and (1)'. Then these two pairs possess the same sets of eigenvalues defined above if and only if we have four nonlinear equalities with respect to the coefficients of the equations and the constants of the boundary conditions in the two pairs of boundary value problems. The nonlinearity of these equalities corresponds to the above-mentioned one. Moreover these equalities are crucial in the succeeding chapters.

In Chapter 3, we consider the following three types of identification problems : Let us assume that in the coefficient matrix $\begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}$ of the equations in (1), one diagonal component and one nondiagonal component, two nondiagonal components and two diagonal components are known, respectively. In the respective three cases, we discuss the identification problems of uniquely determining the other components from the sets of eigenvalues defined in Chapter 2. For convenience, we will call these three problems Type-I, Type-II and Type-III, respectively.

The reason why we have to restrict our identification problems to the determination of two components of the coefficient matrices is the following characteristic points of systems of ordinary differential equations under consideration :
1) Let us consider two pairs such that each pair is composed of

two boundary value problems in the forms (1) and (1)', and let these two pairs possess the same sets of eigenvalues defined in Chapter 2. Then by the main theorem in Chapter 2, we can obtain at most **four** *equalities in the coefficients, though we have unknown* **eight** *coefficients in both pairs. (We note that each pair has four coefficients.) In this sense, it turns out that the sets of eigenvalues obtained by considering two kinds of boundary conditions at $x = 1$, are not sufficient for the identification of all the coefficients.*

2) *We cannot get more data than data of the sets of eigenvalues as is defined in Chapter 2, even though we further replace boundary conditions to consider a pair composed of three boundary value problems. In this sense, the sets of eigenvalue obtained in Chapter 2 contain the maximum data for the identification problem.*

Now we determine the sets of coefficients and constants which satisfy the four equalities derived in Chapter 2, so that for the uniqueness in the problems of Type-I, Type-II and Type-III, we can get the affirmative result (Theorem 1 in §2.1 in Chapter 3), the partially affirmative one (Theorem 2 in §2.2) and the negative one (Theorem 3 in §2.3), respectively. In other words, these theorems mean that there exist components easy to identify and components difficult to identify in the coefficient matrix. This fact is one of notable features of our inverse problem.

Furthermore, as applications of the results for Type-I and Type-II, we consider the uniqueness in the identification problems concerning proper vibrations of a string with viscous drag and electric oscillations in a transmission line, respectively.

In Chapter 4, we study the continuous dependence problem for the identification problem of Type-I. In view of Theorem 1 in Chapter 3, we can define the mapping which transforms the sets of eigenvalues defined in Chapter 2 into the one diagonal and the one nondiagonal components of the coefficient matrix, and we can show the result (Theorem 2 in $1), which asserts that if we introduce a stronger metric than the one matching with the asymptotic behavior of eigenvalues, then this mapping is continuous with respect to the introduced metric of the specified sets of eigenvalues and to an appropriate norm of the corresponding coefficients. Here as a norm of the coefficients, we take the maximum norm, for instance. This fact is associated with the ill-posedness as is mentioned above.

4. The main features of the method adopted in this paper are the following two.

1) **The integral transformation between eigenfunctions** : We can construct an integral operator which transforms each eigenfunction of a problem (1) to an eigenfunction of a problem of the same type with a different coefficient matrix. Furthermore we find the kernel functions of the integral operator as the solution to a hyperbolic system of partial differential equations of first order. This method is used in the inverse spectral problem for the Sturm - Liouville equation by Gel'fand and Levitan [5], Suzuki [46, 47], etc.

While a single coefficient has to be considerd in the Sturm - Liouville problem, there appear four coefficients maneuvearble in our problem. Therefore we need an appropriate device for its construction. Then the equalities in the main result in Chapter 2 can be derived from the boundary conditions imposed on the above-mentioned hyperbolic system.

2) **The hyperbolic system stated in 1) as the alternative to the Gel'fand - Levitan equation** : For the Sturm - Liouville problem, the kernel function of the integral operator as in 1) is the solution to a hyperbolic differential equation of second order, and at the same time, also the solution to some integral equation called the Gel'fand - Levitan equation. By using this fact, it is possible to solve the continuous dependence problem for the Sturm - Liouville equation (Iwasaki [20]).

On the other hand, for (1), an equation corresponding to the Gel'fand - Levitan equation is not derived. However we can solve our continuous dependence problem by the fact that the solutions to the hyperbolic system, satisfied by the kernel functions, depend upon boundary data continuously in an appropriate sense. All these proof is carried out in Chapter 4. For the proof, the following two facts are crucial :

(i) a priori estimates on solutions to the hyperbolic systems.

(ii) results on perturbation of some basis called a Riesz basis.

Chapter 2

Preparation for the Identification Problem : Characterization of Boundary Value Problems in Terms of Eigenvalues

§1. Formulation and the result. We consider a system of ordinary differential equations of first order in the interval $(0, 1)$:

$$
(1.1) \qquad \begin{cases} \dfrac{du_2(x)}{dx} + p_{11}(x)u_1(x) + p_{12}(x)u_2(x) = \lambda u_1(x) \\[2ex] \dfrac{du_1(x)}{dx} + p_{21}(x)u_1(x) + p_{22}(x)u_2(x) = \lambda u_2(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le x \le 1) \end{cases}
$$

with boundary conditions

$$
(1.2) \qquad u_2(0) + hu_1(0) = 0
$$

and

$$
(1.3) \qquad u_2(1) + Hu_1(1) = 0 \quad .
$$

Here let $p_{ij}(x)$ $(1 \le i,j \le 2)$ be real-valued C^1-functions defined on $[0, 1]$ and, let $h, H \in \mathbb{R} \cup \{ \infty \}$. Throughout this paper, if $h = \infty$ and $H = \infty$ in (1.2) and (1.3), respectively, then we regard (1.2) and (1.3) as $u_1(0) = 0$ and $u_1(1) = 0$, respectively (cf. Remark 3 below).

In this paper, we discuss the following problem called the inverse spectral problem : to determine the coefficients $p_{ij}(x)$ $(1 \le i, j \le 2)$ and the constants h and H from some knowledge of the eigenvalues of the boundary value problem (1.1) - (1.3).

The purpose of Chapters 2 and 3 is to discuss the identification problem, that is, whether the coefficients $p_{ij}(x)$ $(1 \le i,j \le 2)$ in the equation (1.1) and the constants h, H in the boundary conditions (1.2) and (1.3) are uniquely determined from the sets of eigenvalues obtained in the manner stated below. Chapter 2 is devoted to a preparation for Chapter 3 where the identification problems of three types are considered.

As for the inverse Sturm - Liouville problem :

$$(1.4) \qquad -\frac{d^2u(x)}{dx^2} + p(x)u(x) = \lambda u(x) \qquad (0 \le x \le 1)$$

and

$$(1.5) \qquad \begin{cases} \dfrac{du(0)}{dx} - hu(0) = 0 \\ \\ \dfrac{du(1)}{dx} + Hu(1) = 0 \quad , \end{cases}$$

according to the result in Borg [2], Levinson [27], etc., one set of eigenvalues associated with given boundary conditions does not determine the potential $p(x)$ and, if in addition we give a set of eigenvalues associated with another boundary conditions, we can determine $p(x)$ uniquely.

Also for our problem, we adopt a formulation similar to the one in Borg [2], Hochstadt [13], Levinson [27], and Suzuki [46, 51].

Remark 1. For the inverse Sturm-Liouville problem, we have two other formulations. That is, in Hald [10], Hochstadt and Lieberman [17], Suzuki [46, 51], and Willis [55], it is shown that one set of eigenvalues associated with one pair of boundary conditions at $x = 0, 1$ determines $p(x)$ uniquely on the whole interval $[0, 1]$ on condition that $p(x)$ is known on the "half" interval $[0, \frac{1}{2}]$.

On the other hand, in Borg [2], Hald [7, 8], Hochstadt [13], Levinson [27], and Suzuki [46, 48], on the assumption that $p(x)$ in the Sturm-Liouville problem (1.4) is spatially symmetric (that is, $p(x) = p(1 - x)$ for $0 \leq x \leq 1$), the same conclusion as the one above stated holds. As for the spatially symmetric potential, we can further refer to Iwasaki [20], and Suzuki [49, 50].

Also for our system (1.1), under formulations similar to the ones stated above, we can discuss the inverse spectral problem to obtain the corresponding results.

In this section, we give the formulation and state our result, which is crucial for the identification problems in Chapter 3.

Let $L^2(0,1)$ be the Hilbert space of square integrable complex-valued functions in the interval $(0, 1)$ and let $\{ L^2(0,1) \}^2$ be the product space. As is well-known, the space $\{ L^2(0,1) \}^2$ is a Hilbert space with an inner product defined by (1.6) :

$$(1.6) \qquad (u, v) = (u, v)_{\{L^2(0,1)\}^2}$$

$$= \left(\left(\int_0^1 u_1(x)\overline{v_1(x)}\, dx \right)^2 + \left(\int_0^1 u_2(x)\overline{v_2(x)}\, dx \right)^2 \right)^{1/2}$$

$$(u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \{ L^2(0,1) \}^2),$$

where $\overline{\alpha}$ denotes the complex conjugate of $\alpha \in \mathbb{C}$.

Henceforth let p_{ij} and q_{ij} $(1 \leq i, j \leq 2)$ be real-valued C^1-functions defined on $[0, 1]$ and let $h, j, H, H^*, J, J^* \in \mathbb{R} \cup \{ \infty \}$. We set

$$P(x) = (p_{ij}(x))_{1\leq i,j\leq 2} = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix} ,$$

$$Q(x) = (q_{ij}(x))_{1\leq i,j\leq 2} = \begin{pmatrix} q_{11}(x) & q_{12}(x) \\ q_{21}(x) & q_{22}(x) \end{pmatrix} ,$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

We introduce

Definition 1. *We define an operator* $A_{P,h,H}$ *in* $\{ L^2(0,1) \}^2$ *by the realization in* $\{ L^2(0,1) \}^2$ *of a differential operator* $B \frac{d\cdot}{dx} + P(x)\cdot$ *with boundary conditions* $u_2(0) + hu_1(0) = 0$ *and* $u_2(1) + Hu_2(1) = 0$. *That is,*

$$(1.7) \qquad (A_{P,h,H}u)(x) = B \frac{du(x)}{dx} + P(x)u(x) \qquad u \in \mathcal{D}(A_{P,h,H})$$

$$\mathcal{D}(A_{P,h,H}) = \Big\{ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \{ H^1(0,1) \}^2 ;$$
$$u_2(0) + hu_1(0) = 0 , \; u_2(1) + Hu_1(1) = 0 \Big\} .$$

Here $H^1(0,1)$ denotes the Sobolev space and $\{ H^1(0,1) \}^2$ is the product space. Similarly we define the operators A_{P,h,H^*}, $A_{Q,j,J}$, A_{Q,j,J^*}, etc.

Remark 2. Throughout this paper, we can assume that

(1.8) $\quad h \neq \infty \quad ,$

in the boundary condition $u_2(0) + hu_1(0) = 0$.
In fact, let $h = \infty$, that is, the boundary condition at $x = 0$ be $u_1(0) = 0$. Then, defining v by

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \quad ,$$

we have a system

$$(1.1)' \quad \begin{cases} \dfrac{dv_2(x)}{dx} + p_{22}(x)v_1(x) + p_{21}(x)v_2(x) = \lambda\, v_1(x) \\[2em] \dfrac{dv_1(x)}{dx} + p_{12}(x)v_1(x) + p_{11}(x)v_2(x) = \lambda\, v_2(x) \\ \hfill (0 \leq x \leq 1) \end{cases}$$

with the boundary conditions

$$(1.2)' \quad v_2(0) + 0 \cdot v_1(0) = 0$$

and

$$(1.3)' \quad v_2(1) + \frac{1}{H} \cdot v_1(1) = 0 \quad .$$

The system (1.1)' with (1.2)' and (1.3)' is the one obtained by in (1.1) - (1.3) replacing h and H by 0 and $\frac{1}{H}$, respectively.

Henceforth let $\sigma(A_{P,h,H})$ denote the spectrum of the operator $A_{P,h,H}$. As for $\sigma(A_{P,h,H})$, the following result is known (Russell [42, 43], for example).

Proposition 0. *Let* $h \in \mathbb{R} \setminus \{ -1, 1 \}$ *and* $H \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1, 1 \}$.

(I) The spectrum of the operator $A_{P,h,H}$ *consists entirely of countable eigenvalues* λ_n *and the multiplicity of each* λ_n *is one. That is,* $\dim \operatorname{Ker} (\lambda_n - A_{P,h,H}) = 1$.

Let us set

$$(1.9) \qquad \sigma(A_{P,h,H}) = \{ \lambda_n \}_{n \in \mathbb{Z}} .$$

(II) (the asymptotic behavior of the eigenvalues) We put

$$(1.10) \qquad \gamma = \begin{cases} \frac{1}{2} \log \frac{(1+h)(1-H)}{(1-h)(1+H)} , & \text{if } H \neq \infty , \\ \\ \frac{1}{2} \log \frac{h+1}{h-1} , & \text{if } H = \infty , \end{cases}$$

and

$$(1.11) \qquad \theta = \frac{1}{2} \int_0^1 (p_{11}(s) + p_{22}(s))\, ds ,$$

where in (1.10) we take the principal value of the logarithm. Then we have

$$(1.12) \qquad \lambda_n = \gamma + \theta + n\pi\sqrt{-1} + O\left(\frac{1}{n} \right) \qquad (\text{as } |n| \to \infty) .$$

(III) (the completeness of eigenvectors) Let us denote an eigenvector associated with the eigenvalue λ_n *by* $\phi_n(\cdot) = \phi(\cdot, \lambda_n)$. *Then the system* $\{ \phi_n \}_{n \in \mathbb{Z}}$ *is a "Riesz basis" in*

the Hilbert space $\{ L^2(0,1) \}^2$, *that is, each* $\begin{pmatrix} u \\ v \end{pmatrix} \in \{L^2(0,1)\}^2$ *has a unique expansion*

$$(1.13) \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \sum_{n=-\infty}^{\infty} c_n \phi_n$$

and furthermore we have

$$(1.14) \qquad M^{-1} \sum_{n=-\infty}^{\infty} |c_n|^2 \leq \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\{L^2(0,1)\}^2} \leq M \sum_{n=-\infty}^{\infty} |c_n|^2$$

for some positive constant M *independent of* $\begin{pmatrix} u \\ v \end{pmatrix}$.

From now on, for simplicity of our discussion, we suppose the following assumption on the real constants in the boundary conditions.

Assumption.

$$(1.15) \qquad \begin{cases} h, j \in \mathbb{R} \setminus \{ -1, 1 \} , \\ \\ H, H^*, J, J^* \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1, 1 \} , \text{ and } H \neq H^* . \end{cases}$$

In particular, $H \neq H^*$ implies that either H or H^* is finite.

Now our inverse spectral problem or identification problem can be stated as follows :

Problem A. Do the conditions

$$
(1.16)\qquad \begin{cases} \sigma(A_{Q,j,J}) = \sigma(A_{P,h,H}) \\ \\ \sigma(A_{Q,j,J^*}) = \sigma(A_{P,h,H^*}) \end{cases}
$$

imply $Q(x) = P(x)$ $(0 \leq x \leq 1)$, $j = h$, $J = H$, and $J^* = H^*$?

As is seen from Theorem below, the answer for Problem A is negative. Thus in this chapter, we concentrate upon

Problem B. To characterize (Q, j, J, J^*) such that $\sigma(A_{Q,j,J}) = \sigma(A_{P,h,H})$ and $\sigma(A_{Q,j,J^*}) = \sigma(A_{P,h,H^*})$.

In order to consider Problem B, we introduce

Definition 2. *Let* $P = (p_{ij})_{1 \leq i,j \leq 2} \in \{ C^1[0,1] \}^4$ *and* h, H, H^* *be fixed such that* $h \in \mathbb{R} \setminus \{ -1, 1 \}$, $H, H^* \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1, 1 \}$, *and* $H \neq H^*$.

We set

$$
(1.17)\qquad M(P,h,H,H^*)
$$

$$
= \Bigg\{ (Q,j,J,J^*) \ ; \ Q = (q_{ij})_{1 \leq i, j \leq 2} \in \{ C^1[0,1] \}^4,
$$

$$
j \in \mathbb{R} \setminus \{ -1, 1 \}, \quad J, J^* \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1, 1 \} ,
$$

and

$$
\begin{cases} \sigma(A_{Q,j,J}) = \sigma(A_{P,h,H}) \\ \\ \sigma(A_{Q,j,J^*}) = \sigma(A_{P,h,H^*}) \end{cases} \quad . \Bigg\} ,
$$

and

$$(1.18) \qquad \widetilde{M}(P,h,H,H^*)$$

$$= \Big\{ (Q,j,J,J^*) \ ; \ Q = (q_{ij})_{1 \le i,\ j \le 2} \in \{ C^1[0,1] \}^4,$$

$$j \in \mathbb{R} \setminus \{ -1,\ 1 \}, \quad J,\ J^* \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1,\ 1 \},$$

and

$$\left. \begin{cases} \sigma(A_{Q,j,J}) \supset \sigma(A_{P,h,H}) \\ \sigma(A_{Q,j,J^*}) \supset \sigma(A_{P,h,H^*}) \end{cases} . \right\} .$$

In other words, $M(P,h,H,H^*)$ denotes the totality of operators $A_{Q,j,J}$ and A_{Q,j,J^*} whose spectra coincide with the spectra of the operators $A_{P,h,H}$ and A_{Q,j,J^*}, respectively.

It is obvious that $(p,h,H,H^*) \in M(P,h,H,H^*)$. If we had $M(P,h,H,H^*) = \{ (P,h,H,H^*) \}$, then the two sets of eigenvalues would determine the operators $A_{P,h,H}$ and A_{P,h,H^*} uniquely. (That is, the answer for Problem A would be affirmative.) Thus, for the discussion of the uniqueness or the nonuniqueness in our inverse problem, it is sufficient to determine the set $M(P,h,H,H^*)$.

Moreover, throughout this paper, we note

Remark 3. Henceforth we set

$$(1.19) \qquad \frac{1 - H}{1 + H} = \frac{1 + H}{1 - H} = -1 \ , \quad \text{if} \quad H = \infty \ .$$

Then, for example, in (1.10) of Proposition 0, we always get

$$\gamma = \frac{1}{2} \log \frac{(1 + h)(1 - H)}{(1 - h)(1 + H)} \qquad \text{for each} \quad H \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1, 1 \} \ .$$

Furthermore, throughout this paper, we use the following :

(1.20) Let $\alpha, \beta \in \mathbb{R}$. If $H = \infty$, then the equality $\alpha + \beta H = 0$ means $\beta = 0$.

Then, without distinguishing the cases $H = \infty$, $J = \infty$, etc. from the cases $H \neq \infty$, $J \neq \infty$, etc., we can formally write and mathematically follow all our discussion in this paper. For example, $H = \infty$ means $u_1(1) = 0$ in (1.3).

As a characterization of $M(P,h,H,H^*)$, our main theorem in this chapter can be stated as follows.

Theorem . *(I) We have*

$$(1.21) \qquad \tilde{M}(P,h,H,H^*) = M(P,h,H,H^*) \quad .$$

(II) We have

$$(1.22) \qquad (Q,j,J,J^*) \in M(P,h,H,H^*)$$

if and only if (1.23) - (1.26) hold :

$$(1.23)\quad \begin{aligned} &\frac{1-j}{1-h}\big(q_{11}(x) + q_{12}(x) - q_{21}(x) - q_{22}(x) \\ &\qquad - p_{11}(x) + p_{12}(x) - p_{21}(x) + p_{22}(x) \big) \\ &+ \frac{1+j}{1+h}\big(q_{11}(x) - q_{12}(x) + q_{21}(x) - q_{22}(x) \\ &\qquad - p_{11}(x) - p_{12}(x) + p_{21}(x) + p_{22}(x) \big) \\ &\times \exp\Big(\int_0^x (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds\Big) = 0 \\ &\hfill (0 \leq x \leq 1), \end{aligned}$$

$$(1.24)\quad \begin{aligned} &\frac{1-j}{1-h}\big(q_{11}(x) + q_{12}(x) - q_{21}(x) - q_{22}(x) \\ &\qquad + p_{11}(x) - p_{12}(x) + p_{21}(x) - p_{22}(x) \big) \\ &+ \frac{1+j}{1+h}\big(- q_{11}(x) + q_{12}(x) - q_{21}(x) + q_{22}(x) \\ &\qquad - p_{11}(x) - p_{12}(x) + p_{21}(x) + p_{22}(x) \big) \\ &\times \exp\Big(\int_0^x (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds\Big) = 0 \\ &\hfill (0 \leq x \leq 1)\ , \end{aligned}$$

$$(1.25)\quad \begin{cases} \dfrac{(1+h)(1-H)(1-j)(1+J)}{(1-h)(1+H)(1+j)(1-J)} > 0\ , \\[2ex] \log \dfrac{(1+h)(1-H)(1-j)(1+J)}{(1-h)(1+H)(1+j)(1-J)} \\ = \displaystyle\int_0^1 (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds\ , \end{cases}$$

$$(1.26)\quad \begin{cases} \dfrac{(1+h)(1-H^*)(1-j)(1+J^*)}{(1-h)(1+H^*)(1+j)(1-J^*)} > 0, \\[2ex] \log \dfrac{(1+h)(1-H^*)(1-j)(1+J^*)}{(1-h)(1+H^*)(1+j)(1-J^*)} \\ = \displaystyle\int_0^1 (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds\ . \end{cases}$$

From the fact that the equalities (1.23) and (1.24) can be regarded as two nonlinear integral equations of four unknown functions q_{ij} $(1 \leq i, j \leq 2)$, we can show that there are infinitely many $Q = (q_{ij})_{1 \leq i,j \leq 2}$ satisfying (1.23) and (1.24). That is, the answer for Problem A is negative.

This chapter is composed of three sections and three appendixes. In $2, we derive a formula (a "deformation formula" according to the terminology in Suzuki [46]), which is a key in the later discussion. In $3, we prove Theorem.

$2. **Deformation formula.** We begin this section with the following proposition on a system of hyperbolic equations. Let

$$(2.1) \qquad \Omega = \left\{ (x, y) \; ; \; 0 < y < x < 1 \right\} ,$$

and we recall that $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proposition 1. *Let* $P = (p_{ij})_{1 \le i,j \le 2}$ *and* $Q = (q_{ij})_{1 \le i,j \le 2}$ *be given in* $\{ C^1[0,1] \}^4$ *, and let* h *be a given real number satisfying* $|h| \neq 1$ *. Then, for each* r_1, r_2 $\in C^1[0,1]$ *, there exists a unique* $K = K(x,y) = (K_{ij}(x,y))_{1 \le i,j \le 2} \in \{ C^1(\overline{\Omega}) \}^4$ *satisfying (2.2) - (2.5) :*

$$(2.2) \qquad B \frac{\partial K(x,y)}{\partial x} + Q(x)K(x,y) - K(x,y)P(y) = - \frac{\partial K(x,y)}{\partial y} B$$

$$((x,y) \in \overline{\Omega}).$$

$$(2.3) \qquad \begin{cases} K_{12}(x,0) = hK_{11}(x,0) \\ K_{22}(x,0) = hK_{21}(x,0) \qquad (0 \le x \le 1) \end{cases} .$$

$$(2.4) \qquad K_{12}(x,x) - K_{21}(x,x) = r_1(x) \qquad (0 \le x \le 1) .$$

$$(2.5) \qquad K_{11}(x,x) - K_{22}(x,x) = r_2(x) \qquad (0 \le x \le 1) .$$

Proposition 1 is proved by reducing (2.2) - (2.5) to a system of Volterra's integral equations (cf. Petrovsky [40] and Picard [41]) and we carry out its proof in Appendix I.

Let $P = (p_{ij})_{1 \le i,j \le 2}$ and $Q = (q_{ij})_{1 \le i,j \le 2}$ be given in $\{ C^1[0,1] \}^4$ and let $h, j \in \mathbb{R}$ satisfy $|h|, |j| \ne 1$. Then we set

$$\theta_1(x) = \frac{1}{2} \int_0^x (q_{12}(s) + q_{21}(s) - p_{12}(s) - p_{21}(s))ds \qquad (0 \le x \le 1), \tag{2.6}$$

$$\theta_2(x) = \frac{1}{2} \int_0^x (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds \qquad (0 \le x \le 1), \tag{2.7}$$

and

$$\begin{cases} a_1 = \dfrac{1 - j}{1 - h} \\ \\ a_2 = \dfrac{1 + j}{1 + h} \end{cases} . \tag{2.8}$$

Moreover let us put

$$a(x) = \frac{1}{2} \left\{ a_1 \exp(- \theta_1(x) - \theta_2(x)) + a_2 \exp(- \theta_1(x) + \theta_2(x)) \right\} \quad (0 \le x \le 1), \tag{2.9}$$

$$b(x) = \frac{1}{2} \left\{ a_1 \exp(- \theta_1(x) - \theta_2(x)) - a_2 \exp(- \theta_1(x) + \theta_2(x)) \right\} \quad (0 \le x \le 1), \tag{2.10}$$

and

$$R(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & a(x) \end{pmatrix} \qquad (0 \le x \le 1). \tag{2.11}$$

Now we get to state a "deformation formula" in our case ;

Lemma 1. *(I) For given* $P, Q \in \{ C^1[0,1] \}^4$ *and* $h, j \in \mathbb{R}$ *such that* $|h|, |j| \neq 1$, *there exists a unique* $K = K(x,y) = (K_{ij}(x,y))_{1 \leq i,j \leq 2} \in \{ C^1(\overline{\Omega}) \}^4$ *satisfying (2.12) - (2.15) :*

$$(2.12) \qquad B \frac{\partial K(x,y)}{\partial x} + Q(x)K(x,y) - K(x,y)P(y) = - \frac{\partial K(x,y)}{\partial y} B \qquad ((x, y) \in \overline{\Omega}).$$

$$(2.13) \qquad \begin{cases} K_{12}(x,0) = hK_{11}(x,0) \\ \\ K_{22}(x,0) = hK_{21}(x,0) \qquad (0 \leq x \leq 1). \end{cases}$$

$$\begin{aligned} (2.14) \qquad & K_{12}(x,x) - K_{21}(x,x) = \\ & \frac{1}{4} a_1 \exp (- \theta_1(x) - \theta_2(x)) \\ & \times (q_{11}(x) + q_{12}(x) - q_{21}(x) - q_{22}(x) \\ & \quad - p_{11}(x) + p_{12}(x) - p_{21}(x) + p_{22}(x)) \\ & + \frac{1}{4} a_2 \exp (- \theta_1(x) + \theta_2(x)) \\ & \times (q_{11}(x) - q_{12}(x) + q_{21}(x) - q_{22}(x) \\ & \quad - p_{11}(x) - p_{12}(x) + p_{21}(x) + p_{22}(x)) \quad (0 \leq x \leq 1). \end{aligned}$$

$$\begin{aligned} (2.15) \qquad & K_{11}(x,x) - K_{22}(x,x) = \\ & \frac{1}{4} a_1 \exp(- \theta_1(x) - \theta_2(x)) \\ & \times (q_{11}(x) + q_{12}(x) - q_{21}(x) - q_{22}(x) \\ & \quad + p_{11}(x) - p_{12}(x) + p_{21}(x) - p_{22}(x)) \\ & + \frac{1}{4} a_2 \exp(- \theta_1(x) + \theta_2(x)) \end{aligned}$$

$$\times \, (- q_{11}(x) + q_{12}(x) - q_{21}(x) + q_{22}(x)$$
$$- p_{11}(x) - p_{12}(x) + p_{21}(x) + p_{22}(x) \,) \quad (0 \le x \le 1).$$

(II) (a deformation formula) If, for $\lambda \in \mathbb{C}$ *, the function* $\phi(\cdot) = \phi(\cdot,\lambda) \in \{ C^1[0,1] \}^2$ *satisfies*

$$\text{(2.16)} \qquad B\,\frac{d\phi(x)}{dx} + P(x)\phi(x) = \lambda\phi(x) \qquad (0 \le x \le 1)$$

and

$$\text{(2.17)} \qquad \phi(0) = \begin{pmatrix} 1 \\ -h \end{pmatrix} ,$$

then $\psi(\cdot) = \psi(\cdot,\lambda) \in \{ C^1[0,1] \}^2$ *defined by*

$$\text{(2.18)} \qquad \psi(x,\lambda) = R(x)\phi(x,\lambda) + \int_0^x K(x,y)\phi(y,\lambda)dy \qquad (0 \le x \le 1) ,$$

satisfies

$$\text{(2.19)} \qquad B\,\frac{d\psi(x)}{dx} + Q(x)\psi(x) = \lambda\psi(x) \qquad (0 \le x \le 1) ,$$

and

$$\text{(2.20)} \qquad \psi(0) = \begin{pmatrix} 1 \\ -j \end{pmatrix} .$$

Remark 4. As for the Sturm - Liouville problem, a formula of the type of (2.18) is derived in Suzuki and Murayama [52], which is stated as follows ; Let $p, q \in C^1[0,1]$, and $h, j \in \mathbb{R}$ be given. Then there exists a unique $L = L(x,y) \in C^2(\bar{\Omega})$ such that

$$\frac{\partial^2 L(x,y)}{\partial x^2} - \frac{\partial^2 L(x,y)}{\partial y^2} + p(y)L(x,y) = q(x)L(x,y)$$

$$((x, y) \in \bar{\Omega}),$$

$$L(x,x) = j - h + \frac{1}{2}\int_0^x (q(s) - p(s))ds \qquad (0 \leq x \leq 1) ,$$

and

$$\frac{\partial L(x,0)}{\partial y} = hL(x,0) \qquad (0 \leq x \leq 1) .$$

Furthermore, if $f(\cdot) = f(\cdot,\lambda) \in C^2[0, 1]$ satisfies

$$(p(x) - \frac{d^2}{dx^2})f(x) = \lambda f(x) \qquad (0 \leq x \leq 1)$$

$$f(0) = 1, \qquad \frac{df(0)}{dx} = h \qquad , \text{ for } \lambda \in \mathbb{R} ,$$

then $g(\cdot) = g(\cdot,\lambda) \in C^2[0, 1]$ defined by

$$(2.21) \qquad g(x) = f(x) + \int_0^x L(x,y)f(y)dy \qquad (0 \leq x \leq 1)$$

satisfies

$$(q(x) - \frac{d^2}{dx^2})g(x) = \lambda g(x) \qquad (0 \leq x \leq 1)$$

$$g(0) = 1 , \qquad \frac{dg(0)}{dx} = j \qquad .$$

In particular case, the formula (2.21) is shown in Marchenko [30], or Gel'fand and Levitan [5].

The formula (2.21) is a key for the inverse Sturm - Liouville problem.

On the other hand, in Gasymov and Levitan [4] (cf. Levitan and Sargsjan [29]), a similar formula is shown for the one - dimensional Dirac's system.

Remark 5. In our systems (2.16) and (2.19), we note that there are four coefficients to be "deformed" (namely, $p_{ij} \longrightarrow q_{ij}$ for $1 \leq i,j \leq 2$). Thus we need to introduce $R(x)$ in our formula (2.18) as a modification factor.

Moreover, also as for more general systems involving N functions on the interval $[0, 1]$, we can show formulae similar to (2.18).

Proof of Lemma 1. The part (I) of this lemma is seen by Proposition 1.

The part (II) follows by elementary computations including integration by parts. In fact, we can prove the part (II) as follows.

In (2.18), we get

$$(2.22) \qquad B\frac{dR(x)}{dx}$$

$$= B\frac{dR(x)}{dx}\phi(x) + BR(x)\frac{d\phi(x)}{dx}$$

$$+ BK(x,x)\phi(x) + \int_0^x B\frac{\partial K(x,y)}{\partial x}\phi(y)dy$$

$$= B\frac{dR(x)}{dx}\phi(x) + R(x)B\frac{d\phi(x)}{dx} + BK(x,x)\phi(x)$$

$$+ \int_0^x \left(-\frac{\partial K(x,y)}{\partial y}B\phi(y) + (K(x,y)P(y) - Q(x)K(x,y))\phi(y)\right)dy$$

$$(\text{by } BR(x) = R(x)B \text{ and } (2.12))$$

$$= B\frac{dR(x)}{dx}\phi(x) + R(x)B\frac{d\phi(x)}{dx} + BK(x,x)\phi(x) +$$

$$+ \left[- K(x,y)B\phi(y) \right]_{y=0}^{y=x} + \int_0^x K(x,y)B\frac{d\phi(y)}{dy}dy$$

$$+ \int_0^x K(x,y)P(y)\phi(y)dy - Q(x)\int_0^x K(x,y)\phi(y)dy$$

$$(\text{by integration by parts})$$

$$= B\frac{dR(x)}{dx}\phi(x) + R(x)(\lambda - P(x))\phi(x)$$

$$+ (BK(x,x) - K(x,x)B)\phi(x) + K(x,0)B\phi(0)$$

$$+ \int_0^x K(x,y)(\lambda - P(y))\phi(y)dy + \int_0^x K(x,y)P(y)\phi(y)dy$$

$$- Q(x)\int_0^x K(x,y)\phi(y)\,dy \qquad (\text{by } (2.16))$$

$$= \left(B\frac{dR(x)}{dx} - R(x)P(x) + BK(x,x) - K(x,x)B \right)\cdot\phi(x)$$

$$+ \lambda R(x)\phi(x)$$

$$+ (\lambda - Q(x))\int_0^x K(x,y)\phi(y)\,dy \qquad .$$

In the last equality, we use $K(x,0)B\phi(0) = 0$ by (2.3) and (2.17).

Having

$$Q(x)\psi(x) - \lambda\psi(x) = Q(x)R(x)\phi(x) + (Q(x) - \lambda)\int_0^x K(x,y)\phi(y)dy$$

$$- \lambda R(x)\phi(x) \qquad \text{by } (2.18)\ ,$$

we obtain

$$B\frac{d\psi(x)}{dx} + Q(x)\psi(x) - \lambda\psi(x)$$

$$= \left(B \frac{dR(x)}{dx} + Q(x)R(x) - R(x)P(x) + BK(x,x) - K(x,x)B \right) \cdot \phi(x) \ .$$

Since, by (2.9) - (2.11), we see

$$B \frac{dR(x)}{dx} + Q(x)R(x) - R(x)P(x) + BK(x,x) - K(x,x)B = 0 \ ,$$

we reach the equality (2.19).

Finally, as to the initial condition (2.20), we have only to note that $a(0) = \dfrac{1 - jh}{1 - h^2}$, $b(0) = \dfrac{h - j}{1 - h^2}$, and therefore,

$$R(0) \begin{pmatrix} 1 \\ -h \end{pmatrix} = \begin{pmatrix} a(0) & b(0) \\ b(0) & a(0) \end{pmatrix} \begin{pmatrix} 1 \\ -h \end{pmatrix} = \begin{pmatrix} 1 \\ -j \end{pmatrix} \ .$$

§3. **Proof of Theorem .** In order to prove Theorem, we have only to show

Proposition 2. We have

$$(3.1)\qquad (Q,j,J,J^*) \in \tilde{M}(P,h,H,H^*)$$

if and only if the equalities (1.23) - (1.26) hold.

In fact, let Proposition 2 be proved. Then we can deduce Theorem as follows :

As for the part (II) of Theorem, firstly assuming that $(Q,j,J,J^*) \in M(P,h,H,H^*)$, we have to show (1.23) - (1.26). To this end, we note that $M(P,h,H,H^*) \subset \tilde{M}(P,h,H,H^*)$ by the definitions (1.17) and (1.18). Hence we have $(Q,j,J,J^*) \in \tilde{M}(P,h,H,H^*)$, which implies the equalities (1.23) - (1.26) by Proposition 2.

Conversely we suppose that the equalities (1.23) - (1.26) hold. Then we have to show that $(Q,j,J,J^*) \in M(P,h,H,H^*)$. Its proof is carried out in the following manner.

By Proposition 2, we get $(Q,j,J,J^*) \in \tilde{M}(P,h,H,H^*)$. On the other hand, we can easily derive the equalities (3.2) - (3.5) from the equalities (1.23) - (1.26), respectively :

$$(3.2)\qquad \begin{aligned} &\frac{1-h}{1-j}(p_{11}(x) + p_{12}(x) - p_{21}(x) - p_{22}(x) \\ &\qquad - q_{11}(x) + q_{12}(x) - q_{21}(x) + q_{22}(x)) \\ &+ \frac{1+h}{1+j}(p_{11}(x) - p_{12}(x) + p_{21}(x) - p_{22}(x) \\ &\qquad - q_{11}(x) - q_{12}(x) + q_{21}(x) + q_{22}(x)) \\ &\times \exp\left(\int_0^x (p_{11}(s) + p_{22}(s) - q_{11}(s) - q_{22}(s))ds\right) = 0 \\ &\hspace{8cm} (0 \leq x \leq 1) , \end{aligned}$$

$$(3.3)\quad \frac{1-h}{1-j}\,(p_{11}(x) + p_{12}(x) - p_{21}(x) - p_{22}(x)$$

$$+ q_{11}(x) - q_{12}(x) + q_{21}(x) - q_{22}(x)\ )$$

$$+ \frac{1+h}{1+j}\,(-p_{11}(x) + p_{12}(x) - p_{21}(x) + p_{22}(x)$$

$$- q_{11}(x) - q_{12}(x) + q_{21}(x) + q_{22}(x)\ )$$

$$\times \exp\left(\int_0^x (p_{11}(s) + p_{22}(s) - q_{11}(s) - q_{22}(s))ds\right) = 0$$

$$(0 \le x \le 1),$$

$$(3.4)\quad \log \frac{(1+j)(1-J)(1-h)(1+H)}{(1-j)(1+J)(1+h)(1-H)}$$

$$= \int_0^1 (p_{11}(s) + p_{22}(s) - q_{11}(s) - q_{22}(s))ds\ ,$$

and

$$(3.5)\quad \log \frac{(1+j)(1-J^*)(1-h)(1+H^*)}{(1-j)(1+J^*)(1+h)(1-H^*)}$$

$$= \int_0^1 (p_{11}(s) + p_{22}(s) - q_{11}(s) - q_{22}(s))ds\ .$$

Since (3.2) - (3.5) are nothing but the equalities obtained in (1.23) - (1.26) by replacing p_{ij} $(1 \le i,j \le 2)$, h, H and H^* by q_{ij} $(1 \le i,j \le 2)$, j, J and J^*, respectively, we get $(P,h,H,H^*) \in \tilde{M}(Q,j,J,J^*)$, again by Proposition 2.

Therefore we prove that $(Q,j,J,J^*) \in M(P,h,H,H^*)$, which shows the "if" part of (II) of Theorem.

Finally the part (I) of Theorem follows from the part (II) and Proposition 2.

Next we proceed to

Proof of the "if" part of Proposition 2. Let us assume that the equalities (1.23) - (1.26) hold. Then we have to show that $(Q,j,J,J^*) \in \tilde{M}(P,h,H,H^*)$.

We recall that $\phi(\cdot) = \phi(\cdot,\lambda) \in \{ C^1[0,1] \}^2$ satisfies

$$B \frac{d\phi(x)}{dx} + P(x)\phi(x) = \lambda\phi(x) \qquad (0 \le x \le 1) \tag{3.6}$$

and

$$\phi(0) = \begin{pmatrix} 1 \\ -h \end{pmatrix} . \tag{3.7}$$

For $\lambda_n \in \sigma(A_{P,h,H})$, we set

$$\psi_n(x) = \begin{pmatrix} \psi_n^{(1)}(x) \\ \psi_n^{(2)}(x) \end{pmatrix} \equiv R(x)\phi(x,\lambda_n) , \tag{3.8}$$

where $R(x)$ is the 2×2 matrix given by (2.11). By (1.23) - (1.26) we can see by direct computations

$$B \frac{d\psi_n(x)}{dx} + Q(x)\psi_n(x) = \lambda_n\psi_n(x) \qquad (0 \le x \le 1) ,$$

$$\psi_n(0) = \begin{pmatrix} 1 \\ -j \end{pmatrix} ,$$

and

$$\psi_n^{(2)}(1) + J\psi_n^{(1)}(1) = 0 .$$

Here we note also Remark 3 stated in $1. These imply $\lambda_n \in \sigma(A_{Q,j,J})$. That is, we see that $\sigma(A_{P,h,H}) \subset \sigma(A_{Q,j,J})$. We can similarly show that $\sigma(A_{P,h,H^*}) \subset \sigma(A_{Q,j,J^*})$ and therefore, we see that $(Q,j,J,J^*) \in \tilde{\mathbb{M}}(P,h,H,H^*)$.

Proof of the "only if" part of Proposition 2. We can show the "only if" part along the line of the argument in Suzuki [46, 51].

Assume that

$$\sigma(A_{P,h,H}) \subset \sigma(A_{Q,j,J}) \tag{3.9}$$

and

$$\sigma(A_{P,h,H^*}) \subset \sigma(A_{Q,j,J^*}) \quad . \tag{3.10}$$

Then we have to prove the equalities (1.23) - (1.26).

Let us set

$$\sigma(A_{P,h,H}) = \{ \lambda_n \}_{n \in \mathbb{Z}} \tag{3.11}$$

and

$$\sigma(A_{Q,j,J}) = \{ \mu_n \}_{n \in \mathbb{Z}} \quad . \tag{3.12}$$

Firstly we see, by (3.9), that for each $n \in \mathbb{Z}$, there exists some $m(n) \in \mathbb{Z}$ such that $\lim_{|n|\to\infty} |m(n)| = \infty$ and

$$\lambda_n = \mu_{m(n)} \qquad (n \in \mathbb{Z}) \quad . \tag{3.13}$$

Therefore it follows from (3.13) and the asymptotic behavior of the eigenvalues ((II) of Proposition 0 in $1) that we get

$$\begin{aligned} &\frac{1}{2} \log \frac{(1 + h)(1 - H)}{(1 - h)(1 + H)} + \frac{1}{2} \int_0^1 (p_{11}(s) + p_{22}(s))ds \\ &+ n\pi \sqrt{-1} + O(\frac{1}{n}) \\ &= \frac{1}{2} \log \frac{(1 + j)(1 - J)}{(1 - j)(1 + J)} + \frac{1}{2} \int_0^1 (q_{11}(s) + q_{22}(s))ds \end{aligned} \tag{3.14}$$

$$+ m(n)\,\pi\sqrt{-1} \quad + O\left(\frac{1}{m(n)}\right) \qquad (n \in \mathbb{Z}) .$$

Since we have

$$\frac{1}{2}\int_0^1 (p_{11}(s) + p_{22}(s))ds\ , \quad \frac{1}{2}\int_0^1 (q_{11}(s) + q_{22}(s))ds \quad \in \quad \mathbb{R}$$

and

$$\mathrm{Im}\ \frac{1}{2}\log\frac{(1+h)(1-H)}{(1-h)(1+H)}\ , \quad \mathrm{Im}\ \frac{1}{2}\log\frac{(1+j)(1-J)}{(1-j)(1+J)} = 0\ ,$$

or $\frac{1}{2}\pi$, we see that

$$\lim_{|n|\to\infty} (m(n) - n)\pi\sqrt{-1} = 0\ ,$$

which implies

$$\text{(3.15)} \qquad \frac{1}{2}\log\frac{(1+h)(1-H)}{(1-h)(1+H)} + \frac{1}{2}\int_0^1 (p_{11}(s) + p_{22}(s))ds$$

$$= \frac{1}{2}\log\frac{(1+j)(1-J)}{(1-j)(1+J)} + \frac{1}{2}\int_0^1 (q_{11}(s) + q_{22}(s))ds\ .$$

This equality is nothing but the equality (1.25). Similarly we can see (1.26).

Secondly, in order to show (1.23) and (1.24), we apply Lemma 1 in §2. Let $K = K(x,y) \in \{\ C^1(\bar{\Omega})\ \}^4$ be the solution to (2.12) - (2.15) in Lemma 1 and let $R(x)$ be defined by (2.11). We put

$$\text{(3.16)} \qquad \psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$$

$$= R(x)\phi(x,\lambda) + \int_0^x K(x,y)\phi(y,\lambda)dy \quad (0 \le x \le 1)\ .$$

Then, by Lemma 1 , we have

$$
(3.17) \quad \begin{cases} B \dfrac{d\psi(x,\lambda)}{dx} + Q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda) \qquad (0 \le x \le 1) \\[2ex] \psi(0,\lambda) = \begin{pmatrix} 1 \\ -j \end{pmatrix} \end{cases} .
$$

On the other hand, by the assumption (3.9) and the fact that each eigenvalue λ_n is simple, we see that $\psi(\cdot,\lambda_n)$ is an eigenvector of $A_{Q,j,J}$ associated with λ_n. Therefore, from the boundary condition at $x = 1$, we get

$$
(3.18) \qquad \psi_2(1,\lambda_n) + J\psi_1(1,\lambda_n) = 0 \qquad (n \in \mathbb{Z}) .
$$

Here we recall that (3.18) means $\psi_1(1,\lambda_n) = 0$, if $J = \infty$.

Henceforth let

$$
(3.19) \qquad H,\ H^*,\ J,\ J^* \ne \infty \quad .
$$

Otherwise we can similarly proceed and in Appendix II, we give the discussion in the case of $H \ne \infty$, $H^* = \infty$, $J = \infty$, and $J^* \ne \infty$, for example.

Substituting (3.16) into (3.18), we obtain

$$
(3.20) \qquad (J - H)a(1) + (1 - JH)b(1) + K_2(\lambda_n) + JK_1(\lambda_n) = 0 \qquad (n \in \mathbb{Z}) .
$$

Here and henceforth, we put

$$
(3.21) \qquad K_1(\lambda) = \int_0^1 (K_{11}(1,y)\phi_1(y,\lambda) + K_{12}(1,y)\phi_2(y,\lambda))dy
$$

and

$$
(3.22) \qquad K_2(\lambda) = \int_0^1 (K_{21}(1,y)\phi_1(y,\lambda) + K_{22}(1,y)\phi_2(y,\lambda))dy \quad ,
$$

and we recall that $a(x)$ and $b(x)$ are given by (2.9) and (2.10), respectively.

By the assumption (3.10), we similarly get

$$(3.23) \qquad (J^* - H^*)a(1) + (1 - J^*H^*)b(1) + K_2(\lambda_n^*) + J^*K_1(\lambda_n^*) = 0 \qquad (n \in \mathbb{Z}) .$$

Here we set $\sigma(A_{P,h,H^*}) = \{ \lambda_n^* \}_{n \in \mathbb{Z}}$.

As is easily seen, the equalities (1.25) and (1.26) imply

$$(3.24) \quad \begin{cases} (J - H)a(1) + (1 - JH)b(1) = 0 \\ (J^* - H^*)a(1) + (1 - J^*H^*)b(1) = 0 \end{cases} .$$

Hence by (3.20), (3.23) and (3.24), we reach

$$(3.25) \qquad K_2(\lambda_n) + JK_1(\lambda_n) = 0 \qquad (n \in \mathbb{Z}) ,$$

and

$$(3.26) \qquad K_2(\lambda_n^*) + J^*K_1(\lambda_n^*) = 0 \qquad (n \in \mathbb{Z}) .$$

Since the system $\{ \phi(\cdot,\lambda_n) \}_{n \in \mathbb{Z}}$ and $\{ \phi(\cdot,\lambda_n^*) \}_{n \in \mathbb{Z}}$ are complete in $\{ L^2(0,1) \}^2$, respectively ((III) of Proposition 0 in $1), the equalities (3.25) and (3.26) imply

$$(3.27) \quad \begin{cases} K_{21}(1,y) + J\cdot K_{11}(1,y) = 0 \\ K_{21}(1,y) + J^*K_{11}(1,y) = 0 \qquad (0 \leq y \leq 1) \end{cases}$$

and

$$(3.28)\quad \begin{cases} K_{22}(1,y) + J\cdot K_{12}(1,y) = 0 \\ K_{22}(1,y) + J^{*}K_{12}(1,y) = 0 \qquad (0 \le y \le 1) \end{cases} .$$

Now, by (1.25) and (1.26), we have

$$(3.29)\quad \frac{(1 - H)(1 + J)}{(1 + H)(1 - J)} = \frac{(1 - H^{*})(1 + J^{*})}{(1 + H^{*})(1 - J^{*})} .$$

Since we assume that $H \neq H^{*}$ (Definition 2 in $1), the equality (3.29) implies

$$(3.30)\quad J \neq J^{*} .$$

Therefore by (3.27),(3.28) and (3.30), we obtain

$$(3.31)\quad K_{ij}(1,y) = 0 \qquad (1 \le i,\ j \le 2,\ \ 0 \le y \le 1) .$$

As is proved in Appendix III, we have a result on uniqueness of solutions to a hyperbolic system :

Lemma 2. *Let* $K = K(x,y)$ *satisfy*

$$(3.32)\quad B\,\frac{\partial K(x,y)}{\partial x} + Q(x)K(x,y) - K(x,y)P(y) = -\frac{\partial K(x,y)}{\partial y}\,B$$

$$((x,\ y) \in \overline{\Omega}) ,$$

$$(3.33)\quad \begin{cases} K_{12}(x,0) = hK_{11}(x,0) \\ K_{22}(x,0) = hK_{21}(x,0) \qquad (0 \le x \le 1) , \end{cases}$$

and

$$(3.34)\quad K_{ij}(1,y) = 0 \qquad (1 \le i,\ j \le 2,\ 0 \le y \le 1).$$

Then the identity

$$K(x,y) = 0 \qquad ((x, y) \in \overline{\Omega}) \tag{3.35}$$

holds.

We return to the proof of the "only if" part. By Lemma 2 and (3.31), noting (2.12) and (2.13), we reach

$$K(x,y) = 0 \qquad ((x, y) \in \overline{\Omega}) \quad . \tag{3.36}$$

Thus (3.36), (2.14) and (2.15) imply (1.23) and (1.24).

The proof of Proposition 2 is completed.

Appendix I. Proof of Proposition 1. Setting

$$(\mathrm{I}.1)\quad \begin{cases} L_1(x,y) = K_{12}(x,y) - K_{21}(x,y) \\ L_2(x,y) = K_{11}(x,y) - K_{22}(x,y) \\ L_3(x,y) = K_{11}(x,y) + K_{22}(x,y) \\ L_4(x,y) = K_{12}(x,y) + K_{21}(x,y) \end{cases},$$

we can rewrite (2.2) - (2.5), so that we get

$$(\mathrm{I}.2)\quad \begin{cases} \dfrac{\partial L_1(x,y)}{\partial x} - \dfrac{\partial L_1(x,y)}{\partial y} = f_1(x,y,\ L_1,\ L_2,\ L_3,\ L_4) \\ \dfrac{\partial L_2(x,y)}{\partial x} - \dfrac{\partial L_2(x,y)}{\partial y} = f_2(x,y,\ L_1,\ L_2,\ L_3,\ L_4), \end{cases} \qquad (\ (x,\ y) \in \bar{\Omega}\ ),$$

$$(\mathrm{I}.3)\quad \begin{cases} \dfrac{\partial L_3(x,y)}{\partial x} + \dfrac{\partial L_3(x,y)}{\partial y} = f_3(x,y,\ L_1,\ L_2,\ L_3,\ L_4) \\ \dfrac{\partial L_4(x,y)}{\partial x} + \dfrac{\partial L_4(x,y)}{\partial y} = f_4(x,y,\ L_1,\ L_2,\ L_3,\ L_4) \end{cases} \qquad (\ (x,\ y) \in \bar{\Omega}\ ),$$

$$(\mathrm{I}.4)\quad \begin{cases} L_3(x,0) = k\cdot L_1(x,0) + \ell\cdot L_2(x,0) \\ L_4(x,0) = -\ell\cdot L_1(x,0) - k\cdot L_2(x,0) \qquad (0 \le x \le 1) \end{cases},$$

and

$$(\text{I.5}) \quad \begin{cases} L_1(x,x) = r_1(x) \\ \\ L_2(x,x) = r_2(x) & (0 \leq x \leq 1) \end{cases} .$$

Here and henceforth we put

$$(\text{I.6}) \quad \begin{cases} f_1(x,y,\ L_1,\ L_2,\ L_3,\ L_4) = \\ -\frac{1}{2}(\ \ p_{12}(y) + p_{21}(y) + q_{12}(x) + q_{21}(x))L_1(x,y) \\ -\frac{1}{2}(\ \ p_{11}(y) + p_{22}(y) - q_{11}(x) - q_{22}(x))L_2(x,y) \\ -\frac{1}{2}(\ \ p_{11}(y) - p_{22}(y) - q_{11}(x) + q_{22}(x))L_3(x,y) \\ -\frac{1}{2}(-p_{12}(y) + p_{21}(y) - q_{12}(x) + q_{21}(x))L_4(x,y) \\ \\ f_2(x,y,\ L_1,\ L_2,\ L_3,\ L_4) = \\ -\frac{1}{2}(\ \ p_{11}(y) + p_{22}(y) - q_{11}(x) - q_{22}(x))L_1(x,y) \\ -\frac{1}{2}(\ \ p_{12}(y) + p_{21}(y) + q_{12}(x) + q_{21}(x))L_2(x,y) \\ +\frac{1}{2}(-p_{12}(y) + p_{21}(y) + q_{12}(x) - q_{21}(x))L_3(x,y) \\ -\frac{1}{2}(-p_{11}(y) + p_{22}(y) - q_{11}(x) + q_{22}(x))L_4(x,y) \\ \\ f_3(x,y,\ L_1,\ L_2,\ L_3,\ L_4) = \\ \ \ \frac{1}{2}(-p_{11}(y) + p_{22}(y) - q_{11}(x) + q_{22}(x))L_1(x,y) \\ +\frac{1}{2}(\ \ p_{12}(y) - p_{21}(y) + q_{12}(x) - q_{21}(x))L_2(x,y) \\ +\frac{1}{2}(\ \ p_{12}(y) + p_{21}(y) - q_{12}(x) - q_{21}(x))L_3(x,y) \\ +\frac{1}{2}(\ \ p_{11}(y) + p_{22}(y) - q_{11}(x) - q_{22}(x))L_4(x,y) \end{cases}$$

$$
\left\{
\begin{aligned}
& f_4(x,y, L_1, L_2, L_3, L_4) = \\
& \quad \tfrac{1}{2} (- p_{12}(y) + p_{21}(y) + q_{12}(x) - q_{21}(x))L_1(x,y) \\
& + \tfrac{1}{2} (p_{11}(y) - p_{22}(y) - q_{11}(x) + q_{22}(x))L_2(x,y) \\
& + \tfrac{1}{2} (p_{11}(y) + p_{22}(y) - q_{11}(x) - q_{22}(x))L_3(x,y) \\
& + \tfrac{1}{2} (p_{12}(y) + p_{21}(y) - q_{12}(x) - q_{21}(x))L_4(x,y) \\
& \qquad\qquad\qquad\qquad\qquad\qquad ((x, y) \in \overline{\Omega}),
\end{aligned}
\right.
$$

and

$$
(\mathrm{I}.7) \quad \left\{
\begin{aligned}
k &= \frac{-2h}{1 - h^2} \\
\ell &= \frac{1 + h^2}{1 - h^2} \quad .
\end{aligned}
\right.
$$

First we can integrate (I.2) with (I.5) along the characteristic curve $y + x = \text{const.}$, so that we get integral equations (I.8) :

$$
(\mathrm{I}.8) \quad \left\{
\begin{aligned}
L_1(x,y) &= \int_y^{\frac{x+y}{2}} f_1(-s + x + y, s, L_1, L_2, L_3, L_4)\,ds \\
&\quad + r_1\left(\frac{x + y}{2} \right)
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
L_2(x,y) &= \int_y^{\frac{x+y}{2}} f_2(-s + x + y, s, L_1, L_2, L_3, L_4)ds \\
&\quad + r_2\left(\frac{x + y}{2} \right) \qquad ((x, y) \in \overline{\Omega}).
\end{aligned}
\right.
$$

Second we integrate (I.3) along the characteristic curve $y - x = \text{const.}$, so that we obtain

$$
L_i(x,y) = \int_0^y f_i(s+x-y, s, L_1, L_2, L_3, L_4)ds + L_i(x-y, 0) \qquad (i = 3,4).
$$

Therefore by (I.4) and (I.8), we get

$$
\text{(I.9)} \quad
\left\{
\begin{aligned}
L_3(x,y) &= \int_0^y f_3(s + x - y, s, L_1, L_2, L_3, L_4)ds \\
&\quad + \int_0^{\frac{x-y}{2}} (k\cdot f_1(-s + x - y, s, L_1, L_2, L_3, L_4) \\
&\qquad + \ell\cdot f_2(-s + x - y, s, L_1, L_2, L_3, L_4))ds \\
&\quad + k\cdot r_1\left(\frac{x - y}{2} \right) + \ell\cdot r_2\left(\frac{x - y}{2} \right) \\
L_4(x,y) &= \int_0^y f_4(s + x - y, s, L_1, L_2, L_3, L_4)ds \\
&\quad - \int_0^{\frac{x-y}{2}} (\ell\cdot f_1(- s + x - y, s, L_1, L_2, L_3, L_4) \\
&\qquad + k\cdot f_2(-s + x - y, s, L_1, L_2, L_3, L_4))ds \\
&\quad - \ell\cdot r_1\left(\frac{x - y}{2} \right) - k\cdot r_2\left(\frac{x - y}{2} \right) \\
&\qquad\qquad ((x, y) \in \overline{\Omega}) .
\end{aligned}
\right.
$$

Thus, on condition that $r_1, r_2 \in C^1[0,1]$ and $P, Q \in \{ C^1[0,1] \}^4$, the problem (I.2) - (I.5) is equivalent to the Volterra's integral equations (I.8) and (I.9), if $L \in \{C^1(\overline{\Omega})\}^4$ is proved.

The unique solution $L = L(x,y) \in \{ C^1(\overline{\Omega}) \}^4$ to (I.8) and (I.9) is given by the following iteration method : Let us define approximation sequences $\{ L_i^{(n)}(x,y) \}_{n \geq 0}$ $(1 \leq i \leq 4)$ by (I.10) - (I.12) :

$$\text{(I.10)} \qquad L_i^{(0)}(x,y) = 0 \qquad ((x, y) \in \overline{\Omega}, \quad 1 \leq i \leq 4),$$

$$\text{(I.11)} \quad \begin{cases} L_1^{(n)}(x,y) = \\ \displaystyle\int_y^{\frac{x+y}{2}} f_1(-s+x+y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)}) ds \\ + r_1\left(\dfrac{x+y}{2} \right), \\ \\ L_2^{(n)}(x,y) = \\ \displaystyle\int_y^{\frac{x+y}{2}} f_2(-s+x+y, s, L_1^{(n-1)}, L_2^{(n-1)}, L_3^{(n-1)}, L_4^{(n-1)}) ds \\ + r_2\left(\dfrac{x+y}{2} \right) \qquad ((x,y) \in \overline{\Omega}, \quad n \geq 1), \end{cases}$$

and

$$
(\text{I}.12)\quad
\begin{cases}
L_3^{(n)}(x,y) = \\
\quad \displaystyle\int_0^y f_3(s+x-y,s,\ L_1^{(n-1)},\ L_2^{(n-1)},\ L_3^{(n-1)},\ L_4^{(n-1)})ds \\
\quad + \displaystyle\int_0^{\frac{x-y}{2}} (\ k\cdot f_1(-s+x-y,s,\ L_1^{(n-1)},\ L_2^{(n-1)},\ L_3^{(n-1)},\ L_4^{(n-1)}) \\
\quad + \ell\cdot f_2(-s+x-y,s,\ L_1^{(n-1)},\ L_2^{(n-1)},\ L_3^{(n-1)},\ L_4^{(n-1)})\ )ds \\
\quad + k\cdot r_1(\ \frac{x-y}{2}\ ) + \ell\cdot r_2(\ \frac{x-y}{2}\ ), \\
\\
L_4^{(n)}(x,y) = \\
\quad \displaystyle\int_0^y f_4(s+x-y,s,\ L_1^{(n-1)},\ L_2^{(n-1)},\ L_3^{(n-1)},\ L_4^{(n-1)})ds \\
\quad - \displaystyle\int_0^{\frac{x-y}{2}} (\ \ell\cdot f_1(-s+x-y,s,\ L_1^{(n-1)},\ L_2^{(n-1)},\ L_3^{(n-1)},\ L_4^{(n-1)}) \\
\quad + k\cdot f_2(-s+x-y,s,\ L_1^{(n-1)},\ L_2^{(n-1)},\ L_3^{(n-1)},\ L_4^{(n-1)})\ )ds \\
\quad - \ell\cdot r_1(\ \frac{x-y}{2}\ ) - k\cdot r_2(\ \frac{x-y}{2}\ ) \\
\qquad\qquad\qquad\qquad (\quad (x,y) \in \overline{\Omega}\ ,\quad n \geq 1\quad).
\end{cases}
$$

We set

$$
M = 8(|k| + |\ell| + 1)\cdot\max\left\{ \max_{1 \leq i,j \leq 2} \| p_{ij} \|_{C^0[0,1]}\ ,\ \max_{1 \leq i,j \leq 2} \| q_{ij} \|_{C^0[0,1]} \right\}.
$$

Then, by induction, we can see the estimates

$$(\text{I}.13) \qquad | L_i^{(n)}(x,y) - L_i^{(n-1)}(x,y) |$$

$$\leq \frac{M^{n-1}(1+x)^{n-1}}{(n-1)!} \cdot (|k| + |\ell| + 1)(\| r_1 \|_{C^0[0,1]} +$$

$$+ \| r_2 \|_{C^0[0,1]}) \qquad ((x,y) \in \overline{\Omega} , \ 1 \leq i \leq 4),$$

for each $n \geq 1$.

Thus $L_i(x,y) = \lim_{n \to \infty} L_i^{(n)}(x,y) \quad (1 \leq i \leq 4)$ exist uniformly in $(x,y) \in \overline{\Omega}$ and we see that $L_i(x,y) \quad (1 \leq i \leq 4)$ satisfy (I.8) and (I.9). Furthermore we can get similar estimates on $\left| \dfrac{\partial L_i^{(n)}(x,y)}{\partial x} - \dfrac{\partial L_i^{(n-1)}(x,y)}{\partial x} \right|$ and $\left| \dfrac{\partial L_i^{(n)}(x,y)}{\partial y} - \dfrac{\partial L_i^{(n-1)}(x,y)}{\partial y} \right|$, by induction, and therefore we see also that $L \in \{C^1(\overline{\Omega})\}^4$.

The uniqueness of solutions is also shown by (I.13).

Thus the proof of Proposition 1 is completed.

Appendix II. Derivation of (1.23) and (1.24) in the case of $H \neq \infty$, $H^* = \infty$, $J = \infty$ and $J^* \neq \infty$ in the proof of Proposition 2. In this appendix, assuming that

$$(II.1) \qquad H \neq \infty , \; H^* = \infty , \; J = \infty , \; \text{and} \; J^* \neq \infty ,$$

we derive (1.23) and (1.24) in the proof of the "if" part of Proposition 2.

Here we recall that

$$(II.2) \qquad \psi(x,\lambda) = \begin{pmatrix} \psi_1(x,\lambda) \\ \psi_2(x,\lambda) \end{pmatrix}$$

$$= R(x)\phi(x,\lambda) + \int_0^x K(x,y)\phi(y,\lambda)dy \qquad (0 \leq x \leq 1),$$

$$(II.3) \qquad \begin{cases} \sigma(A_{P,h,H}) = \{ \lambda_n \}_{n \in \mathbb{Z}} \\ \sigma(A_{P,h,\infty}) = \{ \lambda_n^* \}_{n \in \mathbb{Z}} , \end{cases}$$

and

$$(II.4) \qquad \begin{cases} K_1(\lambda) = \int_0^1 (K_{11}(1,y)\phi_1(y,\lambda) + K_{12}(1,y)\phi_2(y,\lambda))dy \\ K_2(\lambda) = \int_0^1 (K_{21}(1,y)\phi_1(y,\lambda) + K_{22}(1,y)\phi_2(y,\lambda))dy \end{cases}$$

for $\lambda \in \mathbb{C}$,

and by the boundary conditions for ϕ_1 and ϕ_2 , and $H \neq \infty$,

$H^* = \infty$, we note that

$$(\text{II.5}) \qquad \phi_2(1,\lambda_n) + H\cdot\phi_1(1,\lambda_n) = 0 \qquad (n \in \mathbb{Z})$$

and

$$(\text{II.6}) \qquad \phi_1(1,\lambda_n^*) = 0 \qquad (n \in \mathbb{Z}) \quad .$$

Furthermore we have already derived (1.25) and (1.26), and we have obtained

$$(\text{II.7}) \qquad \psi_1(1,\lambda_n) = 0 \qquad (n \in \mathbb{Z}) \quad ,$$

and

$$(\text{II.8}) \qquad \psi_2(1,\lambda_n^*) + J^*\cdot\psi_1(1,\lambda_n^*) = 0 \qquad (n \in \mathbb{Z}) \quad .$$

Then, in order to derive (1.23) and (1.24), we have only to prove

$$(\text{II.9}) \qquad K_{ij}(1,y) = 0 \qquad (1 \leq i,j \leq 2 \ , \quad 0 \leq y \leq 1) \ ,$$

in view of Lemma 2.

Substituting (II.2) into (II.7) and using (II.5), we get

$$(\text{II.10}) \qquad (a(1) - Hb(1))\phi_1(1,\lambda_n) + K_1(\lambda_n) = 0 \qquad (n \in \mathbb{Z}) \ .$$

On the other hand, as is easily checked, the equality (1.25) implies $a(1) - Hb(1) = 0$. Therefore by (II.10), we obtain

$$(\text{II.11}) \qquad K_1(\lambda_n) = 0 \qquad (n \in \mathbb{Z}) \ .$$

Next, substituting (II.2) into (II.8) and using (II.6), we get

$$(\mathrm{II}.12) \qquad (a(1) + J^{*}b(1))\phi_2(1,\lambda_n^{*}) + (K_2(\lambda_n^{*}) + J^{*}K_1(\lambda_n^{*})) = 0 \qquad (n \in \mathbb{Z}) .$$

Since we see by direct computations that the equality (1.26) implies $a(1) + J^{*}b(1) = 0$, we obtain

$$(\mathrm{II}.13) \qquad K_2(\lambda_n^{*}) + J^{*}K_1(\lambda_n^{*}) = 0 \qquad (n \in \mathbb{Z}) .$$

by (II.12).

Since the systems $\{\phi(\cdot,\lambda_n)\}_{n \in \mathbb{Z}}$ and $\{\phi(\cdot,\lambda_n^{*})\}_{n \in \mathbb{Z}}$ are complete in $\{ L^2(0,1) \}^2$, respectively, the equalities (II.11) and (II.13) imply

$$(\mathrm{II}.14) \qquad K_{11}(1,y) = K_{12}(1,y) = 0 \qquad (0 \le y \le 1) ,$$

and

$$(\mathrm{II}.15) \quad \begin{cases} K_{21}(1,y) + J^{*}\cdot K_{11}(1,y) = 0 \\ \\ K_{22}(1,y) + J^{*}\cdot K_{12}(1,y) = 0 \qquad (0 \le y \le 1) , \end{cases}$$

respectively. Thus we prove (II.9).

Appendix III. Proof of Lemma 2. By an argument similar to the one in Appendix I, we have only to show the following : If L_i $(1 \leq i \leq 4)$ satisfy (I.2) - (I.4) and

$$L_i(1,y) = 0 \quad (0 \leq y \leq 1,\ 1 \leq i \leq 4) \tag{III.1}$$

hold, then we have $L_i(x,y) = 0 \quad ((x,y) \in \bar{\Omega},\ 1 \leq i \leq 4)$.

Let us set

$$\begin{cases} \Omega_1 = \left\{ (x,y)\ ;\ 1 - x < y < x\ ,\ \frac{1}{2} < x < 1 \right\} \\ \Omega_2 = \Omega \setminus \Omega_1 \setminus \left\{ (x,y)\ ;\ 1 - x = y \right\} . \end{cases} \tag{III.2}$$

Then, by a result on uniqueness of solutions to the Cauchy problem for hyperbolic systems (for example, Petrovsky [40, p.68]), we see

$$L_i(x,y) = 0 \qquad ((x,y) \in \bar{\Omega}_1\ ,\ 1 \leq i \leq 4) \tag{III.3}$$

from (I.2), (I.3) and (III.1).

By (III.3), we have

$$L_i(x,1-x) = 0 \quad \left(\frac{1}{2} \leq x \leq 1\ ,\ 1 \leq i \leq 4 \right) . \tag{III.4}$$

Since L_i $(1 \leq i \leq 4)$ satisfy (I.2) - (I.4) and (III.4), we obtain the integral equations (III.5) and (III.6) in L_i by a discussion similar to the one in Appendix I :

$$
\text{(III.5)} \begin{cases}
L_1(x,y) = & - \displaystyle\int_0^y f_1(-s+x+y,s,\ L_1,\ L_2,\ L_3,\ L_4)\,ds \\
& + \displaystyle\int_0^{\frac{1-x-y}{2}} (k\cdot f_3(s+x+y,s,\ L_1,\ L_2,\ L_3,\ L_4) \\
& \qquad + \ell\cdot f_4(s+x+y,s,\ L_1,\ L_2,\ L_3,\ L_4)\ )ds \\
L_2(x,y) = & - \displaystyle\int_0^y f_2(-s+x+y,s,\ L_1,\ L_2,\ L_3,\ L_4)\,ds \\
& - \displaystyle\int_0^{\frac{1-x-y}{2}} (\ell\cdot f_3(s+x+y,s,\ L_1,\ L_2,\ L_3,\ L_4) \\
& \qquad + k\cdot f_4(s+x+y,s,\ L_1,\ L_2,\ L_3,\ L_4)\ )ds \\
& \qquad\qquad (\ (x,y) \in \overline{\Omega_2}\ )\ ,
\end{cases}
$$

and

$$
\text{(III.6)} \begin{cases}
L_3(x,y) = - \displaystyle\int_y^{\frac{1-x+y}{2}} f_3(s+x-y,s,\ L_1,\ L_2,\ L_3,\ L_4)ds \\
L_4(x,y) = - \displaystyle\int_y^{\frac{1-x+y}{2}} f_4(s+x-y,s,\ L_1,\ L_2,\ L_3,\ L_4)ds \\
\qquad\qquad (\ (x,y) \in \overline{\Omega_2}\ )\ .
\end{cases}
$$

Setting

$$
m(x,y) = \max_{1 \le i,j \le 2} |\ L_{ij}(x,y)\ |\ ,
$$

and

$$M = 8(|k| + |\ell| + 1) \max \left\{ \max_{1 \leq i,j \leq 2} \| p_{ij} \|_{C^0[0,1]}, \max_{1 \leq i,j \leq 2} \| q_{ij} \|_{C^0[0,1]} \right\} ,$$

we inductively deduce the estimates

$$\text{(III.7)} \qquad m(x,y) \leq \frac{M^{n-1}(1 - x)^{n-1}}{(n - 1)!} \cdot \| m \|_{C^0(\overline{\Omega_2})} \qquad ((x,y) \in \overline{\Omega_2}) ,$$

for each $n \geq 1$. Since n is arbitrary, the estimates (III.7) prove

$$m(x,y) = 0 \qquad ((x,y) \in \overline{\Omega_2}) .$$

Thus the proof of Lemma 2 is completed.

Chapter 3

Identification Problems of Boundary Value Problems in Terms of Eigenvalues

§1. **Introduction.** Let us consider

$$(1.1)\quad \begin{cases} \dfrac{du_2(x)}{dx} + p_{11}(x)u_1(x) + p_{12}(x)u_2(x) = \lambda u_1(x) \\ \\ \dfrac{du_1(x)}{dx} + p_{21}(x)u_1(x) + p_{22}(x)u_2(x) = \lambda u_2(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \leq x \leq 1)\ , \end{cases}$$

with boundary conditions

$$(1.2)\qquad u_2(0) + h\cdot u_1(0) = 0$$

at $x = 0$, and either

$$(1.3)\qquad u_2(1) + H\cdot u_1(1) = 0\quad ,$$

or

$$(1.4)\qquad u_2(1) + H^*\cdot u_1(1) = 0 \qquad (H \neq H^*)$$

at $x = 1$.

In the preceding chapter, we show the theorem which characterizes the coefficients $p_{ij}(x)$ $(1 \leq i,j \leq 2)$, etc. of (1.1) by the two sets of eigenvalues associated with two boundary value problems (1.1) - (1.3) and (1.1),(1.2),(1.4), respectively. In virtue of Theorem in Chapter 2, we can get the following proposition. Here let us recall that $A_{P,h,H}$ is the operator given in Definition 1 of Chapter 2 and that $\sigma(A_{P,h,H})$ denotes the spectrum of $A_{P,h,H}$.

Proposition 1. *Let* $h \in \mathbb{R} \setminus \{-1, 1\}$ *and* $H, H^* \in \mathbb{R} \cup \{\infty\} \setminus \{-1, 1\}$. *If* $Q = (q_{ij})_{1 \leq i,j \leq 2} \in \{C^1[0,1]\}^4$ *satisfies*

$$(1.5) \quad \begin{cases} \sigma(A_{Q,h,H}) = \sigma(A_{P,h,H}) \\ \\ \sigma(A_{Q,h,H^*}) = \sigma(A_{P,h,H^*}) \end{cases},$$

then the relation

$$\sigma(A_{Q,\alpha,\beta}) = \sigma(A_{P,\alpha,\beta})$$

holds for each $\alpha \in \mathbb{R} \setminus \{-1, 1\}$ *and* $\beta \in \mathbb{R} \cup \{\infty\} \setminus \{-1, 1\}$.

In Appendix I, we will prove Proposition 1.

This proposition asserts that in our inverse spectral problem, only two sets of eigenvalues obtained above give "independent" knowledge in the determination of the coefficients of (1.1).

On the other hand, as is seen by Theorem in Chapter 2, we cannot uniquely determine all the four coefficients of (1.1) by those two sets of the eigenvalues. (In fact, we get only two equations (1.23) and (1.24) in Chapter 2, whereas there are four unknown functions q_{ij} $(1 \leq x \leq 2)$.)

Thus, in this chapter, we are restricted to three types of identification problems to determine the other coefficients and the boundary conditions at $x = 1$ provided that some of the four coefficients $p_{ij}(x)$ $(1 \leq i,j \leq 2)$ and the boundary condition at $x = 0$ are known in advance. That is, we consider the identification problems of the following three types :

Type-I : *Determination of "one diagonal" component and "one nondiagonal" component of* $P = (p_{ij})_{1 \le i,j \le 2}$ *and of two boundary conditions at* $x = 1$.

Type-II : *Determination of "two diagonal" components of* $P = (p_{ij})_{1 \le i,j \le 2}$ *and of two boundary conditions at* $x = 1$.

Type-III : *Determination of "two nondiagonal" components of* $P = (p_{ij})_{1 \le i,j \le 2}$ *and of two boundary conditions at* $x = 1$.

This chapter is composed of six sections and three appendixes. In §2, we state our results for the three types and, in §§ 3 and 4, we prove them. In §§ 5 and 6, as application of the results in §2, we consider identification problems for a vibration of a string and for an ellectric oscillation.

§2. The three types of identification problems

§2.1. The result for Type-I. In this case we can answer Yes for the uniqueness. That is, as is stated in Theorem 1, the two sets $\sigma(A_{P,h,H})$ and $\sigma(A_{P,h,H^*})$ ($h \in \mathbb{R} \setminus \{-1, 1\}$, $H, H^* \in \mathbb{R} \cup \{\infty\} \setminus \{-1, 1\}$ and $H \neq H^*$) uniquely determine one diagonal component and one nondiagonal component of P and H, H^*.

Throughout this chapter, let

$$(2.1) \qquad \begin{cases} h \in \mathbb{R} \setminus \{-1, 1\}, \\ \\ H, H^*, J, J^* \in \mathbb{R} \cup \{\infty\} \setminus \{-1, 1\}, \quad H \neq H^*, \end{cases}$$

and

$$(2.2) \qquad P, Q \in \{C^1[0, 1]\}^4, \quad \text{real-valued.}$$

Theorem 1. *Let us assume that*

$$(2.3) \qquad \begin{cases} \sigma(A_{Q,h,J}) = \sigma(A_{P,h,H}) \\ \\ \sigma(A_{Q,h,J^*}) = \sigma(A_{P,h,H^*}) \end{cases} .$$

If either of (2.6) - (2.9) holds, then we have

$$(2.4) \qquad Q(x) = P(x) \qquad (0 \leq x \leq 1)$$

and

$$(2.5) \qquad J = H \quad \text{and} \quad J^* = H^* \quad ;$$

$$(2.6) \qquad q_{21}(x) = p_{21}(x) \quad \text{and} \quad q_{22}(x) = p_{22}(x) \qquad (0 \leq x \leq 1) .$$

$$(2.7) \qquad q_{12}(x) = p_{12}(x) \quad \text{and} \quad q_{22}(x) = p_{22}(x) \qquad (0 \leq x \leq 1) .$$

$$(2.8) \qquad q_{11}(x) = p_{11}(x) \quad \text{and} \quad q_{21}(x) = p_{21}(x) \qquad (0 \leq x \leq 1) .$$

$$(2.9) \qquad q_{11}(x) = p_{11}(x) \quad \text{and} \quad q_{12}(x) = p_{12}(x) \qquad (0 \leq x \leq 1) .$$

In §3, we will prove Theorem 1.

§2.2. **The result for Type-II.** In this case we can answer Yes under some additional conditions, for the uniqueness.

For the statement of Theorem 2, our main result for Type-II, let $P = (p_{ij})_{1 \le i,j \le 2} \in \{C^1[0, 1]\}^4$ be known and let us introduce the notation used throughout this chapter ;

$$(2.10) \qquad M(p_{11}, p_{22} ; H, H^*)$$

$$= \left\{ (Q, h, J, J^*) ; Q = \begin{pmatrix} q_{11} & p_{12} \\ p_{21} & q_{22} \end{pmatrix} \right.$$

$q_{11}, q_{22} \in C^1[0, 1]$, real-valued ,

$J, J^* \in \mathbb{R} \cup \{ \infty \} \setminus \{-1, 1\}$

and

$$\left. \begin{cases} \sigma(A_{Q,h,J}) = \sigma(A_{P,h,H}) \\ \sigma(A_{Q,h,J^*}) = \sigma(A_{P,h,H^*}) \end{cases} . \right\} ,$$

$$(2.11) \qquad \begin{cases} \alpha(x) = p_{11}(x) - p_{22}(x) \\ \\ \beta(x) = p_{12}(x) - p_{21}(x) \qquad (0 \le x \le 1) , \end{cases}$$

$$(2.12) \qquad S = \{ x \in [0, 1] ; |\alpha(x)| = |\beta(x)| \} ,$$

$$(2.13) \qquad r(x) = \frac{\frac{d\alpha(x)}{dx} \cdot \beta(x) - \alpha(x) \cdot \frac{d\beta(x)}{dx}}{\alpha^2(x) - \beta^2(x)} \qquad \text{for } x \notin S,$$

and

$$(2.14) \qquad \tilde{P}(x) = \begin{pmatrix} p_{22}(x) + r(x) & p_{12}(x) \\ p_{21}(x) & p_{11}(x) + r(x) \end{pmatrix}$$

for $x \notin S$.

Here we note that if α, $\beta \in C^2[0, 1]$, then $r \in C^1([0, 1] \setminus S)$ and therefore, $P \in \{ C^1([0, 1] \setminus S) \}^4$.

If we had $M(p_{11}, p_{22} ; H, H^*) = \{ (P, h, H, H^*) \}$, then the two sets of eigenvalues would uniquely determine the two diagonal components of P. However, in general, it is impossible as is shown in the main result for Type-II :

Theorem 2. *Let us assume that*

(2.15) $\quad S$ consists entirely of isolated points.

Let $[\delta, \delta'] \subset [0, 1]$.

Then each $(Q, j, J, J^*) \in M(p_{11}, p_{22} ; H, H^*)$ *satisfies (1) - (4) :*

(1) *Let*

(2.16) $\quad \alpha(x) \neq 0 \qquad (\delta \leq x \leq \delta')$.

Then the following facts (a) and (b) hold :

(a) *We have either*

(2.17) $\quad Q(x) = P(x) \qquad (\delta \leq x \leq \delta')$,

or

(2.18) $\quad [\delta, \delta'] \cap S = \emptyset \qquad$ *and*

$\qquad Q(x) = \tilde{P}(x) \qquad (\delta \leq x \leq \delta')$.

Moreover we get

(2.19) $\quad P(x) \neq \tilde{P}(x) \qquad (x \in [\delta, \delta'] \setminus S)$.

(b) *If (2.18) holds, then we get*

(2.20) $\quad | \beta(x) | > | \alpha(x) | \qquad (\delta \leq x \leq \delta')$,

and

$$(2.21) \qquad \frac{d\alpha}{dx}\cdot\beta - \alpha\cdot\frac{d\beta}{dx} \in C^1[\delta, \delta'] \ .$$

(2) **(the property at a zero of α)** *Let* $\gamma \in (0, 1)$ *be an isolated point in a set* $\{ x \in [0, 1] \ ; \ \alpha(x) = 0 \}$. *Then we have the following three cases :*

(a) If for some $\varepsilon > 0$, *we have*

$$(2.22) \qquad Q(x) = \tilde{P}(x) \qquad (\gamma - \varepsilon < x < \gamma + \varepsilon) \ ,$$

then $r \in C^1(\gamma - \varepsilon, \gamma + \varepsilon)$.

(b) If for some $\varepsilon > 0$, *we have*

$$(2.23) \qquad P(x) = \begin{cases} P(x) & (\gamma - \varepsilon < x \leq \gamma) \\ \\ \tilde{P}(x) & (\gamma \leq x < \gamma + \varepsilon) \ , \end{cases}$$

then $r \in C^1[\gamma, \gamma + \varepsilon)$ *and moreover we have*

$$(2.24) \qquad \lim_{x \to \gamma + 0} \frac{d^i r(x)}{dx^i} = 0 \qquad (i = 0, 1) \ .$$

(c) If for some $\varepsilon > 0$, *we have*

$$(2.25) \qquad Q(x) = \begin{cases} \tilde{P}(x) & (\gamma - \varepsilon < x \leq \gamma) \\ \\ P(x) & (\gamma \leq x < \gamma + \varepsilon) \ , \end{cases}$$

then $r \in C^1(\gamma - \varepsilon, \gamma]$ *and moreover we have*

$$(2.26) \qquad \lim_{x \to \gamma - 0} \frac{d^i r(x)}{dx^i} = 0 \qquad (i = 0, 1) \ .$$

(3) *Let*

$$(2.27) \qquad \alpha(x) = 0 \qquad (\delta \leq x \leq \delta') \quad .$$

Then we get

$$(2.28) \qquad Q(x) = P(x) \qquad (\delta \leq x \leq \delta') \ ,$$

and further

$$(2.29) \qquad \int_0^x (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds = 0$$

$$(\delta \leq x \leq \delta') \quad .$$

(4) **(the property of Q near $x = 0$)** *If*

$$(2.30) \qquad \alpha(x) \neq 0 \qquad (0 \leq x < \delta') \ ,$$

then we get

$$(2.31) \qquad Q(x) = P(x) \qquad (0 \leq x \leq \delta') \ .$$

(5) **(the property of Q near $x = 1$)** *If*

$$(2.32) \qquad \alpha(x) \neq 0 \qquad (\delta < x \leq 1)$$

and, either

$$(2.33) \qquad J = H$$

or

$$(2.34) \qquad J^* = H^* \ ,$$

then we get

$$(2.35) \qquad Q(x) = P(x) \qquad (\delta \leq x \leq 1) \quad .$$

(6) (the determination of J and J^*)

(a) Let us assume that for any sequence $\{ x_n \}_{n \geq 1} \subset [0, 1]$ *such that* $\lim_{n\to\infty} x_n = 1$ *and* $\beta(x_n) + \alpha(x_n) \neq 0$, *the limit*

$$(2.36) \qquad \lim_{n \to \infty} \frac{\beta(x_n) - \alpha(x_n)}{\beta(x_n) + \alpha(x_n)}$$

does not exist, or that even if the limit k *of (2.36) exists for some sequence* $\{ x_n \}_{n\geq 1} \subset [0, 1]$ *such that* $\lim_{n\to\infty} x_n = 1$ *and* $\beta(x_n) + \alpha(x_n) \neq 0$, *we have either*

$$(2.37.1) \qquad k \leq 0 \quad ,$$

or

$$(2.37.2) \qquad k = 1 \quad .$$

Then we get

$$(2.38) \qquad \begin{cases} J = H \\ \\ J^* = H^* \end{cases} \quad .$$

Furthermore, if

$$(2.39) \qquad \alpha(1) \neq 0 \quad ,$$

then we have

$$(2.40) \qquad Q(x) = P(x) \quad (1 - \varepsilon \leq x \leq 1)$$

for sufficiently small $\varepsilon > 0$.

(b) Let us assume that the limit k *of (2.36) exists for some sequence* $\{ x_n \}_{n\geq 1} \subset [0, 1]$ *such that* $\lim_{n\to\infty} x_n = 1$ *and* $\beta(x_n) + \alpha(x_n) \neq 0$, *and that*

(2.41) $\quad k > 0$ *and* $k \neq 1$.

Then we get either $\begin{cases} J = H \\ \qquad\qquad\qquad \text{or} \\ J^* = H^* \end{cases}$

$$(2.42.1) \qquad J = \begin{cases} \dfrac{k(1+H) + H - 1}{k(1+H) + 1 - H} , & \text{if} \quad k \neq \dfrac{H-1}{H+1} \\ \\ \infty , & \text{if} \quad k = \dfrac{H-1}{H+1} \end{cases}$$

and

$$(2.42.2) \qquad J^* = \begin{cases} \dfrac{k(1+H^*) + H^* - 1}{k(1+H^*) + 1 - H^*} , & \text{if} \quad k \neq \dfrac{H^*-1}{H^*+1} \\ \\ \infty , & \text{if} \quad k = \dfrac{H^*-1}{H^*+1} \end{cases} .$$

Furthermore, if

(2.43) $\quad \alpha(1) \neq 0$,

then we have the two cases :

(i) *(2.38) and* $Q(x) = P(x) \quad (1 - \varepsilon \leq x \leq 1)$.

(ii) *(2.38) or (2.42), and* $Q(x) = \tilde{P}(x) \quad (1 - \varepsilon \leq x \leq 1)$.

Here $\varepsilon > 0$ *is sufficiently small.*

Here, if $H = \infty$ *and* $H^* = \infty$, *then the equalities (2.42.1) and (2.42.2) mean* $J = \frac{k+1}{k-1}$ *and* $J^* = \frac{k+1}{k-1}$, *respectively.*

In §4, we prove Theorem 2.

Now we give several remarks on this theorem.

Remark 1. In Theorem 2, the assumption (2.15) is essential. That is, if S contains an interior point, then in S, there are infinitely many Q such that $(Q, j, J, J^*) \in M(p_{11}, p_{22}\ ;\ H, H^*)$. For example, let given $P(x)$ have the form

$$\begin{pmatrix} p(x) & q(x) \\ q(x) & p(x) \end{pmatrix}$$

for any $p, q \in C^1[0, 1]$.

Let us take any $g \in C^1[0, 1]$ such that

$$\exp\left(2\int_0^1 (p(s) - g(s))ds \right) \neq \frac{H-1}{H+1}, \neq \frac{H^*-1}{H^*+1}$$ and let us set $k = \exp\left(2\int_0^1 (p(s) - g(s))ds \right)$. Then we have

$$(2.44)\quad \left(\begin{pmatrix} g(x) & q(x) \\ q(x) & g(x) \end{pmatrix}, h, \frac{k(1+H)+H-1}{k(1+H)+1-H}, \frac{k(1+H^*)+H^*-1}{k(1+H^*)+1-H^*} \right)$$

$$\in M(p_{11}, p_{22}\ ;\ H, H^*) .$$

In fact, the relation (2.44) follows directly by checking that each element at the left hand side of (2.44) satisfies the equalities (1.23) - (1.26) of Theorem in Chapter 2.

Remark 2. In view of the part (6) of Theorem 2, we see the following facts :

(1) Let us assume that

$$(2.45)\quad \beta(1) + \alpha(1) \neq 0 .$$

and let us set $k = \frac{\beta(1) - \alpha(1)}{\beta(1) + \alpha(1)}$.

(a) If we have either

(2.46) $\quad k \leq 0 \quad \text{or} \quad k = 1$,

then we get (2.38).

(b) If (2.46) does not hold, then we get either (2.38) or (2.42).

(2) Let us assume that

(2.47) $\quad \beta(1) + \alpha(1) = 0$.

Then,

(a) The condition $\beta(1) - \alpha(1) \neq 0$ implies (2.38).

(b) If $\beta(1) - \alpha(1) = 0$, then we have no knowledge of J and J^* .

Remark 3. We give an explanation of Theorem 2 with an illustration. Let $P(x) = (p_{ij}(x))_{1 \leq i,j \leq 2}$ be given such that the number of zeros of $\alpha(x) = p_{11}(x) - p_{22}(x)$ is finite. We denote the set of zeros of α by $\{ \gamma_k \}_{k=1}^{m}$.

Let $(Q,j,J,J^*) \in M(p_{11},p_{22} ; H,H^*)$. Then, according to the part (1) - (a) of Theorem 2, the two branches $P(x)$ and $\tilde{P}(x)$ cross each other only at $x = \gamma_k$ $(1 \leq k \leq m)$ (in fact, $P(x) \neq \tilde{P}(x)$ for each $x \neq \gamma_k$).
The part (1) - (a) asserts also that the possible branches of Q are either P and $\tilde{P}$ in the intervals (γ_k, γ_{k+1}) $(1 \leq k \leq m - 1)$, $(0, \gamma_1)$ and $(\gamma_m , 1)$.

Moreover, the coefficient matrix Q changes the branches from P to $\tilde{P}$, or from $\tilde{P}$ to P , if and only if at $x = \gamma_k$, the two branches P and $\tilde{P}$ connect smoothly in the sense stated in the part (2) of this theorem.

If $\alpha(0) \neq 0$, then it follows from the part (4) that $Q(x) = P(x) \quad (0 \leq x \leq \gamma_1)$.

For example, let $m = 6$ and we consider P and $\tilde{P}$ illustrated in Fig.1, where the bold line and the broken line indicate P and $\tilde{P}$, respectively. Let P and $\tilde{P}$ connect only at $x = \gamma_2$, γ_4 and γ_6 smoothly in the sense of the part (2). Then the bold line in Fig.2 indicates a possible branch as Q , while the broken line in Fig.3 indicates an impossible one.

Furthermore, in Appendix II, we determine the set $M(p_{11}, p_{22} ; H, H^*)$ on condition that α has a single zero in the open interval $(0, 1)$.

Now we conclude this subsection with the following corollary, which is easily derived by the parts (1) and (4) of Theorem 2 :

Corollary 1. *Let us assume that*

$$S = \emptyset \quad . \tag{2.48}$$

and that the set $\{ x \in [0, 1] ; \alpha(x) = 0 \}$ *consists of finite points* $\gamma_k \quad (1 \leq k \leq m)$. *Let us set* $\gamma_{m+1} = 1$. *If for* $1 \leq k \leq m$, *there is some* $x_k \in (\gamma_k, \gamma_{k+1})$ *satisfying*

$$Q(x_k) = P(x_k) \quad , \tag{2.49}$$

then $Q(x) = P(x) \quad (0 \leq x \leq 1)$ *follows.*

Fig. 1 P and $\tilde{P}$

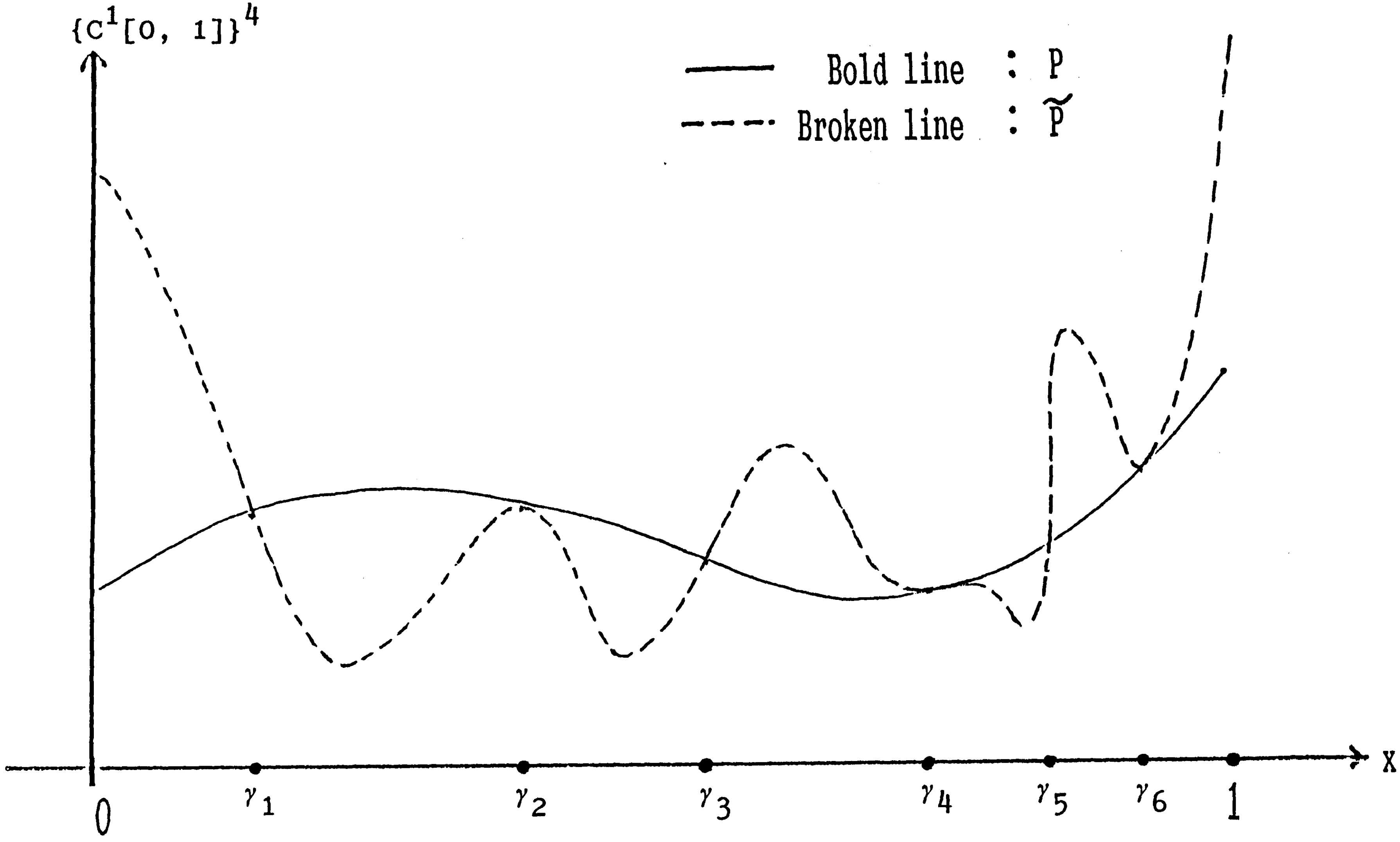


Fig. 2 Example of a possible branch as Q

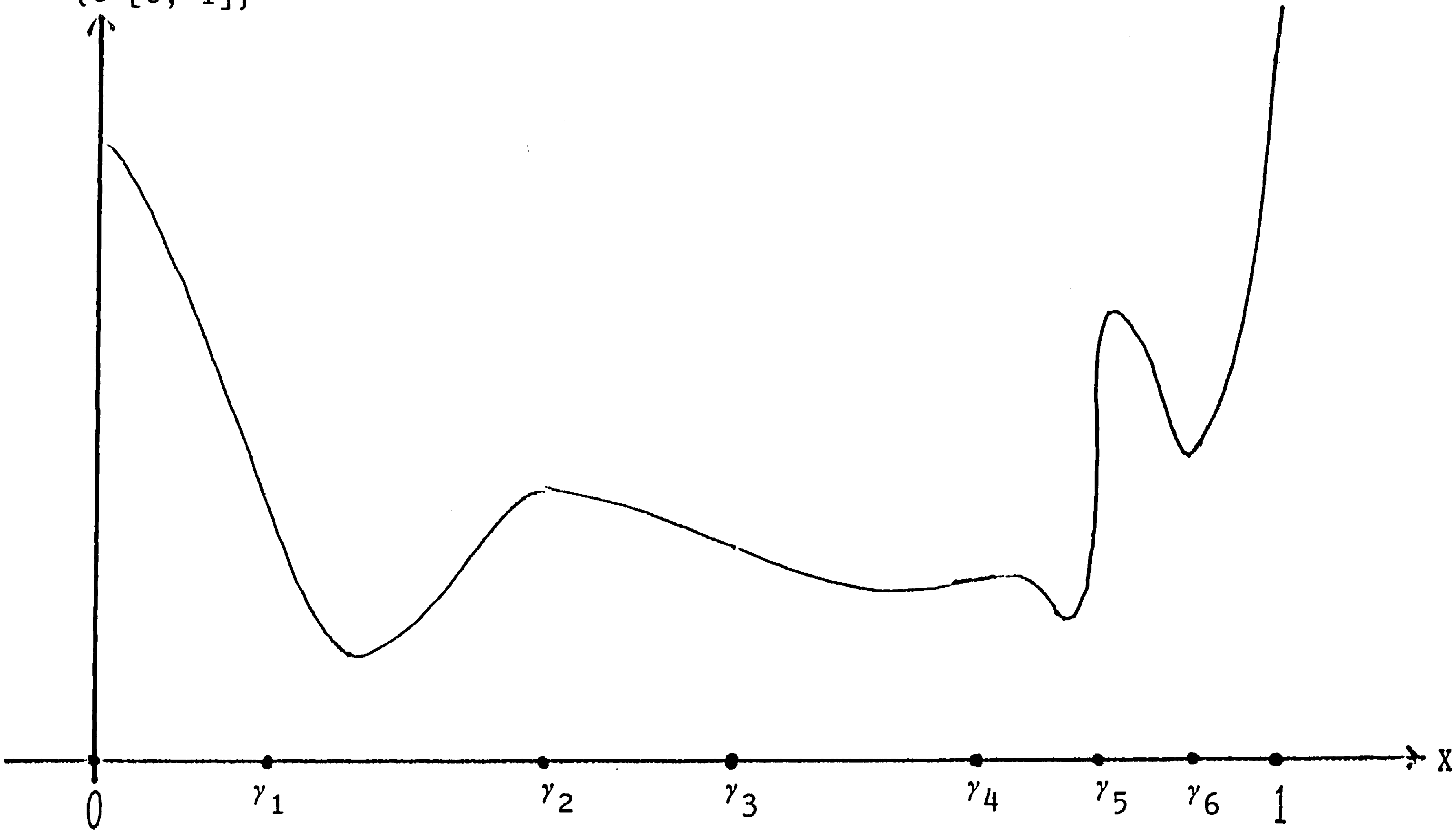
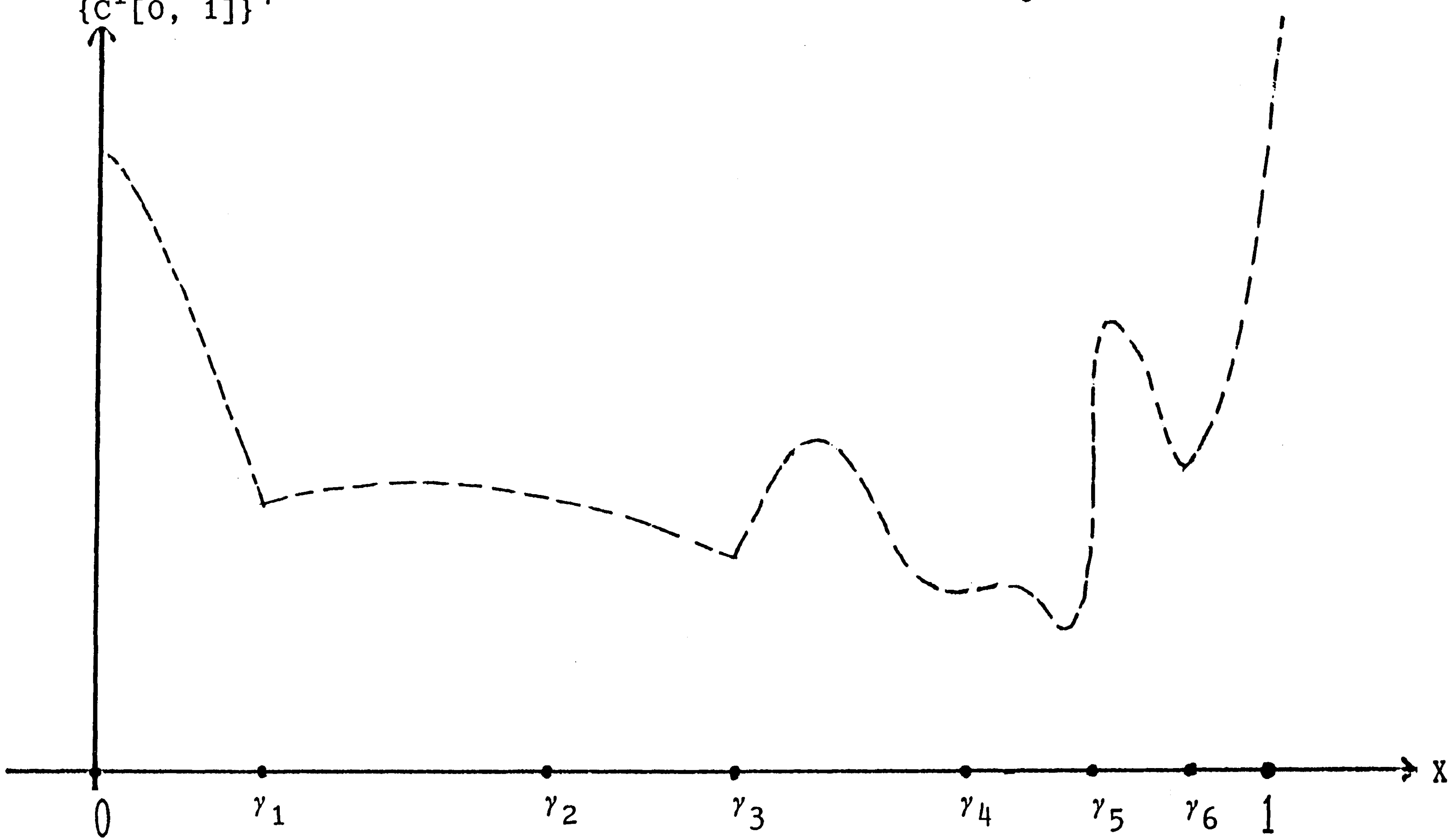


Fig. 3 Example of an impossible branch as Q



\$2.3. **The result for Type-III.** In this case we can completely answer No for the uniqueness.

Setting

$$
(2.50)\qquad M(p_{12},p_{21}\ ;\ H,H^*)
$$
$$
= \left\{ (Q,h,J,J^*)\ ;\ Q = \begin{pmatrix} p_{11} & q_{12} \\ q_{21} & p_{12} \end{pmatrix} \right.
$$
$$
q_{12},\ q_{21} \in C^1[0,\ 1],\ \text{real-valued}\ ,
$$
$$
J,\ J^* \in \mathbb{R} \cup \{\ \infty\ \} \setminus \{-1,\ 1\}
$$
and
$$
\left. \begin{cases} \sigma(A_{Q,h,J}) = \sigma(A_{P,h,H}) \\ \sigma(A_{Q,h,J^*}) = \sigma(A_{P,h,H^*}) \end{cases} .\ \right\}\ ,
$$

we get

Theorem 3. *The relation*

$$
(2.51)\qquad M(p_{12},p_{21}\ ;\ H,H^*)
$$
$$
= \left\{ \left(\begin{pmatrix} p_{11} & d + p_{12} - p_{21} \\ d & p_{22} \end{pmatrix} ,h,H,H^* \right)\ ; \right.
$$
$$
\left. d \in C^1[0,\ 1],\quad \text{real-valued} \right\}
$$

holds.

We note that the set $M(p_{12},p_{21}\ ;\ H,H^*)$ contains an indeterminate element d in the coefficient matrix, while the boundary conditions at $x = 1$ are uniquely determined.
A proof of Theorem 3 is given in \$3.

Remark 4. We can discuss identification problems of other types. For example, let us set

$$(2.52)\quad M(p_{12}\ ;\ h,H,H^*)$$
$$= \left\{ (Q,j,J,J^*)\ ;\ Q = \begin{pmatrix} p_{11} & q_{12} \\ p_{21} & p_{22} \end{pmatrix} \right.$$
$$q_{12} \in C^1[0,\ 1],\ \text{real-valued}\ ,$$
$$j \in \mathbb{R} \setminus \{-1,\ 1\}\ ,\quad J,\ J^* \in \mathbb{R} \cup \{\ \infty\ \} \setminus \{-1,\ 1\}\ .$$
$$\text{and}$$
$$\left. \begin{cases} \sigma(A_{Q,h,J}) = \sigma(A_{P,h,H}) \\ \sigma(A_{Q,h,J^*}) = \sigma(A_{P,h,H^*})\ . \end{cases} \right\}.$$

Then $M(p_{12}\ ;\ h,H,H^*)$ can be given in Table 1. Here let us recall that the functions α and β are defined by (2.11) and, as for the determination of J and J^*, we understand that $\frac{\lambda - x}{\lambda x - 1} = -\frac{1}{\lambda}$ and $\frac{1}{x} = 0$ if $x = \infty$, and $\frac{1}{x} = \infty$ if $x = 0$.

In a forthcoming paper, we fully discuss identification problems of other types including the present one.

Table 1. Determination of $M(p_{12}\ ;\ h,H,H^*)$.

$\alpha \neq 0$	There exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 0, \neq 1, \neq -1$, $\lambda h \neq 1$, and $\beta = \lambda\alpha$	$\beta \neq 0$	$M(p_{12}\ ;\ h,H,H^*)$
YES	YES	YES or NO	$\left\{(P,h,H,H^*),\ \left(\begin{pmatrix} p_{11} & 2p_{21}-p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \frac{\lambda-h}{\lambda h-1}, \frac{\lambda-H}{\lambda H-1}, \frac{\lambda-H^*}{\lambda H^*-1}\right)\right\}$
	NO	YES or NO	$\left\{(P,h,H,H^*)\right\}$
NO	YES or NO	YES	$\left\{(P,h,H,H^*),\ \left(\begin{pmatrix} p_{11} & 2p_{21}-p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \frac{1}{h}, \frac{1}{H}, \frac{1}{H^*}\right)\right\}$, if $h \neq 0$. $\left\{(P,h,H,H^*)\right\}$, if $h = 0$.
	YES or NO	NO	$\left\{\left(P, j, \frac{H - h + j(1 - hH)}{j(H - h) + 1 - hH}, \frac{H^* - h + j(1 - hH^*)}{j(H^* - h) + 1 - hH^*}\right)\ ;\ \lvert j\rvert \neq 1\right\}$

§3. Proof of Theorems 1 and 3. Let us set

$$(3.1)\qquad M(p_{12},p_{21},p_{22}\ ;\ H,H^*)$$
$$= \left\{ (Q,h,J,J^*)\ ;\ Q = \begin{pmatrix} p_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \right.$$

$q_{12},\ q_{21},\ q_{22} \in C^1[0,\ 1]$, real-valued,

$J,\ J^* \in \mathbb{R} \cup \{\ \infty\ \} \setminus \{-1,\ 1\}$

and

$$\left. \begin{cases} \sigma(A_{Q,h,J}) = \sigma(A_{P,h,H}) \\ \sigma(A_{Q,h,J^*}) = \sigma(A_{P,h,H^*}) \end{cases} . \right\} ,$$

and

$$(3.2)\qquad M(p_{11},p_{12},p_{21}\ ;\ H,H^*)$$
$$= \left\{ (Q,h,J,J^*)\ ;\ Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & p_{22} \end{pmatrix} \right.$$

$q_{11},\ q_{12},\ q_{21} \in C^1[0,\ 1]$, real-valued,

$J,\ J^* \in \mathbb{R} \cup \{\ \infty\ \} \setminus \{-1,\ 1\}$

and

$$\left. \begin{cases} \sigma(A_{Q,h,J}) = \sigma(A_{P,h,H}) \\ \sigma(A_{Q,h,J^*}) = \sigma(A_{P,h,H^*}) \end{cases} . \right\} .$$

Then, for the proof of Theorems 1 and 3, it is sufficient to verify

Proposition 2. *We have*

$$(3.3)\qquad M(p_{12}, p_{21}, p_{22} \; ; \; H, H^{*})$$
$$= \left\{ \left(\begin{pmatrix} p_{11} & d + p_{12} - p_{21} \\ d & p_{22} \end{pmatrix}, h, H, H^{*} \right) ; \; d \in C^{1}[0, 1], \textit{ real-valued.} \right\},$$

and

$$(3.4)\qquad M(p_{11}, p_{12}, p_{21} \; ; \; H, H^{*})$$
$$= \left\{ \left(\begin{pmatrix} p_{11} & d + p_{12} - p_{21} \\ d & p_{22} \end{pmatrix}, h, H, H^{*} \right) ; \; d \in C^{1}[0, 1], \textit{ real-valued.} \right\} .$$

Proof of Proposition 2. Let us denote the set given by the right hand side in (3.3) by M_1 .

Let $(Q, j, J, J^{*}) \in M(p_{12}, p_{21}, p_{22} \; ; \; H, H^{*})$ and let us set $d = q_{21}$. Then, by Theorem in Chapter 2, we have

$$(3.5)\qquad (q_{12}(x)-d(x)+p_{12}(x)-p_{21}(x)-(q_{22}(x)-p_{22}(x)))$$
$$+ (-(q_{12}(x)-d(x)+p_{12}(x)-p_{21}(x))-(q_{22}(x)-p_{22}(x)))$$
$$\cdot \exp\left(\int_0^x (q_{22}(s) - p_{22}(s))ds \right) = 0 \qquad (0 \le x \le 1) ,$$

$$(3.6)\qquad (q_{12}(x)-(d(x)+p_{12}(x)-p_{21}(x))+2p_{11}(x)-q_{22}(x)-p_{22}(x))$$
$$+ (q_{12}(x)-(d(x)+p_{12}(x)-p_{21}(x))-2p_{11}(x)+q_{22}(x)+p_{22}(x))$$
$$\cdot \exp\left(\int_0^x (q_{22}(s) - p_{22}(s))ds \right) = 0 \qquad (0 \le x \le 1) ,$$

$$(3.7) \qquad \log \frac{(1 - H)(1 + J)}{(1 + H)(1 - J)} = \int_0^1 (q_{22}(s) - p_{22}(s))ds ,$$

and

$$(3.8) \qquad \log \frac{(1 - H^*)(1 + J^*)}{(1 + H^*)(1 - J^*)} = \int_0^1 (q_{22}(s) - p_{22}(s))ds .$$

Setting

$$(3.9) \qquad f(x) = q_{22}(x) - p_{22}(x)$$

and

$$(3.10) \qquad a(x) = q_{12}(x) - d(x) + p_{12}(x) - p_{21}(x) ,$$

we can rewrite (3.5) as

$$(a(x) - f(x)) + (-a(x) - f(x)) \cdot \exp\left(\int_0^x f(s)ds \right) = 0 \qquad (0 \le x \le 1) ,$$

which is equivalent to

$$(3.11) \qquad f(x) = a(x) \cdot \frac{1 - \exp\left(\int_0^x f(s)ds\right)}{1 + \exp\left(\int_0^x f(s)ds\right)} \qquad (0 \le x \le 1) .$$

Since q_{ij} , $p_{ij} \in C^1[0, 1]$ $(1 \le i,j \le 2)$, we have

$m_1 \le f(x) \le m_2 \qquad (0 \le x \le 1)$

and

$| \ a(x) \ | \le m_3 \qquad (0 \le x \le 1)$

for some constants m_1, m_2 and m_3 . Therefore we obtain

$$(3.12) \qquad \left|\exp\left(\int_0^x f(s)ds\right) - 1\right| \le e^{|m_1|+|m_2|} \cdot \int_0^x |f(s)|ds \qquad (0 \le x \le 1)$$

by the mean value theorem, and

$$(3.13) \qquad \exp\left(\int_0^x f(s)ds\right) + 1 > 1 \quad .$$

Estimating the right hand side in (3.11) in view of (3.12) and (3.13), we have

$$|f(x)| \le m_3 \cdot e^{|m_1|+|m_2|} \cdot \int_0^x |f(s)|ds \qquad (0 \le x \le 1),$$

which implies

$$(3.14) \qquad f(x) = 0 \qquad (0 \le x \le 1)$$

by Gronwall's inequality. Thus we see

$$(3.15) \qquad q_{22}(x) = p_{22}(x) \qquad (0 \le x \le 1) \ .$$

Substituting (3.15) into (3.6) - (3.8), we obtain

$$(3.16) \qquad q_{12}(x) = d(x) + p_{12}(x) - p_{21}(x) \quad ,$$

$$(3.17) \qquad J = H \quad ,$$

and

$$(3.18) \qquad J^* = H^* \ .$$

Here in the derivation of (3.17) and (3.18), we note also that $\frac{1+J}{1-J} = -1$ and $\frac{1+J^*}{1-J^*} = -1$ imply $J = \infty$ and $J^* = \infty$, respectively.

The equalities (3.15) - (3.18) imply that $M(p_{12}, p_{21}, p_{22} ; H, H^*) \subset M_1$.

On the other hand, by direct computations, we can show that each $(Q, j, J, J^*) \in M_1$ satisfies (1.23) - (1.26) in Theorem of Chapter 2, provided that $p_{11}(x) = q_{11}(x)$ $(0 \leq x \leq 1)$. Therefore we see that $M_1 \subset M(p_{12}, p_{21}, p_{22} ; H, H^*)$. This proves (3.3).

As for (3.4), we can proceed similarly.

This completes the proof of Proposition 2, so that both of Theorems 1 and 3 are proved.

§4. **Proof of Theorem 2.** For the proof, Theorem in Chapter 2 is a key.

Let us assume that $(Q, j, J, J^*) \in M(p_{11}, p_{22}\ ;\ H, H^*)$. Then, by (1.23) and (1.24) in Theorem of Chapter 2, we obtain

$$(4.1)\qquad (m(x) - \alpha(x) + 2\beta(x)) + (m(x) - \alpha(x) - 2\beta(x))\cdot e^{\theta(x)} = 0 \qquad (0 \leq x \leq 1),$$

and

$$(4.2)\qquad (m(x) + \alpha(x))(e^{\theta(x)} - 1) = 0 \qquad (0 \leq x \leq 1).$$

Here we set

$$(4.3)\qquad m(x) = q_{11}(x) - q_{22}(x) \qquad (0 \leq x \leq 1),$$

and

$$(4.4)\qquad \theta(x) = \int_0^x (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds \qquad (0 \leq x \leq 1),$$

and we recall that α and β are given by

$$(4.5)\qquad \begin{cases} \alpha(x) = p_{11}(x) - p_{22}(x) \\ \beta(x) = p_{12}(x) - p_{21}(x) \qquad (0 \leq x \leq 1). \end{cases}$$

Further we set

$$(4.6)\begin{cases} K_1 = \{\ x \in [0, 1]\ ;\ m(x) + \alpha(x) = 0,\ m(x) - \alpha(x) \neq 0\ \}, \\ K_2 = \{\ x \in [0, 1]\ ;\ m(x) + \alpha(x) \neq 0,\ m(x) - \alpha(x) = 0\ \}, \\ K_3 = \{\ x \in [0, 1]\ ;\ m(x) + \alpha(x) = 0,\ m(x) - \alpha(x) = 0\ \}. \end{cases}$$

We prove Theorem 2 by applying (4.1) and (4.2). To this end, we prepare Lemmas 4.1 - 4.7, which constitute essential parts of Theorem 2.

Lemma 4.1. *If* $y \in K_1$, *then we have*

(4.7) $\quad y \notin S$,

and

(4.8) $\quad (\alpha(y) - \beta(y)) + (\alpha(y) + \beta(y)) \cdot e^{\theta(y)} = 0$.

Proof of Lemma 4.1. Since $y \in K_1$ implies $m(y) + \alpha(y) = 0$, we have (4.8) by (4.1).

Next we proceed to the proof of (4.7). To this end, let us assume that

(4.9) $\quad \alpha(y) + \beta(y) = 0$.

Then in view of (4.8), we get

(4.10) $\quad \alpha(y) - \beta(y) = 0$.

The equalities (4.9) and (4.10) imply $\alpha(y) = 0$, which contradicts $y \in K_1$. Here, by the definition (4.6) of K_1 , we note that $\alpha(y) \neq 0$ for $y \in K_1$.

This contradiction shows

(4.11) $\quad \alpha(y) + \beta(y) \neq 0$.

Hence, again by (4.8), we get

(4.12) $\quad \alpha(y) - \beta(y) \neq 0$.

By (4.11) and (4.12), we see (4.7).

Lemma 4.2. *Let* $[\delta, \delta'] \subset [0, 1]$.

(1) If

$$(4.13) \qquad [\delta, \delta'] \subset K_1 \quad ,$$

then we have

$$(4.14) \qquad [\delta, \delta'] \cap S = \emptyset$$

and

$$(4.15) \qquad Q(x) = \tilde{P}(x) \qquad (\delta \leq x \leq \delta') \ .$$

(2) If

$$(4.16) \qquad [\delta, \delta'] \subset K_2 \quad ,$$

then we have

$$(4.17) \qquad Q(x) = P(x) \qquad (\delta \leq x \leq \delta')$$

and

$$(4.18) \qquad \theta(x) = 0 \qquad (\delta \leq x \leq \delta') \quad .$$

Proof of Lemma 4.2. (1) Let (4.13) hold. First by (4.7) in Lemma 4.1, we see (4.14).

Next we have to show (4.15). By (4.8) in Lemma 4.1, we have

$$(4.19) \qquad (\alpha(x) - \beta(x)) + (\alpha(x) + \beta(x))\cdot e^{\theta(x)} = 0 \quad (\delta \leq x \leq \delta').$$

Differentiating (4.19), we get

$$(4.20) \qquad (\alpha'(x) - \beta'(x)) + \\ (\alpha'(x) + \beta'(x) + \theta'(x)\cdot(\alpha(x) + \beta(x)))\cdot e^{\theta(x)} = 0 \\ (\delta \leq x \leq \delta').$$

Here and henceforth, we use the notation $\alpha'(x)$, $\beta'(x)$, etc. in place of $\frac{d\alpha(x)}{dx}$, $\frac{d\beta(x)}{dx}$.

By eliminating $e^{\theta(x)}$ in (4.19) and (4.20), we reach

$$(4.21) \qquad \theta'(x)(\alpha^2(x) - \beta^2(x)) = 2(\alpha'(x)\beta(x) - \alpha(x)\beta'(x)) \qquad (\delta \le x \le \delta').$$

Since $\theta'(x) = q_{11}(x) + q_{22}(x) - p_{11}(x) - p_{22}(x) \quad (0 \le x \le 1)$, the equality (4.21) implies

$$(4.22) \qquad q_{11}(x) + q_{22}(x) = p_{11}(x) + p_{22}(x) + 2r(x) \quad (\delta \le x \le \delta').$$

Here we note that $[\delta, \delta'] \cap S = \emptyset$ (by (4.14)), and therefore r is well-defined on $[\delta, \delta']$.

On the other hand, by (4.13), we have $m(x) + \alpha(x) = 0$ $(\delta \le x \le \delta')$, namely,

$$(4.23) \qquad q_{11}(x) - q_{22}(x) = p_{22}(x) - p_{11}(x) \qquad (\delta \le x \le \delta').$$

Combining (4.23) with (4.22), we see (4.15).

This completes the proof of (1) of this lemma.

(2) Let (4.16) hold. Then, since $m(x) + \alpha(x) \neq 0$ $(\delta \le x \le \delta')$, we get

$$e^{\theta(x)} - 1 = 0 \qquad (\delta \le x \le \delta')$$

by (4.2) and hence we obtain (4.18).

Next we have to show (4.17). By (4.18), we have

$$(4.24) \qquad q_{11}(x) + q_{22}(x) = p_{11}(x) + p_{22}(x) \quad (\delta \le x \le \delta') .$$

On the other hand, (4.16) implies $m(x) - \alpha(x) = 0$ $(\delta \le x \le \delta')$, namely,

$$(4.25) \qquad q_{11}(x) - q_{22}(x) = p_{11}(x) - p_{22}(x) \qquad (\delta \le x \le \delta').$$

Combining (4.25) with (4.24), we reach (4.17).

Thus the proof of Lemma 4.2 is completed.

Lemma 4.3. *We have*

$$K_1 \cup K_2 \cup K_3 = [0, 1] \quad . \tag{4.26}$$

Further K_1 *and* K_2 *are open sets in* $[0, 1]$.

Proof of Lemma 4.3. By eliminating $e^{\theta(x)}$ in (4.1) and (4.2), we get

$$(m(x) + \alpha(x))(m(x) - \alpha(x)) = 0 \qquad (0 \le x \le 1) \ . \tag{4.27}$$

Therefore the relation (4.26) follows.

Next we have to show that K_1 is an open set. Let us assume that there exists $x_0 \in K_1$ such that x_0 is not an interior point of K_1 . Then there exists $x_n \notin K_1$ $(n \ge 1)$ satisfying

$$\lim_{n \to \infty} x_n = x_0 \quad . \tag{4.28}$$

Further, by $x_n \notin K_1$, we have

$$m(x_n) - \alpha(x_n) = 0 \qquad (n \ge 1) \ . \tag{4.29}$$

Hence by the continuity of $m - \alpha$, we get $m(x_0) - \alpha(x_0) = \lim_{n \to \infty} (m(x_n) - \alpha(x_n)) = 0$, which contradicts $x_0 \in K_1$. This contradiction shows the openness of K_1. Similarly we can prove the openness of K_2 .

Lemma 4.4. *Let* $[\delta, \delta'] \subset [0, 1]$ *and let*

$$\alpha(x) \neq 0 \qquad (\delta \le x \le \delta'). \tag{4.30}$$

Then we have either

$$[\delta, \delta'] \subset K_1 \setminus S \ , \tag{4.31}$$

or

(4.32) $\quad [\delta, \delta'] \subset K_2 \quad .$

Proof of Lemma 4.4. Since (4.30) implies $[\delta, \delta'] \cap K_3 = \emptyset$, we have

(4.33) $\quad [\delta, \delta'] \subset K_1 \cup K_2 \quad .$

by Lemma 4.3.

Then δ is an interior point of either K_1 or K_2 by the openness of K_1 and K_2. Let us assume that δ is an interior point of K_1. Then we have to show (4.31).

To this end, let us put

(4.34) $\quad s(\delta) = \sup \{ \eta \ ; \ (\delta, \eta) \subset K_1 \ , \ \eta < \delta' \} \ .$

Since K_1 is an open set by Lemma 4.3, the number $s(\delta)$ is well-defined. Then we can show

(4.35) $\quad s(\delta) = \delta' \quad .$

In fact, assume that $s(\delta) < \delta'$. By (4.33), there exist two sequences $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ such that

(4.36) $$\begin{cases} \lim\limits_{n \to \infty} x_n = s(\delta), \text{ and } x_n \in K_1 \ , \\ \delta < x_n \leq s(\delta) \qquad (n \geq 1) \ , \end{cases}$$

and

(4.37) $$\begin{cases} \lim\limits_{n \to \infty} y_n = s(\delta), \text{ and } y_n \in K_2 \ , \\ s(\delta) \leq y_n < \delta' \qquad (n \geq 1) \ . \end{cases}$$

By (4.36) and (4.37), we get $m(x_n) + \alpha(x_n) = 0$ and $m(y_n) - \alpha(y_n) = 0$ $(n \geq 1)$, which imply $m(s(\delta)) + \alpha(s(\delta)) = 0$ and $m(s(\delta)) - \alpha(s(\delta)) = 0$ in view of the continuity of m and α. Thus we show that $\alpha(s(\delta)) = 0$ for $\delta < s(\delta) < \delta'$, which contradicts our assumption (4.30). Hence we get (4.35).

Now we return to the proof of (4.31). In virtue of (4.35), we obtain

$$(4.38) \qquad (\delta,\ s(\delta)) = (\delta,\ \delta') \subset K_1 \quad .$$

On the other hand, we can see that

$$(4.39) \qquad \delta' \in K_1 \quad .$$

In fact, since $\delta' \in K_3$ implies $\alpha(\delta') = 0$, which contradicts the assumption (4.30).
Further, if $\delta' \in K_2$, then the openness of K_2 contradicts (4.38). Thus we see (4.39).

Combining (4.39) with (4.38) and recalling $\delta \in K_1$, we show that $[\delta,\ \delta'] \subset K_1$. Moreover noting $K_1 \cap S = \emptyset$ by Lemma 4.1, we see (4.31).

In the case where δ is an interior point of K_2, we can similarly proceed to see (4.32).

Thus the proof of Lemma 4.4 is completed.

Lemma 4.5. *Let*

$$(4.40) \qquad \alpha(x) = 0 \qquad (\delta \le x \le \delta') \ .$$

Then we get

$$(4.41) \qquad Q(x) = P(x) \quad (\delta \le x \le \delta')$$

and

$$(4.42) \qquad \theta(x) \equiv \int_0^x (q_{11}(s) + q_{22}(s) - p_{11}(s) - p_{22}(s))ds = 0 \qquad (\delta \le x \le \delta') \quad .$$

Proof of Lemma 4.5. In view of (4.40), the equalities (4.1) and (4.2) are seen to be

(4.43) $\quad (m(x) + 2\beta(x)) + (m(x) - 2\beta(x))\cdot e^{\theta(x)} = 0$

$$(\delta \leq x \leq \delta'),$$

and

(4.44) $\quad m(x)(e^{\theta(x)} - 1) = 0 \qquad (\delta \leq x \leq \delta')$.

If $m(x) \neq 0$, then we have

(4.45) $\quad e^{\theta(x)} = 1$,

in view of (4.44).

On the other hand, if $m(x) = 0$, then we get

(4.46) $\quad \beta(x)(e^{\theta(x)} - 1) = 0$

by (4.43).

Since $\{ x \in [\delta, \delta'] ; \beta(x) = 0 \} =$ $\{ x \in [\delta, \delta'] ; \alpha(x) = \beta(x) = 0 \}$ in view of (4.40), we have $\{ x \in [\delta, \delta'] ; \beta(x) = 0 \} \subset S$, so that $\{ x \in [\delta, \delta'] ; \beta(x) = 0 \}$ also consists entirely of isolated points by the assumption (2.15) of Theorem 2.

Therefore, by (4.46) and the continuity of $e^{\theta(x)} - 1$, we again get (4.45).

Thus we see that (4.40) implies (4.42).

By substituting (4.42) into (4.43), we obtain $m(x) = 0$ $(\delta \leq x \leq \delta')$, namely,

(4.47) $\quad q_{11}(x) = q_{22}(x) \qquad (\delta \leq x \leq \delta')$.

In view of (4.42), we have

(4.48) $\quad q_{11}(x) + q_{22}(x) = p_{11}(x) + p_{22}(x) \qquad (\delta \leq x \leq \delta')$,

and in virtue of (4.47), (4.48), and (4.40), we reach $q_{11}(x) = p_{11}(x)$ and $q_{22}(x) = p_{22}(x) \quad (\delta \leq x \leq \delta')$.
This shows (4.41).

Lemma 4.6. *(1) Let us assume that there exists some sequence* $\{ x_n \}_{n \geq 1} \subset [0, 1]$ *such that*

$$x_n \in K_1 \quad (n \geq 1) \quad \text{and} \quad \lim_{n \to \infty} x_n = 1 \quad . \tag{4.49}$$

Then we have

$$\beta(x_n) + \alpha(x_n) \neq 0 \ , \tag{4.50}$$

$$\text{The limit} \quad k = \lim_{n \to \infty} \frac{\beta(x_n) - \alpha(x_n)}{\beta(x_n) + \alpha(x_n)} \quad \text{exists,} \tag{4.51}$$

$$k > 0 \ , \tag{4.52}$$

and

$$e^{\theta(1)} = k \quad . \tag{4.53}$$

(2) In particular, if $1 \in K_1$ *, then we have*

$$k = \frac{\beta(1) - \alpha(1)}{\beta(1) + \alpha(1)} \quad , \tag{4.51$'$}$$

as well as (4.52) and (4.53), and furthermore

$$Q(x) = \tilde{P}(x) \qquad (1 - \varepsilon \leq x \leq 1) \tag{4.54}$$

for some $\varepsilon > 0$.

Proof of Lemma 4.6. (1) Since $x_n \in K_1$ $(n \geq 1)$, we have (4.50) and

$$(\alpha(x_n) - \beta(x_n) + (\alpha(x_n) + \beta(x_n))\cdot e^{\theta(x_n)} = 0 \quad (n \geq 1) \tag{4.55}$$

in virtue of Lemma 4.1.

Hence we get

$$e^{\theta(x_n)} = \frac{\beta(x_n) - \alpha(x_n)}{\beta(x_n) + \alpha(x_n)} \qquad (n \geq 1) \ . \tag{4.56}$$

Since in (4.56), $\lim_{n \to \infty} e^{\theta(x_n)}$ exists, we see that also

$\lim_{n \to \infty} \dfrac{\beta(x_n) - \alpha(x_n)}{\beta(x_n) + \alpha(x_n)}$ exists and that its limit is equal

to $e^{\theta(1)}$ for every sequence $\{ x_n \}_{n \geq 1}$ satisfying (4.49). This fact shows (4.51) - (4.53).

(2) Let $1 \in K_1$. Setting $x_n = 1$ $(n \geq 1)$, we see that this sequence $\{ x_n \}_{n \geq 1}$ satisfies the condition (4.49) and therefore we have (4.50) - (4.53).

On the other hand, since $1 \in K_1$ implies $1 \notin S$ by Lemma 4.1, we see $\beta(1) + \alpha(1) \neq 0$. Thus (4.51) implies (4.51)' in the case of $1 \in K_1$.

Finally we have to show (4.54). In view of $1 \in K_1$ and the openness of K_1, we have $[1 - \varepsilon, 1] \subset K_1$ for some $\varepsilon > 0$. Therefore by Lemma 4.2 (1), we see (4.54).

Thus we complete the proof of Lemma 4.6.

Lemma 4.7. *(1) Let us assume that there exists a sequence* $\{ x_n \}_{n \geq 1} \subset [0, 1]$ *such that*

(4.57) $\quad x_n \in K_2 \quad (n \geq 1)$ *and* $\lim_{n \to \infty} x_n = 1$.

Then we have

(4.58) $\quad J = H$ *and* $J^* = H^*$.

(2) In particular, if $1 \in K_2$ *, then we have (4.58) and therefore*

(4.59) $\quad Q(x) = P(x) \quad (1 - \varepsilon \leq x \leq 1)$

for some $\varepsilon > 0$.

Proof of Lemma 4.7. (1) Putting $x = x_n$ in (4.2) and noting that $m(x_n) + \alpha(x_n) \neq 0$ by $x_n \in K_2$, we get $e^{\theta(x_n)} = 1$ $(n \geq 1)$, namely, $\theta(x_n) = 0$. Hence we have

$$\theta(1) = \lim_{n \to \infty} \theta(x_n) = 0 .$$

In view of (1.25) and (1.26) in Theorem of Chapter 2 with $j = h$, we obtain (4.58).

(2) Let $1 \in K_2$. Then, setting $x_n = 1$ $(n \geq 1)$, we see that this sequence $\{ x_n \}_{n \geq 1}$ satisfies the condition (4.57), and therefore (4.58) follows.

Finally, $1 \in K_2$ and the openness of K_2 imply $[1 - \varepsilon, 1] \subset K_2$ for some $\varepsilon > 0$. Therefore, in view of Lemma 4.2 (2), we get (4.59).

Thus we complete the proof of Lemma 4.7.

Under these preparations, we proceed to a proof of Theorem 2.

Proof of (1) - (a). We have to prove that either

$$(4.60) \qquad Q(x) = P(x) \qquad (\delta \leq x \leq \delta')$$

or

$$(4.61) \qquad [\delta, \delta'] \cap S = \emptyset \quad \text{and} \quad Q(x) = \tilde{P}(x) \quad (\delta \leq x \leq \delta').$$

Since $\alpha(x) \neq 0 \quad (\delta \leq x \leq \delta')$, we get either

$$(4.62) \qquad [\delta, \delta'] \subset K_1 \qquad ,$$

or

$$(4.63) \qquad [\delta, \delta'] \subset K_2 \qquad ,$$

in view of Lemma 4.4.

Firstly let (4.62) hold. Then we have (4.61) by Lemma 4.2 (1).

Secondly let (4.63) hold. Then we have (4.60) by Lemma 4.2 (2).

Finally the relation (2.19) :

$$P(x) \neq \tilde{P}(x) \qquad (x \in [\delta, \delta'] \setminus S\ )$$

follows from $\alpha(x) \neq 0 \quad (\delta \leq x \leq \delta')$. In fact, assume that $P(x_0) = \tilde{P}(x_0)$, namely, $p_{11}(x_0) = p_{22}(x_0) + r(x_0)$ and $p_{22}(x_0) = p_{11}(x_0) + r(x_0)$ for some $x_0 \in [\delta, \delta'] \setminus S$. Then we have $p_{11}(x_0) = p_{22}(x_0)$, which contradicts $\alpha(x_0) \neq 0$.

Proof of (1) - (b). Let us assume (2.18), that is, let $[\delta, \delta'] \cap S = \emptyset$ and $Q(x) = \tilde{P}(x) \quad (\delta \leq x \leq \delta')$.

Then we have to show (2.20) and (2.21) :

$$|\ \beta(x)\ | > |\ \alpha(x)\ | \qquad (\delta \leq x \leq \delta')$$

and

$$\frac{d\alpha}{dx}\cdot\beta - \alpha\cdot\frac{d\beta}{dx} \in C^1[\delta, \delta'] \quad .$$

Since $Q(x) = \tilde{P}(x)$ implies $m(x) = q_{11}(x) - q_{22}(x)$ $= p_{22}(x) - p_{11}(x) = -\alpha(x)$ $(\delta \leq x \leq \delta')$, we get

(4.64) $\quad m(x) + \alpha(x) = 0 \quad (\delta \leq x \leq \delta')$.

By substituting (4.64) into (4.1), we obtain

(4.65) $\quad (\alpha(x) - \beta(x)) + (\alpha(x) + \beta(x))\cdot e^{\theta(x)} = 0 \quad (\delta \leq x \leq \delta').$

Then, since $[\delta, \delta'] \cap S = \emptyset$ by (2.18), we have $(\alpha(x) + \beta(x))(\alpha(x) - \beta(x)) \neq 0 \quad (\delta \leq x \leq \delta')$, so that we obtain

(4.66) $\quad e^{\theta(x)} = \dfrac{\beta(x) - \alpha(x)}{\beta(x) + \alpha(x)} \quad (\delta \leq x \leq \delta')$.

Hence we get (2.20) in view of $e^{\theta(x)} > 0$.

Finally (2.21) follows from (2.18) and $r \in C^1[\delta, \delta']$.

Proof of (2). First let us consider the case (2) - (a). That is, assume (2.22):

$$Q(x) = \tilde{P}(x) \qquad (\gamma - \varepsilon < x < \gamma + \varepsilon) .$$

Then, since $Q \in \{ C^1[0, 1] \}^4$, we see that $\tilde{P} \in \{ C^1(\gamma - \varepsilon, \gamma + \varepsilon) \}^4$ and, in particular, also that $r \in C^1(\gamma - \varepsilon, \gamma + \varepsilon)$.

Next let us consider the case (2) - (b). That is, assume (2.23):

$$Q(x) = \begin{cases} P(x) & \gamma - \varepsilon < x \leq \gamma \\ \tilde{P}(x) & \gamma \leq x < \gamma + \varepsilon \quad . \end{cases}$$

Then, in view of $Q \in \{ C^1[0, 1] \}^4$, we see that $r \in C^1[\gamma, \gamma + \varepsilon)$. Moreover, since Q is 1-time continuously differentiable also at $x = \gamma$, we get $\lim_{x \to \gamma+0} \frac{d^i \tilde{P}(x)}{dx^i} = \lim_{x \to \gamma-0} \frac{d^i P(x)}{dx^i}$ $(i = 0, 1)$. The last equality is equivalent to (2.24):

$$\lim_{x \to \gamma+0} \frac{d^i r(x)}{dx^i} = 0 \qquad (i = 0, 1) .$$

Thus we see the case (2) - (b).

As for the case (2) - (c), we can similarly proceed.

Proof of (3). This part is nothing but Lemma 4.5.

Proof of (4). Let us assume (2.30):

$$\alpha(x) \neq 0 \qquad (0 \leq x < \delta') .$$

Then we have to show (2.31):

$$Q(x) = P(x) \qquad (0 \leq x \leq \delta') .$$

First for arbitrarily small $\varepsilon > 0$, we can see

(4.67) $\quad [0, \delta' - \varepsilon] \subset K_2$.

In fact, let us assume that (4.67) does not hold. Then, since $\alpha(x) \neq 0$ $(0 \leq x \leq \delta' - \varepsilon)$, we have

(4.68) $\quad [0, \delta' - \varepsilon] \subset K_1 \setminus S$

in view of Lemma 4.4. Therefore by the definition of K_1 , we have

(4.69) $\quad m(0) + \alpha(0) = 0$,

and by setting $x = 0$ in (4.1) and noting $\theta(0) = 0$, we obtain

(4.70) $\quad m(0) - \alpha(0) = 0$.

The equalities (4.69) and (4.70) imply $\alpha(0) = 0$, which contradicts our assumption (2.30). Thus we see (4.67).

Therefore, by Lemma 4.2 (2), we get

$$Q(x) = P(x) \qquad (0 \le x \le \delta' - \varepsilon) .$$

Since $\varepsilon > 0$ is arbitrarily small, we reach (2.31).

Proof of (5). Let us assume (2.33) : $J = H$ or (2.34) : $J^* = H^*$. Then, on the assumption (2.32) : $\alpha(x) \neq 0$ $(\delta < x \le 1)$, we have to prove (2.35) : $Q(x) = P(x)$ $(\delta \le x \le 1)$.

Since either (2.33) or (2.34) holds, we get by (1.25) or (1.26) in Theorem of Chapter 2,

(4.71) $\quad \theta(1) = 0$.

In view of (4.71), we can show this part by the same way as the one in the proof of the part (4).
In fact, assume that

(4.72) $\quad [\delta + \varepsilon, 1] \subset K_1 \setminus S$

for arbitrarily small $\varepsilon > 0$. Then we have

(4.73) $\quad m(1) + \alpha(1) = 0$,

and by setting $x = 1$ in (4.1) and noting (4.71), we obtain

(4.74) $\quad m(1) - \alpha(1) = 0$.

The equalities (4.73) and (4.74) imply $\alpha(1) = 0$, which contradicts our assumption that $\alpha(x) \neq 0$ $(\delta < x \le 1)$.

Thus in view of Lemma 4.4, we have $[\delta + \varepsilon, 1] \subset K_2$, so that by Lemma 4.2 (2) we see $Q(x) = P(x)$ $(\delta + \varepsilon \leqq x \leqq 1)$. Taking $\varepsilon \to 0+0$, we reach (2.35), our conclusion.

Now we prove the part (6) only in the case of

(4.75) $\quad H \neq \infty$ and $H^{*} \neq \infty$,

because we can similarly proceed also in the two cases of $H = \infty$ and $H^{*} \neq \infty$, $H \neq \infty$ and $H^{*} = \infty$.

Proof of (6) - (a). Let us assume that for any sequence $\{ x_n \}_{n\geq 1} \subset [0, 1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\beta(x_n) + \alpha(x_n) \neq 0$, the limit

$$(4.76) \qquad k = \lim_{n \to \infty} \frac{\beta(x_n) - \alpha(x_n)}{\beta(x_n) + \alpha(x_n)}$$

does not exist, or that even if k exists for some sequence $\{ x_n \}_{n\geq 1} \subset [0, 1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\beta(x_n) + \alpha(x_n) \neq 0$, either $k \leq 0$ or $k = 1$ holds. Then we have to show (2.38) :

$$\begin{cases} J = H \\ J^* = H^* \end{cases} .$$

In virtue of Lemma 4.3, we have either

(6) - (i) $\qquad 1 \in K_1$,

(6) - (ii) $\qquad 1 \in K_2$,

or

(6) - (iii) $\qquad 1 \in K_3$.

Therefore we consider the respective three cases ;

Case (6) - (i). By Lemma 4.6 (2), the limit k in (4.76) exists, and we obtain

$$(4.77) \qquad k > 0 \quad \text{and} \quad e^{\theta(1)} = k .$$

Thus the condition $k \leq 0$ in the case (6) - (a) impliy $k = 1$. Hence (4.77) implies $\theta(1) = 0$.

However $\theta(1) = 0$ and $1 \in K_1$ are not compatible. In fact, by substituting $\theta(1) = 0$ into (4.1), we obtain

$m(1) - \alpha(1) = 0$, which contradicts $1 \in K_1$ by the definition (4.6) of K_1 .

Thus we see that Case (6) - (i) is impossible in (6) - (a).

Case (6) - (ii). By Lemma 4.7 (2), we immediately obtain (2.38).

Case (6) - (iii). By the definition (4.6) of K_3 , we get either (4.78) or (4.79) :

(4.78) $\alpha(x) = m(x) = 0 \quad (1 - \varepsilon \leq x \leq 1)$ for some sufficiently small $\varepsilon > 0$,

or

(4.79) there exists $\{ x_n \}_{n \geq 1} \subset [0, 1]$ such that $x_n \notin K_3$ and $\lim_{n \to \infty} x_n = 1$.

First let (4.78) hold. Then, since $\alpha(x) = 0$ $(1 - \varepsilon \leq x \leq 1)$, we obtain $\theta(1) = 0$ by (4.42) in Lemma 4.5. Therefore in virtue of (1.25) and (1.26) in Theorem of Chapter 2 with $j = h$, we reach (2.38).

Next let (4.79) hold. Then, if necessary, by taking a subsequence of $\{ x_n \}_{n \geq 1}$, we can assume either

(4.79.1) $\quad x_n \in K_1 \quad (n \geq 1)$ and $\lim_{n \to \infty} x_n = 1$,

or

(4.79.2) $\quad x_n \in K_2 \quad (n \geq 1)$ and $\lim_{n \to \infty} x_n = 1$.

Firstly let (4.79.1) hold. Then, by Lemma 4.6 (1), we see that the limit k in (4.76) exists for $\{ x_n \}_{n\geq1}$ and $k > 0$. Hence, by means of the assumption (2.37) :

$k \leq 0$ or $k = 1$

we see $k = 1$. That is, we get

(4.80) $\theta(1) = 0$.

Therefore (4.80) with (1.25) and (1.26) in Theorem of Chapter 2 imply (2.38).

On the other hand, let (4.79.2) hold. Then we immediately obtain (2.38) by Lemma 4.7 (1).

Hence, in this case, we see that $1 \in K_3$ implies (2.38).

Thus, in the case (6) - (a), we always obtain (2.38).

Proof of (6) - (b). Let us assume that the limit k in (4.76) exists for some sequence $\{ x_n \}_{n\geq 1} \subset [0, 1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\beta(x_n) + \alpha(x_n) \neq 0$, and that $k > 0$ and $k \neq 1$.

Then we have to show either (2.38) :

$$\begin{cases} J = H \\ J^* = H^* \end{cases}$$

or (2.42) :

$$J = \begin{cases} \dfrac{k(1 + H) + H - 1}{k(1 + H) + 1 - H}, & \text{if } k \neq \dfrac{H - 1}{H + 1} \\ \\ \infty , & \text{if } k = \dfrac{H - 1}{H + 1} \end{cases}$$

and

$$J^* = \begin{cases} \dfrac{k(1 + H^*) + H^* - 1}{k(1 + H^*) + 1 - H^*}, & \text{if } k \neq \dfrac{H^* - 1}{H^* + 1} \\ \\ \infty , & \text{if } k = \dfrac{H^* - 1}{H^* + 1} \end{cases} .$$

In the same way as the proof of (6) - (a), we have either

(6) - (i) $1 \in K_1$,

(6) - (ii) $\quad 1 \in K_2$,

or

(6) - (iii) $\quad 1 \in K_3$.

Now we consider

Case (6) - (i). In view of Lemma 4.6 (2), we obtain

$$(4.81) \qquad e^{\theta(1)} = k \quad .$$

On the other hand, in view of (1.25) in Theorem of Chapter 2, we have

$$(4.82) \qquad e^{\theta(1)} = \frac{(1 - H)(1 + J)}{(1 + H)(1 - J)} \quad .$$

Here if $J = \infty$, then we set $\frac{1 + J}{1 - J} = -1$.

By (4.82) and (4.81), we see

$$(4.83) \qquad k = \frac{(1 - H)(1 + J)}{(1 + H)(1 - J)} \quad .$$

First let $k \neq \frac{H - 1}{H + 1}$. Then, noting the conditions $|H| \neq 1$, $k > 0$ and $k \neq 1$, we see $J = \frac{k(1 + H) + H - 1}{k(1 + H) + 1 - H} \in \mathbb{R} \setminus \{-1, 1\}$, from (4.83).

Next let $k = \frac{H - 1}{H + 1}$. Then by (4.83), we have $\frac{1 + J}{1 - J} = -1$, which means $J = \infty$.

This shows (2.42.1). Similarly we can obtain (2.42.2).

Thus we obtain (2.42) in Case (6) - (i).

Case (6) - (ii). We immediately see (2.38) in view of Lemma 4.7 (2).

Case (6) - (iii). As in Case (6) - (iii) of the part (6) - (a), we have either (4.78), (4.79.1) or (4.79.2).

Firstly let (4.78) hold. Then we can proceed in the same way as the one in the corresponding case of the part (6) - (a). Actually, since $\theta(1) = 0$ by Lemma 4.5, the equalities (1.25) and (1.26) of Theorem in Chapter 2 imply (2.38).

Secondly let (4.79.1) hold. Then, by Lemma 4.6 (1), we see $e^{\theta(1)} = k$. Therefore we can proceed in the same way as the one in Case (6) - (i) of this part and we can see (2.42).

Thirdly let (4.79.2) hold. Then we obtain (2.38) in view of Lemma 4.7 (1).

Thus in (6) - (b), we get either (2.38) or (2.42).

Finally we have to show the latter part of (6). Assume that $\alpha(1) \neq 0$. The condition $\alpha(1) \neq 0$ implies $1 \notin K_3$. Therefore, by applying Lemma 4.6 (2) and Lemma 4.7 (2), we can show the latter part of (6) along the line of the proof of the former part.

Thus we complete the proof of the part (6) of Theorem 2.

§5. **The identification problem for a modeled equation describing small vibrations of a flexible string : an application of the result for Type I.** A small transverse vibration of a flexible string is governed by

$$(5.1) \qquad \rho(x)\cdot\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left(E(x)\frac{\partial u(x,t)}{\partial x}\right) + k(x)\cdot\frac{\partial u(x,t)}{\partial t} = 0$$

$$(0 \leq x \leq 1 \ , \ t \geq 0 \ ),$$

with the displacement $u(x,t)$. Here x and t denote the space variable and the time variable, respectively. Furthermore ρ and E are the linear mass density and the modulus of elasticity, respectively and the term $k(x)\cdot\frac{\partial u(x,t)}{\partial t}$ corresponds to some viscous drag (Timoshenko [53], for example).

Let us assume that

$$(5.2) \qquad \begin{cases} \rho,\ E \in C^2[0,\ 1] \\ k \in C^1[0,\ 1] \\ \rho(x) > 0,\ E(x) > 0 \qquad (0 \leq x \leq 1) \ . \end{cases}$$

We consider (5.1) with either

$$(5.3) \qquad u(0,t) = u(1,t) = 0$$

or

$$(5.4) \qquad u(0,t) = 0 \ , \ \frac{\partial u(0,t)}{\partial x} = 0 \quad ,$$

as boundary conditions.

(The boundary condition (5.3) corresponds to fixed ends at $x = 0,\ 1$, whereas (5.4) corresponds to a fixed end at $x = 0$ and an end free to move in the direction of the u-axis.)

Here we consider proper vibrations for (5.1) with (5.3) or

(5.4). That is, we suppose that $u(x,t)$ has the form

$$(5.5) \qquad u(x,t) = e^{\lambda t} v(x) \qquad (\lambda \in \mathbb{C}) .$$

Then $v = v(x)$ is the solution to $(5.1)_p$ with $(5.3)_p$ or $(5.4)_p$.

$$(5.1)_p \qquad \frac{d}{dx}(E(x)\frac{dv}{dx}) - \lambda k(x)v = \lambda^2 \rho(x)v \qquad (0 \leq x \leq 1) ,$$

with

$$(5.3)_p \qquad v(0) = v(1) = 0 \quad ,$$

or

$$(5.4)_p \qquad v(0) = \frac{dv(1)}{dx} = 0 \quad .$$

Now, by a change of independent variable

$$(5.6) \qquad z = \int_0^x \left(\frac{\rho(\xi)}{E(\xi)}\right)^{1/2} d\xi \quad ,$$

the systems $(5.1)_p$ with $(5.3)_p$ and $(5.1)_p$ with $(5.4)_p$ are transformed to certain standard forms (5.7) with (5.8) and (5.7) with (5.9), respectively :

$$(5.7) \qquad \frac{d^2u(z)}{dz^2} + a(z)\cdot\frac{du(z)}{dz} + \lambda b(z)u(z) = \lambda^2 u(z) \quad (0 \leq z \leq \ell),$$

with

$$(5.8) \qquad u(0) = u(\ell) = 0 \quad ,$$

or

$$(5.9) \qquad u(0) = \frac{du(\ell)}{dz} = 0 \quad .$$

Here we set

$$(5.10) \qquad \ell = \int_0^1 \left(\frac{\rho(\xi)}{E(\xi)}\right)^{1/2} d\xi \quad ,$$

and we define $u(z)$, $a(z)$ and $b(z)$ by

$$(5.11) \qquad u(z) = v(x)$$

$$(5.12) \qquad a(z) = \frac{\frac{dE(x)}{dx}}{(E(x)\rho(x))^{1/2}} + \frac{E(x)\frac{d\rho(x)}{dx} - \rho(x)\frac{dE(x)}{dx}}{2(\rho^3(x)E(x))^{1/2}} \quad ,$$

and

$$(5.13) \qquad b(z) = -\frac{k(x)}{\rho(x)}$$

for $z = \int_0^x \left(\frac{\rho(\xi)}{E(x)}\right)^{1/2} d\xi$.

Furthermore we note that

$$(5.14) \qquad a, b \in C^1[0, \ell]$$

by the assumption (5.2).

In order to avoid technical difficulties, we are restricted to discussing the equation of the form (5.7) with (5.8) or (5.9). In a forthcoming paper, we will consider an identification problem for the original form (5.1).

Remark 5. We understand an eigenvalue of (5.7) with (5.8) by λ for which there exists a non-zero function $u(z)$ satisfying (5.7) with (5.8). An eigenvalue of (5.7) with (5.9) is understood similarly.

Then, as application of Theorem 1, we have

Theorem 4. *If all the eigenvalues of (5.7) for each of the boundary conditions (5.8) and (5.9) are given, then* $a(z)$ *and* $b(z)$ *are uniquely determined on* $[0, \ell]$ *provided that* $a, b \in C^1[0, \ell]$ *. That is, let*

$$(5.15) \qquad a_1, b_1, a_2, b_2 \in C^1[0, \ell] ,$$

and let us consider the system (5.15.j) with (5.16.j) or (5.17.j) (j = 1, 2) :

$$(5.15.j) \qquad \frac{d^2u_j(z)}{dz^2} + a_j(z)\frac{du_j(z)}{dz} + \lambda b_j(z)u_j(z) = \lambda^2 u_j(z) \qquad (0 \leq z \leq \ell) ,$$

with

$$(5.16.j) \qquad u_j(0) = u_j(\ell) = 0 ,$$

or

$$(5.17.j) \qquad u_j(0) = \frac{du_j(\ell)}{dz} = 0 .$$

If all the eigenvalues of (5.15.1) with each of (5.16.1) and (5.17.1) coincide with the ones of (5.15.2) with each of (5.16.2) and (5.17.2), respectively, then

$$(5.18) \qquad \begin{cases} a_1(z) = a_2(z) \\ \\ b_1(z) = b_2(z) \qquad (0 \leq z \leq \ell) \end{cases}$$

holds.

Proof. We prove this theorem by reducing (5.15.j) with (5.16.j) and (5.15.j) with (5.17.j) to systems in the form (1.1) with boundary conditions (1.2) and (1.3), and (1.1) with (1.2) and (1.4), respectively.

Without loss of generality, we assume that

$$(5.19) \qquad \ell = 1 \quad ,$$

and, for simplicity, we denote the independent variable again by x, instead of z. Then we set

$$(5.20) \qquad \phi_j(x) = \begin{pmatrix} \phi_j^{(1)}(x) \\ \phi_j^{(2)}(x) \end{pmatrix} = \begin{pmatrix} \dfrac{du_j(x)}{dx} \\ \lambda u_j(x) \end{pmatrix} \qquad (j = 1, 2) ,$$

and

$$(5.21) \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

so that the systems (5.15.j) with (5.16.j) and (5.15.j) with (5.17.j) are transformed to systems (5.22.j) with (5.23.j) and (5.22.j) with (5.24.j), respectively :

$$(5.22.j) \quad B\cdot\frac{d\phi_j(x)}{dx} + \begin{pmatrix} 0 & 0 \\ a_j(x) & b_j(x) \end{pmatrix} \phi_j(x) = \lambda\phi_j(x)$$

$$(0 \leq x \leq 1) .$$

$$(5.23.j) \quad \phi_j^{(2)}(0) = \phi_j^{(2)}(1) = 0 \quad .$$

$$(5.24.j) \quad \phi_j^{(2)}(0) = \phi_j^{(1)}(1) = 0 \quad .$$

The system (5.22.j) is nothing but a special form of (1.1).

Furthermore we get (5.23.j) by putting $h = H = 0$ in (1.2) and (1.3), while we obtain (5.24.j) by putting $h = 0$ and $H^* = \infty$ in (1.2) and (1.4), respectively.

Therefore, in order to show Theorem 4, we have only to prove the following lemma. (In fact, then the result on the determination of $M(p_{21}, p_{22} ; H, H^*)$ implies Theorem 4, because the condition (2.9) in Theorem 1 holds.)

Lemma 5.1. *Let us assume that all the eigenvalues of (5.15.1) with (5.16.1) coincide with the ones of (5.15.2) with (5.16.2). Then also all the eigenvalues of (5.22.1) with (5.23.1) coincide with the ones of (5.22.2) with (5.23.2). As for the eigenvalues of (5.15.j) with (5.17.j) and (5.22.j) with (5.24.j), the same result holds.*

A proof of this lemma is given in Appendix III.

$6. The identification problem for a telegraphic equation : an application of the result for Type-II. From a telegraphic equation (Johnson [22], for example), in a way similar to the one in $5, we can deduce an equation describing proper vibrations for an ellectric oscillation in a transmission line.

The proper vibrations are governed by

$$(6.1) \quad \begin{cases} \dfrac{dv(x)}{dx} + R(x)i(x) = -\lambda L(x)i(x) \\[2ex] \dfrac{di(x)}{dx} + C(x)v(x) = -\lambda G(x)v(x) \qquad (0 \leq x \leq 1) \end{cases}$$

with the voltage $v(x)$ and the electric current $i(x)$. Here we assume that the length of a transmission line is 1 under appropriate normalization, and x denotes the space variable. Further $R(x)$, $L(x)$, $C(x)$ and $G(x)$ are the electric resistance, the inductance, the electric capacity, and the conductance of the line, respectively.

Here we consider two types of boundary conditions :

$$(6.2) \qquad i(0) = i(1) = 0 \ ,$$

and

$$(6.3) \qquad i(0) = v(1) = 0 \ .$$

Remark 6. The boundary condition (6.2) corresponds to insulated ends at $x = 0, 1$, whereas the boundary condition

(6.3) corresponds to an insulated end at $x = 0$ and an earthed end at $x = 1$.

Throughout $6, let us assume that

(6.4) $\quad R, L, C, G \in C^2[0, 1]$.

On condition that L and C are known, we consider determination of R and G from a pair of eigenvalues of (6.1) with (6.2), and (6.1) with (6.3). That is, we obtain

Theorem 5. *Let*

(6.5) $\quad R_j, L_j, C_j, G_j \in C^2[0, 1]$ and $L_j, C_j > 0 \quad (j = 1, 2)$

and let us consider the systems (6.6.j) with (6.7.j) and (6.6.j) with (6.8.j) (j = 1, 2) :

$$(6.6.j) \begin{cases} \dfrac{dv_j(x)}{dx} + R_j(x)i_j(x) = -\lambda L_j(x)i_j(x) \\[2ex] \dfrac{di_j(x)}{dx} + C_j(x)v_j(x) = -\lambda G_j(x)v_j(x) \end{cases} \qquad (0 \le x \le 1) ,$$

with

(6.7.j) $\quad i_j(0) = i_j(1) = 0$,

or

(6.8.j) $\quad i_j(0) = v_j(1) = 0$.

Let us assume that

$$
(6.9) \quad \begin{cases} L_1(x) = L_2(x) \\ \\ C_1(x) = C_2(x) \qquad (0 \leq x \leq 1) \end{cases} ,
$$

and furthermore that

(6.10) *all the eigenvalues of (6.6.1) with each of (6.7.1)* and (6.8.1) coincide with the ones of (6.6.2) with each of (6.7.2) and (6.8.2), respectively.

Then we get all the conclusions of Theorem 2 in §2.2, where p_{11}, p_{22}, q_{11} *and* q_{22} *are replaced by* $-\frac{G_1}{C_1}$, $-\frac{R_1}{L_1}$, $-\frac{G_2}{C_2}$ *and* $-\frac{R_2}{L_2}$, *respectively and the definition (2.11) of* α *and* β *is replaced by*

$$
(6.11) \quad \begin{cases} \alpha(x) = \dfrac{R_1(x)}{L_1(x)} - \dfrac{G_1(x)}{C_1(x)} \\ \\ \beta(x) = \dfrac{1}{2(L_1(x)C_1(x))^{1/2}} \cdot \left\{ \dfrac{1}{C_1(x)} \dfrac{dC_1(x)}{dx} - \dfrac{1}{L_1(x)} \dfrac{dL_1(x)}{dx} \right\} \\ \qquad\qquad\qquad\qquad (0 \leq x \leq 1) . \end{cases}
$$

Let us recall that

$$
(6.12) \quad S = \{ x \in [0, 1] ; |\alpha(x)| = |\beta(x)| \} .
$$

Moreover we have

Corollary 2. *Let us assume that*

$$
(6.13) \quad S = \emptyset ,
$$

and the number of the zeros of α *is finite. Let us set all the* *zeros of* a *by* $\{\gamma_k\}_{k=1}^{m}$. *If we have (6.10) and moreover, if for* $1 \le k \le m-1$, *there is some* $x_k \in (\gamma_k, \gamma_{k+1})$ *such that*

$$(6.14) \quad \begin{cases} R_1(x_k) = R_2(x_k) \\ \\ G_1(x_k) = G_2(x_k) \quad , \end{cases}$$

then we have

$$(6.15) \quad \begin{cases} R_1(x) = R_2(x) \\ \\ G_1(x) = G_2(x) \qquad (0 \le x \le 1) \ . \end{cases}$$

Corollary 3. *Let*

$$(6.16) \qquad R_1(x)C_1(x) = L_1(x)G_1(x) \qquad (0 \le x \le 1) \quad .$$

If we assume (6.10), then we obtain (6.15).

Remark 7. A transmission line satisfying (6.16) is called a distorsionless line (Johnson [22, pp.48 - 50], for instance).

Proof of Theorem 5. We prove the theorem by reducing (6.6.j) to a system considered in §2.2.

Introducing a change of independent variable

$$(6.17) \qquad z = \int_0^x (L_1(\xi)C_1(\xi))^{1/2}\, d\xi \quad ,$$

under the assumption (6.9), and setting

$$(6.18)\qquad \phi_j(z) = \begin{pmatrix} \phi_j^{(1)}(z) \\ \phi_j^{(2)}(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} v_j(x)\cdot L_j(x)^{-1/2} \\ -\frac{1}{2} i_j(x)\cdot C_j(x)^{-1/2} \end{pmatrix}$$

$$(j = 1,\ 2)\quad ,$$

and

$$(6.19)\qquad \ell = \int_0^1 (L_1(\xi)C_1(\xi))^{1/2} d\xi \quad ,$$

the systems (6.6.j) with (6.7.j) and (6.6.j) with (6.8.j) are transformed to (6.20.j) with (6.21.j) and (6.20.j) with (6.22.j), respectively :

$$(6.20.j)\quad B\frac{d\phi_j(z)}{dz} + P_j(z)\phi_j(z) = \lambda\phi_j(z) \qquad (0 \le z \le \ell)\quad ,$$

with

$$(6.21.j)\quad \phi_j^{(2)}(0) = \phi_j^{(2)}(\ell) = 0 \quad ,$$

or

$$(6.22.j)\quad \phi_j^{(2)}(0) = \phi_j^{(1)}(\ell) = 0 \qquad (j = 1,\ 2)\quad .$$

Here we put

$$(6.23.j) \quad P_j(z) = \begin{pmatrix} -\dfrac{G_j(x)}{C_j(x)} & \dfrac{1}{2(L_j(x)C_j(x)^3)^{1/2}}\cdot\dfrac{dC_j(x)}{dx} \\ \dfrac{1}{2(L_j(x)^3C_j(x))^{1/2}}\cdot\dfrac{dL_j(x)}{dx} & -\dfrac{R_j(x)}{L_j(x)} \end{pmatrix}$$

$$(j = 1,\ 2) \quad .$$

Now, without loss of generality, we can assume $\ell = 1$.

Furthermore, the boundary condition (6.21.j) is obtained by setting $h = H = 0$ in (1.2) and (1.3), while (6.22.j) is obtained by setting $h = 0$ and $H^* = \infty$ in (1.2) and (1.4). Thus, in view of Theorem 2 in $2.2, we show Theorem 5.

Noting that (2.33) and (2.34) hold in the part (5) of Theorem 2 (in fact, $J = H = 0$ and $J^* = H^* = \infty$), we see Corollary 2 by Corollary 1 in $2.2. Moreover Corollary 3 follows from the part (3) of Theorem 2.

Appendix I. Proof of Proposition 1. Let us assume that $Q = (q_{ij})_{1 \leq i,j \leq 2} \in \{ C^1[0, 1] \}^4$ satisfies

$$(I.1) \qquad \begin{cases} \sigma(A_{Q,h,H}) = \sigma(A_{P,h,H}) \\ \\ \sigma(A_{Q,h,H^*}) = \sigma(A_{P,h,H^*}) \end{cases} .$$

Then we have to show

$$(I.2) \qquad \sigma(A_{Q,\alpha,\beta}) = \sigma(A_{P,\alpha,\beta}) .$$

for each $\alpha \in \mathbb{R} \setminus \{ -1, 1 \}$ and $\beta \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1, 1 \}$.

Since by (I.1), the equalities (1.23) - (1.26) hold by Theorem in Chapter 2 with $j = h$, $J = H$ and $J^* = H^*$,

$$(I.3) \quad (q_{11}(x)+q_{12}(x)-q_{21}(x)-q_{22}(x)-p_{11}(x)+p_{12}(x)-p_{21}(x)+p_{22}(x))$$
$$+ (q_{11}(x)-q_{12}(x)+q_{21}(x)-q_{22}(x)-p_{11}(x)-p_{12}(x)+p_{21}(x)+p_{22}(x))$$
$$\times \exp(\int_0^x (q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds) = 0$$
$$(0 \leq x \leq 1),$$

and

$$(I.4) \quad (q_{11}(x)+q_{12}(x)-q_{21}(x)-q_{22}(x)+p_{11}(x)-p_{12}(x)+p_{21}(x)-p_{22}(x))$$
$$+ (-q_{11}(x)+q_{12}(x)-q_{21}(x)+q_{22}(x)-p_{11}(x)-p_{12}(x)+p_{21}(x)+p_{22}(x))$$
$$\times \exp(\int_0^x (q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds) = 0$$
$$(0 \leq x \leq 1),$$

and

$$\text{(I.5)} \qquad \int_0^1 (q_{11}(s)+q_{22}(s)-p_{11}(s)-p_{22}(s))ds = 0 \quad .$$

Let α, $\beta \in \mathbb{R}$ be given such that $\alpha \in \mathbb{R} \setminus \{ -1, 1 \}$, $\beta \in \mathbb{R} \cup \{ \infty \} \setminus \{ -1, 1 \}$, and $\beta \neq H$. Then, applying the sufficiency part of Theorem in Chapter 2, we have, by (I.3) - (I.5),

$$\text{(I.6)} \qquad (Q,\alpha,\beta,H) \in M(P,\alpha,\beta,H) \quad ,$$

that is, $\sigma(A_{Q,\alpha,H}) = \sigma(A_{P,\alpha,H})$ and $\sigma(A_{Q,\alpha,\beta}) = \sigma(A_{P,\alpha,\beta})$ ($\beta \neq H$). This proves (I.2).

Thus we complete the proof of Proposition 1.

Appendix II. Example of $M(p_{11}, p_{22} ; H, H^*)$. In this appendix, we determine the set $M(p_{11}, p_{22} ; H, H^*)$ in the case where (II.1) - (II.3) hold :

$$\text{(II.1)} \quad \begin{cases} \alpha(x) \neq 0 \quad (0 < x < \gamma_0 \ , \ \gamma_0 < x < 1) \ , \\ \alpha(0) = 0 \ , \\ \alpha(\gamma_0) = 0, \ \text{ and } \ \alpha \in C^2[0, 1] \ . \end{cases}$$

Here $\gamma_0 \in (0, 1)$ is fixed.

$$\text{(II.2)} \quad \beta(x) > 0 \quad (0 \leq x < 1) \ \text{ and } \ \beta \in C^2[0, 1] \ ,$$

and

$$\text{(II.3)} \quad H \neq \infty \ \text{ and } \ H^* \neq \infty \ .$$

Here we recall that

$$\text{(II.4)} \quad \begin{cases} \alpha(x) = p_{11}(x) - p_{22}(x) \quad (0 \leq x \leq 1) \\ \\ \beta(x) = p_{12}(x) - p_{21}(x) \quad (0 \leq x \leq 1) \ , \end{cases}$$

$$\text{(II.5)} \quad r(x) = \frac{\frac{d\alpha(x)}{dx}\beta(x) - \alpha(x)\frac{d\beta(x)}{dx}}{\alpha^2(x) - \beta^2(x)}$$

for $x \notin S = \{ x \in [0, 1] \ ; \ |\alpha(x)| = |\beta(x)| \}$,

and

$$\text{(II.6)} \quad \tilde{P}(x) = \begin{pmatrix} p_{22}(x) + r(x) & p_{12}(x) \\ \\ p_{21}(x) & p_{11}(x) + r(x) \end{pmatrix}$$

for $x \notin S$.

Henceforth we set

$$\text{(II.7)} \qquad P_1(x) = \begin{cases} \tilde{P}(x) & (0 \leq x \leq \gamma_0) \\ P(x) & (\gamma_0 \leq x \leq 1) \end{cases},$$

$$\text{(II.8)} \qquad P_2(x) = \begin{cases} P(x) & (0 \leq x \leq \gamma_0) \\ \tilde{P}(x) & (\gamma_0 \leq x \leq 1) \end{cases},$$

and

$$\text{(II.9)} \qquad P_3(x) = \tilde{P}(x) \qquad (0 \leq x \leq 1)$$

provided that the right hand sides can be defined. Further we put

$$\text{(II.10)} \qquad k = \lim_{\substack{x \to 1-0 \\ \beta(x)+\alpha(x) \neq 0}} \frac{\beta(x) - \alpha(x)}{\beta(x) + \alpha(x)},$$

if the limit exists, and we set

$$\text{(II.11)} \quad \begin{cases} \tilde{H} = \begin{cases} \dfrac{k(1 + H) + H - 1}{k(1 + H) + 1 - H}, & \text{if } k \neq \dfrac{H - 1}{H + 1} \\ \infty, & \text{if } k = \dfrac{H - 1}{H + 1}, \end{cases} \\ \tilde{H}^* = \begin{cases} \dfrac{k(1 + H^*) + H^* - 1}{k(1 + H^*) + 1 - H^*}, & \text{if } k \neq \dfrac{H^* - 1}{H^* + 1} \\ \infty, & \text{if } k = \dfrac{H^* - 1}{H^* + 1}. \end{cases} \end{cases}$$

Then we can give possible elements of $M(p_{11}, p_{22} ; H, H^*)$ in view of Theorem 2 - (1), and furthermore, by checking the conditions (1.23) - (1.26) in Theorem in Chapter 2, we can show that those possible elements actually belong to

$M(p_{11}, p_{22} ; H, H^*)$.

Now the result can be stated in Table 2.

Table 2. Determination of $M(p_{11}, p_{22}\ ;\ H, H^*)$.

$\lvert\beta(x)\rvert > \lvert\alpha(x)\rvert$ $(0 \le x \le \gamma_0)$	$\lvert\beta(x)\rvert > \lvert\alpha(x)\rvert$ [1] $(\gamma_0 \le x < 1)$	$\alpha'(\gamma_0) = \alpha''(\gamma_0) = 0$ [2]	The limit k in (I.10) exists and $k > 0$. [3]	$M(p_{11}, p_{22}\ ;\ H, H^*)$
YES	YES	YES	[4] YES	[5] $\{(P, h, H, H^*), (P_1, h, H, H^*), (P_2, h, \tilde{H}, \tilde{H}^*), (P_3, h, \tilde{H}, \tilde{H}^*)\}$
			NO	$\{(P, h, H, H^*), (P_1, h, H, H^*)\}$
		NO	YES	$\{(P, h, H, H^*), (P_3, h, \tilde{H}, \tilde{H}^*)\}$
			NO	$\{(P, h, H, H^*)\}$
	NO	YES	YES or NO	$\{(P, h, H, H^*), (P_1, h, H, H^*)\}$
		NO	YES or NO	$\{(P, h, H, H^*)\}$
NO	YES	YES	YES	$\{(P, h, H, H^*), (P_2, h, \tilde{H}, \tilde{H}^*)\}$
			NO	$\{(P, h, H, H^*)\}$
		NO	YES or NO	
	NO	YES or NO	YES or NO	

Notes to Table 2.

1) The condition $|\beta(x)| > |\alpha(x)|$ is necessary for $Q(x) = P(x)$ (Theorem 2 - (1) - (b)).

2) The equalities $\alpha'(\gamma_0) = \alpha''(\gamma_0) = 0$ are the "continuation conditions" (2.24) and (2.26) at the zero γ_0 of α (Theorem 2 - (2)). Here $\alpha'(x)$ and $\alpha''(x)$ denote $\frac{d\alpha(x)}{dx}$ and $\frac{d^2\alpha(x)}{dx^2}$.

3) This condition corresponds to (2.41) in Theorem 2 - (6).

4) Firstly let

$$\text{(II.12)} \quad \begin{cases} \beta(1) + \alpha(1) \neq 0 \\ \\ \dfrac{\beta(1) - \alpha(1)}{\beta(1) + \alpha(1)} > 0 \text{ , and } \neq 1 \end{cases}$$

hold. Then we have "Yes" in this column and hence we get either

$$\begin{cases} J = H \\ \\ J^* = H^* \end{cases} \text{, or} \quad \begin{cases} J = \tilde{H} \\ \\ J^* = \tilde{H}^* \end{cases} .$$

Here we note that (II.12) is equivalent to the case (1) - (b) in Remark 2 in $2.2.

Secondly, if either

$$\text{(II.13)} \quad \begin{cases} \beta(1) + \alpha(1) \neq 0 \\ \\ \dfrac{\beta(1) - \alpha(1)}{\beta(1) + \alpha(1)} \leq 0 \end{cases} ,$$

or

$$\text{(II.14)} \quad \begin{cases} \beta(1) + \alpha(1) = 0 \\ \\ \beta(1) - \alpha(1) \neq 0 \end{cases}$$

holds, then we have "No" in this column, that is, we have

$$\begin{cases} J = H \\ J^* = H^* \end{cases} .$$

Finally, if

$$\text{(II.15)} \qquad \frac{\beta(1) - \alpha(1)}{\beta(1) + \alpha(1)} = 1$$

holds true, then we obtain

$$\begin{cases} J = H \\ J^* = H^* \end{cases} .$$

In fact, by (II.15), we have $\tilde{H} = H$ and $\tilde{H}^* = H^*$.

We see that either (II.13), (II.15) or (II.14) holds if and only if we have either the case (1) - (a) or (2) - (a) in Remark 2 in $2.2.

5) Since, in view of $\alpha(0) = 0$, the condition (2.30) in Theorem 2 - (4) does not hold for $x = 0$, both P_1 and P_3 are possible in $(0, \gamma_0)$.

Appendix III. Proof of Lemma 5.1. (1) First, we prove

Assertion I. *Let* $\lambda \neq 0$. *Then, for* $j = 1, 2$, *a complex number* λ *is an eigenvalue of (5.15.j) with (5.16.j) if and only if* λ *is an eigenvalue of (5.22.j) with (5.23.j). For the eigenvalues of (5.15.j) with (5.17.j) and (5.22.j) with (5.24.j), the same assertion holds.*

Proof of Assertion I. Assume that λ is an eigenvalue of (5.22.j) with either (5.23.j) or (5.24.j).

Let us denote an eigenvector associated with λ by $\phi_{j,\lambda} = \phi_j = \begin{pmatrix} \phi_j^{(1)} \\ \phi_j^{(2)} \end{pmatrix}$. Then, since ϕ_j satisfies

(III.1) $$\frac{d\phi_j^{(2)}(x)}{dx} = \lambda\phi_j^{(1)}(x)$$

in view of (5.22.j), we see $\phi_j^{(2)} \neq 0$. (In fact, if $\phi_j^{(2)}(x) = 0$ $(0 \leq x \leq 1)$, then by (III.1) and $\lambda \neq 0$, we have $\phi_j^{(1)}(x) = 0$ $(0 \leq x \leq 1)$. This contradicts that ϕ_j is an eigenvector.)

Therefore, putting $u_j(x) = \frac{1}{\lambda}\phi_j^{(2)}(x)$, we see

(III.2) $$u_j \neq 0 \quad .$$

Further we get

(III.3) $$u_j(0) = u_j(1) = 0$$

in the case where (5.23.j) is considered, whereas we get by (III.1)

$$(\text{III}.4) \qquad u_j(0) = \frac{du_j(1)}{dx} = 0 \quad ,$$

in the case where (5.24.j) is considered.

By eliminating $\phi_j^{(1)}$ in (5.22.j), we obtain

$$\frac{1}{\lambda}\cdot\frac{d^2\phi_j^{(2)}(x)}{dx^2} + \frac{1}{\lambda}a_j(x)\cdot\frac{d\phi_j^{(2)}(x)}{dx} + b_j(x)\phi_j^{(2)}(x) = \lambda\phi_j^{(2)}(x) \qquad (0 \leq x \leq 1) \quad ,$$

which is equivalent to

$$(\text{III}.5) \qquad \frac{d^2u_j(x)}{dx^2} + a_j(x)\cdot\frac{du_j(x)}{dx} + \lambda b_j(x)u_j(x) = \lambda^2 u_j(x) \qquad (0 \leq x \leq 1) \ .$$

When (5.23.j) is considered as the boundary condition, in view of (III.2), (III.3) and (III.5), u_j is an eigenvector of (5.15.j) with (5.16.j) associated with the eigenvalue λ . On the other hand, when (5.24.j) is considered, in view of (III.2), (III.4) and (III.5), u_j is an eigenvector (5.15.j) with (5.17.j) associated with the eigenvalue λ .

This shows the "if" part of Assertion I.

On the other hand, a proof of the "only if" part of Assertion I is straightforward by (5.20) introduced in deriving (5.22.j).

(2) We show

Assertion II. *(i) Zero is an eigenvalue of (5.22.j) with (5.23.j).*

(ii) Zero is not an eigenvalue of (5.22.j) with (5.24.j).

Proof of Assertion II. (i) Putting

$$\phi_j(x) = \begin{pmatrix} \exp\left(-\int_0^x a_j(s)ds\right) \\ \\ 0 \end{pmatrix} \qquad (0 \leq x \leq 1) ,$$

we see by direct computations that ϕ_j is an eigenvector of (5.22.j) with (5.23.j) associated with the eigenvalue 0 .

(ii) Let $\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$ satisfy

$$\text{(III.6)} \qquad B \cdot \frac{d\phi(x)}{dx} + \begin{pmatrix} 0 & 0 \\ a_j(x) & b_j(x) \end{pmatrix} \phi(x) = 0 \qquad (0 \leq x \leq 1)$$

and

$$\text{(III.7)} \qquad \phi_2(0) = \phi_1(1) = 0 \quad .$$

Then we have only to show $\phi(x) = 0$ $(0 \leq x \leq 1)$ for the proof. The equalities (III.6) are equivalent to

$$\frac{d\phi_2(x)}{dx} = 0 \qquad (0 \leq x \leq 1)$$

and

$$\frac{d\phi_1(x)}{dx} + a_j(x)\phi_1(x) + b_j(x)\phi_2(x) = 0 \quad (0 \le x \le 1) ,$$

which imply $\phi_2(x) = 0 \quad (0 \le x \le 1)$ by $\phi_2(0) = 0$ and

$$\text{(III.8)} \quad \begin{cases} \dfrac{d\phi_1(x)}{dx} + a_j(x)\phi_1(x) = 0 & (0 \le x \le 1) \\ \\ \phi_1(1) = 0 & . \end{cases}$$

Thus we see $\phi_1(x) = 0 \quad (0 \le x \le 1)$ in virtue of the uniqueness of solutions to (III.8). This shows that $\phi(x) = 0 \quad (0 \le x \le 1)$, so that zero cannot be an eigenvalue.

(3) In view of Assertions I and II, we can see Lemma 5.1. In fact, let us assume that all the eigenvalues of (5.15.1) with (5.16.1) coincide with the ones of (5.15.2) with (5.16.2). Then Assertion I implies that all the non-zero eigenvalues of (5.22.1) with (5.23.1) coincide with the non-zero ones of (5.22.2) with (5.23.2). Furthermore, by Assertion II, zero is an eigenvalue of (5.22.j) with (5.23.j), for $j = 1, 2$.

Therefore all the eigenvalues of (5.22.1) with (5.23.1) coincide with the ones of (5.22.2) with (5.23.2).

Similarly we can prove the latter part of this lemma and we omit its proof.

Thus the proof of Lemma 5.1 is completed.

Chapter 4

Continuous Dependence of the Boundary Value Problem on Eigenvalues

§1. Formulation and the main result. We consider a system (1.1) of ordinary differential equations of first order in the interval $(0,\ 1)$ with boundary conditions (1.2) and (1.3) :

$$(1.1)\quad \begin{cases} \dfrac{du_2(x)}{dx} + p_{11}(x)u_1(x) + p_{12}(x)u_2(x) = \lambda u_1(x) \\[2ex] \dfrac{du_1(x)}{dx} + p_{21}(x)u_1(x) + p_{22}(x)u_2(x) = \lambda u_2(x) \end{cases}$$

$$(0 \le x \le 1) .$$

$$(1.2)\qquad u_2(0) + hu_1(0) = 0 \quad .$$

$$(1.3)\qquad u_2(1) + Hu_1(1) = 0 \quad .$$

In Chapters 2 and 3, we consider another boundary condition at $x = 1$

$$(1.4)\qquad u_2(1) + H^*u_1(1) = 0 \qquad (H \ne H^*) ,$$

and discuss the inverse problem to determine the coefficients $p_{ij}(x)$ $(1 \le i,j \le 2)$, etc. from the two pairs of eigenvalues of (1.1) - (1.3) and (1.1), (1.2), (1.4), so that we obtain the results on the uniqueness of coefficients and boundary conditions.

Next we will study a problem on the well-posedness. That is, the purpose of this chapter is to discuss

Problem. *Let*

$$(1.5)\quad \begin{cases} \sigma(A_{P_i,h_i,H_i}) = \{\lambda_n^{(i)}\}_{n\in\mathbb{Z}} \\ \\ \sigma(A_{P_i,h_i,H_i^*}) = \{\mu_n^{(i)}\}_{n\in\mathbb{Z}} \qquad (i = 1,\ 2)\ . \end{cases}$$

Then, in order to assure that $\| P_1 - P_2 \|_{\{C^1[0,1]\}^4} + |h_1 - h_2| + |H_1 - H_2| + |H_1^* - H_2^*|$ *is small, in what sense should* $\left\{ \{\lambda_n^{(1)}\}_{n\in\mathbb{Z}},\ \{\mu_n^{(1)}\}_{n\in\mathbb{Z}} \right\}$ *be close to* $\left\{ \{\lambda_n^{(2)}\}_{n\in\mathbb{Z}},\ \{\mu_n^{(2)}\}_{n\in\mathbb{Z}} \right\}$ *?*

Here and henceforth, we define

$$(1.6)\quad \begin{cases} \| P \|_{\{C^0[0,1]\}^4} = \| P \|_{C^0} = \max\limits_{\substack{1\le i,j\le 2\\ 0\leqq x\leqq 1}} | p_{ij}(x) | \\ \\ \| P \|_{\{C^1[0,1]\}^4} = \| P \|_{C^1} \\ = \max\left(\max\limits_{\substack{1\le i,j\le 2\\ 0\leqq x\leqq 1}} | p_{ij}(x) |\ ,\ \max\limits_{\substack{1\le i,j\le 2\\ 0\leqq x\leqq 1}} \left| \dfrac{dp_{ij}(x)}{dx} \right| \right), \end{cases}$$

for $P = (p_{ij})_{1\le i,j\le 2} \in \{C^1[0,\ 1]\}^4$. As for $p \in \{C^1[0,\ 1]\}^2$, etc., we adopt similar notation.

Remark 1. Let us recall that $A_{P,h,H}$ is given in Definition 1 in Chapter 2. Moreover, for the operator $A_{P,h,H}$, we can define $A^*_{P,h,H}$, the adjoint operator of $A_{P,h,H}$, by

$$(1.7)\qquad \mathcal{D}(A^*_{P,h,H}) = \left\{ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \{ H^1(0,1) \}^2 ;\right.$$
$$\left. v_2(0) - hv_1(0) = 0 \quad\text{and}\quad v_2(1) - Hv_1(1) = 0 \right\}$$

and

$$(1.8)\qquad (A^*_{P,h,H}v)(x) = -B\,\frac{dv(x)}{dx} + {}^tP(x)v(x)$$
$$\text{for each}\quad v \in \mathcal{D}(A^*_{P,h,H}) .$$

Here we recall $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and ${}^tP(x)$ denotes the transpose of the matrix $P(x)$ (that is,
${}^tP(x) = \begin{pmatrix} p_{11}(x) & p_{21}(x) \\ p_{12}(x) & p_{22}(x) \end{pmatrix}$ for $P(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix}$).

Furthermore we obtain

$$(1.9)\qquad (A_{P,h,H}u,\ v)_{\{L^2(0,1)\}^2} = (u,\ A^*_{P,h,H}v)_{\{L^2(0,1)\}^2}$$

for each $u \in \mathcal{D}(A_{P,h,H})$ and $v \in \mathcal{D}(A^*_{P,h,H})$.

Remark 2. For the Sturm - Liouville equation, a problem on the well-posedness is considered in Hochstadt [16] and Iwasaki [20]. In the latter paper, potentials under consideration are assumed to be spatially symmetric. Further we can refer also to Mizutani [34].

When we formulate our problem on the well-posedness for the system (1.1), as is seen from Theorem in Chapter 2, we encounter the following difficulty : we cannot uniquely determine P, h, H and H^* by the two sets of eigenvalues obtained by (1.1) - (1.3) and by (1.1), (1.2), (1.4). This difficulty is because of the fact that there are too many coefficients that should be determined, compared with the data on eigenvalues. This is not the case in the Sturm - Liouville problem.

On the other hand, as is shown by Proposition 1 in Chapter 3, we see that only the two sets of eigenvalues obtained above give "independent" data.

Therefore, in this chapter, we are restricted to the coefficient $P(x)$ in the form

$$P(x) = \begin{pmatrix} a(x) & b(x) \\ p_1(x) & p_2(x) \end{pmatrix} \qquad (0 \le x \le 1) , \tag{1.10}$$

where a and b are fixed, and furthermore the boundary condition at $x = 0$ is known. That is, we introduce

Definition 1. *Let us arbitrarily fix* $a, b \in C^1[0, 1]$, *and let us define a set* $A(a,b)$ *by*

$$A(a,b) = \left\{ \begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix} ; p_1, p_2 \in C^1[0, 1] \text{ and } p_1, p_2 : \textit{real-valued} \right\} . \tag{1.11}$$

Throughout this chapter, let us assume that all the coefficients of equations under consideration belong to $A(a,b)$.

Then the system is determined uniquely by the two sets of eigenvalues, in view of Theorem 1 in Chapter 3. That is,

Theorem 1. *Let* $h, H, H^* \in \mathbb{R} \setminus \{-1, 1\}$ *and* $H \neq H^*$, $P \in A(a,b)$. *If*

$$\begin{cases} \sigma(A_{Q,h,J}) = \sigma(A_{P,h,H}) \\ \sigma(A_{Q,h,J^*}) = \sigma(A_{P,h,H^*}) \end{cases}$$

holds for $Q \in A(a,b)$ *and* $J, J^* \in \mathbb{R} \setminus \{-1, 1\}$, *then we have*

$$Q(x) = P(x) \qquad (0 \leq x \leq 1)$$

and

$$J = H \text{ and } J^* = H^* \quad .$$

In order to state our main result, we prepare

Proposition 1. *Let* $\sigma(A_{P,h,H}) = \{\lambda_n\}_{n\in\mathbb{Z}}$. *Then,*

(I) *The set* $\sigma(A_{P,h,H}) \cap \mathbb{R}$ *is a finite set.*

(II) $\lambda_n \in \sigma(A_{P,h,H})$ *if and only if* $\overline{\lambda_n} \in \sigma(A_{P,h,H})$.

We recall that $\overline{\alpha}$ denotes the complex conjugate of $\alpha \in \mathbb{C}$. In Appendix I, we prove this proposition.

Throughout this chapter, let $P \in A(a,b)$ and $h, H, H^* \in \mathbb{R} \setminus \{-1, 1\}$ be arbitrarily fixed.

By Proposition 1, we can number all the eigenvalues of $\sigma(A_{P,h,H})$ in the following manner :

Let us denote the number of elements of the set $\sigma(A_{P,h,H}) \cap \mathbb{R}$ by N_0 . Then we set

(1.12.1) $\sigma(A_{P,h,H}) \cap \mathbb{R} = \{ \lambda_{-N_0/2}, \dots, \lambda_{-1}, \lambda_1, \dots, \lambda_{N_0/2} \}$.

(1.12.2) $\sigma(A_{P,h,H}) \cap \{ z ; \operatorname{Im} z > 0 \} = \{ \lambda_n \}_{n \geq N_0/2 + 1}$.

(1.12.3) $\lambda_n = \overline{\lambda_{-n}} \qquad (n \leq -N_0/2 - 1)$

(1.12.4) $\quad \operatorname{Im} \lambda_{n+1} \geq \operatorname{Im} \lambda_n \quad (n \geq N_0/2)$,

if N_0 is even,

and

(1.13.1) $\quad \sigma(A_{P,h,H}) \cap \mathbb{R} = \{ \lambda_{-(N_0-1)/2}, \dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_{(N_0-1)/2} \}$.

(1.13.2) $\quad \sigma(A_{P,h,H}) \cap \{ z \,;\, \operatorname{Im} z > 0 \} = \{ \lambda_n \}_{n \geq (N_0+1)/2}$.

(1.13.3) $\quad \lambda_n = \overline{\lambda_{-n}} \quad (n \leq -(N_0 + 1)/2)$

(1.13.4) $\quad \operatorname{Im} \lambda_{n+1} \geq \operatorname{Im} \lambda_n \quad (n \geq (N_0 - 1)/2)$,

if N_0 is odd.

The numbering is not unique for the whole $\{ \lambda_n \}_{n \in \mathbb{Z}}$, but it is so for sufficiently large n , in view of the asymptotic behavior of λ_n (Proposition 0 in Chapter 2).

For $\sigma(A_{P,h,H^*}) = \{ \mu_n \}_{n \in \mathbb{Z}}$, we can number all the elements in a similar manner. To sum up, for $\sigma(A_{P,h,H}) = \{ \lambda_n \}_{n \in \mathbb{Z}}$ and $\sigma(A_{P,h,H^*}) = \{ \mu_n \}_{n \in \mathbb{Z}}$, there exist some $N_1, N_2 \in \mathbb{N} \cup \{0\}$ such that

(1.14.1) $\quad \lambda_n \in \mathbb{R} \quad (-N_1 \leq n \leq N_1)$

(1.14.2) $\quad \lambda_n = \overline{\lambda_{-n}} \quad (n \leq -N_1 - 1)$

(1.14.3) $\quad \operatorname{Im} \lambda_{n+1} \geq \operatorname{Im} \lambda_n \quad (n \geq N_1)$,

and

(1.15.1) $\quad \mu_n \in \mathbb{R} \quad (-N_2 \leq n \leq N_2)$

(1.15.2) $\mu_n = \overline{\mu_{-n}}$ $(n \leq -N_2 - 1)$

(1.15.3) $\mathrm{Im}\,\mu_{n+1} \geq \mathrm{Im}\,\mu_n$ $(n \geq N_2)$.

Here and henceforth, if N_1 and N_2 are even, then the suffix "0" of $\lambda_.$ and $\mu_.$ is skipped, respectively.
Similarly let us number $\sigma(A_{Q,h,J})$, $\sigma(A_{Q,h,J^*})$, etc.

Now we are ready to state our main result :

Theorem 2. *Let* P, Q ∈ A(a,b), h, H, H^*, J, J^* ∈ $\mathbb{R} \setminus \{-1, 1\}$ *and let us set*

$$(1.16)\quad \begin{cases} \sigma(A_{P,h,H}) = \{ \lambda_n \}_{n\in\mathbb{Z}} \\ \\ \sigma(A_{P,h,H^*}) = \{ \mu_n \}_{n\in\mathbb{Z}} \end{cases}$$

and

$$(1.17)\quad \begin{cases} \sigma(A_{Q,h,J}) = \{ \lambda_n^* \}_{n\in\mathbb{Z}} \\ \\ \sigma(A_{Q,h,J^*}) = \{ \mu_n^* \}_{n\in\mathbb{Z}} \end{cases} .$$

If $\sum_{n=-\infty}^{\infty}(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$ *is sufficiently small for* P, h, H *and* H^* , *then we have the estimates*

$$(1.18)\qquad |H - J| + |H^* - J^*| \leq C\cdot\sum_{n=-\infty}^{\infty}(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$$

and

$$(1.19)\qquad \| P - Q \|_{\{C^j[0,1]\}^4}$$

$$\le C\cdot\sum_{n=-\infty}^{\infty}(|n|^j + 1)(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$$

$$(j = 0,\ 1)\ ,$$

where C *is some positive constant depending on* P, h, H *and* H^* .

Actually we can obtain Theorem 2 from Proposition 2, which assures the existence of $Q \in A(a,b)$, J, $J^* \in \mathbb{R} \setminus \{-1, 1\}$ satisfying $\sigma(A_{Q,h,J}) = \{ \lambda_n^* \}_{n\in\mathbb{Z}}$ and $\sigma(A_{Q,h,J^*}) = \{ \mu_n^* \}_{n\in\mathbb{Z}}$ provided that $\sum_{n=-\infty}^{\infty}(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$ is sufficiently small. That is,

Proposition 2. *Let* $P \in A(a,b)$ *and* h, H, $H^* \in \mathbb{R} \setminus \{-1, 1\}$ *be fixed, and let* $\sigma(A_{P,h,H}) = \{ \lambda_n \}_{n\in\mathbb{Z}}$ *and* $\sigma(A_{P,h,H^*}) = \{ \mu_n \}_{n\in\mathbb{Z}}$ *satisfy (1.14) and (1.15), respectively.*
If two sets of complex numbers $\{ \lambda_n^* \}_{n\in\mathbb{Z}}$ *and* $\{ \mu_n^* \}_{n\in\mathbb{Z}}$ *satisfy*

(1.20.1) $\lambda_n^* \in \mathbb{R}$ $\quad(-N_1 \le n \le N_1)$

(1.20.2) $\lambda_n^* = \overline{\lambda_{-n}^*}$ $\quad(n \le -N_1 - 1)$

(1.20.3) $\operatorname{Im} \lambda_{n+1}^* \ge \operatorname{Im} \lambda_n^*$ $\quad(n \ge N_1)$,

and

(1.21.1) $\mu_n^* \in \mathbb{R}$ $\quad(-N_2 \le n \le N_2)$

(1.21.2) $\mu_n^* = \overline{\mu_{-n}^*}$ $(n \le -N_2 - 1)$

(1.21.3) $\operatorname{Im} \mu_{n+1}^* \ge \operatorname{Im} \mu_n^*$ $(n \ge N_2)$,

respectively, and the inequality

$$(1.22) \qquad \sum_{n=-\infty}^{\infty} |n| \cdot (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|) < \infty$$

holds, then there exists a unique $(Q, J, J^*) \in$ $A(a,b) \times (\mathbb{R} \setminus \{-1, 1\})^2$ *such that*

$$(1.23) \qquad \begin{cases} \sigma(A_{Q,h,J}) = \{ \lambda_n^* \}_{n \in \mathbb{Z}} \\ \\ \sigma(A_{Q,h,J^*}) = \{ \mu_n^* \}_{n \in \mathbb{Z}} \end{cases} ,$$

provided that $\sum_{n=-\infty}^{\infty} (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$ *is sufficiently small for* P, h, H *and* H^* .

If $P \in \{C^1[0, 1]\}^4$ and $|h|, |H| \ne 1$, then according to Proposition 0 in Chapter 2, for $\lambda_n \in \sigma(A_{P,h,H})$, we have the asymptotic behavior

$$(1.24) \qquad \lambda_n = \gamma + \theta + n\pi\sqrt{-1} + O(\tfrac{1}{n}) \quad (\text{as } |n| \to \infty) .$$

Here γ and θ are the constants given in the proposition. Therefore, in general, the relation $P, Q \in \{C^1[0, 1]\}^4$ does not imply the convergence of the series at the right hand side of (1.19). In other words, Theorem 2 suggests that our inverse problem is **ill-posed** by the fact that the topology in $\sigma(A_{P,h,H}) \times \sigma(A_{P,h,H^*})$ introduced in order to assure the continuity of the mapping

$$\Big\{ \{\lambda_n\}_{n \in \mathbb{Z}} , \{\mu_n\}_{n \in \mathbb{Z}} \Big\} \longmapsto P \in A(a,b) \subset \{C^1[0, 1]\}^4 ,$$

is too strong in comparison with the asymptotic behavior (1.24).

We can observe the ill-posedness of this kind also in the results for the Sturm - Liouville problem (Hochstadt [16] and Iwasaki [20]).

This chapter is composed of three sections and eight appendixes. In §2, we prove Proposition 2, while we postpone proofs of the technical lemmas required there to Appendixes II - VIII. In §3, we give a proof of Theorem 2 on the basis of Proposition 2. Appendix I is devoted to a proof of Proposition 1 in §1.

§2. Proof of Proposition 2. In this section, we prove Proposition 2, by construction of $(Q, J, J^*) \in A(a,b) \times (\mathbb{R} \setminus \{ -1, 1 \})^2$ satisfying (1.23) as a fixed point of a contraction mapping. To this end, in subsections §§2.1 - 2.5, we define a contraction mapping G.

First let us consider a domain where our operator is defined.

§2.1. Definition of the domain. We can choose a small constant M so that

$$\frac{H - 1}{H + 1}, \ \frac{H^* - 1}{H^* + 1} \notin [\, e^{-M}, e^{M} \,] \qquad . \tag{2.1}$$

In fact, since $\dfrac{H - 1}{H + 1}, \ \dfrac{H^* - 1}{H^* + 1} \neq 1$, we can see the existence of M satisfying (2.1).

Henceforth let us fix M satisfying (2.1).

As the domain, we define a set $\mathscr{A}_M$ by

$$\mathscr{A}_M = \Big\{ (q_1, q_2) \in \{ C^0[0, 1] \}^2 \ ; \ \| q_1 - p_1 \|_{C^0[0,1]}, \ \| q_2 - p_2 \|_{C^0[0,1]} \leq M \Big\} . \tag{2.2}$$

Here in $\mathscr{A}_M$, we introduce the same norm as the one in $\{ C^0[0, 1] \}^2$:

$$\| (u_1, u_2) \| = \max \{ \| u_1 \|_{C^0[0,1]}, \ \| u_2 \|_{C^0[0,1]} \} . \tag{2.3}$$

Next let us define an operator on $\mathscr{A}_M$ by composing G_i $(1 \le i \le 4)$ given in §§2.2 - 2.5, that is, the operator to be constructed is defined by

$$(2.4)\qquad \begin{aligned} G &= G_4 \circ G_3 \circ G_2 \circ G_1 \\ \mathscr{D}(G) &= \mathscr{A}_M \end{aligned} \qquad .$$

Notation. Let $\sigma(A_{P,h,H}) = \{ \lambda_n \}_{n\in\mathbb{Z}}$ and $\sigma(A_{P,h,H^*}) = \{ \mu_n \}_{n\in\mathbb{Z}}$, and let $\{ \lambda_n^* \}_{n\in\mathbb{Z}}$ and $\{ \mu_n^* \}_{n\in\mathbb{Z}}$ be given such that (1.20) - (1.22) are satisfied.

Then we set

$$(2.5)\qquad \delta_0 = \sum_{n=-\infty}^{\infty} (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$$

and

$$(2.6)\qquad \delta = \sum_{n=-\infty}^{\infty} (|n| + 1)(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|) \ .$$

Moreover let us denote the solutions to (2.7) and to (2.8) by $\phi(\cdot,\lambda) = \begin{pmatrix} \phi_1(\cdot,\lambda) \\ \phi_2(\cdot,\lambda) \end{pmatrix} \in \{ C^1[0, 1] \}^2$ and

$\phi^*(\cdot,\lambda) = \begin{pmatrix} \phi_1^*(\cdot,\lambda) \\ \phi_2^*(\cdot,\lambda) \end{pmatrix} \in \{ C^1[0, 1] \}^2$, respectively ;

$$(2.7)\qquad \begin{cases} B\dfrac{d\phi(x,\lambda)}{dx} + P(x)\phi(x,\lambda) = \lambda\phi(x,\lambda) \ , \quad 0 \le x \le 1 \\[2ex] \phi(0,\lambda) = \begin{pmatrix} 1 \\ -h \end{pmatrix} \end{cases} \qquad .$$

$$
(2.8)\quad \begin{cases} B\dfrac{d\phi^*(x,\lambda)}{dx} - {}^tP(x)\phi^*(x,\lambda) = \lambda\phi^*(x,\lambda)\ , \ 0 \le x \le 1 \\[2ex] \phi^*(0,\lambda) = \begin{pmatrix} 1 \\ h \end{pmatrix} \end{cases} .
$$

Here we recall that $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and ${}^tP(x)$ is the transpose matrix of $P(x)$ (that is, ${}^tP(x) = \begin{pmatrix} a(x) & p_1(x) \\ b(x) & p_2(x) \end{pmatrix}$).

For example, $\phi(\cdot,\lambda_n)$ is an eigenvector of $A_{P,h,H}$ associated with $\lambda_n \in \sigma(A_{P,h,H})$.

Henceforth M_i $(1 \le i \le 32)$ denote positive constants depending on P, h, H, H^*, δ_0, δ and M , and further each M_i is bounded as $\delta_0 \downarrow 0$.

For simplicity, we adopt notation $\| P \|_{C^0}$ etc. in place of $\| P \|_{\{C^0[0,1]\}^4}$.

§2.2. **Definition of G_1.** We define G_1 which transforms each element $q = (q_1, q_2)$ of $\mathcal{A}_M$ to two sequences $\{ a_n(q), b_n(q) \}_{n\in\mathbb{Z}}$ of complex numbers in the following manner :

$$(2.9) \qquad a_n(q) = \frac{-2\exp\left(\frac{1}{2} \int_0^1 (q_2(s)-p_2(s)+p_1(s)-q_1(s))ds\right)}{(H + 1)\exp\left(\int_0^1 (q_2(s)-p_2(s))ds\right) + 1 - H} \times (\phi_2(1,\lambda_n^*) + H\phi_1(1,\lambda_n^*)) \quad (n \in \mathbb{Z})$$

and

$$(2.10) \qquad b_n(q) = \frac{-2\exp\left(\frac{1}{2} \int_0^1 (q_2(s)-p_2(s)+p_1(s)-q_1(s))ds\right)}{(H^* + 1)\exp\left(\int_0^1 (q_2(s)-p_2(s))ds\right) + 1 - H^*} \times (\phi_2(1,\mu_n^*) + H^*\phi_1(1,\mu_n^*)) \quad (n \in \mathbb{Z}) .$$

Then we set

$$(2.11) \qquad G_1(q) = \{ a_n(q), b_n(q) \}_{n\in\mathbb{Z}} .$$

In (2.9) and (2.10), in view of (2.1), we can see that there exists some positive constant M_1 such that

$$(2.12) \quad \begin{cases} \left|(H + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds\right) + 1 - H\right| \geq M_1 \\ \\ \left|(H^* + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds\right) + 1 - H^*\right| \geq M_1 \end{cases}$$

for each $(q_1, q_2) \in \mathcal{A}_M$.

In fact, since $e^{-M} \leq \exp(\int_0^1 (q_2(s) - p_2(s))ds) \leq e^M$, we have

$$\left|(H + 1)\exp(\int_0^1 (q_2(s) - p_2(s))ds) + 1 - H\right|$$

$$\geq \ |H + 1|\cdot\min\left\{ \left|e^M - \frac{H - 1}{H + 1}\right| , \left|e^{-M} - \frac{H - 1}{H + 1}\right| \right\} > 0 .$$

Similarly we can see that

$$\left|(H^* + 1)\exp(\int_0^1 (q_2(s) - p_2(s))ds) + 1 - H^*\right|$$

$$\geq \ |H^* + 1|\cdot\min\left\{ \left|e^M - \frac{H^* - 1}{H^* + 1}\right| , \left|e^{-M} - \frac{H^* - 1}{H^* + 1}\right| \right\} > 0 .$$

Moreover we have

Lemma 1. *Let* $q \in \mathcal{A}_M$ *and let* $a_n(q)$ *and* $b_n(q)$ $(n \in \mathbb{Z})$ *be defined by (2.9) and (2.10), respectively. Then we get*

$$(2.13) \qquad \sum_{n=-\infty}^{\infty} (|a_n(q)| + |b_n(q)|) \leq M_2\delta_0$$

and

$$(2.14) \qquad \sum_{n=-\infty}^{\infty} (|n| + 1)(|a_n(q)| + |b_n(q)|) \leq M_2\delta$$

for some positive constant M_2 .

Here we recall that δ_0 and δ are given by (2.5) and (2.6), respectively.

In Appendix II, we prove this lemma.

§2.3. **Definition of** G_2. For the definition, we prepare Lemmas 2 and 3.

Lemma 2. *Under all the assumptions of Proposition 2, we have the following facts on* $\{ \phi(\cdot, \lambda_n^*) \}_{n\in\mathbb{Z}}$.

(I) (the completeness of $\{ \phi(\cdot, \lambda_n^*) \}_{n\in\mathbb{Z}}$ *) The system* $\{ \phi(\cdot, \lambda_n^*) \}_{n\in\mathbb{Z}}$ *is a Riesz basis in* $\{ L^2(0, 1) \}^2$.

(II) (the existence of a complete biorthogonal system to $\{ \phi(\cdot, \lambda_n^*) \}_{n\in\mathbb{Z}}$ *) There exists some system* $\{ \psi_n^{(1)} \}_{n\in\mathbb{Z}}$ *satisfying (2.15) - (2.19).*

$$\psi_n^{(1)} \in \{ C^1[0, 1] \}^2 \qquad (n \in \mathbb{Z}) . \tag{2.15}$$

$$\begin{cases} \psi_n^{(1)} \ : \textit{real-valued} & (-N_1 \le n \le N_1) \\ \overline{\psi_n^{(1)}} = \psi_{-n}^{(1)} & (n \ge N_1 + 1) \end{cases} . \tag{2.16}$$

$$\begin{cases} \| \psi_n^{(1)} \|_{\{C^0[0,1]\}^2} \le M_3 & \\ \| \psi_n^{(1)} \|_{\{C^1[0,1]\}^2} \le M_3(|n| + 1) & (n \in \mathbb{Z}) , \end{cases} \tag{2.17}$$

for some positive constant M_3 .

$$(2.18) \quad (\phi(\cdot,\lambda_n^*),\ \psi_m^{(1)})_{\{L^2(0,1)\}^2}$$

$$= \delta_{nm} = \begin{cases} 0, & \text{if} \quad n \neq m \\ 1, & \text{if} \quad n = m \end{cases} .$$

$$(2.19) \quad u = \sum_{n=-\infty}^{\infty} (u,\ \overline{\phi(\cdot,\lambda_n^*)}) \cdot \overline{\psi_n^{(1)}} ,$$

for each $u \in \{L^2(0,\ 1)\}^2$.

Here the series at the right hand side of (2.19) is convergent in $\{L^2(0,\ 1)\}^2$.

Furthermore, as for the system $\{\phi(\cdot,\mu_n^*)\}_{n\in\mathbb{Z}}$, *similar facts hold. That is,*

(I)' The system $\{\phi(\cdot,\mu_n^*)\}_{n\in\mathbb{Z}}$ *is a Riesz basis.*

(II)' There exists some $\{\psi_n^{(2)}\}_{n\in\mathbb{Z}}$ *satisfying*

$$(2.15)' \quad \psi_n^{(2)} \in \{C^1[0,\ 1]\}^2 \qquad (n \in \mathbb{Z}) .$$

$$(2.16)' \begin{cases} \psi_n^{(2)} \ : \text{real-valued} & (-N_2 \leq n \leq N_2) \\ \overline{\psi_n^{(2)}} = \psi_{-n}^{(2)} & (n \geq N_2 + 1) \end{cases} .$$

$$(2.17)' \begin{cases} \| \psi_n^{(2)} \|_{\{C^0[0,1]\}^2} \leq M_3 & \\ \| \psi_n^{(2)} \|_{\{C^1[0,1]\}^2} \leq M_3(|n| + 1) & (n \in \mathbb{Z}) , \end{cases}$$

for some positive constant M_3 .

(2.18)' $\quad (\phi(\cdot,\mu_n^*),\ \psi_m^{(2)})_{\{L^2(0,1)\}^2} = \delta_{nm}$.

(2.19)' $\quad u = \sum_{n=-\infty}^{\infty} (u,\ \overline{\phi(\cdot,\mu_n^*)}\ )\cdot\overline{\psi_n^{(2)}}$

in the topology in $\{L^2(0,\ 1)\}^2$,

for each $u \in \{L^2(0,\ 1)\}^2$.

Lemma 3. *Let* M *satisfy (2.1) and let* $q = (q_1,\ q_2) \in \mathcal{A}_M$. *Then, setting*

(2.20)
$$J = J(q) = \left((H + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds\right) + H - 1\right) \times \left((H + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds\right) + 1 - H\right)^{-1}$$

(2.21)
$$J^* = J^*(q) = \left((H^* + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds\right) + H^* - 1\right) \times \left((H^* + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds\right) + 1 - H^*\right)^{-1},$$

we have

(2.22) $\quad J \in \mathbb{R} \setminus \{-1,\ 1\}$,

(2.23) $\quad J^* \in \mathbb{R} \setminus \{-1,\ 1\}$,

and

(2.24) $\quad J \neq J^*$.

In Appendixes III and IV, we prove Lemmas 2 and 3, respectively.

Now, as is seen by Lemma 4 stated below, for $\{ a_n(q),\ b_n(q) \}_{n \in \mathbb{Z}}$ given by (2.9) and (2.10), we can set

$$(2.25)\qquad \begin{pmatrix} c_{11}(y,q) \\ c_{12}(y,q) \end{pmatrix} = \frac{1}{J - J^*} \cdot \sum_{n=-\infty}^{\infty} (a_n(q)\overline{\psi_n^{(1)}(y)} - b_n(q)\overline{\psi_n^{(2)}(y)})$$

and

$$(2.26)\qquad \begin{pmatrix} c_{21}(y,q) \\ c_{22}(y,q) \end{pmatrix} = \frac{1}{J^* - J} \cdot \sum_{n=-\infty}^{\infty} (J^* a_n(q)\overline{\psi_n^{(1)}(y)} - Jb_n(q)\overline{\psi_n^{(2)}(y)}) ,$$

where $J = J(q)$ and $J^* = J^*(q)$ are given by (2.20) and (2.21). Let us set $C(\cdot,q) = (c_{ij}(\cdot,q))_{1 \le i,j \le 2}$.

Lemma 4. *(I) On all the assumptions of proposition 2, we have*

(2.27) $\quad C(\cdot,q)$: *real-valued.*

(2.28) $\quad C(\cdot,q) \in \{C^1[0, 1]\}^4$.

(2.29) $\quad \| C(\cdot,q) \|_{\{C^0[0,1]\}^4} \le M_4\delta_0$.

(2.30) $\quad \| C(\cdot,q) \|_{\{C^1[0,1]\}^4} \le M_4\delta$,

for some positive constant M_4 .

(II) For $q^{(i)} = (q_1^{(i)},\ q_2^{(i)}) \in \mathscr{A}_M$ $(i = 1, 2)$, *we have the estimate*

$$(2.31)\qquad \| C(\cdot,q^{(1)}) - C(\cdot,q^{(2)}) \|_{\{C^0[0,1]\}^4} \le M_4\delta_0 \| q^{(1)} - q^{(2)} \|_{\{C^0[0,1]\}^2} .$$

In Appendix V, we prove this lemma.

Then we define an operator G_2 by

$$(2.32)\qquad G_2(\{a_n(q),\ b_n(q)\}_{n\in\mathbb{Z}}) = C(\cdot,q) .$$

By Lemma 4, the operator G_2 sends $\{a_n(q),\ b_n(q)\}_{n\in\mathbb{Z}}$ to the four real-valued C^1-functions $c_{ij}(\cdot,q)$ $(1 \le i,j \le 2)$ on $[0,\ 1]$.

§2.4. Definition of G_3. Let us recall $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and henceforth let us set

$$\Omega = \{ (x, y) \; ; \; 0 < y < x < 1 \} \quad . \tag{2.33}$$

For the definition of G_3 , we need

Lemma 5. *Let*

$$P \in A(a,b) \subset \{C^1[0, 1]\}^4 \quad , \tag{2.34}$$

and

$$Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix} \quad , \textit{ where } \quad (q_1, q_2) \in \mathscr{A}_M \; . \tag{2.35}$$

(I) For given $D = (d_{ij})_{1 \le i,j \le 2} \in \{C^1[0, 1]\}^4$ *, there exists a unique solution* $K = K(\cdot,\cdot, P,Q,D) \in \{C^1(\overline{\Omega})\}^4$ *to (2.36) - (2.38) :*

$$B \frac{\partial K(x,y)}{\partial x} + Q(x)K(x,y) - K(x,y)P(y) = - \frac{\partial K(x,y)}{\partial y} B \qquad ((x, y) \in \overline{\Omega}) \quad . \tag{2.36}$$

$$\begin{cases} K_{12}(x,0) = hK_{11}(x,0) \\ \\ K_{22}(x,0) = hK_{21}(x,0) \qquad (0 \le x \le 1) \quad . \end{cases} \tag{2.37}$$

$$K(1,y) = D(y) \qquad (0 \le y \le 1) \quad . \tag{2.38}$$

Furthermore the estimates

(2.39) $$\| K \|_{\{C^0(\bar{\Omega})\}^4} \le M_5 \| D \|_{\{C^0[0,\ 1]\}^4}$$

and

(2.40) $$\| K \|_{\{C^1(\bar{\Omega})\}^4} \le M_5 \| D \|_{\{C^1[0,\ 1]\}^4}$$

hold for some positive constant M_5 .

(II) For given Q_1 , Q_2 *in the form (2.35) and* D_1, $D_2 \in \{C^1[0,\ 1]\}^4$, *we have the estimate*

(2.41) $$\| K(\cdot,\cdot,P,Q_1,D_1) - K(\cdot,\cdot,P,Q_2,D_2) \|_{\{C^0(\bar{\Omega})\}^4}$$
$$\le M_5(\| D_2 \|_{\{C^0[0,\ 1]\}^4} \cdot \| Q_1 - Q_2 \|_{\{C^0[0,\ 1]\}^4} + \| D_1 - D_2 \|_{\{C^0[0,\ 1]\}^4}) .$$

In Appendix VI, we prove this lemma.

In (2.38), let us substitute $C(\cdot,q) = (c_{ij}(\cdot,q))_{1 \le i,j \le 2}$ given by (2.25) and (2.26) into $D(\cdot)$. Then, by Lemma 5, we can define G_3 by

(2.42) $$G_3(C(\cdot,q)) = K(\cdot,\cdot,P,Q,C) .$$

§2.5. Definition of G_4. Let us set

$$(2.43) \qquad A(x) =$$

$$\frac{1}{2}\begin{pmatrix} -a(x)-b(x)+p_1(x)+p_2(x) & a(x)+b(x)-p_1(x)-p_2(x) \\ -a(x)+b(x)-p_1(x)+p_2(x) & a(x)-b(x)+p_1(x)-p_2(x) \end{pmatrix}$$

$$(0 \leq x \leq 1)$$

and let us consider the initial value problem (2.44) and (2.45) for ordinary differential equations :

$$(2.44) \qquad \frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = A(x)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} +$$

$$\begin{pmatrix} K_{11}(x,x)-K_{22}(x,x)+K_{12}(x,x)-K_{21}(x,x) \\ K_{11}(x,x)-K_{22}(x,x)+K_{21}(x,x)-K_{12}(x,x) \end{pmatrix} \quad (0 \leq x \leq 1) .$$

$$(2.45) \qquad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Here $K = G_3(C(\cdot,q))$.

Then we have

Lemma 6. *(I) There exists some positive constant* M_6 *such that, if*

$$(2.46) \qquad \delta_0 \leq M_6 ,$$

then the solution $\begin{pmatrix} u \\ v \end{pmatrix}$ *to (2.44) and (2.45) satisfies*

$$(2.47) \qquad u(x),\ v(x) \geq \frac{1}{2} \qquad (0 \leq x \leq 1) \quad .$$

(II) On the condition (2.46), we can define real-valued C^1*-functions* r_1, r_2 *by*

$$(2.48) \quad \begin{cases} r_1(x) = p_1(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} - \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \\ \\ r_2(x) = p_2(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} + \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \end{cases}$$

$$(0 \leq x \leq 1) \quad .$$

Furthermore the following estimates hold :

$$(2.49) \qquad \| p_1 - r_1 \|_{C^0[0,\ 1]},\ \| p_2 - r_2 \|_{C^0[0,\ 1]}$$

$$\leq M_7 \max_{\substack{1 \leq i,j \leq 2 \\ 0 \leqq x \leqq 1}} | K_{ij}(x,x) | \quad .$$

$$(2.50) \qquad \| p_1 - r_1 \|_{C^1[0,\ 1]},\ \| p_2 - r_2 \|_{C^1[0,\ 1]}$$

$$\leq M_7 \max \left(\max_{\substack{1 \leq i,j \leq 2 \\ 0 \leqq x \leqq 1}} | K_{ij}(x,x) | \ , \right.$$

$$\left. \max_{\substack{1 \leq i,j \leq 2 \\ 0 \leqq x \leqq 1}} \left| \frac{dK_{ij}(x,x)}{dx} \right| \right) \quad .$$

(III) For $q^{(i)} = (q_1{}^{(i)},\ q_2{}^{(i)}) \in \mathscr{A}_M$ $(i = 1,2)$, *let us put*

$$K^{(i)} = (G_3 \circ G_2 \circ G_1) q^{(i)} \qquad (i = 1,2) \ .$$

On the assumptions (2.46), let $(u^{(i)},\ v^{(i)})$ $(i = 1,2)$ *be the solution to*

$$(2.51) \quad \frac{d}{dx}\begin{pmatrix} u^{(i)}(x) \\ v^{(i)}(x) \end{pmatrix} = A(x)\begin{pmatrix} u^{(i)}(x) \\ v^{(i)}(x) \end{pmatrix} +$$

$$\begin{pmatrix} K_{11}^{(i)}(x,x)-K_{22}^{(i)}(x,x)+K_{12}^{(i)}(x,x)-K_{21}^{(i)}(x,x) \\ K_{11}^{(i)}(x,x)-K_{22}^{(i)}(x,x)+K_{21}^{(i)}(x,x)-K_{12}^{(i)}(x,x) \end{pmatrix}$$

$$(i = 1,2, \quad 0 \leq x \leq 1)$$

and

$$(2.52) \quad \begin{pmatrix} u^{(i)}(0) \\ v^{(i)}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (i = 1, 2) ,$$

and moreover, let us set

$$(2.53) \quad \begin{cases} r_1^{(i)}(x) = p_1(x) - \dfrac{1}{u^{(i)}(x)}\dfrac{du^{(i)}(x)}{dx} - \dfrac{1}{v^{(i)}(x)}\dfrac{dv^{(i)}(x)}{dx} \\ \\ r_2^{(i)}(x) = p_2(x) - \dfrac{1}{u^{(i)}(x)}\dfrac{du^{(i)}(x)}{dx} + \dfrac{1}{v^{(i)}(x)}\dfrac{dv^{(i)}(x)}{dx} \end{cases}$$

$$(i = 1, 2, \quad 0 \leq x \leq 1) \quad .$$

Then the estimate

$$(2.54) \quad \| r_1^{(1)} - r_1^{(2)} \|_{C^0[0,\ 1]} + \| r_2^{(1)} - r_2^{(2)} \|_{C^0[0,\ 1]}$$

$$\leq M_7 \max_{\substack{1 \leq i,j \leq 2 \\ 0 \leq x \leq 1}} | K_{ij}^{(1)}(x,x) - K_{ij}^{(2)}(x,x) |$$

holds.

In Appendix VII, we carry out the proof of Lemma 6.

Now we proceed to the definition of G_4 . Under the assumption (2.46) of Lemma 6 (I), let us define G_4 by

$$G_4(K(\cdot,\cdot,P,Q,C)) = (r_1, r_2) \quad , \tag{2.55}$$

where $K(\cdot,\cdot,P,Q,C)$ and (r_1, r_2) are given by (2.42) and (2.48), respectively.

Thus we complete the definitions of G_1, G_2, G_3, and G_4 .

§2.6. **Reduction to a fixed point.** In this subsection, we show Lemma 7 which asserts that a fixed point (q_1, q_2) of the mapping G gives the functions satisfying all the conditions in Proposition 2. That is,

Lemma 7. *On all the assumptions of Proposition 2, let* $q = (q_1, q_2) \in \{ C^0[0, 1]\}^2$ *satisfying*

$$(2.56) \qquad q = Gq \quad .$$

Then we have

$$(2.57) \qquad q_i \in C^1[0, 1] \qquad (i = 1, 2) \quad ,$$

and furthermore

$$(2.58) \quad \begin{cases} \sigma(A_{Q,h,J}) = \{ \lambda_n^* \}_{n\in\mathbb{Z}} \\ \\ \sigma(A_{Q,h,J^*}) = \{ \mu_n^* \}_{n\in\mathbb{Z}} \end{cases}$$

for

$$(2.59) \qquad Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}$$

and J, J^* *defined by (2.20) and (2.21).*

Proof of Lemma 7. Let us assume that $q = (q_1, q_2) \in \{ C^0[0, 1]\}^2$ satisfies (2.56).
First we prove (2.57). To this end, we have only to prove

$$(2.60) \qquad Gq \in \{ C^1[0, 1]\}^2 \quad ,$$

in view of (2.56). Since, by Lemma 5 (I), the relation (2.28) implies that $K(\cdot,\cdot,P,Q,C) \in \{C^1(\bar{\Omega})\}^4$, where $P = \begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix}$ and $Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}$, we see (2.57) from the definition of G and $a, b, p_1, p_2 \in C^1[0, 1]$.

Now, for the unique solution $K = K(\cdot,\cdot,P,Q,C)$ to (2.36) - (2.38), the solution (u, v) to (2.44) and (2.45) satisfies

$$
(2.61) \quad \begin{cases} q_1(x) = p_1(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} - \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \\ q_2(x) = p_2(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} + \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \end{cases} \quad (0 \leq x \leq 1) .
$$

Noting (2.47) and (2.45), we integrate (2.61) with respect to x, so that we get

$$
\int_0^x (p_1(s) - q_1(s))ds = \log u(x) + \log v(x)
$$

$$
\int_0^x (p_2(s) - q_2(s))ds = \log u(x) - \log v(x)
$$

which imply

$$
(2.62) \quad \begin{cases} u(x) = \exp\left(\frac{1}{2} \displaystyle\int_0^x (p_1(s)+p_2(s)-q_1(s)-q_2(s)ds \right) \\ v(x) = \exp\left(\frac{1}{2} \displaystyle\int_0^x (p_1(s)-p_2(s)-q_1(s)+q_2(s)ds \right) \end{cases} \quad (0 \leq x \leq 1) .
$$

Substituting (2.62) into (2.44), we obtain

$$(2.63)\qquad K_{12}(x,x) - K_{21}(x,x)$$

$$= \frac{1}{4} e^{-\theta_1(x)-\theta_2(x)}(2b(x)-q_1(x)-q_2(x)-p_1(x)+p_2(x))$$

$$+ \frac{1}{4} e^{-\theta_1(x)+\theta_2(x)}(-2b(x)+q_1(x)-q_2(x)+p_1(x)+p_2(x))$$

$$(0 \leq x \leq 1)$$

and

$$(2.64)\qquad K_{11}(x,x) - K_{22}(x,x)$$

$$= \frac{1}{4} e^{-\theta_1(x)-\theta_2(x)}(2a(x)-q_1(x)-q_2(x)+p_1(x)-p_2(x))$$

$$+ \frac{1}{4} e^{-\theta_1(x)+\theta_2(x)}(-2a(x)-q_1(x)+q_2(x)+p_1(x)+p_2(x))$$

$$(0 \leq x \leq 1) \quad .$$

Here and henceforth, we put

$$(2.65)\qquad \theta_1(x) = \frac{1}{2}\int_0^x (q_1(s) - p_1(s))ds \quad (0 \leq x \leq 1) \ ,$$

$$(2.66)\qquad \theta_2(x) = \frac{1}{2}\int_0^x (q_2(s) - p_2(s))ds \quad (0 \leq x \leq 1) \ ,$$

and

$$(2.67)\qquad R(x) = e^{-\theta_1(x)} \begin{pmatrix} \cosh \theta_2(x) & -\sinh \theta_2(x) \\ -\sinh \theta_2(x) & \cosh \theta_2(x) \end{pmatrix}$$

$$(0 \leq x \leq 1) \quad .$$

Defining $\psi(\cdot,\lambda) = \begin{pmatrix} \psi_1(\cdot,\lambda) \\ \psi_2(\cdot,\lambda) \end{pmatrix}$ by

$$(2.68) \qquad \psi(x,\lambda) = R(x)\phi(x,\lambda) + \int_0^x K(x,y)\phi(y,\lambda)dy \qquad (0 \le x \le 1),$$

in virtue of (2.36), (2.37), (2.63) and (2.64), we can apply Lemma 1 (II) in Chapter 2, so that we see that

$$(2.69) \qquad \begin{cases} B \dfrac{d\psi(x,\lambda)}{dx} + Q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda) \qquad (0 \le x \le 1) \\ \psi(0,\lambda) = \begin{pmatrix} 1 \\ -h \end{pmatrix} \end{cases} .$$

Then we can prove that for J and J^* defined by (2.20) and (2.21),

$$(2.70) \qquad \psi_2(1,\lambda_n^*) + J\psi_1(1,\lambda_n^*) = 0 \qquad (n \in \mathbb{Z})$$

and

$$(2.71) \qquad \psi_2(1,\mu_n^*) + J^*\psi_1(1,\mu_n^*) = 0 \qquad (n \in \mathbb{Z}) \quad ,$$

in the following way.

First we have

$$(\overline{\psi_m^{(1)}(\cdot)} \ , \ \overline{\phi(\cdot,\lambda_n^*)} \ ) = \overline{(\overline{\phi(\cdot,\lambda_n^*)} \ , \ \overline{\psi_m^{(1)}(\cdot)} \ )}$$

$$= (\phi(\cdot,\lambda_n^*) \ , \ \psi_m^{(1)}(\cdot) \ ) = \delta_{nm} \qquad \text{by (2.18)} \ ,$$

that is, we get

$$(2.72) \qquad (\overline{\psi_m^{(1)}(\cdot)} \ , \ \overline{\phi(\cdot,\lambda_n^*)} \ ) = \delta_{nm} \qquad .$$

Similarly we get

$$(2.72)' \qquad (\overline{\psi_m^{(2)}(\cdot)} \ , \ \overline{\phi(\cdot,\mu_n^*)} \ ) = \delta_{nm} \qquad .$$

Second by the boundary condition (2.38) for K, we see

$$(2.73)\qquad \begin{pmatrix} K_{11}(1,y) \\ K_{12}(1,y) \end{pmatrix} = \frac{1}{J - J^*} \sum_{n=-\infty}^{\infty} (a_n(q)\overline{\psi_n^{(1)}(y)} - b_n(q)\overline{\psi_n^{(2)}(y)})$$

and

$$(2.74)\qquad \begin{pmatrix} K_{21}(1,y) \\ K_{22}(1,y) \end{pmatrix} = \frac{1}{J^* - J} \sum_{n=-\infty}^{\infty} (J^* a_n(q)\overline{\psi_n^{(1)}(y)} - Jb_n(q)\overline{\psi_n^{(2)}(y)}) ,$$

where the right hand sides of (2.73) and (2.74) are convergent in $\{L^2(0, 1)\}^2$.

Thus, by using (2.72), the equalities (2.73) and (2.74) imply

$$(2.75)\qquad \left(\begin{pmatrix} K_{11}(1,\cdot) \\ K_{12}(1,\cdot) \end{pmatrix} , \overline{\phi(\cdot,\lambda_n^*)} \right)_{\{L^2(0,1)\}^2}$$

$$= \frac{a_n(q)}{J - J^*} - \frac{1}{J - J^*} \sum_{m=-\infty}^{\infty} b_m(q) (\overline{\psi_m^{(2)}(\cdot)} , \overline{\phi(\cdot,\lambda_n^*)}) \qquad (n \in \mathbb{Z}) ,$$

and

$$(2.76)\qquad \left(\begin{pmatrix} K_{21}(1,\cdot) \\ K_{22}(1,\cdot) \end{pmatrix} , \overline{\phi(\cdot,\lambda_n^*)} \right)_{\{L^2(0,1)\}^2}$$

$$= \frac{J^* a_n(q)}{J^* - J} - \frac{J}{J^* - J} \sum_{m=-\infty}^{\infty} b_m(q) (\overline{\psi_m^{(2)}(\cdot)} , \overline{\phi(\cdot,\lambda_n^*)}) \qquad (n \in \mathbb{Z}) .$$

Now we have, for $n \in \mathbb{Z}$,

$$\psi_2(1,\lambda_n^*) + J\psi_1(1,\lambda_n^*)$$

$$= e^{-\theta_1(1)} (J\cdot\cosh\theta_2(1) - \sinh\theta_2(1))\ \phi_1(1,\lambda_n^*)$$
$$+ e^{-\theta_1(1)} (\cosh\theta_2(1) - J\cdot\sinh\theta_2(1))\ \phi_2(1,\lambda_n^*)$$
$$+ \left(\begin{pmatrix} K_{21}(1,\cdot) \\ K_{22}(1,\cdot) \end{pmatrix} , \overline{\phi(\cdot,\lambda_n^*)} \right)_{L^2} + J\cdot\left(\begin{pmatrix} K_{11}(1,\cdot) \\ K_{12}(1,\cdot) \end{pmatrix} , \overline{\phi(\cdot,\lambda_n^*)} \right)_{L^2}$$

by (2.68)

$$= \frac{2e^{\theta_2(1)-\theta_1(1)}}{(H + 1)e^{2\theta_2(1)} + 1 - H} \cdot (\phi_2(1,\lambda_n^*) + H\phi_1(1,\lambda_n^*)) + a_n(q)$$

by (2.20) and (2.75), (2.76)

$$= 0$$ by the definition (2.9) of $a_n(q)$.

That is, we see (2.70).

As for (2.71), we can similarly proceed in view of (2.72)', (2.73), (2.74), (2.68), (2.21) and (2.10).

Thus we complete the proof of (2.70) and (2.71).

Since $\psi(\cdot,\lambda)$ satisfies (2.69), the relations (2.70) and (2.71) imply

(2.77) $$\{ \lambda_n^* \}_{n\in\mathbb{Z}} \subset \sigma(A_{Q,h,J})$$

and

(2.78) $$\{ \mu_n^* \}_{n\in\mathbb{Z}} \subset \sigma(A_{Q,h,J^*}) ,$$

respectively.

Finally we have to prove that

$$(2.79) \qquad \{ \lambda_n^* \}_{n\in\mathbb{Z}} \supset \sigma(A_{Q,h,J})$$

and

$$(2.80) \qquad \{ \mu_n^* \}_{n\in\mathbb{Z}} \supset \sigma(A_{Q,h,J^*}) \quad .$$

To this end, we have only to show

Lemma 8. *Let us denote the solution to (2.81) by*

$$\psi^*(\cdot,\lambda) = \begin{pmatrix} \psi_1^*(\cdot,\lambda) \\ \psi_2^*(\cdot,\lambda) \end{pmatrix} \quad :$$

$$(2.81) \quad \begin{cases} B\,\dfrac{d\psi^*(x,\lambda)}{dx} - {}^tQ(x)\psi^*(x,\lambda) = \lambda\psi^*(x,\lambda) & (0 \leq x \leq 1) \\ \psi^*(0,\lambda) = \begin{pmatrix} 1 \\ h \end{pmatrix} & . \end{cases}$$

Then,

(I) The equalities

$$(2.82) \qquad \psi_2^*(1, \overline{-\lambda_n^*}) - J\psi_1^*(1, \overline{-\lambda_n^*}) = 0 \qquad (n \in \mathbb{Z})$$

hold. That is, $\psi^*(\cdot, \overline{-\lambda_n^*})$ *is an eigenvector of* $A^*_{Q,h,J}$ *, the adjoint operator given as in Remark 1 in §1, associated with the eigenvalue* $\overline{\lambda_n^*}$.

(II) For each $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \{L^2(0, 1)\}^2$ *, we get*

$$(2.83) \qquad u = \sum_{n=-\infty}^{\infty} \frac{(u, \psi^*(\cdot, \overline{-\lambda_n^*}))}{\alpha_n} \psi(\cdot,\lambda_n^*) \quad ,$$

where the right hand side is convergent in $\{L^2(0,\ 1)\}^2$ *and we set*

$$\alpha_n = (\psi(\cdot,\lambda_n^*),\ \psi^*(\cdot,\ \overline{-\lambda_n^*})) \quad .$$

Moreover, as for $\{\ \psi(\cdot,\mu_n^*)\ \}_{n\in\mathbb{Z}}$ *, we obtain similar results :*

(I)' $\psi^*(\cdot,\ \overline{-\mu_n^*})$ *is an eigenvector of* A^*_{Q,h,J^*} *associated with the eigenvalue* $\overline{\mu_n^*}$.

(II)' For each $u \in \{L^2(0,\ 1)\}^2$ *, we get*

$$(2.83)' \qquad u = \sum_{n=-\infty}^{\infty} \frac{(u,\ \psi^*(\cdot,\ \overline{-\mu_n^*}))}{\beta_n}\ \psi(\cdot,\mu_n^*) \quad ,$$

in $\{L^2(0,\ 1)\}^2$. *Here we set*

$$\beta_n = (\psi(\cdot,\mu_n^*),\ \psi^*(\cdot,\ \overline{-\mu_n^*})) \quad .$$

In Appendix VIII, a proof of this lemma is given.

Now we return to the proof of (2.79) and (2.80). Assume that there exists $\chi \in \sigma(A_{Q,h,J})$ such that

$$(2.84) \qquad \chi \neq \lambda_n^* \qquad\qquad (n \in \mathbb{Z}) \quad .$$

Let $u \neq 0$ be an eigenvector of $A_{Q,h,J}$ associated with χ . Since, for $n \in \mathbb{Z}$,

$$\chi(u,\ \psi^*(\cdot,\ \overline{-\lambda_n^*})\ )$$

$$= (A_{Q,h,J}u,\ \psi^*(\cdot,\ \overline{-\lambda_n^*})\ ) \qquad (\text{by } A_{Q,h,J}u = \chi u\ )$$

$$= (u,\ A^*_{Q,h,J}\psi^*(\cdot,\ \overline{-\lambda_n^*})\ )$$

$= \lambda_n^*(u, \psi^*(\cdot, \overline{-\lambda_n^*}))$ (by Lemma 8 (I)) ,

we get

$$(2.85) \qquad (u, \psi^*(\cdot, \overline{-\lambda_n^*})) = 0 \qquad (n \in \mathbb{Z})$$

by (2.84). Therefore it follows form (2.83) that $u = 0$, which contradicts that $u \neq 0$.

Thus we see (2.79).

Similarly we can prove (2.80).

This completes the proof of Lemma 7.

§2.7. Completion of the proof of Proposition 2. Applying the principle of contraction mappings (Kolmogorov and Fomin [25, p.66], for instance), we complete the proof of Proposition 2. To this end, provided that δ_0 is sufficiently small, we have only to verify

(I) G *is a contraction mapping, that is, there exists some constant* $0 \le \varkappa < 1$ *such that*

$$\| Gq^{(1)} - Gq^{(2)} \|_{\{C^0[0,1]\}^2} \le \varkappa \| q^{(1)} - q^{(2)} \|_{\{C^0[0,1]\}^2},$$

for each $q^{(1)}, q^{(2)} \in \mathscr{A}_M$.

(II) $G\mathscr{A}_M \subset \mathscr{A}_M$.

In fact, if (I) and (II) are proved, then since $\mathscr{A}_M$ is a closed set in $\{C^0[0, 1]\}^2$ by the definition (2.2), we can apply the principle of contraction mappings, so that we see the unique existence of fixed point $q = (q^{(1)}, q^{(2)})$ of G . Therefore, in view of Lemma 7, $Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}$ and J, J^* given by (2.20) and (2.21) for the q_1, q_2 satisfy (1.23).

Proof of (I). For $q^{(i)} = (q_1^{(i)}, q_2^{(i)}) \in \mathscr{A}_M$, let $r^{(i)} = Gq^{(i)}$ where $r^{(i)} = (r_1^{(i)}, r_2^{(i)})$ $(i = 1, 2)$. Henceforth we set

$$Q_i = \begin{pmatrix} a & b \\ q_1^{(i)} & q_2^{(i)} \end{pmatrix} \qquad (i = 1, 2).$$

Then, by the estimate (2.41) of Lemma 5, we have

$$\| K(\cdot,\cdot,P,Q_1,C(\cdot,q^{(1)})) - K(\cdot,\cdot,P,Q_2,C(\cdot,q^{(2)})) \|_{\{C^0(\bar{\Omega})\}^4}$$
$$\leq M_5(\|C(\cdot,q^{(2)})\|_{\{C^0[0,\ 1]\}^4} \cdot \|Q_1 - Q_2\|_{\{C^0[0,\ 1]\}^4} +$$
$$\|C(\cdot,q^{(1)}) - C(\cdot,q^{(2)})\|_{\{C^0[0,\ 1]\}^4}) .$$

Thus, by the estimates (2.29) and (2.31) in Lemma 4, we get

$$\| K^{(1)} - K^{(2)} \|_{\{C^0(\bar{\Omega})\}^4} \tag{2.86}$$
$$\leq 2M_4M_5\delta_0\| q^{(1)} - q^{(2)}\|_{\{C^0[0,\ 1]\}^2} .$$

Here and henceforth, for brevity, we put

$$K^{(i)}(x,y) = K(x,y,P,Q_i,C(\cdot,q^{(i)}))$$
$$K^{(i)}_{jk}(x,y) = K_{jk}(x,y,P,Q_i,C(\cdot,q^{(i)}))$$
$$(i = 1,\ 2,\quad 1 \leq j,k \leq 2,\quad (x,y) \in \bar{\Omega})$$
$$r^{(i)} = (r^{(i)}_1,\ r^{(i)}_2) = G_4(K^{(i)}) \qquad (i = 1,\ 2) .$$

By Lemma 6 (III), we have the estimate

$$\| r^{(1)}_1 - r^{(2)}_1 \|_{C^0} + \| r^{(1)}_2 - r^{(2)}_2 \|_{C^0}$$
$$\leq M_7\| K^{(1)} - K^{(2)} \|_{\{C^0(\bar{\Omega})\}^4} ,$$

with which we combine (2.86), so that we arrive at

$$\| r^{(1)} - r^{(2)} \|_{\{C^0[0,\ 1]\}^2} \tag{2.87}$$
$$\leq 2M_4M_5M_7\delta_0\| q^{(1)} - q^{(2)} \|_{\{C^0[0,\ 1]\}^2} .$$

Therefore, if δ_0 is sufficiently small so that

$$\varkappa = 2M_4M_5M_7\delta_0 < 1 , \tag{2.88}$$

then we see that G is a contraction mapping.

This shows the assertion (I).

Proof of (II). Let $q = (q_1, q_2) \in \mathscr{A}_M$. Since c_{ij} $(i \le i,j \le 2)$ are real-valued functions by (2.27), the solution K to (2.36) - (2.38) with $D = C$ is also real-valued. Therefore we see that Gq is real-valued.

Let us assume that $\delta_0 = \sum_{n=-\infty}^{\infty} (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$ is sufficiently small. Let $(r_1, r_2) = Gq$. Then we have to prove that

$$
(2.89) \quad \begin{cases} \| r_1 - p_1 \|_{C^0[0,\ 1]} \le M \\ \| r_2 - p_2 \|_{C^0[0,\ 1]} \le M \end{cases} .
$$

Firstly, by the definition (2.55) of G_4 and the inequality (2.49) of Lemma 6, we see

$$
(2.90) \quad \max \{ \| r_1 - p_1 \|_{C^0[0,\ 1]}, \| r_2 - p_2 \|_{C^0[0,\ 1]} \}
$$
$$
\le M_7 \max_{\substack{1 \le i,j \le 2 \\ 0 \leqq x \leqq 1}} |K_{ij}(x,x,P,Q,C)| .
$$

Here we put $P = \begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix}$ and $Q = \begin{pmatrix} a & b \\ q_1 & q_2 \end{pmatrix}$.

Secondly, by the definition (2.42) of G_3 and the inequality (2.39) of Lemma 5, we have

$$
(2.91) \quad \| K(\cdot,\cdot,P,Q,C) \|_{\{C^0(\bar{\Omega})\}^4}
$$
$$
\le M_5 \| C(\cdot,q) \|_{\{C^0[0,\ 1]\}^4} .
$$

Finally, by the definition (2.32) of G_2 and the estimate (2.29) of Lemma 4, we have

(2.92) $\quad \| C(\cdot,q) \|_{\{C^0[0,\ 1]\}^4} \le M_4\delta_0$.

Therefore, combining (2.90), (2.91) and (2.92), we reach

(2.93) $\quad \max \{ \| r_1 - p_1 \|_{C^0[0,\ 1]}, \| r_2 - p_2 \|_{C^0[0,\ 1]} \}$

$\le M_4M_5M_7\delta_0$.

Thus, taking a sufficiently small δ_0 , so that

(2.94) $\quad M_4M_5M_7\delta_0 \le M$,

we see (2.89).

Therefore we complete the proof of the assertions (I) and (II), so that Proposition 2 is proved.

§3. **Proof of Theorem 2.** We consider a proof separately in the two cases :

Case 1. $\delta = \sum_{n=-\infty}^{\infty} (|n| + 1)(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|) < \infty$.

Case 2. $\delta = \infty$.

Proof in Case 1. Let $Q \in A(a,b)$, J, $J^* \in \mathbb{R} \setminus \{-1, 1\}$ and let

$$(3.1) \qquad \begin{cases} \sigma(A_{Q,h,J}) = \{ \lambda_n^* \}_{n\in\mathbb{Z}} \\ \sigma(A_{Q,h,J^*}) = \{ \mu_n^* \}_{n\in\mathbb{Z}} \end{cases} .$$

If

$$(3.2) \qquad \delta_0 = \sum_{n=-\infty}^{\infty} (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$$

is sufficiently small for P, h, H and H^*, then in virtue of Proposition 2, there exist $\mathfrak{Q} = \begin{pmatrix} a & b \\ \tilde{q}_1 & \tilde{q}_2 \end{pmatrix} \in A(a,b)$ and $\mathfrak{J}$, $\mathfrak{J}^* \in \mathbb{R} \setminus \{-1, 1\}$ such that

$$(3.1)' \qquad \begin{cases} \sigma(A_{\mathfrak{Q},h,\mathfrak{J}}) = \{ \lambda_n^* \}_{n\in\mathbb{Z}} \\ \sigma(A_{\mathfrak{Q},h,\mathfrak{J}^*}) = \{ \mu_n^* \}_{n\in\mathbb{Z}} \end{cases} .$$

Moreover, in view of the estimate (2.93), we get

$$(3.3) \qquad \| \tilde{q}_1 - p_1 \|_{C^0[0,1]} + \| \tilde{q}_2 - p_2 \|_{C^0[0,1]} \le M_8 \delta_0 ,$$

and, combining the estimates (2.30), (2.40) and (2.50), and proceeding in a way analogous to the one in getting (3.3), we can obtain

$$(3.4)\qquad \| \tilde{q}_1 - p_1 \|_{C^1[0,1]} + \| \tilde{q}_2 - p_2 \|_{C^1[0,1]} \le M_8 \delta \quad .$$

On the other hand, since $\mathfrak{J}$ and $\mathfrak{J}^*$ are given by (2.20) and (2.21), by noting (2.12), it follows from (3.3) that

$$(3.5)\qquad |H - \mathfrak{J}| + |H^* - \mathfrak{J}^*| \le M_8 \delta_0 \quad .$$

Now, by Theorem 1 in §1, the relations (3.1) and (3.1)' imply

$$(3.6)\qquad \begin{cases} \tilde{q}_1(x) = q_1(x), \quad \tilde{q}_2(x) = q_2(x) & (0 \le x \le 1) \\ \mathfrak{J} = J \ , \quad \mathfrak{J}^* = J^* & . \end{cases}$$

Thus we can obtain (1.18) and (1.19) by (3.3) - (3.6). This completes the proof of Theorem 2 in Case 1.

Proof in Case 2. In this case, we have to prove

$$|H - J| + |H^* - J^*| \leq M_8\delta_0 \tag{3.7}$$

and

$$\| q_1 - p_1 \|_{C^0[0,1]} + \| q_2 - p_2 \|_{C^0[0,1]} \leq M_8\delta_0 . \tag{3.8}$$

Without loss of generality, we may assume that $\delta_0 < \infty$. Along the line of the proof of Proposition 2, we can carry out the proof as follows.

1) In this step, we derive

$$\left\{\begin{aligned} J &= \left((H + 1)\exp(\int_0^1 (q_2(s) - p_2(s))ds) + H - 1 \right) \\ &\quad \times \left((H + 1)\exp(\int_0^1 (q_2(s) - p_2(s))ds) + 1 - H \right)^{-1} \\ J^* &= \left((H^* + 1)\exp(\int_0^1 (q_2(s) - p_2(s))ds) + H^* - 1 \right) \\ &\quad \times \left((H^* + 1)\exp(\int_0^1 (q_2(s) - p_2(s))ds) + 1 - H^* \right)^{-1} . \end{aligned}\right. \tag{3.9}$$

Derivation of (3.9). Since $\sum_{n=-\infty}^{\infty} (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|) < \infty$, we have $\lim_{|n| \to \infty} (\lambda_n^* - \lambda_n) = 0$ and $\lim_{|n| \to \infty} (\mu_n^* - \mu_n) = 0$, which imply

$$\lim_{|n| \to \infty} (\theta + \gamma_1 + (n - m_1(n))\pi\sqrt{-1}) = 0 \tag{3.10}$$

and

$$\lim_{|n| \to \infty} (\theta + \gamma_2 + (n - m_2(n))\pi\sqrt{-1}) = 0 \tag{3.11}$$

in view of Proposition 0 (II) in Chapter 2.

Here we set

$$(3.12) \qquad \theta = \frac{1}{2}\int_0^1 (q_2(s) - p_2(s))ds$$

and

$$(3.13) \qquad \begin{cases} \gamma_1 = \frac{1}{2}\left\{ \log \frac{(1+h)(1-J)}{(1-h)(1+J)} - \log \frac{(1+h)(1-H)}{(1-h)(1+H)} \right\} \\ \gamma_2 = \frac{1}{2}\left\{ \log \frac{(1+h)(1-J^*)}{(1-h)(1+J^*)} - \log \frac{(1+h)(1-H^*)}{(1-h)(1+H^*)} \right\} \end{cases}$$

and $m_j(n)$ $(j = 1,2)$ denote some strictly increasing sequences of integers, and moreover, in (3.13), we take the principal value of the logarithm.

Since $h, H, H^*, J, J^* \in \mathbb{R} \setminus \{-1, 1\}$, and $\theta \in \mathbb{R}$, we have

$$\mathrm{Im}\ (\theta + \gamma_1) = 0 \quad \text{or} \quad \tfrac{1}{2}\pi, \quad \text{or} \quad -\tfrac{1}{2}\pi$$

$$\mathrm{Im}\ (\theta + \gamma_2) = 0 \quad \text{or} \quad \tfrac{1}{2}\pi, \quad \text{or} \quad -\tfrac{1}{2}\pi \quad .$$

Therefore (3.10) and (3.11) imply $\theta + \gamma_1 = \theta + \gamma_2 = 0$, that is, we reach (3.9).

2) In this step, we prove Lemma 9, which is a converse of Lemma 7 :

Lemma 9. *Let*

$$\delta_0 = \sum_{n=-\infty}^{\infty} (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$$

be sufficiently small. If $q = (q_1, q_2) \in \{C^1[0, 1]\}^2$ *satisfies*

$$(3.14) \qquad \begin{cases} \sigma(A_{Q,h,J}) = \{ \lambda_n^* \}_{n\in\mathbb{Z}} \\ \sigma(A_{Q,h,J^*}) = \{ \mu_n^* \}_{n\in\mathbb{Z}} \end{cases} ,$$

then q *is a fixed point of the mapping* G *defined in §2.*

Proof of Lemma 9. By Lemma 1 (I) in Chapter 2, there exists a unique $K = K(x,y) \in \{C^1(\overline{\Omega})\}^4$ satisfying (3.15) - (3.18) :

$$(3.15) \qquad B\,\frac{\partial K(x,y)}{\partial x} + Q(x)K(x,y) - K(x,y)P(y) = -\frac{\partial K(x,y)}{\partial y}\,B \qquad ((x,y) \in \overline{\Omega}) .$$

$$(3.16) \qquad \begin{cases} K_{12}(x,0) = hK_{11}(x,0) \\ K_{22}(x,0) = hK_{21}(x,0) \qquad (0 \le x \le 1) . \end{cases}$$

$$(3.17) \qquad \begin{aligned} &K_{12}(x,x) - K_{21}(x,x) \\ &= \frac{1}{4}\, e^{-\theta_1(x)-\theta_2(x)}(2b(x) - q_1(x) - q_2(x) - p_1(x) + p_2(x)) \\ &+ \frac{1}{4}\, e^{-\theta_1(x)+\theta_2(x)}(-2b(x) + q_1(x) - q_2(x) + p_1(x) + p_2(x)) \\ &\qquad (0 \le x \le 1) . \end{aligned}$$

$$(3.18) \qquad \begin{aligned} &K_{11}(x,x) - K_{22}(x,x) \\ &= \frac{1}{4}\, e^{-\theta_1(x)-\theta_2(x)}(2a(x) - q_1(x) - q_2(x) + p_1(x) - p_2(x)) \\ &+ \frac{1}{4}\, e^{-\theta_1(x)+\theta_2(x)}(-2a(x) - q_1(x) + q_2(x) + p_1(x) + p_2(x)) \\ &\qquad (0 \le x \le 1) . \end{aligned}$$

Here we recall that θ_1, θ_2 and R are defined by (2.65), (2.66) and (2.67), respectively.

Then, $\psi(\cdot,\lambda) = \begin{pmatrix} \psi_1(\cdot,\lambda) \\ \psi_2(\cdot,\lambda) \end{pmatrix}$ defined by (2.68) satisfies

$$\begin{cases} B\,\dfrac{d\psi(x,\lambda)}{dx} + Q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda) & (0 \leq x \leq 1) \\ \psi(0,\lambda) = \begin{pmatrix} 1 \\ -h \end{pmatrix} & , \end{cases}$$

in view of Lemma 1 (II) in Chapter 2.

Therefore, since $\sigma(A_{Q,h,J}) = \{ \lambda_n^* \}_{n\in\mathbb{Z}}$ and λ_n^* is a simple eigenvalue, we see that $\psi(\cdot,\lambda_n^*)$ is an eigenvector of $A_{Q,h,J}$ associated with λ_n^*, so that we get

$$(3.19) \qquad \psi_2(1,\lambda_n^*) + J\psi_1(1,\lambda_n^*) = 0 \qquad (n \in \mathbb{Z}) \quad .$$

Similarly we can get

$$(3.20) \qquad \psi_2(1,\mu_n^*) + J^*\psi_1(1,\mu_n^*) = 0 \qquad (n \in \mathbb{Z}) \quad .$$

Now, substituting (2.68) into (3.19) and (3.20), we obtain

$$(3.21) \qquad \left(\begin{pmatrix} (K_{21} + JK_{11})(1,\cdot) \\ (K_{22} + JK_{12})(1,\cdot) \end{pmatrix} , \phi(\cdot,\lambda_n^*) \right)_{\{L^2(0,\ 1)\}^2} = a_n(q\ ) \qquad (n \in \mathbb{Z})$$

and

$$(3.22) \qquad \left(\begin{pmatrix} (K_{21} + J^*K_{11})(1,\cdot) \\ (K_{22} + J^*K_{12})(1,\cdot) \end{pmatrix} , \phi(\cdot,\mu_n^*) \right)_{\{L^2(0,\ 1)\}^2} = b_n(q\ ) \qquad (n \in \mathbb{Z}) \quad .$$

Since $c_{ij}(\cdot,q)$ $(1 \leq i,j \leq 2)$ are given by (2.25) and (2.26), the equalities (3.21) and (3.22) imply

$$(3.23) \qquad K_{ij}(1,y) = c_{ij}(y,q) \qquad (1 \le i,j \le 2,\ 0 \le y \le 1),$$

by direct computations. Here we note that since δ_0 is sufficiently small, Lemma 2 holds true.

Considering the problem (3.23) with (3.15) and (3.16), in view of the uniqueness of solutions to the problem (Lemma 5), we see that

$$(3.24) \qquad K = (G_3 \circ G_2 \circ G_1)(q) \quad .$$

Here we recall that G_1, G_2 and G_3 are defined by (2.11), (2.32) and (2.42), respectively.

As is seen by direct computations, the equalities (3.17) and (3.18) imply the following : $\begin{pmatrix} u \\ v \end{pmatrix}$ given by

$$(3.25) \qquad \begin{cases} u(x) = \exp\left(\frac{1}{2} \int_0^x (p_1(s)+p_2(s)-q_1(s)-q_2(s))ds \right) \\ v(x) = \exp\left(\frac{1}{2} \int_0^x (p_1(s)-p_2(s)-q_1(s)+q_2(s))ds \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad (0 \le x \le 1) \end{cases}$$

is the solution to

$$(3.26) \qquad \frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = A(x)\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} + \begin{pmatrix} K_{11}(x,x)-K_{22}(x,x)+K_{12}(x,x)-K_{21}(x,x) \\ K_{11}(x,x)-K_{22}(x,x)+K_{21}(x,x)-K_{12}(x,x) \end{pmatrix} \qquad (0 \le x \le 1)$$

and

$$(3.27) \qquad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Here let us recall that $A(x)$ is defined by (2.43).

By (3.25), we get

$$(3.28) \qquad \begin{cases} q_1(x) = p_1(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} - \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \\ q_2(x) = p_2(x) - \dfrac{1}{u(x)} \dfrac{du(x)}{dx} + \dfrac{1}{v(x)} \dfrac{dv(x)}{dx} \end{cases} \qquad (0 \leq x \leq 1) .$$

Therefore we get

$$(3.29) \qquad q = (q_1, q_2) = G_4 K .$$

Here G_4 is defined by (2.55).

The relations (3.24) and (3.29) imply that $q = (q_1, q_2)$ is a fixed point of $G = G_4 \circ G_3 \circ G_2 \circ G_1$.

Thus we complete the proof of Lemma 9.

3) By the final stage of the proof of Proposition 2, we see that if δ_0 is sufficiently small, then G possesses a unique fixed point $(\tilde{q}_1, \tilde{q}_2)$ and furthermore, the estimate

$$(3.30) \qquad \| \tilde{q}_1 - p_1 \|_{C^0} + \| \tilde{q}_2 - p_2 \|_{C^0} \leq M_8 \delta_0$$

holds.

Lemma 9 and the uniqueness of fixed points imply $\tilde{q}_1 = q_1$ and $\tilde{q}_2 = q_2$, so that we obtain (1.19) for $j = 0$.

On the other hand, the estimate (1.19) for $j = 1$ is trivial by $\delta = \infty$.

In a way similar to Case 1, we can prove (3.7) by using (3.8).

Thus we complete the proof of Theorem 2.

Appendix I. Proof of Proposition 1. In view of Proposition 0 in Chapter 2, we see that for $\lambda_n \in \sigma(A_{P,h,H})$, we have

$$(I.1) \qquad \lambda_n = \gamma + \theta_0 + n\pi\sqrt{-1} + O\left(\frac{1}{n}\right) \quad (\text{as } |n| \to \infty) .$$

Here we set

$$\gamma = \frac{1}{2} \log \frac{(1+h)(1-H)}{(1-h)(1+H)}$$

(the principal value of the logarithm)

and

$$\theta_0 = \frac{1}{2}\int_0^1 (a(s) + p_2(s))ds \quad .$$

The relation (I.1) implies $\lambda_n \notin \mathbb{R}$ for sufficiently large $|n|$, so that we can immediately see the part (I) of Proposition 1.

Now we proceed to a proof of the part (II) of this proposition. Let us assume that $\lambda_n \in \sigma(A_{P,h,H})$, and let $\phi(\cdot,\lambda_n) = \begin{pmatrix} \phi_1(\cdot,\lambda_n) \\ \phi_2(\cdot,\lambda_n) \end{pmatrix}$ be an eigenvector of $A_{P,h,H}$ associated with λ_n. That is, we have

$$(I.2) \quad \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \dfrac{d\phi(x,\lambda_n)}{dx} + P(x)\phi(x,\lambda_n) = \lambda_n \cdot \phi(x,\lambda_n) \\ \qquad\qquad\qquad\qquad\qquad\qquad (0 \le x \le 1) \\ \phi_2(0,\lambda_n) + h\cdot\phi_1(0,\lambda_n) = 0 \\ \phi_2(1,\lambda_n) + H\cdot\phi_1(1,\lambda_n) = 0 \end{cases} .$$

Since P is real-valued and $h, H \in \mathbb{R}$, the equalities (I.2) are equivalent to

$$
\text{(I.2)}' \quad \left\{
\begin{aligned}
&\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d\overline{\phi(x,\lambda_n)}}{dx} + P(x)\overline{\phi(x,\lambda_n)} = \overline{\lambda}_n \cdot \overline{\phi(x,\lambda_n)} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 \le x \le 1) \\
&\overline{\phi_2(0,\lambda_n)} + h \cdot \overline{\phi_1(0,\lambda_n)} = 0 \\
&\overline{\phi_2(1,\lambda_n)} + H \cdot \overline{\phi_1(1,\lambda_n)} = 0
\end{aligned}
\right. ,
$$

which implies that $\overline{\phi(\cdot,\lambda_n)}$ is an eigenvector of $A_{P,h,H}$ associated with $\overline{\lambda_n}$. That is, we see that $\lambda_n \in \sigma(A_{P,h,H})$ implies $\overline{\lambda_n} \in \sigma(A_{P,h,H})$.

Similarly we can show that $\overline{\lambda_n} \in \sigma(A_{P,h,H})$ implies $\lambda_n \in \sigma(A_{P,h,H})$.

Thus we complete the proof of Proposition 1.

Appendix II. Proof of Lemma 1. First, we show the following Lemmas II.1 and II.2, which are useful also in Appendix III.

Let us recall that

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Omega = \{\ (x,y)\ ;\ 0 < y < x < 1\ \}\ .$$

Lemma II.1. *(I) For given* $P = \begin{pmatrix} a & b \\ p_1 & p_2 \end{pmatrix} \in \{C^1[0,\ 1]\}^4$ *and* $h \in \mathbb{R} \setminus \{-1,\ 1\}$ *, there exists a unique* $U = U(x,y) = (\ U_{ij}(x,y)\ )_{1 \le i,j \le 2} \in \{C^1(\overline{\Omega})\}^4$ *satisfying (II.1) - (II.4) :*

$$\text{(II.1)} \qquad B\,\frac{\partial U(x,y)}{\partial x} + P(x)U(x,y) = -\,\frac{\partial U(x,y)}{\partial y}\,B \qquad (\ (x,y) \in \overline{\Omega}\ )\ .$$

$$\text{(II.2)} \qquad \begin{cases} U_{12}(x,0) = hU_{11}(x,0) \\ \\ U_{22}(x,0) = hU_{21}(x,0) \qquad (0 \le x \le 1)\ . \end{cases}$$

$$\text{(II.3)} \qquad U_{12}(x,x) - U_{21}(x,x) =$$

$$\frac{1}{4}\exp(-\eta_1(x)-\eta_2(x))(a(x)+b(x)-p_1(x)-p_2(x)) +$$

$$\frac{1}{4}\exp(-\eta_1(x)+\eta_2(x))(a(x)-b(x)+p_1(x)-p_2(x)) \quad (0 \le x \le 1).$$

(II.4) $$U_{11}(x,x) - U_{22}(x,x) =$$

$$\frac{1}{4} \exp(-\eta_1(x)-\eta_2(x))(a(x)+b(x)-p_1(x)-p_2(x)) +$$

$$\frac{1}{4} \exp(-\eta_1(x)+\eta_2(x))(-a(x)+b(x)-p_1(x)+p_2(x))$$

$$(0 \leq x \leq 1) .$$

Here and henceforth, we put

(II.5) $$\begin{cases} \eta_1(x) = \frac{1}{2} \int_0^x (b(s) + p_1(s))ds \\ \\ \eta_2(x) = \frac{1}{2} \int_0^x (a(s) + p_2(s))ds \quad (0 \leq x \leq 1). \end{cases}$$

(II) For the solution $U = (U_{ij})_{1 \leq i,j \leq 2}$ *, we have the estimates*

(II.6) $$\| U \|_{\{C^0(\bar{\Omega})\}^4} \leq M_9(\| P \|_{\{C^0[0,\ 1]\}^4} , h) ,$$

and

(II.7) $$\| U \|_{\{C^1(\bar{\Omega})\}^4} \leq M_{10}(\| P \|_{\{C^1[0,\ 1]\}^4} , h) .$$

(III) For $\lambda \in \mathbb{C}$ *, let us set*

(II.8) $$f(x,\lambda) = \begin{pmatrix} f_1(x,\lambda) \\ f_2(x,\lambda) \end{pmatrix} = \begin{pmatrix} \cosh \lambda x - h \sinh \lambda x \\ \sinh \lambda x - h \cosh \lambda x \end{pmatrix} ,$$

and

$$(\text{II.9}) \qquad S(x) = e^{-\eta_1(x)} \begin{pmatrix} \cosh \eta_2(x) & -\sinh \eta_2(x) \\ -\sinh \eta_2(x) & \cosh \eta_2(x) \end{pmatrix} \qquad (0 \le x \le 1) .$$

Then $\phi(\cdot,\lambda) = \begin{pmatrix} \phi_1(\cdot,\lambda) \\ \phi_2(\cdot.\lambda) \end{pmatrix}$ *defined by*

$$(\text{II.10}) \qquad \phi(x,\lambda) = S(x)f(x,\lambda) + \int_0^x U(x,y)f(y,\lambda)dy \qquad (0 \le x \le 1)$$

belongs to $\{C^1[0,\ 1]\}^2$ *and furthermore satisfies*

$$(\text{II.11}) \qquad B\,\frac{d\phi(x,\lambda)}{dx} + P(x)\phi(x,\lambda) = \lambda\phi(x,\lambda) \qquad (0 \le x \le 1)$$

and

$$(\text{II.12}) \qquad \phi(0,\lambda) = \begin{pmatrix} 1 \\ -h \end{pmatrix} .$$

In particular, $\phi(\cdot,\lambda)$ *is nothing but the function given by (2.7).*

Similarly the following facts hold :

(I)' There exists a unique $V = V(x,y) = (\ V_{ij}(x,y)\ )_{1 \le i,j \le 2} \in \{C^1(\bar{\Omega})\}^4$ *satisfying (II.1)' - (II.4)' :*

$$(\text{II.1})' \qquad B\,\frac{\partial V(x,y)}{\partial x} - {}^tP(x)V(x,y) = -\frac{\partial V(x,y)}{\partial y}B \qquad (\ (x,y) \in \bar{\Omega}\ ) .$$

$$(\text{II.2})' \begin{cases} V_{12}(x,0) = -hV_{11}(x,0) \\ V_{22}(x,0) = -hV_{21}(x,0) \qquad (0 \le x \le 1) . \end{cases}$$

(II.3)' $V_{12}(x,x) - V_{21}(x,x) =$

$$\frac{1}{4}\exp(\eta_1(x)+\eta_2(x))(-a(x)+b(x)-p_1(x)+p_2(x)) +$$

$$\frac{1}{4}\exp(\eta_1(x)-\eta_2(x))(-a(x)-b(x)+p_1(x)+p_2(x))$$

$$(0 \le x \le 1) .$$

(II.4)' $V_{11}(x,x) - V_{22}(x,x) =$

$$\frac{1}{4}\exp(\eta_1(x)+\eta_2(x))(-a(x)+b(x)-p_1(x)+p_2(x)) +$$

$$\frac{1}{4}\exp(\eta_1(x)-\eta_2(x))(a(x)+b(x)-p_1(x)-p_2(x))$$

$$(0 \le x \le 1) .$$

(II)' The estimates

(II.6)' $$\| V \|_{\{C^0(\bar{\Omega})\}^4} \le M_9(\| P \|_{\{C^0[0,\ 1]\}^4} , h) ,$$

and

(II.7)' $$\| V \|_{\{C^1(\bar{\Omega})\}^4} \le M_{10}(\| P \|_{\{C^1[0,\ 1]\}^4} , h)$$

hold.

(III)' Let us set

(II.8)' $$f^*(x,\lambda) = \begin{pmatrix} f_1^*(x,\lambda) \\ f_2^*(x,\lambda) \end{pmatrix} = \begin{pmatrix} \cosh \lambda x + h \sinh \lambda x \\ \sinh \lambda x + h \cosh \lambda x \end{pmatrix} ,$$

and

(II.9)' $$T(x) = e^{\eta_1(x)} \begin{pmatrix} \cosh \eta_2(x) & \sinh \eta_2(x) \\ \sinh \eta_2(x) & \cosh \eta_2(x) \end{pmatrix}$$

$$(0 \le x \le 1) .$$

Then $\phi^*(\cdot,\lambda) = \begin{pmatrix} \phi_1^*(\cdot,\lambda) \\ \phi_2^*(\cdot,\lambda) \end{pmatrix}$ *defined by*

$$\text{(II.10)}' \quad \phi^*(x,\lambda) = T(x)f^*(x,\lambda) + \int_0^x V(x,y)f^*(y,\lambda)dy \qquad (0 \leq x \leq 1)$$

belongs to $\{C^1[0, 1]\}^2$ *and furthermore satisfies*

$$\text{(II.11)}' \quad B\frac{d\phi^*(x,\lambda)}{dx} - {}^tP(x)\phi^*(x,\lambda) = \lambda\phi^*(x,\lambda) \quad (0 \leq x \leq 1) ,$$

and

$$\text{(II.12)}' \quad \phi^*(0,\lambda) = \begin{pmatrix} 1 \\ h \end{pmatrix} .$$

In particular, $\phi^*(\cdot,\lambda)$ *is nothing but the function given by (2.8).*

Proof of Lemma II.1. The parts (I) and (III) follow directly from the parts (I) and (II) of Lemma 1 in Chapter 2, respectively.

On the other hand, we can show the estimate (II.6) by means of the inequalities for the iterative approximate solutions for (II.1) - (II.4) derived in the course of the proof of Proposition 1 in Chapter 2. (See Appendix I to Chapter 2.) Similarly the estimate (II.7) can be obtained and so, we omit the detail.

Lemma II.2. *For* $n \in \mathbb{Z}$, *we have*

$$\text{(II.13)} \qquad \| \phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n) \|_{\{C^0[0,\ 1]\}^2} \le M_{11}|\lambda_n^* - \lambda_n|$$

and

$$\text{(II.14)} \qquad \| \phi(\cdot,\mu_n^*) - \phi(\cdot,\mu_n) \|_{\{C^0[0,\ 1]\}^2} \le M_{11}|\mu_n^* - \mu_n| \quad .$$

Here M_{11} *is a positive constant depending on* $\| P \|_{\{C^0[0,\ 1]\}^4}$, h, H, H^*, δ_0 , *and moreover* M_{11} *remains bounded as* δ_0 *is bounded.*

Proof of Lemma II.2. First we prove

$$\text{(II.15)} \qquad \| f(\cdot,\lambda_n^*) - f(\cdot,\lambda_n) \|_{\{C^0[0,\ 1]\}^2} \le M_{11}'|\lambda_n^* - \lambda_n| \qquad (n \in \mathbb{Z})$$

for a positive constant M_{11}' with a property similar to M_{11}. To this end, we have only to prove

$$\text{(II.16)} \qquad \left| e^{\lambda_n^* x} - e^{\lambda_n x} \right| \le M_{11}' \, |\lambda_n^* - \lambda_n| \qquad (n \in \mathbb{Z}) \quad .$$

By the mean value theorem, we get

$$\text{(II.17)} \qquad \left| e^{\lambda_n^* x} - e^{\lambda_n x} \right| \le 2 \max_{0 \le t \le 1} \left| e^{t\lambda_n^* x + (1-t)\lambda_n x} \right| . \times |\lambda_n^* - \lambda_n| \ .$$

On the other hand, we can obtain

$$\text{(II.18)} \qquad \lambda_n = \gamma_H + n\pi\sqrt{-1} + \frac{\alpha_n}{n} \qquad (\text{as } |n| \to \infty),$$

where $\alpha_n \in \mathbb{C}$ $(n \in \mathbb{Z})$ satisfy

$$\text{(II.19)} \qquad \alpha \equiv \sup_{n\in\mathbb{Z}} |\alpha_n| < \infty$$

and $\gamma_H = \frac{1}{2} \log \frac{(1 + h)(1 - H)}{(1 - h)(1 + H)}$ (the principal value of the logarithm) by Proposition 0 in Chapter 2.

Furthermore, by $\delta_0 = \sum_{n=-\infty}^{\infty} (|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$, we note

$$\text{(II.20)} \qquad |\lambda_n^* - \lambda_n| \le \delta_0 \qquad (n \in \mathbb{Z}) \qquad .$$

From (II.18) - (II.20), we can see $\left| e^{t\lambda_n^* x + (1-t)\lambda_n x} \right| \le \exp(2|\gamma_H| + 2\alpha + \delta_0)$ $(0 \le t, x \le 1)$, and therefore, we see (II.16) by (II.17).

Now, by applying the estimates (II.15) and (II.6) in (II.10), we reach (II.13), the conclusion.

Similarly we can prove (II.14).

Thus Lemma II.2 is proved.

Now we return to the proof of Lemma 1. Since $\lambda_n \in \sigma(A_{P,h,H})$, we have

$$\phi_2(1,\lambda_n) + H\phi_1(1,\lambda_n) = 0 \qquad (n \in \mathbb{Z}) \qquad .$$

Therefore we get, for $n \in \mathbb{Z}$,

$$|\phi_2(1,\lambda_n^*) + H\phi_1(1,\lambda_n^*)|$$

$$= |\phi_2(1,\lambda_n^*) - \phi_2(1,\lambda_n) + H(\phi_1(1,\lambda_n^*) - \phi_1(1,\lambda_n))|$$

$$\leq |\phi_2(1,\lambda_n^*) - \phi_2(1,\lambda_n)| + |H||\phi_1(1,\lambda_n^*) - \phi_1(1,\lambda_n)|$$

$$\leq (1 + |H|)\| \phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n) \|_{\{C^0[0,\ 1]\}^2} .$$

Thus Lemma II.2 implies

$$\text{(II.21)} \qquad |\phi_2(1,\lambda_n^*) + H\phi_1(1,\lambda_n^*)| \leq M_{11}|\lambda_n^* - \lambda_n| \qquad (n \in \mathbb{Z}) .$$

Similarly we can get

$$\text{(II.22)} \qquad |\phi_2(1,\mu_n^*) + H^*\phi_1(1,\mu_n^*)| \leq M_{11}|\mu_n^* - \mu_n| \qquad (n \in \mathbb{Z}).$$

Noting the inequalities (2.12) and recalling that

$\delta_0 = \sum_{n=-\infty}^{\infty}(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$ and

$\delta = \sum_{n=-\infty}^{\infty}(|n| + 1)(|\lambda_n^* - \lambda_n| + |\mu_n^* - \mu_n|)$, we see that the inequalities (II.21) and (II.22) imply the estimates (2.13) and (2.14), the conclusion of Lemma 1.

Appendix III. Proof of Lemma 2.

Proof of the part (I). A theorem on perturbation of Riesz bases by K.Bari (Gohberg and Kreĭn [6]) is a key. That is, in order to prove that $\{\phi(\cdot,\lambda_n^*)\}_{n\in\mathbb{Z}}$ is a Riesz basis in $\{L^2(0, 1)\}^2$, we have to show the following two facts :

(III.1) $$\sum_{n=-\infty}^{\infty} \| \phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n) \|^2_{\{L^2(0,\ 1)\}^2} < \infty .$$

(III.2) $$\sum_{n=-\infty}^{\infty} c_n\phi(x,\lambda_n^*) = 0 \quad \text{almost everywhere in } [0,\ 1]$$

implies $c_n = 0 \quad (n \in \mathbb{Z})$.

Proof of (III.1). In view of (II.13) of Lemma II.2 in Appendix II, we get

$$\sum_{n=-\infty}^{\infty} \| \phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n) \|^2_{\{L^2(0,\ 1)\}^2}$$

$$\leq \sum_{n=-\infty}^{\infty} \| \phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n) \|^2_{\{C^0(0,\ 1)\}^2}$$

$$\leq M_{11}^2 \sum_{n=-\infty}^{\infty} |\lambda_n^* - \lambda_n|^2 \leq M_{11}^2 \left(\sum_{n=-\infty}^{\infty} |\lambda_n^* - \lambda_n| \right)^2 \leq M_{11}^2 \delta_0^2 .$$

This proves (III.1).

Proof of (III.2). Let us assume that

$$\sum_{n=-\infty}^{\infty} c_n\phi(x,\lambda_n^*) = 0 \quad \text{almost everywhere in } [0,\ 1] .$$

Then we have

$$(\text{III.3}) \qquad - \sum_{n=-\infty}^{\infty} c_n \phi(x, \lambda_n) = \sum_{n=-\infty}^{\infty} c_n (\phi(x, \lambda_n^*) - \phi(x, \lambda_n))$$

almost everywhere in $[0, 1]$.

On the other hand, since $\{ \phi(\cdot, \lambda_n) \}_{n \in \mathbb{Z}}$ is a Riesz basis in $\{L^2(0, 1)\}^2$ by Proposition 0 in Chapter 2, there exists some positive constant M_{12} such that

$$(\text{III.4}) \qquad M_{12} \| \sum_{n=-\infty}^{\infty} c_n \phi(x, \lambda_n) \|_{\{L^2(0, 1)\}^2} \geq \left(\sum_{n=-\infty}^{\infty} |c_n|^2 \right)^{1/2} .$$

Applying (III.4) in (III.3), we have

$$M_{12}^{-1} \left(\sum_{n=-\infty}^{\infty} |c_n|^2 \right)^{1/2} \leq \| \sum_{n=-\infty}^{\infty} c_n \phi(\cdot, \lambda_n) \|_{\{L^2(0, 1)\}^2}$$

$$= \| \sum_{n=-\infty}^{\infty} c_n \, (\phi(\cdot, \lambda_n^*) - \phi(\cdot, \lambda_n)) \|_{\{L^2(0, 1)\}^2}$$

$$\leq \sum_{n=-\infty}^{\infty} |c_n| \| \phi(\cdot, \lambda_n^*) - \phi(\cdot, \lambda_n) \|_{\{C^0[0, 1]\}^2}$$

$$\leq M_{11} \sum_{n=-\infty}^{\infty} |c_n| |\lambda_n^* - \lambda_n| \qquad \text{(by (II.13))}$$

$$\leq M_{11} \left(\sum_{n=-\infty}^{\infty} |c_n|^2 \right)^{1/2} \left(\sum_{n=-}^{\infty} |\lambda_n^* - \lambda_n|^2 \right)^{1/2}$$

(by Schwarz's inequality)

$$\leq M_{11} \delta_0 \left(\sum_{n=-\infty}^{\infty} |c_n|^2 \right)^{1/2} .$$

Therefore if δ_0 is so small that $M_{11} M_{12} \delta_0 < 1$, then we see $\left(\sum_{n=-\infty}^{\infty} |c_n|^2 \right)^{1/2} = 0$, which implies (III.2).

Proof of Part (II). We divide the proof into the following five steps.

1) In this step, we prove

Lemma III.1. *Let us set*

$$\text{(III.5)} \qquad \rho_n = \left(\phi(\cdot,\lambda_n),\ \phi^*(\cdot,\overline{-\lambda_n}) \right)_{\{L^2(0,1)\}^2} \qquad (n \in \mathbb{Z}).$$

Then we have

$$\text{(III.6)} \qquad \rho_n = 1 - h^2 + O\left(\frac{1}{n} \right) \qquad (\text{as } |n| \to \infty),$$

and

$$\text{(III.7)} \qquad |\rho_n| \geq M_{13}$$

for some positive constant M_{13} .

Proof of Lemma III.1. First we note that $\phi(\cdot,\lambda_n)$ and $\phi^*(\cdot,\overline{-\lambda_n})$ are given by (II.10) and (II.10)', respectively.

In (II.10), by integration by parts, we can get

$$\text{(III.8)} \qquad \int_0^x (U_{i1}(x,y)f_1(y,\lambda_n) + U_{i2}(x,y)f_2(y,\lambda_n))dy$$

$$= \frac{1}{2} \int_0^x \left((1 - h)(U_{i1}(x,y) + U_{i2}(x,y))e^{\lambda_n y} + (1 + h)(U_{i1}(x,y) - U_{i2}(x,y))e^{-\lambda_n y} \right)dy$$

$$= \frac{1}{\lambda_n} d_n^{(i)}(x) \qquad (i = 1,\ 2,\ \ n \in \mathbb{Z}) .$$

Here and henceforth we set

(III.9) $d_n^{(i)}(x)$

$$= \frac{1}{2}\Bigg((1-h)(U_{i1}(x,x)+U_{i2}(x,x))e^{\lambda_n x} -$$

$$(1+h)(U_{i1}(x,x)-U_{i2}(x,x))e^{-\lambda_n x} -$$

$$(1-h)(U_{i1}(x,0)+U_{i2}(x,0)) + (1+h)(U_{i1}(x,0)-U_{i2}(x,0))$$

$$+\int_0^x\Big((1+h)\Big(\frac{\partial U_{i1}(x,y)}{\partial y}-\frac{\partial U_{i2}(x,y)}{\partial y}\Big)e^{-\lambda_n y} -$$

$$(1-h)\Big(\frac{\partial U_{i1}(x,y)}{\partial y}+\frac{\partial U_{i2}(x,y)}{\partial y}\Big)e^{\lambda_n y}\Big)dy\Bigg) .$$

Similarly we can get

(III.10) $$\int_0^x (V_{i1}(x,y)f_1^*(y,\overline{-\lambda_n}) + V_{i2}(x,y)f_2^*(y,\overline{-\lambda_n}))dy$$

$$= \frac{1}{\overline{\lambda_n}}\, e_n^{(i)}(x) \qquad (i=1,\ 2,\ \ n\in\mathbb{Z}) ,$$

where

(III.11) $e_n^{(i)}(x)$

$$= \frac{1}{2}\Bigg((1-h)(V_{i1}(x,x)-V_{i2}(x,x))e^{\overline{\lambda_n}x} -$$

$$(1+h)(V_{i1}(x,x)+V_{i2}(x,x))e^{-\overline{\lambda_n}x} -$$

$$(1-h)(V_{i1}(x,0)-V_{i2}(x,0)) + (1+h)(V_{i1}(x,0)+V_{i2}(x,0))$$

$$+\int_0^x\Big((1+h)\Big(\frac{\partial V_{i1}(x,y)}{\partial y}+\frac{\partial V_{i2}(x,y)}{\partial y}\Big)e^{-\overline{\lambda_n}y}$$

$$-(1-h)\Big(\frac{\partial V_{i1}(x,y)}{\partial y}-\frac{\partial V_{i2}(x,y)}{\partial y}\Big)e^{\overline{\lambda_n}y}\Big)dy\Bigg) .$$

Substituting (III.8) and (III.10) into (II.10) and (II.10)', respectively, we have

$$\text{(III.12)} \quad \int_0^1 {}^t\phi(x,\lambda_n)\overline{\phi^*(x,\overline{-\lambda_n})}dx$$

$$= 1 - h^2 + \frac{1}{\lambda_n}\int_0^1 c_{1,n}(x)dx + \frac{1}{\lambda_n^2}\int_0^1 c_{2,n}(x)dx ,$$

where

$$\text{(III.13)} \quad c_{1,n}(x)$$

$$= e^{-\eta_1(x)}(f_1(x,\lambda_n)\cosh \eta_2(x)-f_2(x,\lambda_n)\sinh \eta_2(x))\overline{e_n^{(1)}(x)}$$

$$+ e^{-\eta_1(x)}(-f_1(x,\lambda_n)\sinh \eta_2(x)+f_2(x,\lambda_n)\cosh \eta_2(x))\overline{e_n^{(2)}(x)}$$

$$+ e^{\eta_1(x)}(f_1^*(x,-\lambda_n)\cosh \eta_2(x)+f_2^*(x,-\lambda_n)\sinh \eta_2(x))d_n^{(1)}(x)$$

$$+ e^{\eta_1(x)}(f_1^*(x,-\lambda_n)\sinh \eta_2(x)+f_2^*(x,-\lambda_n)\cosh \eta_2(x))d_n^{(2)}(x)$$

$$(0 \le x \le 1)$$

and

$$\text{(III.14)} \quad c_{2,n}(x) = d_n^{(1)}(x)\overline{e_n^{(1)}(x)} + d_n^{(2)}(x)\overline{e_n^{(2)}(x)} .$$

On the other hand, in view of the asymptotic behavior of λ_n (Proposition 0 in Chapter 2), we have

$$\text{(III.15)} \quad \sup_{n\in\mathbb{Z}} \|f(\cdot,\lambda_n)\|_{\{C^0[0,1]\}^2} < \infty \quad \text{and}$$

$$\sup_{n\in\mathbb{Z}} \|f^*(\cdot,\overline{-\lambda_n})\|_{\{C^0[0,1]\}^2} < \infty \quad .$$

Therefore, by means of the estimates (II.6), (II.7), (II.6)' and (II.7)' in Lemma II.1, and the asymptotic behavior of λ_n , we see

$$\sup_{n\in\mathbb{Z}} \| d_n^{(i)}(\cdot) \|_{C^0[0,1]} < \infty$$

$$\sup_{n\in\mathbb{Z}} \| e_n^{(i)}(\cdot) \|_{C^0[0,1]} < \infty \qquad (i = 1,\ 2) \ ,$$

that is,

$$\text{(III.16)} \quad \sup_{n\in\mathbb{Z}} | c_{i,n}(\cdot) |_{C^0[0,1]} < \infty \qquad (i = 1,\ 2) \quad .$$

Again, in view of the asymptotic behavior of λ_n , we have

$$\text{(III.17)} \quad \frac{1}{\lambda_n} = O(\ \frac{1}{n}\ ) \qquad (\text{as} \quad |n| \to \infty) \quad .$$

Applying (III.16) and (III.17) in (III.12), we reach

$$\int_0^1 {}^t\phi(x,\lambda_n)\overline{\phi^*(\cdot,\overline{-\lambda_n})}dx = 1 - h^2 + O(\ \frac{1}{n}\ ) \quad ,$$

which proves (III.6).

Next we proceed to a proof of (III.7). To this end, we note

$$\text{(III.18)} \quad (\phi(\cdot,\lambda_n),\ \phi^*(\cdot,\overline{-\lambda_n})) \neq 0 \qquad (n \in \mathbb{Z}) \ .$$

In fact, assume that

$$\text{(III.19)} \quad (\phi(\cdot,\lambda_{n_0}),\ \phi^*(\cdot,\overline{-\lambda_{n_0}})) = 0 \quad \text{for some} \quad n_0 \in \mathbb{Z} \ .$$

Then, by means of the definition (II.10) and (II.10)' of $\phi(\cdot,\lambda)$ and $\phi^*(\cdot,\lambda)$, and $\lambda_n \in \sigma(A_{P,h,H})$, $\overline{\lambda_{n_0}} \in \sigma(A^*_{P,h,H})$, we see

$$\lambda_m(\phi(\cdot,\lambda_m),\ \phi^*(\cdot,\overline{-\lambda_{n_0}}))$$
$$= (A_{P,h,H}\phi(\cdot,\lambda_m),\ \phi^*(\cdot,\overline{-\lambda_{n_0}})) = (\phi(\cdot,\lambda_m),\ A^*_{P,h,H}\phi^*(\cdot,\overline{-\lambda_{n_0}}))$$
$$= (\phi(\cdot,\lambda_m),\ \overline{\lambda_{n_0}}\ \phi^*(\cdot,\overline{-\lambda_{n_0}})) ,$$

so that we get

$$\text{(III.20)} \quad (\phi(\cdot,\lambda_m),\ \phi^*(\cdot,\overline{-\lambda_{n_0}})) = 0$$

for each $m \neq n_0$. Combining (III.19) with (III.20), we have

$$\text{(III.21)} \quad (\phi(\cdot,\lambda_m),\ \phi^*(\cdot,\overline{-\lambda_{n_0}})) = 0$$

for each $m \in \mathbb{Z}$.

Since $\{\ \phi(\cdot,\lambda_m)\ \}_{m\in\mathbb{Z}}$ is a Riesz basis in $\{L^2(0,1)\}^2$, the equalities (III.21) $(m \in \mathbb{Z})$ imply $\phi^*(\cdot,\overline{-\lambda_{n_0}}) = 0$, which is a contradiction.

Thus we see (III.18).

We return to the proof of (III.7). The asymptotic behavior (III.6) and $|h| \neq 1$ imply that

$$\text{(III.22)} \quad |\rho_n| \geq M'_{13} > 0$$

for each $|n| \geq N_0$, where N_0 is sufficiently large.

Since $M''_{13} = \min\limits_{|n|<N_0} |\rho_n| > 0$ by (III.18), we have only to set $M_{13} = \min\{\ M'_{13}\ ,\ M''_{13}\ \}$.

This completes the proof of Lemma III.1.

2) In this step, we show

Lemma III.2. *For* $n \in \mathbb{Z}$, *we have*

(III.23) $\| \phi(\cdot,\lambda_n) \|_{\{C^0[0,1]\}^2} \le M_{14}$

(III.24) $\| \phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n) \|_{\{C^0[0,1]\}^2} \le M_{14}|\lambda_n^* - \lambda_n|$

(III.25) $\| \phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n) \|_{\{C^1[0,1]\}^2}$

$\le M_{14}(|n| + 1)|\lambda_n^* - \lambda_n|$

(III.26) $\| \phi^*(\cdot,-\lambda_n) \|_{\{C^0[0,1]\}^2} \le M_{14}$

and

(III.27) $\| \phi^*(\cdot,-\lambda_n) \|_{\{C^1[0,1]\}^2} \le M_{14}(|n| + 1)$.

Proof of (III.23). First we recall (II.10). Then we immediately get

$\| \phi(\cdot,\lambda_n) \|_{\{C^0[0,1]\}^2}$

$\le 2(\| S \|_{\{C^0[0,1]\}^4} + \| K \|_{\{C^0(\bar{\Omega})\}^4})\| f(\cdot,\lambda_n) \|_{\{C^0[0,1]\}^2}$

($n \in \mathbb{Z}$), and therefore

(III.28) $\|\phi(\cdot,\lambda_n)\|_{\{C^0[0,1]\}^2} \le M_{14}\|f(\cdot,\lambda_n)\|_{\{C^0[0,1]\}^2}$ ($n \in \mathbb{Z}$),

in view of the definition (II.9) of $S(x)$ and the estimate (II.6). Combining (III.28) with (III.15), we obtain (III.23).

Proof of (III.24) and (III.25). The inequality (III.24) has been proved in Lemma II.2.

Since $\phi(\cdot,\lambda)$ satisfies the differential equations in (2.7), we see

$$
\text{(III.29)}\begin{cases}
\frac{d}{dx}(\phi_1(x,\lambda_n^*) - \phi_1(x,\lambda_n)) \\
= -p_1(x)(\phi_1(x,\lambda_n^*)-\phi_1(x,\lambda_n))-p_2(x)(\phi_2(x,\lambda_n^*)-\phi_2(x,\lambda_n)) \\
+ \lambda_n^*(\phi_2(x,\lambda_n^*)-\phi_2(x,\lambda_n)) + (\lambda_n^* - \lambda_n)\phi_2(x,\lambda_n) \\
\frac{d}{dx}(\phi_2(x,\lambda_n^*) - \phi_2(x,\lambda_n)) \\
= -a(x)(\phi_1(x,\lambda_n^*)-\phi_1(x,\lambda_n))-b(x)(\phi_2(x,\lambda_n^*)-\phi_2(x,\lambda_n)) \\
+ \lambda_n^*(\phi_1(x,\lambda_n^*)-\phi_1(x,\lambda_n)) + (\lambda_n^* - \lambda_n)\phi_1(x,\lambda_n) \\
\qquad\qquad\qquad\qquad\qquad\qquad (n \in \mathbb{Z}) \quad .
\end{cases}
$$

Furthermore λ_n^* $(n \in \mathbb{Z})$ have the forms

$$
\text{(III.30)} \quad \lambda_n^* = \beta_n + n\pi\sqrt{-1} + \frac{\alpha_n}{n} \qquad ,
$$

where

$$
\text{(III.31)}\begin{cases}
\sup_{n\in\mathbb{Z}} |\alpha_n| < \infty \\
\beta_n = \gamma_H + \lambda_n^* - \lambda_n \\
\sup_{n\in\mathbb{Z}} |\beta_n| \le \gamma_H + \delta_0 \qquad ,
\end{cases}
$$

as is seen by the asymptotic behavior of (II.18) of λ_n and (II.20).

Applying (III.23), (III.24) and (III.30) at the right hand side of (III.29), we get

$$\left\| \frac{d}{dx} \begin{pmatrix} \phi_1(\cdot,\lambda_n^*) - \phi_1(\cdot,\lambda_n) \\ \phi_2(\cdot,\lambda_n^*) - \phi_2(\cdot,\lambda_n) \end{pmatrix} \right\|_{\{C^0[0,1]\}^2}$$

$$\leq M_{14}(|n| + 1)|\lambda_n^* - \lambda_n| \qquad (n \in \mathbb{Z}) ,$$

from which we see (III.25).

Proof of (III.26) and (III.27). We can prove (III.26) by a way analogous with the one in the proof of (III.23), noting (II.10)' and (II.6)'.

Next we proceed to a proof of (III.27). Since $\phi^*(x,-\lambda_n)$ satisfies (II.11)', the equalities

$$\text{(III.32)} \begin{cases} \dfrac{d\phi_1^*(x,-\lambda_n)}{dx} = b(x)\phi_1^*(x,-\lambda_n)+p_2(x)\phi_2^*(x,-\lambda_n)-\lambda_n\phi_2^*(x,-\lambda_n) \\ \\ \dfrac{d\phi_2^*(x,-\lambda_n)}{dx} = a(x)\phi_1^*(x,-\lambda_n)+p_1(x)\phi_2^*(x,-\lambda_n)-\lambda_n\phi_2^*(x,-\lambda_n) \\ \qquad\qquad\qquad\qquad\qquad\qquad (n \in \mathbb{Z}) \end{cases}$$

hold. In (III.32), we apply (III.26) and the asymptotic behavior (II.18) of λ_n , so that we have

$$\left\| \frac{d}{dx} \begin{pmatrix} \phi_1^*(\cdot,-\lambda_n) \\ \phi_2^*(\cdot,-\lambda_n) \end{pmatrix} \right\|_{\{C^0[0,1]\}^2} \leq M_{14}(|n| + 1) \quad .$$

This completes the proof of (III.27), and so, Lemma III.2 is proved.

3) We set

$$\text{(III.33)} \quad Z(x,y) = (Z_{ij}(x,y))_{1 \le i,j \le 2}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\phi(x,\lambda_n^*) - \phi(x,\lambda_n)}{\rho_n} \cdot {}^t\phi^*(y,-\lambda_n)$$

$$((x,y) \in [0, 1]^2) .$$

Then $Z(x,y)$ is well-defined and has the following properties :

$$\text{(III.34)} \quad Z(\cdot,\cdot) \in \{C^1([0,1]^2)\}^4 \quad .$$

$$\text{(III.35)} \quad Z(x,y) : \text{real-valued.}$$

$$\text{(III.36)} \quad \phi(x,\lambda_n^*) = \phi(x,\lambda_n) + \int_0^1 Z(x,y)\phi(y,\lambda_n)dy$$

$$(n \in \mathbb{Z},\ 0 \le x \le 1) \quad .$$

The purpose of this step is to verify (III.34), (III.35) and (III.36).

First by (III.7) of Lemma III.1, we see that

$$\text{(III.37)} \quad \left| \frac{1}{\rho_n} \right| \le \frac{1}{M_{13}} \quad .$$

Verification of (III.34). From (III.25), (III.27) and (III.37), we have

$$\left\| \frac{(\phi(\cdot,\lambda_n^*) - \phi(\cdot,\lambda_n)) \ {}^t\phi^*(\cdot,-\lambda_n)}{\rho_n} \right\|_{\{C^1([0,1]^2)\}^4}$$

$$= \max \left(\max_{\substack{1\le i,j\le 2 \\ 0\le x,y\le 1}} \left| \frac{(\phi_i(x,\lambda_n^*) - \phi_i(x,\lambda_n)) \ {}^t\phi_j^*(y,-\lambda_n)}{\rho_n} \right| ,\right.$$

$$\max_{\substack{1\le i,j\le 2 \\ 0\le x,y\le 1}} \left| \frac{d}{dx} \left(\frac{\phi_i(x,\lambda_n^*) - \phi_i(x,\lambda_n)}{\rho_n} \right) \cdot {}^t\phi_j^*(y,-\lambda_n) \right| ,$$

$$\max_{\substack{1 \leq i,j \leq 2 \\ 0 \leqq x,y \leqq 1}} \left| \frac{\phi_i(x,\lambda_n^*) - \phi_i(x,\lambda_n)}{\rho_n} \; \frac{d\; {}^t\phi_j^*(y,-\lambda_n)}{dy} \right| \Bigg)$$

$$\leq M_{14}^2 M_{13}^{-1} (|n| + 1)|\lambda_n^* - \lambda_n| \qquad (n \in \mathbb{Z}) \quad .$$

Thus the relation $\sum_{n=-\infty}^{\infty} (|n| + 1)|\lambda_n^* - \lambda_n| \leq \delta < \infty$ means that the majorant series for the right hand side of (III.33) is convergent. This proves (III.34)

Verification of (III.35). We prepare Lemma III.3, which is useful later.

Lemma III.3. *The equalities*

$$\text{(III.38)} \quad \overline{\phi(x,\lambda)} = \phi(x,\overline{\lambda})$$

and

$$\text{(III.39)} \quad \overline{\phi^*(x,\lambda)} = \phi^*(x,\overline{\lambda})$$

hold for $\lambda \in \mathbb{C}$.

Proof of Lemma III.3. Let us recall (II.11) and (II.12). That is, $\phi(x,\lambda)$ satisfies

$$\begin{cases} B \dfrac{d\phi(x,\lambda)}{dx} + P(x)\phi(x,\lambda) = \lambda\phi(x,\lambda) & (0 \leq x \leq 1) \\ \phi(0,\lambda) = \begin{pmatrix} 1 \\ -h \end{pmatrix} & . \end{cases}$$

Since $P(x)$ is real-valued and $h \in \mathbb{R}$, we have

$$\text{(III.40)} \begin{cases} B \dfrac{d\overline{\phi(x,\lambda)}}{dx} + P(x)\overline{\phi(x,\lambda)} = \overline{\lambda}\; \overline{\phi(x,\lambda)} & (0 \leq x \leq 1) \\ \overline{\phi(0,\lambda)} = \begin{pmatrix} 1 \\ -h \end{pmatrix} & . \end{cases}$$

On the other hand, by the definition, also $\phi(x,\overline{\lambda})$ satisfies (III.40). Therefore the uniqueness of solutions to the initial value problem (III.40) means (III.38).

Similarly we can prove (III.39).

Now we return to the verification of (III.35). By Lemma III.3, we have

$$\text{(III.41)} \quad \overline{\phi(\cdot,\lambda_n^*)} = \phi(\cdot,\overline{\lambda_n^*}) \qquad (n \geq N_1 + 1) ,$$

$$\text{(III.42)} \quad \overline{\phi(\cdot,\lambda_n)} = \phi(\cdot,\overline{\lambda_n}) \qquad (n \geq N_1 + 1) ,$$

and

$$\text{(III.43)} \quad \overline{\phi^*(\cdot,-\lambda_n)} = \phi^*(\cdot,\overline{-\lambda_n}) \qquad (n \geq N_1 + 1) .$$

Since $\overline{\lambda_n} = \lambda_{-n}$ and $\overline{\lambda_n^*} = \lambda_{-n}^*$ $(n \geq N_1 + 1)$ by the assumptions (1.14.2) and (1.20.2), the equalities (III.41) - (III.43) imply

$$\text{(III.41)'} \quad \overline{\phi(\cdot,\lambda_n^*)} = \phi(\cdot,\lambda_{-n}^*) \qquad (n \geq N_1 + 1) ,$$

$$\text{(III.42)'} \quad \overline{\phi(\cdot,\lambda_n)} = \phi(\cdot,\lambda_{-n}) \qquad (n \geq N_1 + 1) ,$$

and

$$\text{(III.43)'} \quad \overline{\phi^*(\cdot,-\lambda_n)} = \phi^*(\cdot,-\lambda_{-n}) \qquad (n \geq N_1 + 1) ,$$

respectively. Further since $\rho_n = (\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_n}))$, we see by (III.42)' and (III.43)' that

$$\text{(III.44)} \quad \overline{\rho_n} = \rho_{-n} \qquad (n \geq N_1 + 1) .$$

On the other hand, it follows from (1.14.1) and (1.20.1) that $\phi(x,\lambda_n^*)$, $\phi(x,\lambda_n)$ and $\phi^*(x,-\lambda_n)$ $(-N_1 \leq n \leq N_1)$ are real-valued, and, again by $\lambda_n \in \mathbb{R}$ $(-N_1 \leq n \leq N_1)$, we see that $\rho_n \in \mathbb{R}$ $(-N_1 \leq n \leq N_1)$.

Since the series at the right hand side of (III.33) is absolutely convergent as is seen in the course of the verification of (III.34), we have

$$Z(x,y) = \sum_{n=-N_1}^{N_1} \frac{(\phi(x,\lambda_n^*) - \phi(x,\lambda_n))\ {}^t\phi^*(y,-\lambda_n)}{\rho_n}$$

$$+ \lim_{N\to\infty} \sum_{n=N_1+1}^{N} \left(\frac{(\phi(x,\lambda_n^*) - \phi(x,\lambda_n))\ {}^t\phi^*(y,-\lambda_n)}{\rho_n} + \frac{\overline{(\phi(x,\lambda_n^*) - \phi(x,\lambda_n))}\ \overline{{}^t\phi^*(y,-\lambda_n)}}{\overline{\rho_n}} \right)$$

(by (III.41)' - (III.43)' and (III.44)),

from which we can verify (III.35), noting also that the functions $\dfrac{(\phi(x,\lambda_n^*) - \phi(x,\lambda_n))\ {}^t\phi^*(y,-\lambda_n)}{\rho_n}$ are real-valued for $-N_1 \leq n \leq N_1$.

Verification of (III.36). We have

$$\int_0^1 Z(x,y)\phi(y,\lambda_n)dy$$

$$= \sum_{m=-\infty}^{\infty} \frac{(\phi(x,\lambda_m^*) - \phi(x,\lambda_m))}{\rho_m} \int_0^1 {}^t\phi^*(y,-\lambda_m)\phi(y,\lambda_n)dy$$

(because the right hand side of (III.33) is uniformly convergent in $\{C^1([0,1]^2)\}^4$.)

$$= (\phi(x,\lambda_n^*) - \phi(x,\lambda_n))\left(\frac{1}{\rho_n} \int_0^1 {}^t\phi^*(y,-\lambda_n)\phi(y,\lambda_n)dy \right) \quad (n \in \mathbb{Z}).$$

Here we use the equalities

$$\int_0^1 {}^t\phi^*(y,-\lambda_m)\phi(y,\lambda_n)dy = (\ \phi(\cdot,\lambda_n),\ \phi^*(x,\overline{-\lambda_m})\ )_{\{L^2(0,1)\}^2}$$

$$= 0 \qquad (n \neq m)$$

which are nothing but (III.20).

Noting (III.5), we obtain

$$\text{(III.45)} \quad \int_0^1 Z(x,y)\phi(y,\lambda_n)dy = \phi(x,\lambda_n^*) - \phi(x,\lambda_n)$$

$$(n \in \mathbb{Z},\ 0 \le x \le 1) .$$

This proves (III.36).

4) Let us define an operator F from $\{C^0[0,1]\}^2$ into itself by

$$\text{(III.46)} \quad (Fu)(x) = \int_0^1 {}^tZ(y,x)u(y)dy \qquad (0 \le x \le 1) .$$

Henceforth, $\mathfrak{L}(X)$ denotes the set of bounded linear operators defined on a Banach space X to itself.

The purpose of this step is to show that $(1 + F)^{-1} \in \mathfrak{L}(\{C^1[0,1]\}^2)$. To this end, we prove the facts :

(III.47) F *is a compact operator in* $\{C^1[0,1]\}^2$.

(III.48) -1 *is not an eigenvalue of* F .

Verification of (III.47). Let us consider any sequence $\{u_n\}_{n\ge1} \subset \{C^1[0,1]\}^2$ such that

$$\text{(III.49)} \quad \| u_n \|_{\{C^1[0,1]\}^2} \le M_{15} .$$

Then we have to prove that $\{Fu_n\}_{n\ge1}$ contains a subsequence convergent in $\{C^1[0,1]\}^2$. To this end, by the Ascoli - Arzelà theorem, it is sufficient to verify

(III.50) $\quad \| Fu_n \|_{\{C^1[0,1]\}^2} \leq M'_{15} \qquad (n \geq 1)$

and

(III.51) $\quad \dfrac{d(Fu_n)(x)}{dx}$ is equi-continuous in n, that is,

$$\lim_{\varepsilon\to 0} \sup_{\substack{n\geq 1 \\ |x-x'|<\varepsilon}} \left| \frac{d(Fu_n)(x)}{dx} - \frac{d(Fu_n)(x')}{dx} \right| = 0 .$$

By ${}^tZ \in \{C^1([0,1]^2)\}^4$, we immediately see (III.50).

As for (III.51), we have

$$\frac{d(Fu_n)(x)}{dx} - \frac{d(Fu_n)(x')}{dx} = \int_0^1 \left(\frac{\partial\, {}^tZ(y,x)}{\partial x} - \frac{\partial\, {}^tZ(y,x')}{\partial x} \right) u_n(y)dy ,$$

and so

(III.52) $$\max_{i=1,2} \left| \frac{d(Fu_n)_i(x)}{dx} - \frac{d(Fu_n)_i(x')}{dx} \right|$$

$$\leq 2M_{15} \max_{\substack{1\leq i,j\leq 2 \\ 0 \leqq y \leqq 1}} \left| \frac{\partial\, {}^tZ_{ij}(y,x)}{\partial x} - \frac{\partial\, {}^tZ_{ij}(y,x')}{\partial x} \right|$$

$$(n \geq 1) ,$$

where we set $Fu_n = \begin{pmatrix} (Fu_n)_1 \\ (Fu_n)_2 \end{pmatrix}$.

Since $\dfrac{\partial\, {}^tZ(y,x)}{\partial x}$ is uniformly continuous in x by ${}^tZ \in \{C^1([0,1]^2)\}^4$, the left hand side of (III.52) tends to 0 uniformly in n as $|x - x'| \to 0$.

This proves (III.51) and therefore (III.47) is verified.

Verification of (III.48). Let us define an operator E from $\{L^2(0,1)\}^2$ into itself by

$$(\text{III.53}) \quad (Eu)(x) = \int_0^1 Z(x,y)u(y)dy \quad .$$

Obviously $E \in \mathfrak{L}(\{L^2(0,1)\}^2)$. Regarding F defined by (III.46) as an operator on $\{L^2(0,1)\}^2$, we can easily see that $E^* = F$, where E^* is the adjoint of $E \in \mathfrak{L}(\{L^2(0,1)\}^2)$. Therefore we have $\overline{\sigma(E)} = \sigma(F)$. Thus it is sufficient to show that

$$(\text{III.54}) \quad u + Eu = 0 \quad \text{implies} \quad u = 0 \quad ,$$

in order to verify (III.48).

Since $\{ \phi(\cdot,\lambda_n) \}_{n\in\mathbb{Z}}$ is a Riesz basis in $\{L^2(0,1)\}^2$ (Proposition 0 in Chapter 2), we get

$$(\text{III.55}) \quad u = \sum_{n=-\infty}^{\infty} c_n \phi(\cdot,\lambda_n)$$

for some $c_n \in \mathbb{C}$ $(n \in \mathbb{Z})$. Here the series in (III.55) is convergent in $\{L^2(0,1)\}^2$. Therefore we have

$$(\text{III.56}) \quad 0 = (1 + E)u = \sum_{n=-\infty}^{\infty} c_n (1 + E)\phi(\cdot,\lambda_n)$$

(by the boundedness of $1 + E$)

$$= \sum_{n=-\infty}^{\infty} c_n \phi(\cdot,\lambda_n^*) \qquad \text{(by (III.36))}.$$

Since as is proved in the part (I) of this lemma, $\{\phi(\cdot,\lambda_n^*)\}_{n\in\mathbb{Z}}$ is also a Riesz basis in $\{L^2(0,1)\}^2$, it follows from (III.56) that $c_n = 0$ $(n \in \mathbb{Z})$, which imply $u = 0$ by (III.55).

This proves (III.54).

Now by the Riesz-Schauder theorem (Yosida [58, p.283], for instance), these facts (III.47) and (III.48) imply

(III.57) $(1 + F)^{-1} \in \mathfrak{L}(\{C^1[0,1]\}^2)$.

Similarly we can prove that

(III.57)' $(1 + F)^{-1} \in \mathfrak{L}(\{C^0[0,1]\}^2)$.

5) Let us set

(III.58) $\psi_n^{(1)} = \frac{1}{\overline{\rho_n}} (1 + F)^{-1}\phi^*(\cdot, \overline{-\lambda_n}) \qquad (n \in \mathbb{Z}).$

Then we can verify that $\psi_n^{(1)}$ satisfies (2.15) - (2.19).

Verification of (2.15) and (2.17). In view of the estimates (III.26) and (III.27) in Lemma III.2 and (III.7), (III.57), (III.57)', we immediately see (2.15) and (2.17).

Verification of (2.16). Let us consider $\{C^1[0,1]\}^2$ as a real Banach space of all **real-valued** C^1-functions. Then, by (III.35), we can regard F as an operator from the real Banach space $\{C^1[0,1]\}^2$ to itself. Therefore $(1 + F)^{-1}u$ is real-valued for real-valued $u \in \{C^1[0,1]\}^2$. In particular, since $\rho_n \in \mathbb{R}$ and $\phi^*(x, \overline{-\lambda_n}) \in \mathbb{R}^2$ $(-N_1 \le n \le N_1,\ 0 \le x \le 1)$, also $\psi_n^{(1)}$ is real-valued for $-N_1 \le n \le N_1$.

On the other hand, since $\frac{1}{\overline{\rho_n}} \phi^*(\cdot, \overline{-\lambda_n}) + \frac{1}{\overline{\rho_{-n}}}\phi^*(\cdot, \overline{-\lambda_{-n}})$ is real-valued for $n \ge N_1 + 1$ in virtue of (III.43)' and (III.44), we see that $\psi_n^{(1)} + \psi_{-n}^{(1)} = (1 + F)^{-1}\left(\frac{1}{\overline{\rho_n}} \phi^*(\cdot, \overline{-\lambda_n}) + \frac{1}{\overline{\rho_{-n}}}\phi^*(\cdot, \overline{-\lambda_{-n}}) \right)$ $(n \ge N_1 + 1)$ is also real-valued and so, we get (2.16).

Verification of (2.18). Since we can easily see that $F^* = E$, we get

$$((1 + F)^{-1})^* = (1 + E)^{-1} , \tag{III.59}$$

(Kato [23, p.169], for example). Now we have

$$\begin{aligned}
&(\phi(\cdot,\lambda_n^*), \psi_m^{(1)})_{\{L^2(0,1)\}^2} \\
&= (\phi(\cdot,\lambda_n^*), \frac{1}{\overline{\rho_m}} (1 + F)^{-1}\phi^*(\cdot,\overline{-\lambda_m})) \\
&= \frac{1}{\rho_m}((1 + E)^{-1}\phi(\cdot,\lambda_n^*), \phi^*(\cdot,\overline{-\lambda_m})) \qquad \text{(by (III.59))} \\
&= \frac{1}{\rho_m}(\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_m})) \qquad \text{(by (III.53) and (III.36)).}
\end{aligned}$$

Therefore, since $\rho_n = (\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_n}))$ ((III.5)), in order to verify (2.18), we have only to prove

$$(\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_m})) = 0 \qquad (n \neq m) . \tag{III.60}$$

As is seen by the definition (2.7), (2.8) and $\lambda_n \in \sigma(A_{P,h,H})$, $\overline{\lambda_m} \in \sigma(A^*_{P,h,H})$, we have

$$\begin{aligned}
A_{P,h,H}\phi(\cdot,\lambda_n) &= \lambda_n\phi(\cdot,\lambda_n) \\
A^*_{P,h,H}\phi^*(\cdot,\overline{-\lambda_m}) &= \overline{\lambda_m}\phi^*(\cdot,\overline{-\lambda_m}) \quad .
\end{aligned}$$

Thus we get

$$\lambda_n(\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_m})) = (A_{P,h,H}\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_m}))$$

$= (\phi(\cdot,\lambda_n), A^*_{P,h,H}\phi^*(\cdot,\overline{-\lambda_m})) = \lambda_m(\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_m}))$, that is, $(\lambda_n - \lambda_m)(\phi(\cdot,\lambda_n), \phi^*(\cdot,\overline{-\lambda_m})) = 0$. Since $n \neq m$ implies $\lambda_n - \lambda_m \neq 0$, we obtain (III.60).

Verification of (2.19). Since $\{ \phi(\cdot,\lambda_n^*) \}_{n\in\mathbb{Z}}$ is a Riesz basis in $\{L^2(0,1)\}^2$ from the part (I) of Lemma 2, the biorthogonality (2.18) implies that $\{ \psi_n^{(1)} \}_{n\in\mathbb{Z}}$ is also a Riesz basis in $\{L^2(0,1)\}^2$ (Gohberg and Kreĭn [6, p.310]). Therefore, for each $u \in \{L^2(0,1)\}^2$, we have

$$\text{(III.61)} \quad u = \sum_{m=-\infty}^{\infty} c_m \overline{\psi_m^{(1)}}$$

for appropriate $c_m \in \mathbb{C}$ $(m \in \mathbb{Z})$. Here the series at the right hand side of (III.61) is convergent in $\{L^2(0,1)\}^2$.
Then we get

$$\begin{aligned}
(u, \overline{\phi(\cdot,\lambda_n^*)}) &= \sum_{m=-\infty}^{\infty} c_m (\overline{\psi_m^{(1)}}, \overline{\phi(\cdot,\lambda_n^*)}) \\
&= \sum_{m=-\infty}^{\infty} c_m \delta_{mn} \qquad \text{(by (2.18))} \\
&= c_n \quad .
\end{aligned}$$

This proves (2.19).

For the results (I)' and (II)' of Lemma 2, we can proceed similarly.

Thus we complete the proof of Lemma 2.

Appendix IV. Proof of Lemma 3. First we have

$$(H + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds \right) + 1 - H \neq 0 \quad .$$

in view of the inequalities (2.12). Consequently we see that $H^* \in \mathbb{R}$. Moreover, since $|H| \neq 1$, we can show by direct computations that $|J| \neq 1$. Thus we prove (2.22).
Similarly we can prove (2.23).
Moreover we have

$$\begin{aligned}
& J - J^* \\
&= 4\cdot\left((H + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds \right) + 1 - H \right)^{-1} \\
&\quad \cdot\left((H^* + 1)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds \right) + 1 - H^* \right)^{-1} \\
&\quad \cdot(H - H^*)\exp\left(\int_0^1 (q_2(s) - p_2(s))ds \right) \\
&\neq 0 \qquad\qquad (\text{by } H \neq H^*) ,
\end{aligned}$$

which is (2.24).

Thus we complete the proof of Lemma 3.

Appendix V. Proof of Lemma 4. (I) Proof of (2.27) - (2.30). We have

$$\| a_n(q)\overline{\psi_n^{(1)}}(\cdot) - b_n(q)\overline{\psi_n^{(2)}}(\cdot) \|_{\{C^0[0,1]\}^2}$$

$$\leq (|a_n(q)| + |b_n(q)|)\cdot\max\{ \|\psi_n^{(1)}\|_{\{C^0[0,1]\}^2}, \|\psi_n^{(2)}\|_{\{C^0[0,1]\}^2}\}$$

$$\leq M_3(|a_n(q)| + |b_n(q)|)$$

by (2.17) and (2.17)' in Lemma 2.

Therefore we get, for $0 \leq y \leq 1$,

$$\text{(V.1)} \qquad \sum_{n=-\infty}^{\infty} |a_n(q)\overline{\psi_n^{(1)}(y)} - b_n(q)\overline{\psi_n^{(2)}(y)}|$$

$$\leq M_3\cdot\sum_{n=-\infty}^{\infty}(|a_n(q)| + |b_n(q)|) \leq M_2M_3\delta_0$$

by (2.13) in Lemma 1.

On the other hand, since the equality

$$\frac{1}{J - J^*}$$

$$= \frac{1}{4}\left((H + 1)\exp\left(\int_0^1(q_2(s) - p_2(s))ds\right) + 1 - H\right)$$

$$\cdot\left((H^* + 1)\exp\left(\int_0^1(q_2(s) - p_2(s))ds\right) + 1 - H^*\right)$$

$$\cdot\exp\left(-\int_0^1(q_2(s) - p_2(s))ds\right)\cdot\frac{1}{H - H^*}$$

holds, we have

$$\text{(V.2)} \qquad \left|\frac{1}{J - J^*}\right|$$

$$\leq \frac{1}{4}(|H+1|e^M+|1-H|)(|H^*+1|e^M+|1-H^*|)e^M\cdot\left|\frac{1}{H - H^*}\right|$$

$$= M_{16} \qquad ,$$

noting that $\| q_2 - p_2 \|_{C^0[0,1]} \le M$.

Combining (V.1) with (V.2), we see that the inequalities (2.29) hold for c_{11} and c_{12} . Similarly we can show (2.29) for c_{21} and c_{22} .

In a manner analogous with the one in (V.1), we get

$$\text{(V.3)} \qquad \| \sum_{n=-\infty}^{\infty} (a_n(q)\overline{\psi_n^{(1)}(\cdot)} - b_n(q)\overline{\psi_n^{(2)}(\cdot)}) \|_{\{C^1[0,1]\}^2} \le M_2 M_3 \delta \qquad ,$$

in virtue of (2.17), (2.17)' and (2.14).

The inequalities (V.2) and (V.3) imply (2.28) and (2.30) for c_{11} and c_{12} . As to c_{21} and c_{22} , we can proceed similarly.

Next we have to prove (2.27). From (V.1), we see that the series at the right hand sides of (2.25) and (2.26) are absolutely convergent.

Therefore we can rewrite (2.25) as

$$\text{(V.4)} \qquad \begin{pmatrix} c_{11}(y,q) \\ c_{12}(y,q) \end{pmatrix} = \frac{1}{J-J^*} \sum_{n=-N_1}^{N_1} (a_n(q)\overline{\psi_n^{(1)}(y)} - b_n(q)\overline{\psi_n^{(2)}(y)})$$

$$+ \frac{1}{J-J^*} \lim_{N \to \infty} \sum_{n=N_1+1}^{N} \Big((a_n(q)\overline{\psi_n^{(1)}(y)} + a_{-n}(q)\overline{\psi_{-n}^{(1)}(y)})$$

$$- \; (b_n(q)\overline{\psi_n^{(2)}(y)} + b_{-n}(q)\overline{\psi_{-n}^{(2)}(y)} \;) \Big) \quad .$$

Applying (III.41)' in the definition (2.9) of $a_n(q)$, we have

(V.5) $$\overline{a_n(q)} = a_{-n}(q) \qquad (n \geq N_1 + 1) .$$

Similarly we can get

(V.5)' $$\overline{b_n(q)} = b_{-n}(q) \qquad (n \geq N_1 + 1) ,$$

and moreover

(V.6) $$a_n(q), \ b_n(q) \in \mathbb{R} \qquad (-N_1 \leq n \leq N_1) .$$

Using (V.5), (V.5)', (V.6) and (2.16), (2.16)' of Lemma 2, we conclude that c_{11} and c_{12} are real-valued functions.

As for c_{21} and c_{22} , we can proceed similarly, so that we complete the proof of (2.27).

Thus the part (I) of Lemma 4 is proved.

(II) Proof of (2.31). By (2.25), (2.26) and (2.9), (2.10), (II.21), (II.22), (V.2), in order to prove (2.31), we have only to show that

$$\text{(V.7)} \quad \Big| \; 2\exp\Big(\frac{1}{2} \int_0^1 (q_2^{(1)}(s) - p_2(s) + q_1^{(1)}(s) - p_1(s))ds \Big)$$

$$\times \left((H+1)\exp\Big(\int_0^1 (q_2^{(1)}(s) - p_2(s))ds \Big) + 1 - H \right)^{-1}$$

$$- 2\exp\Big(\frac{1}{2} \int_0^1 (q_2^{(2)}(s) - p_2(s) + q_1^{(2)} - p_1(s))ds \Big)$$

$$\times \left((H+1)\exp\Big(\int_0^1 (q_2^{(2)}(s) - p_2(s))ds \Big) + 1 - H \right)^{-1} \Big|$$

$$\le M_{17} \| q^{(1)} - q^{(2)} \|_{\{C^0[0,1]\}^2} ,$$

$$\text{(V.8)} \quad \left| \frac{1}{J(q^{(1)}) - J^*(q^{(1)})} - \frac{1}{J(q^{(2)}) - J^*(q^{(2)})} \right|$$

$$\le M_{17} \| q^{(1)} - q^{(2)} \|_{\{C^0[0,1]\}^2} ,$$

$$\text{(V.9)} \quad |J(q^{(1)}) - J(q^{(2)})|, \; |J^*(q^{(1)}) - J^*(q^{(2)})|$$

$$\le M_{17} \| q^{(1)} - q^{(2)} \|_{\{C^0[0,1]\}^2} ,$$

$$\text{(V.10)} \quad |J(q)|, \; |J^*(q)| \le M_{17}$$

for each $q^{(1)}, q^{(2)}, q \in \mathcal{A}_M$,

and the equality (V.7)' which is obtained by replacing H by H^* in (V.7).

Proof of (V.7). We have

$$\left| \text{[the left hand side of (V.7)]} \right|$$

$$\leq \Bigg| \left((H+1)\exp\left(\int_0^1 (q_2^{(1)}(s) - p_2(s))ds \right) + 1 - H \right)^{-1}$$

$$\times \left((H+1)\exp\left(\int_0^1 (q_2^{(2)}(s) - p_2(s))ds \right) + 1 - H \right)^{-1}$$

$$\times \Bigg(2(H+1)\exp\left(\frac{1}{2} \int_0^1 (q_2^{(1)}(s)+q_2^{(2)}(s)-p_1(s)-3p_2(s))ds\right)$$

$$\times \left\{ \exp\left(\frac{1}{2}\int_0^1 (q_1^{(1)}(s)+q_2^{(2)}(s))ds\right) - \exp\left(\frac{1}{2}\int_0^1 (q_1^{(2)}(s)+q_2^{(1)}(s))ds\right) \right\}$$

$$+ 2(1-H)\exp\left(-\frac{1}{2} \int_0^1 (p_1(s)+p_2(s))ds\right)$$

$$\times \left\{ \exp\left(\frac{1}{2}\int_0^1 (q_1^{(1)}(s)+q_2^{(1)}(s))ds\right) - \exp\left(\frac{1}{2}\int_0^1 (q_1^{(2)}(s)+q_2^{(2)}(s))ds\right) \right\} \Bigg) \Bigg|$$

$$\leq M_1^{-2} \Big(2|H+1|\exp(2M+\|p_1\|_{C^0}+\|p_2\|_{C^0}) + 2|1-H|\exp(M+\|p_1\|_{C^0}+\|p_2\|_{C^0}) \Big)$$
$$\times \| q^{(1)} - q^{(2)} \|_{\{C^0[0,1]\}^2} ,$$

by (2.12) and the mean value theorem for e^x .

This shows (V.7). Similarly we can prove (V.7)'.

Proof of (V.8). By (2.20) and (2.21), we have

$$\left| \frac{1}{J(q^{(1)}) - J^*(q^{(1)})} - \frac{1}{J(q^{(2)}) - J^*(q^{(2)})} \right|$$

$$= \frac{1}{4|H - H^*|} \Bigg| (H+1)(H^*+1)\exp\left(-\int_0^1 p_2(s)ds\right)$$

$$\times \left(\exp\left(\int_0^1 q_2^{(1)}(s)ds\right) - \exp\left(\int_0^1 q_2^{(2)}(s)ds\right) \right)$$

$$+ (H-1)(H^*-1)\exp\left(\int_0^1 p_2(s)ds\right)$$

$$\times \left(\exp\left(-\int_0^1 q_2^{(1)}(s)ds\right)-\exp\left(-\int_0^1 q_2^{(2)}(s)ds\right)\right) \Bigg|$$

$$\leq \frac{1}{4|H - H^*|} \Big(|(H+1)(H^*+1)|\exp(2\|p_2\|_{C^0}+M)\|q_2^{(1)} - q_2^{(2)}\|_{C^0}$$

$$+ |(H-1)(H^*-1)|\exp(2\|p_2\|_{C^0}+M)\|q_2^{(1)} - q_2^{(2)}\|_{C^0} \Big) ,$$

by the mean value theorem.

This shows (V.8).

Proof of (V.9). We have

$$|J(q^{(1)}) - J(q^{(2)})|$$

$$= \left|(H + 1)\exp\left(\int_0^1(q_2^{(1)}(s) - p_2(s))ds \right) + 1 - H \right|^{-1}$$

$$\times \left|(H + 1)\exp\left(\int_0^1(q_2^{(2)}(s) - p_2(s))ds \right) + 1 - H \right|^{-1}$$

$$\times 2\left|(1-H^2)\exp\left(-\int_0^1 p_2(s)ds\right)\left(\exp\left(\int_0^1 q_2^{(1)}(s)ds\right)-\exp\left(\int_0^1 q_2^{(2)}(s)ds\right)\right)\right|$$

$$\leq 2M_1^{-2}|1-H^2|\times\exp(2\|p_2\|_{C^0}+M)\|q_2^{(1)} - q_2^{(2)}\|_{C^0} ,$$

by (2.12) and the mean value theorem.

As for $|J^*(q^{(1)}) - J^*(q^{(2)})|$, we can similarly carry out a proof.

This completes the proof of (V.9).

Proof of (V.10). By using (2.12) in the definition (2.20) of $J(q)$, we get

$$|J(q)| \leq M_1^{-1} \left|(H + 1)\exp\left(\int_0^1(q_2(s) - p_2(s))ds\right) + 1 - H\right|$$

$$\leq M_1^{-1}(|H + 1|e^M + |1 - H|) \qquad ,$$

which implies (V.10) for $|J(q)|$.

As for $|J^*(q)|$, we can proceed similarly.

Thus we complete the proof of Lemma 4.

Appendix V1. Proof of Lemma 5. let us set

$$\Omega_1 = \{ (x,y) \; ; \; 1 - x < y < x, \; \tfrac{1}{2} < x < 1 \} \tag{VI.1}$$

and

$$\Omega_2 = \Omega \setminus \Omega_1 \setminus \{ (x,y) \; ; \; 1 - x = y \} \tag{VI.2}$$

As in Appendix I to Chapter 2, putting

$$\left\{ \begin{aligned} L_1(x,y) &= L_1(x,y,q_1,q_2,D) = K_{12}(x,y) - K_{21}(x,y) \\ L_2(x,y) &= L_2(x,y,q_1,q_2,D) = K_{11}(x,y) - K_{22}(x,y) \\ \\ L_3(x,y) &= L_3(x,y,q_1,q_2,D) = K_{11}(x,y) + K_{22}(x,y) \\ L_4(x,y) &= L_4(x,y,q_1,q_2,D) = K_{12}(x,y) + K_{21}(x,y) \end{aligned} \right. \qquad ((x,y) \in \overline{\Omega}) , \tag{VI.3}$$

we can rewrite (2.36) - (2.38), so that we obtain (VI.4) - (VI.7) :

$$\left\{ \begin{aligned} \frac{\partial L_1(x,y)}{\partial x} - \frac{\partial L_1(x,y)}{\partial y} &= f_1(x,y,L_1,L_2,L_3,L_4) \\ \frac{\partial L_2(x,y)}{\partial x} - \frac{\partial L_2(x,y)}{\partial y} &= f_2(x,y,L_1,L_2,L_3,L_4) \end{aligned} \right. \qquad ((x,y) \in \overline{\Omega}) , \tag{VI.4}$$

$$\left\{ \begin{aligned} \frac{\partial L_3(x,y)}{\partial x} + \frac{\partial L_3(x,y)}{\partial y} &= f_3(x,y,L_1,L_2,L_3,L_4) \\ \frac{\partial L_4(x,y)}{\partial x} + \frac{\partial L_4(x,y)}{\partial y} &= f_4(x,y,L_1,L_2,L_3,L_4) \end{aligned} \right. \qquad ((x,y) \in \overline{\Omega}) , \tag{VI.5}$$

$$\left\{ \begin{aligned} L_1(x,0) &= -kL_3(x,0) - \ell L_4(x,0) \\ \\ L_2(x,0) &= \ell L_3(x,0) + kL_4(x,0) \qquad (0 \le x \le 1) , \end{aligned} \right. \tag{VI.6}$$

and

(VI.7) $\qquad L_i(1,y) = r_i(y,D) \qquad\qquad (1 \leq i \leq 4,\ 0 \leq y \leq 1)$,

where we set (VI.8) - (VI.11) :

(VI.8) $\qquad f_i(x,y,L_1,L_2,L_3,L_4) = f_i(x,y,L_1,L_2,L_3,L_4,q_1,q_2)$

$$= \sum_{j=1}^{4} a_{ij}(x,y)L_j(x,y) \qquad ((x,y) \in \overline{\Omega}) .$$

$$\text{(VI.9)} \begin{cases} a_{11}(x,y) = a_{11}(x,y,q_1,q_2) = -\frac{1}{2}(b(y)+p_1(y)+b(x)+q_1(x)) \\ a_{12}(x,y) = a_{12}(x,y,q_1,q_2) = -\frac{1}{2}(a(y)+p_2(y)-a(x)-q_2(x)) \\ a_{13}(x,y) = a_{13}(x,y,q_1,q_2) = -\frac{1}{2}(a(y)-p_2(y)-a(x)+q_2(x)) \\ a_{14}(x,y) = a_{14}(x,y,q_1,q_2) = -\frac{1}{2}(-b(y)+p_1(y)-b(x)+q_1(x)) \\ a_{21}(x,y) = a_{21}(x,y,q_1,q_2) = a_{12}(x,y,q_1,q_2) \\ a_{22}(x,y) = a_{22}(x,y,q_1,q_2) = a_{11}(x,y,q_1,q_2) \\ a_{23}(x,y) = a_{23}(x,y,q_1,q_2) = \frac{1}{2}(-b(y)+p_1(y)+b(x)-q_1(x)) \\ a_{24}(x,y) = a_{24}(x,y,q_1,q_2) = -\frac{1}{2}(-a(y)+p_2(y)-a(x)+q_2(x)) \\ a_{31}(x,y) = a_{31}(x,y,q_1,q_2) = -a_{24}(x,y,q_1,q_2) \\ a_{32}(x,y) = a_{32}(x,y,q_1,q_2) = a_{14}(x,y,q_1,q_2) \\ a_{33}(x,y) = a_{33}(x,y,q_1,q_2) = \frac{1}{2}(b(y)+p_1(y)-b(x)-q_1(x)) \\ a_{34}(x,y) = a_{34}(x,y,q_1,q_2) = -a_{12}(x,y,q_1,q_2) \\ a_{41}(x,y) = a_{41}(x,y,q_1,q_2) = a_{23}(x,y,q_1,q_2) \\ a_{42}(x,y) = a_{42}(x,y,q_1,q_2) = -a_{13}(x,y,q_1,q_2) \\ a_{43}(x,y) = a_{43}(x,y,q_1,q_2) = -a_{12}(x,y,q_1,q_2) \\ a_{44}(x,y) = a_{44}(x,y,q_1,q_2) = a_{33}(x,y,q_1,q_2) \quad . \end{cases}$$

$$
\text{(VI.10)} \quad \begin{cases} k = \dfrac{-2h}{1 - h^2} \\ \\ \ell = \dfrac{1 + h^2}{1 - h^2} \end{cases} .
$$

$$
\text{(VI.11)} \quad \begin{cases} r_1(y) = r_1(y,D) = d_{12}(y) - d_{21}(y) \\ r_2(y) = r_2(y,D) = d_{11}(y) - d_{22}(y) \\ \\ r_3(y) = r_3(y,D) = d_{11}(y) + d_{22}(y) \\ r_4(y) = r_4(y,D) = d_{12}(y) + d_{21}(y) \qquad (0 \leq y \leq 1) \end{cases} .
$$

We will prove Lemma 5 in each of $\overline{\Omega}_1$ and $\overline{\Omega}_2$. In $\overline{\Omega}_1$, our problem (VI.4), (VI.5) and (VI.7) is a standard Cauchy problem and, as for the unique existence of solutions, we can refer to Petrovsky [40, pp.67 - 73] and Nagumo [36], for instance. However, in this lemma, we have to prove the estimates for solution , and so we will review the argument used in [40], for completeness.

Proof of Lemma 5 in $\overline{\Omega_1}$. First we prepare

Lemma VI.1. *Let* $f(x,y)$ *and* $\dfrac{\partial f(x,y)}{\partial y}$ *be continuous functions on* $\overline{\Omega_1}$ *and satisfy*

$$
\text{(VI.12)} \quad \begin{cases} |f(x,y)| \leq g(x) \\ \\ \left| \dfrac{\partial f(x,y)}{\partial y} \right| \leq h(x) \qquad ((x,y) \in \overline{\Omega_1}) \end{cases}
$$

for some $g,\ h \in C^0[\frac{1}{2},\ 1]$. *Then, for each* $a \in C^1[0,\ 1]$, *there exists a unique solution* $u \in C^1(\overline{\Omega_1})$ *to each of*

$$\text{(VI.13)} \quad \begin{cases} \dfrac{\partial u(x,y)}{\partial x} = \dfrac{\partial u(x,y)}{\partial y} + f(x,y) & (\ (x,y) \in \overline{\Omega_1}\ ) \\ \\ u(1,y) = a(y) & (0 \le y \le 1) \end{cases}$$

and

$$\text{(VI.13)}' \quad \begin{cases} \dfrac{\partial u(x,y)}{\partial x} = -\dfrac{\partial u(x,y)}{\partial y} + f(x,y) & (\ (x,y) \in \overline{\Omega_1}\ ) \\ \\ u(1,y) = a(y) & (0 \le y \le 1) \quad . \end{cases}$$

Moreover the solution to each of (VI.13) and (VI.13)' satisfies

$$\text{(VI.14)} \qquad |u(x,y)| \le \|\ a\ \|_{C^0} + \int_x^1 g(s)ds, \quad (\ (x,y) \in \overline{\Omega_1}\ )$$

and

$$\text{(VI.15)} \qquad \left| \frac{\partial u(x,y)}{\partial y} \right| \le \|\ a\ \|_{C^1} + \int_x^1 h(s)ds, \quad (\ (x,y) \in \overline{\Omega_1}\ ).$$

Proof of Lemma VI.1. Since the solutions u to (VI.13) and and v to (VI.13)' are represented in the forms

$$u(x,y) = a(x+y-1) + \int_1^x f(s,-s+x+y)ds \quad (\ (x,y) \in \overline{\Omega_1}\ )$$

and

$$v(x,y) = a(1-x+y) + \int_1^x f(s,s-x+y)ds \quad (\ (x,y) \in \overline{\Omega_1}\ )\ ,$$

respectively, we can immediately see this lemma.

In $\overline{\Omega_1}$, as is proved below, the solution L_i $(1 \le i \le 4)$ to (VI.4), (VI.5) and (VI.7) is given as the limits of uniformly convergent sequences $L_i^{(n)}$ $(1 \le i \le 4,\ n \ge 0)$ defined inductively by (VI.16) and (VI.17) :

$$\text{(VI.16)} \qquad L_i^{(0)}(x,y) = 0 \qquad (1 \le i \le 4,\ (x,y) \in \overline{\Omega_1}\ )$$

(VI.17) $\quad L_i^{(n)}$ $(1 \le i \le 4)$ is the solution to

$$\frac{\partial L_i^{(n)}(x,y)}{\partial x} + \delta_i \frac{\partial L_i^{(n)}(x,y)}{\partial y}$$

$$= f_i(x,y,L_1^{(n-1)},L_2^{(n-1)},L_3^{(n-1)},L_4^{(n-1)}) \quad (\ (x,y) \in \overline{\Omega_1}\ )$$

$$L_i^{(n)}(1,y) = r_i(y) \qquad (0 \le y \le 1) \qquad .$$

Here and henceforth we set

$$\text{(VI.18)} \qquad \delta_i = \begin{cases} -1, & \text{if } i = 1 \text{ or } 2 \\ 1, & \text{if } i = 3 \text{ or } 4. \end{cases}$$

Obviously the sequences $\{ L_i^{(n)} \}_{n \ge 0} \subset C^1(\overline{\Omega_1})$ $(1 \le i \le 4)$ are well-defined in view of Lemma VI.1. Furthermore for $1 \le i \le 4$, we can see the estimates

$$\text{(VI.19)} \qquad \left| L_i^{(n+1)}(x,y) - L_i^{(n)}(x,y) \right| \le \frac{M_{18}^n (1-x)^n}{n!} \cdot M_{19} \quad (n \ge 0)$$

$$\text{(VI.20)} \qquad \left| \frac{\partial L_i^{(n+1)}(x,y)}{\partial y} - \frac{\partial L_i^{(n)}(x,y)}{\partial y} \right| \le \frac{M_{18}^n (1-x)^n}{n!} \cdot M_{20} \quad (n \ge 0),$$

where we set

$$\text{(VI.21)} \quad \begin{cases} M_{18} = \max\limits_{1 \le i \le 4} \sum\limits_{j=1}^{4} \left(\| a_{ij} \|_{C^0(\overline{\Omega_1})} + \left\| \dfrac{\partial a_{ij}}{\partial y} \right\|_{C^0(\overline{\Omega_1})} \right) \\ M_{19} = \max\limits_{1 \le i \le 4} \| r_i \|_{C^0[0,1]} \\ M_{20} = \max\limits_{1 \le i \le 4} \| r_i \|_{C^1[0,1]} \end{cases} .$$

Here we see that M_{18} is independent of $\left\| \dfrac{dq_1}{dx} \right\|_{C^0[0,1]}$ and $\left\| \dfrac{dq_2}{dx} \right\|_{C^0[0,1]}$ by means of the forms (VI.9) of a_{ij} $(1 \le i,j \le 4)$. That is, for $(q_1, q_2) \in \mathcal{A}_M$, we have

(VI.22) $\quad M_{18} = M_{18}(M, \| P \|_{\{C^1[0,1]\}^4})$.

For $n = 0$, we immediately see (VI.19) and (VI.20). Assume that (VI.19) and (VI.20) hold true for $n = m$. Then since

(VI.23)
$$\frac{\partial(L_i^{(m+2)} - L_i^{(m+1)})(x,y)}{\partial x} + \delta_i \frac{\partial(L_i^{(m+2)} - L_i^{(m+1)})(x,y)}{\partial y}$$
$$= \sum_{j=1}^{4} a_{ij}(x,y)(L_j^{(m+1)}(x,y) - L_j^{(m)}(x,y))$$
$$(1 \le i \le 4, \quad (x,y) \in \overline{\Omega_1})$$

and

$$(L_i^{(m+2)} - L_i^{(m+1)})(1,y) = 0 \quad (1 \le i \le 4,\ 0 \le y \le 1) ,$$

we get, by Lemma VI.1 and (VI.19), (VI.20) for $n = m$,

(VI.24)
$$\left| L_i^{(m+2)}(x,y) - L_i^{(m+1)}(x,y)\right|$$
$$\le \int_x^1 \frac{M_{18}^{m+1}(1-s)^m\ M_{19}}{m!}\,ds = \frac{M_{18}^{m+1}(1-x)^{m+1}}{(m+1)!}\cdot M_{19} ,$$

and

(VI.25)
$$\left| \frac{\partial L_i^{(m+2)}(x,y)}{\partial y} - \frac{\partial L_i^{(m+1)}(x,y)}{\partial y} \right|$$
$$\le \int_x^1 \frac{M_{18}^{m+1}(1-s)^m\ M_{20}}{m!}\,ds = \frac{M_{18}^{m+1}(1-x)^{m+1}}{(m+1)!}\cdot M_{20} .$$

Here we note that $g(x) = \dfrac{M_{18}^{m+1}(1-x)^m\ M_{19}}{m!}$ and $h(x) = \dfrac{M_{18}^{m+1}(1-x)^m M_{20}}{m!}$ in the estimates (VI.14) and (VI.15).

Thus we see that the estimates (VI.19) and (VI.20) hold true also for $n = m + 1$, so that we obtain these estimates for each $n \ge 0$, by induction.

By (VI.19) and (VI.20), for $1 \le i \le 4$, the series

$\sum_{n=0}^{\infty}(L_i^{(n+1)}(x,y) - L_i^{(n)}(x,y))$ and

$\sum_{n=0}^{\infty}\left(\frac{\partial L_i^{(n+1)}(x,y)}{\partial y} - \frac{\partial L_i^{(n)}(x,y)}{\partial y}\right)$ are absolutely convergent to $L_i(x,y)$ and $\frac{\partial L_i(x,y)}{\partial y}$, respectively, and moreover the convergences are uniform in $(x,y) \in \overline{\Omega_1}$. Therefore, we see that $L_i, \frac{\partial L_i}{\partial y} \in C^0(\overline{\Omega_1})$.

By the definition (VI.17) of $L_i^{(n)}$, as $n \to \infty$, also $\frac{\partial L_i^{(n)}(x,y)}{\partial x}$ $(1 \le i \le 4)$ are convergent uniformly in $(x,y) \in \overline{\Omega_1}$, so that $L_i \in C^1(\overline{\Omega_1})$ $(1 \le i \le 4)$ and L_i $(1 \le i \le 4)$ satisfy (VI.4), (VI.5) and (VI.7).

Further, from (VI.19), (VI.20) and (VI.4), (VI.5), for $1 \le i \le 4$, we get the estimates

$$\text{(VI.26)} \qquad \| L_i \|_{C^0(\overline{\Omega_1})} \le e^{M_{18}} \cdot \max_{1\le i\le 4} \| r_i \|_{C^0[0,1]},$$

$$\text{(VI.27)} \qquad \left\| \frac{\partial L_i}{\partial y} \right\|_{C^0(\overline{\Omega_1})} \le e^{M_{18}} \cdot \max_{1\le i\le 4} \| r_i \|_{C^1[0,1]},$$

and

$$\text{(VI.28)} \qquad \left\| \frac{\partial L_i}{\partial x} \right\|_{C^0(\overline{\Omega_1})}$$
$$\le \left\| \frac{\partial L_i}{\partial y} \right\|_{C^0(\overline{\Omega_1})} + \| f_i(\cdot,\cdot,L_1,L_2,L_3,L_4) \|_{C^0(\overline{\Omega_1})}$$
$$\le e^{M_{18}} \max_{1\le i\le 4} \| r_i \|_{C^1[0,1]} + M_{18}e^{M_{18}} \max_{1\le i\le 4} \| r_i \|_{C^0[0,1]}.$$

The estimates (VI.26) - (VI.28) show the corresponding inequalities in $\overline{\Omega_1}$ to (2.39) and (2.40).

Next we proceed to the proof of (2.41). To this end, we have only to prove

(VI.29) $$\left|\left|L_i(\cdot,\cdot,q_1^{(1)},q_2^{(1)},D_1) - L_i(\cdot,\cdot,q_1^{(2)},q_2^{(2)},D_2)\right|\right|_{C^0(\overline{\Omega_1})}$$

$$\le M_5\Big(\|D_2\|_{\{C^0[0,1]\}^4}(\|q_1^{(1)}-q_1^{(2)}\|_{C^0[0,1]} + \|q_2^{(1)}- q_2^{(2)}\|_{C^0})$$
$$+ \|D_1 - D_2\|_{\{C^0[0,1]\}^4} \Big)$$

for $(q_1^{(m)}, q_2^{(m)}) \in \mathcal{A}_M$ $(m = 1,2)$. Here and henceforth, we set

$$D_m = \begin{pmatrix} d_{11}^{(m)} & d_{12}^{(m)} \\ d_{21}^{(m)} & d_{22}^{(m)} \end{pmatrix} \quad (m = 1,2) \text{ , and}$$

(VI.30) $$L_{m,i}(x,y) = L_i(x,y,q_1^{(m)},q_2^{(m)},D_m)$$

$$(1 \le i \le 4,\ m = 1,2,\ \ (x,y) \in \overline{\Omega_1}\ ),$$

for brevity.

Since for $m = 1,2$, the functions $L_{m,i}$ $(1 \le i \le 4)$ satisfy

$$\frac{\partial L_{m,i}(x,y)}{\partial x} + \delta_i \frac{\partial L_{m,i}(x,y)}{\partial y} = \sum_{j=1}^{4} a_{ij}(x,y,q_1^{(m)},q_2^{(m)})L_{m,j}(x,y)$$
$$(1 \le i \le 4,\ \ (x,y) \in \overline{\Omega_1}\ )$$

and

$$L_{m,i}(1,y) = r_i(y,D_m) \qquad (0 \le y \le 1) \text{ ,}$$

we have equations in $L_{1,i} - L_{2,i}$ $(1 \le i \le 4)$

(VI.31) $$\frac{\partial(L_{1,i}- L_{2,i})(x,y)}{\partial x} + \delta_i \frac{\partial(L_{1,i}- L_{2,i})(x,y)}{\partial y}$$

$$= \sum_{j=1}^{4} a_{ij}(x,y,q_1^{(1)},q_2^{(1)})(L_{1,j}- L_{2,j})(x,y)$$

$$+ \sum_{j=1}^{4} (a_{ij}(x,y,q_1^{(1)},q_2^{(1)})-a_{ij}(x,y,q_1^{(2)},q_2^{(2)})L_{2,j}(x,y)$$
$$(1 \le i \le 4,\ \ (x,y) \in \overline{\Omega_1}\ )$$

and

$$\text{(VI.32)} \qquad (L_{1,i} - L_{2,i})(1,y) = r_i(y,D_1) - r_i(y,D_2) \qquad (1 \le i \le 4,\ 0 \le y \le 1) .$$

Thus, by the estimate (VI.14) of Lemma VI.1, we get

$$\text{(VI.33)} \qquad \max_{1-x\le y\le x} | L_{1,i}(x,y) - L_{2,i}(x,y) |$$

$$\le \| r_i(\cdot,D_1) - r_i(\cdot,D_2) \|_{C^0[0,1]}$$

$$+ M_{18}\int_x^1 \max_{1\le j\le 4} \max_{1-s\le y\le s} |L_{1,j}(s,y) - L_{2,j}(s,y)| ds$$

$$+ \sum_{j=1}^{4} \|a_{ij}(\cdot,\cdot,q_1^{(1)},q_2^{(1)})-a_{ij}(\cdot,\cdot,q_1^{(2)},q_2^{(2)})\|_{C^0(\overline{\Omega_1})}$$

$$\times \max_{1\le j\le 4} \| L_{2,j} \|_{C^0(\overline{\Omega_1})} \qquad (1 \le i \le 4) .$$

Here we note that as $g(x)$ in (VI.14), we can choose

$$M_{18} \max_{1\le j\le 4} \max_{1-x\le y\le x} |L_{1,j}(x,y)-L_{2,j}(x,y)|$$

$$+ \sum_{j=1}^{4} \|a_{ij}(\cdot,\cdot,q_1^{(1)},q_2^{(1)})-a_{ij}(\cdot,\cdot,q_1^{(2)},q_2^{(2)})\|_{C^0(\overline{\Omega_1})}$$

$$\times \max_{1\le j\le 4} \| L_{2,j} \|_{C^0(\overline{\Omega_1})} .$$

Let us set $\eta(x) = \max_{1\le i\le 4} \max_{1-x\le y\le x} |L_{1,i}(x,y) - L_{2,i}(x,y)|$.

Applying (2.39) and (2.40) in (VI.33) and noting

$$\|a_{ij}(\cdot,\cdot,q_1^{(1)},q_2^{(1)}) - a_{ij}(\cdot,\cdot,q_1^{(2)},q_2^{(2)})\|_{C^0(\overline{\Omega_1})}$$

$$\le \frac{1}{2} (\|q_1^{(1)}- q_1^{(2)}\|_{C^0[0,1]} + \|q_2^{(1)}- q_2^{(2)}\|_{C^0[0,1]}) ,$$

we get

$$\eta(x) \le \Bigg(\max_{1\le i\le 4} \|r_i(\cdot,D_1) - r_i(\cdot,D_2)\|_{C^0[0,1]} +$$

$$+ 2M_5\|D_2\|_{\{C^0[0,1]\}^4}(\|q_1^{(1)} - q_1^{(2)}\|_{C^0[0,1]} + \|q_2^{(1)}- q_2^{(2)}\|_{C^0} \Bigg)$$

$$+ M_{18} \int_x^1 \eta(s)ds \qquad (\tfrac{1}{2} \le x \le 1) \quad ,$$

which implies by Gronwall's inequality,

$$\text{(VI.34)} \quad \eta(x) \le e^{M_{18}} \Big(\max_{1\le i\le 4} \|r_i(\cdot,D_1) - r_i(\cdot,D_2)\|_{C^0[0,1]} + 2M_5\|D_2\|_{\{C^0[0,1]\}^4}(\|q_1^{(1)} - q_1^{(2)}\|_{C^0[0,1]} + \|q_2^{(1)} - q_2^{(2)}\|_{C^0}\Big) \qquad (\tfrac{1}{2} \le x \le 1).$$

This inequality is equivalent to (VI.29), the conclusion.

Thus in Ω_1 , we complete the proof of Lemma 5.

Proof of Lemma 5 in Ω_2 **.** In Ω_2 , we have to consider a little nonstandard hyperbolic problem (VI.4), (VI.5), (VI.6) and

$$\text{(VI.35)} \quad \begin{cases} L_3(x,1-x) = b_3(x) \\ L_4(x,1-x) = b_4(x) \qquad (\tfrac{1}{2} \le x \le 1) \quad . \end{cases}$$

Here and henceforth we set

$$\text{(VI.36)} \quad b_j(x) = L_j(x,1-x) = \lim_{\substack{x'\to x,\ y'\to 1-x \\ (x',y') \in \Omega_1}} L_j(x',y') \qquad (j = 3,4,\ \tfrac{1}{2} \le x \le 1) \ ,$$

where $L_j \in C^1(\overline{\Omega_1})$ $(1 \le j \le 4)$ is the solution to (VI.4), (VI.5) and (VI.7).

As the approximate sequences for the solution in $\overline{\Omega_2}$, let us inductively define $L_i^{(n)}$ $(1 \leq i \leq 4,\ n \geq 0)$ by (VI.37) - (VI.39) :

(VI.37) $\quad L_i^{(0)}(x,y) = 0 \qquad (1 \leq i \leq 4,\ \ (x,y) \in \overline{\Omega_2}).$

$$
\text{(VI.38)}\begin{cases}
L_3^{(n+1)}(x,y) = b_3\left(\frac{1+x-y}{2}\right) \\
\quad + \displaystyle\int_{\frac{1+x-y}{2}}^{x} f_3(s, s-x+y, L_1^{(n)}, L_2^{(n)}, L_3^{(n)}, L_4^{(n)})\,ds\,, \\
\\
L_4^{(n+1)}(x,y) = b_4\left(\frac{1+x-y}{2}\right) \\
\quad + \displaystyle\int_{\frac{1+x-y}{2}}^{x} f_4(s, s-x+y, L_1^{(n)}, L_2^{(n)}, L_3^{(n)}, L_4^{(n)})\,ds \\
\qquad\qquad (n \geq 0,\ (x,y) \in \overline{\Omega_2})\,.
\end{cases}
$$

$$
\text{(VI.39)}\begin{cases}
L_1^{(n+1)}(x,y) \\
= \displaystyle\int_{\frac{1+x+y}{2}}^{x+y} (-kf_3(s, s-x-y, L_1^{(n)}, L_2^{(n)}, L_3^{(n)}, L_4^{(n)}) \\
\qquad - \ell f_4(s, s-x-y, L_1^{(n)}, L_2^{(n)}, L_3^{(n)}, L_4^{(n)}))\,ds \\
- kb_3\left(\frac{1+x+y}{2}\right) - \ell b_4\left(\frac{1+x+y}{2}\right) \\
+ \displaystyle\int_{x+y}^{x} f_1(s, -s+x+y, L_1^{(n)}, L_2^{(n)}, L_3^{(n)}, L_4^{(n)})\,ds \\
\\
L_2^{(n+1)}(x,y) \\
= \displaystyle\int_{\frac{1+x+y}{2}}^{x+y} (\ell f_3(s, s-x-y, L_1^{(n)}, L_2^{(n)}, L_3^{(n)}, L_4^{(n)}) \\
\qquad + kf_4(s, s-x-y, L_1^{(n)}, L_2^{(n)}, L_3^{(n)}, L_4^{(n)}))\,ds \\
+ \ell b_3\left(\frac{1+x+y}{2}\right) + kb_4\left(\frac{1+x+y}{2}\right)
\end{cases}
$$

$$+\int_{x+y}^{x} f_2(s,-s+x+y,L_1^{(n)},L_2^{(n)},L_3^{(n)},L_4^{(n)})ds$$

$$(n \geq 0,\ (x,y) \in \overline{\Omega_2}\ )\ .$$

Obviously we see that $\{ L_i^{(n)} \}_{n\geq 0}$ $(1 \leq i \leq 4)$ are well-defined and $L_i^{(n)} \in C^1(\overline{\Omega_2})$. Moreover, by induction, we can obtain the estimates :

$$\text{(VI.40)} \qquad \left| L_i^{(n+1)}(x,y) - L_i^{(n)}(x,y) \right| \leq \frac{M_{21}^n(1-x)^n}{n!}\cdot M_{22}$$

$$(1 \leq i \leq 4,\ n \geq 0,\ (x,y) \in \overline{\Omega_2}\ )$$

and

$$\text{(VI.41)} \qquad \left| \frac{\partial L_i^{(n+1)}(x,y)}{\partial y} - \frac{\partial L_i^{(n)}(x,y)}{\partial y} \right| \leq \frac{M_{23}^{n-1}(1-x)^{n-1}}{(n-1)!}\cdot M_{24}$$

$$(1 \leq i \leq 4,\ n \geq 1,\ (x,y) \in \overline{\Omega_2}\ )\ .$$

Here and henceforth, we set

$$M_{21} = (|k|+|\ell|+1)\cdot \max_{1\leq i\leq 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0(\overline{\Omega_2})}\ ,$$

$$M_{22} = (|k|+|\ell|+1)\cdot \max\ \{\ \|b_3\|_{C^0}\ ,\ \|b_4\|_{C^0}\ \}\ ,$$

$$M_{23} = \max \Bigg(M_{21},\quad \left\{ \frac{1}{2} \max_{1\leq i\leq 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0} + \max_{1\leq i\leq 4} \sum_{j=1}^{4} \left\| \frac{\partial a_{ij}}{\partial y} \right\|_{C^0} \right\} M_{21} + \max_{1\leq i\leq 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0}\ ,$$

$$M_{21}(|k|+|\ell|+1)\left\{ \frac{3}{2} \max_{1\leq i\leq 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0} + \max_{1\leq i\leq 4} \sum_{j=1}^{4} \left\| \frac{\partial a_{ij}}{\partial y} \right\|_{C^0} \right\}$$

$$
\text{(VI.42)}\quad
\begin{cases}
\qquad + (|k|+|\ell|+1) \max_{1\le i\le 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0} \Bigg) , \\[2ex]
M_{24} = \max \Bigg(M_{22} , \\
(|k|+|\ell|+1)\Big\{\frac{3}{2} \max_{1\le i\le 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0}\cdot\max\{\|b_3\|_{C^1}, \|b_4\|_{C^1}\} \\
\quad + \max_{1\le i\le 4} \sum_{j=1}^{4} \left\| \frac{\partial a_{ij}}{\partial y} \right\|_{C^0} \max\{\|b_3\|_{C^0}, \|b_4\|_{C^0}\}\Big\} , \\
\Big\{(\frac{3}{2}|k| + \frac{3}{2}|\ell|+1) \max_{1\le i\le 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0} \\
\quad + (|k|+|\ell|+1) \max_{1\le i\le 4} \sum_{j=1}^{4} \left\| \frac{\partial a_{ij}}{\partial y} \right\|_{C^0} \Big\} \\
\quad \times (|k|+|\ell|+1)\cdot\max\{\|b_3\|_{C^0}, \|b_4\|_{C^0}\} \\
\quad + (|k|+|\ell|+1)^2 \max_{1\le i\le 4} \sum_{j=1}^{4} \|a_{ij}\|_{C^0} \max\{\|b_3\|_{C^1}, \|b_4\|_{C^1}\} \Bigg) .
\end{cases}
$$

In (VI.42), by the forms (VI.9) of a_{ij} $(1 \le i,j \le 4)$, the constants M_{21} and M_{23} are independent of $\left\| \frac{dq_1}{dx} \right\|_{C^0}$ and $\left\| \frac{dq_2}{dx} \right\|_{C^0}$. That is, for $(q_1, q_2) \in \mathscr{A}_M$, we see

$$
\text{(VI.43)}\quad
\begin{cases}
M_{21} = M_{21}(M, \|P\|_{\{C^1[0,1]\}^4}, h) \\[2ex]
M_{23} = M_{23}(M, \|P\|_{\{C^1[0,1]\}^4}, h) \quad .
\end{cases}
$$

Thus, the series $\sum_{n=0}^{\infty} (L_i^{(n+1)}(x,y) - L_i^{(n)}(x,y))$ and $\sum_{n=0}^{\infty} \left(\frac{\partial L_i^{(n+1)}(x,y)}{\partial y} - \frac{\partial L_i^{(n)}(x,y)}{\partial y} \right)$ are absolutely convergent to $L_i(x,y)$ and $\frac{\partial L_i(x,y)}{\partial y}$, respectively, and the convergences are uniform in $(x,y) \in \overline{\Omega_2}$, so that L_i , $\frac{\partial L_i}{\partial y} \in C^0(\overline{\Omega_2})$ $(1 \leq i \leq 4)$.

Moreover the definitions (VI.38) and (VI.39) imply $\frac{\partial L_i}{\partial x} \in C^0(\overline{\Omega_2})$, and therefore L_i $(1 \leq i \leq 4)$ satisfy (VI.4) - (VI.6) and (VI.35) in $\overline{\Omega_2}$.

Thus we have constructed L_i $(1 \leq i \leq 4)$ in the respective domains of Ω_1 and Ω_2 , so that we see that there exists a unique solution L_i $(1 \leq i \leq 4)$ to (VI.4) - (VI.7). In fact, all that we have to do is to verify that L_i $(1 \leq i \leq 4)$ are actually C^1 on $\overline{\Omega_1} \cup \overline{\Omega_2}$. In order to verify this fact, we have only to note that L_i $(1 \leq i \leq 4)$ satisfy the following integral equations :

$$\left\{ \begin{array}{l} L_i(x,y) = r_i(x+y-1) + \displaystyle\int_1^x f_i(s,-s+x+y,L_1,L_2,L_3,L_4)ds \\ \qquad\qquad\qquad\qquad (i = 1,2 \ , \ (x,y) \in \overline{\Omega_1}) \ , \\ L_i(x,y) = r_i(-x+y+1) + \displaystyle\int_1^x f_i(s,s-x+y,L_1,L_2,L_3,L_4)ds \\ \qquad\qquad\qquad\qquad (i = 3,4 \ , \ (x,y) \in \overline{\Omega_1}) \ , \end{array} \right.$$

and

$$\left\{ \begin{array}{l} L_1(x,y) = \displaystyle\int_{\frac{x+y+1}{2}}^{x+y} -(kf_3+\ell f_4)(s,s-x-y,L_1,L_2,L_3,L_4)ds \\ \qquad + \displaystyle\int_{x+y}^{x} f_1(s,-s+x+y,L_1,L_2,L_3,L_4)ds \\ \qquad - kb_3\left(\frac{1+x+y}{2}\right) - \ell b_4\left(\frac{1+x+y}{2}\right) \quad ((x,y) \in \overline{\Omega_2}), \end{array} \right.$$

$$
\left\{
\begin{aligned}
L_2(x,y) &= \int_{\frac{x+y+1}{2}}^{x+y} (\ell f_3+kf_4)(s,s-x-y,L_1,L_2,L_3,L_4)ds \\
&\quad + \int_{x+y}^{x} f_2(s,-s+x+y,L_1,L_2,L_3,L_4)ds \\
&\quad + \ell b_3\left(\frac{1+x+y}{2} \right) + kb_4\left(\frac{1+x+y}{2} \right) \quad ((x,y) \in \overline{\Omega_2}), \\
L_i(x,y) &= \int_{\frac{1+x-y}{2}}^{x} f_i(s,s-x+y,L_1,L_2,L_3,L_4)ds + b_i\left(\frac{1+x-y}{2} \right) \\
&\qquad (i = 3,4 \quad (x,y) \in \overline{\Omega_2}).
\end{aligned}
\right.
$$

Using (VI.40) and (VI.41), we see also the estimates (VI.44) and (VI.45) :

$$(\text{VI.44}) \quad \| L_i \|_{C^0(\overline{\Omega_2})} \le M_{25}(M, \|P\|_{\{C^1[0,1]\}^4}, h) \times \max \{\|b_3\|_{C^0}, \|b_4\|_{C^0}\} \quad (1 \le i \le 4) .$$

$$(\text{VI.45}) \quad \| L_i \|_{C^1(\overline{\Omega_2})} \le M_{25}(M, \|P\|_{\{C^1[0,1]\}^4}, h) \times \max \{\|b_3\|_{C^1}, \|b_4\|_{C^1}\} \quad (1 \le i \le 4) .$$

Now, combining (VI.44), (VI.45) with (VI.26) - (VI.28), we obtain

$$(\text{VI.46}) \quad \| L_i \|_{C^0(\overline{\Omega})} \le M_{25}(M, \|P\|_{\{C^1[0,1]\}^4}, h) \times \max_{1\le i\le 4} \| r_i \|_{C^0[0,1]} \quad (1 \le i \le 4) ,$$

and

$$(\text{VI.47}) \quad \| L_i \|_{C^1(\overline{\Omega})} \le M_{25}(M, \|P\|_{\{C^1[0,1]\}^4}, h) \times \max_{1\le i\le 4} \| r_i \|_{C^1[0,1]} \quad (1 \le i \le 4) ,$$

which imply (2.39) and (2.40), respectively.

Finally, in order to complete the proof of Lemma 5, in view of (VI.29), we have to prove

$$\text{(VI.48)} \quad \| L_i(\cdot,\cdot,q_1^{(1)},q_2^{(1)},D_1) - L_i(\cdot,\cdot,q_1^{(2)},q_2^{(2)},D_2) \|_{C^0(\overline{\Omega_2})}$$

$$\leq M_{26} \left(\max_{1\leq i\leq 4} \|r_j^{(2)}\|_{C^0[0,1]} \times \right.$$

$$(\| q_1^{(1)} - q_1^{(2)} \|_{C^0[0,1]} + \| q_2^{(1)} - q_2^{(2)} \|_{C^0[0,1]})$$

$$\left. + \| b_3^{(1)} - b_3^{(2)} \|_{C^0[\frac{1}{2},1]} + \| b_4^{(1)} - b_4^{(2)} \|_{C^0[\frac{1}{2},1]} \right)$$

$$(1 \leq i \leq 4)$$

for each $(q_1^{(m)}, q_2^{(m)}) \in \mathscr{A}_M$ $(m = 1,2)$. Here and henceforth we put

$$b_3^{(m)}(x) = L_3(x,1-x,q_1^{(m)},q_2^{(m)},D_m)$$

$$b_4^{(m)}(x) = L_4(x,1-x,q_1^{(m)},q_2^{(m)},D_m) \qquad (\tfrac{1}{2} \leq x \leq 1)$$

and

$$L_{m,i}(x,y) = L_i(x,y,q_1^{(m)},q_2^{(m)},D_m)$$

$$(1 \leq i \leq 4,\ m = 1,2\ ,\ (x,y) \in \overline{\Omega_2}\ ).$$

We prepare

Lemma VI.2. *Let* $f(x,y)$ *and* $\dfrac{\partial f(x,y)}{\partial y}$ *be continuous functions in* $\overline{\Omega_2}$ *and satisfy*

$$\text{(VI.49)} \quad |f(x,y)| \leq g(x) \qquad (\ (x,y) \in \overline{\Omega_2}\ ),$$

for some $g \in C^0[0, 1]$.

Then,

(I) Let $a \in C^1[\frac{1}{2}, 1]$ *and let* $u \in C^1(\overline{\Omega_2})$ *be a solution to*

$$\text{(VI.50)} \quad \begin{cases} \dfrac{\partial u(x,y)}{\partial x} + \dfrac{\partial u(x,y)}{\partial y} = f(x,y) & ((x,y) \in \overline{\Omega_2}) \\ u(x,1-x) = a(x) & (\frac{1}{2} \le x \le 1). \end{cases}$$

Then we have

$$\text{(VI.51)} \quad |u(x,y)| \le \| a \|_{C^0[\frac{1}{2},1]} + \int_x^{\frac{1+x-y}{2}} g(s)ds \qquad ((x,y) \in \overline{\Omega_2}).$$

(II) Let $a \in C^1[0, 1]$ *and let* $u \in C^1(\overline{\Omega_2})$ *be a solution to*

$$\text{(VI.52)} \quad \begin{cases} \dfrac{\partial u(x,y)}{\partial x} - \dfrac{\partial u(x,y)}{\partial y} = f(x,y) & ((x,y) \in \overline{\Omega_2}) \\ u(x,0) = a(x) & (0 \le x \le 1). \end{cases}$$

Then we have

$$\text{(VI.53)} \quad |u(x,y)| \le |a(x+y)| + \int_x^{x+y} g(s)ds \qquad ((x,y) \in \overline{\Omega_2}).$$

Pooof of Lemma VI.2. By integrating the equations along the characteristic curves, we can easily prove it. That is, since the solutions u to (VI.50) and v to (VI.52) are represented in the forms

$$u(x,y) = a(\frac{1+x-y}{2}) + \int_{\frac{1+x-y}{2}}^{x} f(s,s-x+y)ds \qquad ((x,y) \in \overline{\Omega_2})$$

and

$$v(x,y) = a(x+y) + \int_{x+y}^{x} f(s,-s+x+y)ds \qquad ((x,y) \in \overline{\Omega_2}),$$

respectively, we can immediately see Lemma VI.2.

We return to the proof of (VI.48). Since in $\overline{\Omega_2}$, the functions $L_{1,i} - L_{2,i}$ $(i = 3, 4)$ satisfy the equations (VI.31) and

$$\text{(VI.54)} \quad L_{1,i}(x,1-x) - L_{2,i}(x,1-x) = b_i^{(1)}(x) - b_i^{(2)}(x) \qquad (i = 3,4,\ \tfrac{1}{2} \le x \le 1) \quad ,$$

we can see in view of Lemma VI.2 (I) and (VI.46) that for $i = 3,4$,

$$\text{(VI.55)} \quad \max_{0\le y\le \min\{x,1-x\}} |L_{1,i}(x,y) - L_{2,i}(x,y)|$$

$$\le \max_{i=3,4} \| b_i^{(1)} - b_i^{(2)} \|_{C^0[\frac{1}{2},1]}$$

$$+ M_{27} \int_x^{\frac{1+x-y}{2}} \max_{\substack{1\le j\le 4 \\ 0\leqq s-x+y \leqq \min\{s,1-s\}}} |L_{1,j}(s,s-x+y)-L_{2,j}(s,s-x+y)|ds$$

$$+ M_{27}(\|q_1^{(1)} - q_1^{(2)}\|_{C^0[0,1]} + \|q_2^{(1)} - q_2^{(2)}\|_{C^0}) \max_{1\le j\le 4} \|r_j^{(2)}\|_{C^0} \, .$$

Henceforth we set

$$\text{(VI.56)} \begin{cases} \eta_1(x) = \max\limits_{i=3,4} \ \max\limits_{0\le y\le \min\{x,1-x\}} |L_{1,i}(x,y) - L_{2,i}(x,y)| \\ \\ \eta(x) = \max\limits_{1\le i\le 4} \ \max\limits_{0\le y\le \min\{x,1-x\}} |L_{1,i}(x,y) - L_{2,i}(x,y)| \, . \end{cases}$$

Then we can rewrite (VI.55) as

$$(\mathrm{VI}.57)\qquad \eta_1(x) \le \max_{i=3,4} \| b_i^{(1)} - b_i^{(2)} \|_{C^0[\frac{1}{2},1]}$$

$$+ M_{27}(\|q_1^{(1)} - q_1^{(2)}\|_{C^0} + \|q_2^{(1)} - q_2^{(2)}\|_{C^0}) \cdot \max_{1\le j\le 4} \|r_j^{(2)}\|_{C^0}$$

$$+ M_{27} \int_x^1 \eta(s)\,ds \qquad (0 \le x \le 1) \quad .$$

Next, since in $\overline{\Omega_2}$, the functions $L_{1,i} - L_{2,i}$ $(i = 1,2)$ satisfy the equations (VI.31) and

$$(\mathrm{VI}.58)\begin{cases} L_{1,1}(x,0) - L_{2,1}(x,0) \\ = -k(L_{1,3}(x,0)-L_{2,3}(x,0))-\ell(L_{1,4}(x,0)-L_{2,4}(x,0)) \\ \\ L_{1,2}(x,0) - L_{2,2}(x,0) \\ = \ell(L_{1,3}(x,0)-L_{2,3}(x,0))+k(L_{1,4}(x,0)-L_{2,4}(x,0)) \\ \qquad\qquad\qquad\qquad (0 \le x \le 1) \quad , \end{cases}$$

in virtue of Lemma VI.2 (II) and (VI.46), we get for $i = 1,2$

$$(\mathrm{VI}.59)\qquad \max_{0\le y\le \min\{x,1-x\}} |L_{1,i}(x,y) - L_{2,i}(x,y)|$$

$$\le (|k|+|\ell|) \max_{i=3,4} |L_{1,i}(x+y,0) - L_{2,i}(x+y,0)|$$

$$+ M_{27}\int_x^{x+y} \max_{\substack{1\le j\le 4 \\ 0\leqq -s+x+y \leqq \min\{s,1-s\}}} |L_{1,j}(s,-s+x+y)-L_{2,j}(s,-s+x+y)|\,ds$$

$$+ M_{27} \max_{1\le j\le 4} \|r_j^{(2)}\|_{C^0[0,1]} (\|q_1^{(1)} - q_1^{(2)}\|_{C^0[0,1]}$$

$$+ \|q_2^{(1)} - q_2^{(2)}\|_{C^0[0,1]})$$

$$\begin{aligned}
&\leq (|k|+|\ell|)\eta_1(x+y) \\
&\quad + M_{27}\max_{1\leq j\leq 4}\|r_j^{(2)}\|_{C^0[0,1]}(\|q_1^{(1)}-q_1^{(2)}\|_{C^0}+\|q_2^{(1)}-q_2^{(2)}\|_{C^0}) \\
&\quad + M_{27}\int_x^1 \eta(s)ds \qquad \text{by (VI.56)} \\
&\leq (|k|+|\ell|)\max_{i=3,4}\|b_i^{(1)}-b_i^{(2)}\|_{C^0[0,1]} \\
&\quad + M_{27}(|k|+|\ell|+1)\cdot\max_{1\leq j\leq 4}\|r_j^{(2)}\|_{C^0}(\|q_1^{(1)}-q_1^{(2)}\|_{C^0}+\|q_2^{(1)}-q_2^{(2)}\|_{C^0}) \\
&\quad + M_{27}(|k|+|\ell|+1)\int_x^1 \eta(s)ds \qquad (0\leq x\leq 1)\ .
\end{aligned}$$

In the last inequality, we use

$$\begin{aligned}
\eta_1(x+y) &\leq \max_{i=3,4}\|b_i^{(1)}-b_i^{(2)}\|_{C^0} \\
&\quad + M_{27}\cdot\max_{1\leq j\leq 4}\|r_j^{(2)}\|_{C^0}(\|q_1^{(1)}-q_1^{(2)}\|_{C^0}+\|q_2^{(1)}-q_2^{(2)}\|_{C^0}) \\
&\quad + M_{27}\int_{x+y}^1 \eta(s)ds \qquad \text{by (VI.57)} \\
&\leq \max_{i=3,4}\|b_i^{(1)}-b_i^{(2)}\|_{C^0} \\
&\quad + M_{27}\cdot\max_{1\leq j\leq 4}\|r_j^{(2)}\|_{C^0}(\|q_1^{(1)}-q_1^{(2)}\|_{C^0}+\|q_2^{(1)}-q_2^{(2)}\|_{C^0}) \\
&\quad + M_{27}\int_x^1 \eta(s)ds \qquad \text{by } x+y\geq x\ .
\end{aligned}$$

By (VI.59) and (VI.57), we reach

$$\begin{aligned}
\eta(x) &\leq M_{28}\max_{i=3,4}\|b_i^{(1)}-b_i^{(2)}\|_{C^0[0,1]} \\
&\quad + M_{28}\max_{1\leq j\leq 4}\|r_j^{(2)}\|_{C^0}(\|q_1^{(1)}-q_1^{(2)}\|_{C^0}+\|q_2^{(1)}-q_2^{(2)}\|_{C^0}) \\
&\quad + M_{28}\int_x^1 \eta(s)ds \qquad (0\leq x\leq 1)\ ,
\end{aligned}$$

which implies

$$(\mathrm{VI}.60) \qquad \eta(x) \leq M_{28}\cdot e^{M_{28}} \left(\|b_3^{(1)} - b_3^{(2)}\|_{C^0} + \|b_4^{(1)} - b_4^{(2)}\|_{C^0} + \max_{1\leq j\leq 4}\|r_j^{(2)}\|_{C^0}(\|q_1^{(1)}-q_1^{(2)}\|_{C^0} + \|q_2^{(1)}-q_2^{(2)}\|_{C^0}) \right)$$

$$(0 \leq x \leq 1) \quad ,$$

by Gronwall's inequality.

Since $\max_{0\leq x\leq 1} \eta(x) =$

$$\max_{1\leq i\leq 4}\|L_i(\cdot,\cdot,q_1^{(1)},q_2^{(1)},D_1) - L_i(\cdot,\cdot,q_1^{(2)},q_2^{(2)},D_2)\|_{C^0(\overline{\Omega_2})} \quad ,$$

the inequality (VI.60) means (VI.48), our conclusion.

Thus we complete the proof of Lemma 5.

Appendix VII. Proof of Lemma 6. Let $\Phi(x)$ be a fundamental matrix for the linear homogeneous system

$$\text{(VII.1)} \qquad \frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = A(x)\begin{pmatrix} u \\ v \end{pmatrix} \qquad (0 \leq x \leq 1) \quad .$$

That is, $\Phi(x)$ is a 2×2 matrix and satisfies

$$\frac{d\Phi(x)}{dx} = A(x)\Phi(x) \qquad (0 \leq x \leq 1)$$

and

$$\det \Phi(x) \neq 0 \qquad (0 \leq x \leq 1)$$

(Coddington and Levinson [3, p.69], for example).
Here let us recall that

$$A(x) = \frac{1}{2}\begin{pmatrix} -a(x)-b(x)+p_1(x)+p_2(x) & a(x)+b(x)-p_1(x)-p_2(x) \\ -a(x)+b(x)-p_1(x)+p_2(x) & a(x)-b(x)+p_1(x)-p_2(x) \end{pmatrix}$$
$$(0 \leq x \leq 1) \quad .$$

Then the solution $\begin{pmatrix} u \\ v \end{pmatrix}$ to (2.44) with (2.45) is given by

$$\text{(VII.2)} \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$+ \Phi(x)\int_0^x \Phi^{-1}(y)\begin{pmatrix} K_{11}(y,y)-K_{22}(y,y)+K_{12}(y,y)-K_{21}(y,y) \\ K_{11}(y,y)-K_{22}(y,y)+K_{21}(y,y)-K_{12}(y,y) \end{pmatrix} dy$$
$$(0 \leq x \leq 1) \quad ,$$

([3, p.74], for instance).

Now we proceed to

Proof of the part (I) of Lemma 6. By (VII.2), we have

$$(\mathrm{VII.3}) \qquad |u(x) - 1|,\ |v(x) - 1|$$

$$\leq 16\| \Phi \|_{\{C^0[0,1]\}^4} \| \Phi^{-1} \|_{\{C^0[0,1]\}^4} \cdot \max_{\substack{1\leq i,j\leq 2 \\ 0\leqq x\leqq 1}} \left| K_{ij}(x,x) \right| \qquad (0 \leq x \leq 1) .$$

On the other hand, in view of (2.29) and (2.39), we have

$$(\mathrm{VII.4}) \qquad \| K \|_{\{C^0(\bar{\Omega})\}^4} \leq M_4 M_5 \delta_0 \quad .$$

Therefore, if

$$(\mathrm{VII.5}) \qquad \delta_0 \leq \left(32 M_4 M_5 (\|\Phi\|_{\{C^0[0,1]\}^4} \cdot \|\Phi^{-1}\|_{\{C^0[0,1]\}^4} + 1) \right)^{-1} ,$$

then we obtain

$$(\mathrm{VII.6}) \qquad |u(x) - 1|,\ |v(x) - 1| \leq \frac{1}{2} \qquad (0 \leq x \leq 1) \quad ,$$

which is the conclusion in the part (I).

Proof of the part (II) of Lemma 6. Since by $A(x)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $(0 \leq x \leq 1)$, the equation (2.44) is equivalent to

$$(\mathrm{VII.7}) \qquad \frac{d}{dx}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = A(x)\begin{pmatrix} u(x) - 1 \\ v(x) - 1 \end{pmatrix}$$

$$+ \begin{pmatrix} K_{11}(x,x)-K_{22}(x,x)+K_{12}(x,x)-K_{21}(x,x) \\ K_{11}(x,x)-K_{22}(x,x)+K_{21}(x,x)-K_{12}(x,x) \end{pmatrix} \qquad (0 \leq x \leq 1) ,$$

we have

$$\text{(VII.8)} \qquad \left\| \frac{du}{dx} \right\|_{C^0[0,1]} , \quad \left\| \frac{dv}{dx} \right\|_{C^0[0,1]}$$

$$\leq M_{29} \left(\max \{ \|u-1\|_{C^0} , \|v-1\|_{C^0} \} + \max_{\substack{1\leq i,j\leq 2 \\ 0\leqq x\leqq 1}} \left| K_{ij}(x,x) \right| \right)$$

$$\leq M_{30} \max_{\substack{1\leq i,j\leq 2 \\ 0\leqq x\leqq 1}} \left| K_{ij}(x,x) \right| \qquad \text{by (VII.3)} .$$

Next, differentiating the both hand sides of (VII.7) with respect to x, we get

$$\frac{d^2}{dx^2} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \frac{dA(x)}{dx} \begin{pmatrix} u(x) - 1 \\ v(x) - 1 \end{pmatrix} + A(x) \frac{d}{dx} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

$$+ \frac{d}{dx} \begin{pmatrix} K_{11}(x,x)-K_{22}(x,x)+K_{12}(x,x)-K_{21}(x,x) \\ K_{11}(x,x)-K_{22}(x,x)+K_{21}(x,x)-K_{12}(x,x) \end{pmatrix}$$

$$(0 \leq x \leq 1) .$$

Therefore by a way similar to the one in (VII.8), we have

$$\text{(VII.9)} \qquad \left\| \frac{d^2u}{dx^2} \right\|_{C^0} , \quad \left\| \frac{d^2v}{dx^2} \right\|_{C^0}$$

$$\leq M_{31} \max \left\{ \max_{\substack{1\leq i,j\leq 2 \\ 0\leqq x\leqq 1}} \left| K_{ij}(x,x) \right| , \max_{\substack{1\leq i,j\leq 2 \\ 0\leqq x\leqq 1}} \left| \frac{dK_{ij}(x,x)}{dx} \right| \right\} .$$

Estimating $p_1 - r_1$ and $p_2 - r_2$ in the definition (2.48) by using (VII.6), (VII.8) and (VII.9), we reach (2.49) and (2.50). In obtaining (2.50), we note also that

$$\left\| \frac{du}{dx} \right\|^2_{C^0} , \left\| \frac{dv}{dx} \right\|^2_{C^0}$$

$$\leq M_{30}^2 \max_{\substack{1\leq i,j\leq 2 \\ 0\leqq x\leqq 1}} \left| K_{ij}(x,x) \right|^2 \leq M_{30}^2 \max_{\substack{1\leq i,j\leq 2 \\ 0\leqq x\leqq 1}} \left| K_{ij}(x,x) \right| ,$$

because we see $\left| K_{ij}(x,x) \right| \leq 1 \quad (1 \leq i,j \leq 2,\ 0 \leq x \leq 1)$ by (VII.4) and (VII.5).

Proof of the part (III) of lemma 6. We have only to prove

$$\text{(VII.10)} \quad \left\| \frac{du^{(i)}}{dx} \right\|_{C^0}, \quad \left\| \frac{dv^{(i)}}{dx} \right\|_{C^0} \leq M_{32} \qquad (i = 1,2)$$

and

$$\text{(VII.11)} \quad \| u^{(1)} - u^{(2)} \|_{C^1[0,1]}, \ \| v^{(1)} - v^{(2)} \|_{C^1[0,1]}$$
$$\leq M_{32} \max_{\substack{1 \leq i,j \leq 2 \\ 0 \leqq x \leqq 1}} \left| K_{ij}^{(1)}(x,x) - K_{ij}^{(2)}(x,x) \right| .$$

Then, from (VII.10), (VI.11) and (2.47), we can derive (2.54), the conclusion.

Verification of (VII.10). For $q^{(i)} \in \mathscr{A}_M$ $(i = 1,2)$, we get

$$\text{(VII.12)} \quad \| K^{(i)} \|_{\{C^0(\overline{\Omega})\}^4} \leq M_4 M_5 \delta_0$$

by (2.29) and (2.39). Combining (VII.12) and (VII.8), we immediately obtain (VII.10).

Verification of (VII.11). Since $(u^{(i)}, v^{(i)})$ is the solution to (2.51) and (2.52) $(i = 1,2)$, it follows that $(u^{(1)} - u^{(2)}, v^{(1)} - v^{(2)})$ satisfies

$$\text{(VII.13)} \quad \frac{d}{dx} \begin{pmatrix} (u^{(1)} - u^{(2)})(x) \\ (v^{(1)} - v^{(2)})(x) \end{pmatrix} = A(x) \begin{pmatrix} (u^{(1)} - u^{(2)})(x) \\ (v^{(1)} - v^{(2)})(x) \end{pmatrix}$$
$$+ d(x) \qquad (0 \leq x \leq 1)$$

and

$$
\text{(VII.14)} \quad \begin{pmatrix} (u^{(1)} - u^{(2)})(0) \\ (v^{(1)} - v^{(2)})(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .
$$

Here we set

$$
\text{(VII.15)} \quad d(x) = \begin{pmatrix} K_{11}^{(1)}(x,x)-K_{11}^{(2)}(x,x)-(K_{22}^{(1)}(x,x)-K_{22}^{(2)}(x,x)) \\ K_{11}^{(1)}(x,x)-K_{11}^{(2)}(x,x)-(K_{22}^{(1)}(x,x)-K_{22}^{(2)}(x,x)) \end{pmatrix}
$$

$$
+ \begin{pmatrix} K_{12}^{(1)}(x,x)-K_{12}^{(2)}(x,x)-(K_{21}^{(1)}(x,x)-K_{21}^{(2)}(x,x)) \\ K_{21}^{(1)}(x,x)-K_{21}^{(2)}(x,x)-(K_{12}^{(1)}(x,x)-K_{12}^{(2)}(x,x)) \end{pmatrix}
$$

$$
(0 \leq x \leq 1) .
$$

In view of the fundamental matrix $\Phi(x)$, we have

$$
\begin{pmatrix} (u^{(1)} - u^{(2)})(x) \\ (v^{(1)} - v^{(2)})(x) \end{pmatrix} = \Phi(x) \int_0^x \Phi^{-1}(y) d(y) \, dy \quad (0 \leq x \leq 1),
$$

so that we can easily reach (VII.11) in a manner similar to the one in obtaining (VII.8).

Thus we complete the proof of Lemma 6.

Appendix VIII Proof of Lemma 8. Since $\lambda_n^* \in \sigma(A_{Q,h,J})$ $(n \in \mathbb{Z})$ by (2.70) and $\overline{\sigma(A_{Q,h,J})} = \sigma(A_{Q,h,J}^*)$, we see that $\psi^*(\cdot, -\overline{\lambda_n^*})$ is an eigenvector of $A_{Q,h,J}^*$ associated with the eigenvalue $\overline{\lambda_n^*}$ $(n \in \mathbb{Z})$, by the definition (2.81) of $\psi^*(\cdot,\lambda)$. Here we recall that $(A_{Q,h,J}^* u)(x) = -B\,\dfrac{du(x)}{dx} + {}^tQ(x)u(x)$ and

$$\mathcal{D}(A_{Q,h,J}^*) = \left\{ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \{H^1(0,\ 1)\}^2 \ ;\ u_2(0) - hu_1(0) = 0 \ \text{ and } \ u_2(1) - Ju_1(1) = 0 \right\} .$$

Thus the part (I) of this lemma is proved.

Next we proceed to a proof of the part (II) of this lemma. To this end, we prove that the transformation T in $\{L^2(0,\ 1)\}^2$ given by (VIII.1) has the bounded inverse T^{-1} ;

$$\text{(VIII.1)} \quad (Tf)(x) = R(x)f(x) + \int_0^x K(x,y)f(y)dy \quad (0 \le x \le 1) .$$

Here $R(x)$ and $K(x,y)$ are defined by (2.65) - (2.67) and the solution to the problem (2.36), (2.37), (2.63), (2.64), respectively.

Since $R(x)^{-1}$ exists for $0 \le x \le 1$ and $K \in \{C^0(\overline{\Omega})\}^4$, by applying the routine argument for Volterra's integral equations of second kind (for example, Yosida [57]), we can show that T^{-1} exists and is bounded in $\{L^2(0,\ 1)\}^2$. In fact, setting

$$K^{(1)}(x,y) = -\,R(x)^{-1}K(x,y)$$

$$K^{(n)}(x,y) = -\int_y^x R(x)^{-1}K(x,t)K^{(n-1)}(t,y)dt \quad (n \geq 2) ,$$

we see that the series $\sum_{n=1}^{\infty} K^{(n)}(x,y)$ converges absolutely and uniformly in $(x,y) \in \overline{\Omega}$. Let us put $\Gamma(x,y) = \sum_{n=1}^{\infty} K^{(n)}(x,y)$ $((x,y) \in \overline{\Omega})$. Then we have

$$(T^{-1}f)(x) = R(x)^{-1}f(x) + \int_0^x \Gamma(x,y)R(y)^{-1}f(y)dy \quad (0 \leq x \leq 1).$$

On the other hand, since $\psi(\cdot,\lambda_n^*) = T\phi(\cdot,\lambda_n^*)$ $(n \in \mathbb{Z})$ by (2.68) and $\{ \phi(\cdot,\lambda_n^*) \}_{n\in\mathbb{Z}}$ is a Riesz basis in $\{L^2(0,1)\}^2$ by Lemma 2 (I), it follows from the result in Gohberg and Kreĭn [6, p.309] that $\{ \psi(\cdot,\lambda_n^*) \}_{n\in\mathbb{Z}}$ also forms a Riesz basis in $\{L^2(0,1)\}^2$.

Therefore all that we have to do is to prove the expression (2.83). First we can obtain

$$\text{(VIII.2)} \quad (\psi(\cdot,\lambda_m^*),\ \psi^*(\cdot,\overline{-\lambda_n^*}))_{\{L^2(0,1)\}^2} = 0 ,$$

if $n \neq m$. In fact, since $A_{Q,h,J}\psi(\cdot,\lambda_m^*) = \lambda_m^*\, \psi(\cdot,\lambda_m^*)$ $(m \in \mathbb{Z})$ and $A_{Q,h,J}^*\psi^*(\cdot,\overline{-\lambda_n^*}) = \overline{\lambda_n^*}\, \psi^*(\cdot,\overline{-\lambda_n^*})$ $(n \in \mathbb{Z})$, we have

$$\lambda_m^*(\psi(\cdot,\lambda_m^*),\ \psi^*(\cdot,\overline{-\lambda_n^*})) = (A_{Q,h,J}\ \psi(\cdot,\lambda_m^*),\ \psi^*(\cdot,\overline{-\lambda_n^*}))$$
$$= (\ \psi(\cdot,\lambda_m^*),\ A_{Q,h,J}^*\psi^*(\cdot,\overline{-\lambda_n^*})) = (\ \psi(\cdot,\lambda_m^*),\ \overline{\lambda_n^*}\ \psi^*(\cdot,\overline{-\lambda_n^*}))$$
$$= \lambda_n^*\ (\ \psi(\cdot,\lambda_m^*),\ \psi^*(\cdot,\overline{-\lambda_n^*}))$$

, so that in virtue of $\lambda_n^* \neq \lambda_m^*$ $(m \neq n)$, we get (VIII.2).

Further we have $\alpha_n = (\psi(\cdot,\lambda_n^*),\ \psi^*(\cdot,\overline{-\lambda_n^*})) \neq 0 \quad (n \in \mathbb{Z})$. In fact, assume that $(\psi(\cdot,\lambda_{n_0}^*),\ \psi^*(\cdot,\overline{-\lambda_{n_0}^*})) = 0$ for some $n_0 \in \mathbb{Z}$. Then, by (VIII.2), we have $(\psi(\cdot,\lambda_n^*),\ \psi^*(\cdot,\overline{-\lambda_{n_0}^*})) = 0$ for each $n \in \mathbb{Z}$, which implies $\psi^*(\cdot,\overline{-\lambda_{n_0}^*}) = 0$ by the completeness of $\{\psi(\cdot,\lambda_n^*)\}_{n\in\mathbb{Z}}$. This contradicts the definition (2.81) of $\psi(\cdot,\lambda_n^*)$. Thus we see $\alpha_n \neq 0 \quad (n \in \mathbb{Z})$.

Let us return to the proof of (2.83). Since $\{\psi(\cdot,\lambda_n^*)\}_{n\in\mathbb{Z}}$ is a Riesz basis, we get, for each $u \in \{L^2(0,1)\}^2$,

(VIII.3) $\quad u = \sum_{n=-\infty}^{\infty} c_n \psi(\cdot,\lambda_n^*)$

with appropriate $c_n \in \mathbb{C} \quad (n \in \mathbb{Z})$.

Applying (VIII.2) in (VIII.3), we obtain

$$c_n = (u,\ \psi^*(\cdot,\overline{-\lambda_n^*}))\cdot\alpha_n^{-1} \qquad (n \in \mathbb{Z}) \quad ,$$

which implies (2.83), our conclusion. Thus we complete the proof of the part (II) of Lemma 8.

As for (I)' and (II)' of this lemma, we can proceed similarly.

Therefore Lemma 8 is proved.

Concluding Remarks

We will give four general remarks and views.

1. In Chapter 3, we get the three results on identification problems for our eigenvalue problem and show two applications of them. In order to solve such identification problems, the method based on the integral operator stated in Paragraph 4 in Chapter 1 is very effective. In fact, if such an integral operator can be constructed for systems under consideration, then we can discuss the uniqueness in the identification problem, along the line of the argument in Suzuki [46, 47, 51].

On the other hand, the existence of such an integral operator crucially depends on the order of ordinary differential equations in eigenvalue problems. For example, it is known that in general, there cannot exist such an integral operator for $\frac{d^4u(x)}{dx^4} + p(x)u(x) = \lambda u(x)$, (Macaev, V.I., On the existence of transformation operators for differential equations of higher order (English translation), Soviet Math. Dokl. 1 (1960), 68 - 71).

Thus Lemma 1 in Chapter 2 enables us to expand the effective range of the "integral operator method" to systems of ordinary differential equations of first order.

2. In engineering, it may be more realistic to consider the identification problem for evolutionary problems, as is mentioned

in Paragraph 1 in Chapter 1, rather than for eigenvalue problems. Actually the evolutionary system for our eigenvalue problem

$$(1) \qquad \begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \dfrac{\partial u_2(x,t)}{\partial x} + p_{11}(x)u_1(x,t) + p_{12}(x)u_2(x,t) \\ \dfrac{\partial u_2(x,t)}{\partial t} = \dfrac{\partial u_1(x,t)}{\partial x} + p_{21}(x)u_1(x,t) + p_{22}(x)u_2(x,t) \\ \qquad\qquad\qquad\qquad (0 \le x \le 1,\ t \ge 0) \end{cases} ,$$

contains equations describing evolutions in time t for vibrating systems, for instance, vibrations of a string, and electric oscillations in a transmission line (cf. §§ 5 and 6 in Chapter 3). As for the identification problem for (1), we refer to Yamamoto [56], and moreover a full length paper is in preparation.

3. In Chapter 4, we get the result (Theorem 2) on the continuity of the mapping which transforms the specified sets of eigenvalues to the coefficients and the real constants in our eigenvalue problem. The continuity in the result is the Lipshitz one, while it is "weak" compared with the metric introduced naturally in an appropriate space of eigenvalues, and is "local" in the sense that the Lipshitz constant in the estimate is not uniform in each fixed system. (That is, the constant depends upon the coefficient matrix P and the real constants h, H, H^* in each fixed system.)

The weakness of the continuity is one of inevitable features in the inverse problem as is mentioned in Paragraph 1 in

Chapter 1. On the other hand, the local continuity is probably overcome by constructing a theory similar to the one by Gel'fand and Levitan for the Sturm - Liouville problem [5] and by applying its theory to our case along the line of Levitan and Gasymov [28].

To this end, it is one of essential parts whether for our system we can derive an integral equation corresponding to what is called, the Gel'fand - Levitan equation ([5]) for the Sturm - Liouville problem. As for a version in our case of the Gel'fand - Levitan theory including the Gel'fand - Levitan equation, the author has completed the argument in spite of complication and difficulty because of the facts that our eigenvalue problem involves simultaneous ordinary differential equations and is not symmetric. Here the nonsymmetry is that of the operator associated with our boundary value problem (Definition 1 in Chapter 2). In a forthcoming paper, we will give full construction of the Gel'fand - Levitan theory in our case.

On the other hand, as for the application of the corresponding "Gel'fand - Levitan theory", we are advancing discussion.

4. Finally we note that the eigenvalue problem discussed in this paper is effective also in the consideration of a one dimensional proper longitudinal vibrations in media with discontinuities, as is mentioned in Paragraph 1 in Chapter 1. Such a problem with discontinuities is important also in a field

called the terrestrial spectroscopy of geophysics (cf. Hald [10]).

Let us consider for example, a rod of length 1 with free ends and the rod is composed of several parts with inhomogeneous Young's moduli $E(x) > 0$ $(0 \leq x \leq 1)$. Then eigenfrequencies of proper longitudinal vibrations of the rod are given in terms of the eigenvalues of

$$(2) \qquad -\frac{d}{dx}\left(E(x)\frac{du(x)}{dx}\right) = \lambda u(x)$$

$$0 \leq x \leq 1,\ x \neq d_j \quad (1 \leq j \leq m)$$

$$(3) \qquad \frac{du(0)}{dx} = \frac{du(1)}{dx} = 0$$

$$(4) \qquad u(d_j+) = u(d_j-) \qquad (1 \leq j \leq m)$$

$$(5) \qquad E(d_j+)\frac{du(d_j+)}{dx} = E(d_j-)\frac{du(d_j-)}{dx} \qquad (1 \leq j \leq m)\ .$$

Here we assume that the density is 1 over the whole rod, for simplicity and $d_0 = 0 < d_1 < \cdots < d_m < 1 = d_{m+1}$, $E \in C^2[d_j, d_{j+1}]$ $(0 \leq j \leq m)$, and d_j $(1 \leq j \leq m)$ correspond to the interfaces of the adjacent inhomogeneous parts. Furthermore we set $u(d_j+) = \lim_{\varepsilon \downarrow 0} u(d_j+\varepsilon)$, $u(d_j-) = \lim_{\varepsilon \downarrow 0} u(d_j-\varepsilon)$, etc. We note that the "continuation conditions" (4) and (5) at the discontinuities mean the continuity of the displacement and the stress, respectively.

In the case where there are no real discontinuities, namely, $E \in C^2[0, 1]$, we can reduce (2) - (5) to the usual Sturm - Liouville problem for Liouville's normal form $-\frac{d^2v(z)}{dz^2} + q(z)v(z) = \lambda v(z)$ as is treated in [5], by means of appropriate changes of independent variable and dependent one, called Liouville's transformation. However, if there are really discontinuities, then continuation conditions (5) become more complicated by Liouville's transformation, so that it is very hard to apply the method based on the integral operator to identification problems.

On the other hand, by the following transformation, we can reduce (2) - (5) to a system of ordinary differential equations of first order as considered throughout this paper : Introducing change of independent variable and dependent one

$$z = z(x) = \int_0^x \frac{d\xi}{E(\xi)^{1/2}} , \quad \ell = z(1)$$

$$\phi_\pm(z) = \begin{pmatrix} \pm \frac{1}{2}\lambda\sqrt{-1}\ u(x) \\ \frac{1}{2}\ (E(x))^{1/2}\ \frac{du(x)}{dx} \end{pmatrix}$$

$$\tilde{E}(z) = E(x),$$

$$\tilde{d}_j = z(d_j) \quad (1 \leq j \leq m) ,$$

we can get an equivalent eigenvalue problem to (2) - (5)

$$(6) \quad \left\{ \begin{array}{l} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \dfrac{d\phi_\pm(z)}{dz} + \begin{pmatrix} 0 & \dfrac{d\tilde{E}(z)}{dz}/2\tilde{E}(z) \\ 0 & 0 \end{pmatrix} \phi_\pm(z) \\ = \pm\ \lambda\sqrt{-1}\ \phi_\pm(z) \qquad (0 \leq z \leq \ell,\ z \neq \tilde{d}_j\ ,\ 1 \leq j \leq m) \\ \phi_\pm^{(2)}(0) = \phi_\pm^{(2)}(\ell) = 0 \\ \tilde{E}(\tilde{d}_j+)^{1/2}\ \phi_\pm^{(2)}(\tilde{d}_j+) = \tilde{E}(\tilde{d}_j-)^{1/2}\ \phi_\pm^{(2)}(\tilde{d}_j-)\ (1 \leq j \leq m). \end{array} \right.$$

Then we can modify the continuation conditions (5) so that the reduced system (6) is suitable for the construction of the integral operator.

As for the inverse spectral problem for (2) - (5) of the type as discussed in Hochstadt and Lieberman [17], we can refer to Hald [10] and Willis [55]. In these papers, the reduced system (6) is not adopted and the integral operator is not constructed by our method.

We are preparing a paper where the uniqueness for the inverse spectral problem of the type of Gel'fand and Levitan [5] is solved by means of the integral operator.

Acknowledgements

The subjects of this paper are the interests of the author during a long period, and so I am indebted to a great many persons.

In particular, I would like to express my hearty thanks to Professor Hiroshi Fujita , who has given me a lot of significant advice in both the general and special points of view and has encouraged me kindly and unceasingly ever since I was an undergraduate. I heartily thank Professor Seizô Itô for many valuable suggestions. Also I am so much obliged to Professor Fumio Kikuchi. I owe Professor Takashi Suzuki for giving me a lot of information of inverse problems, and I am grateful a great deal to Professor Śhin-ichi Nakagiri for his hearty and valuable advice. I wish to express my sincere gratitude to Professors Satoshi Kaizu, Yoshio Konishi, Kazuo Ōeda, Hisashi Okamoto, and Masao Yamazaki, and Dr. Shuichi Jimbo, Dr. Chen Yun-Gang, and to many friends in seminars in the graduate school, for their useful discussions. Furthermore I express my sincere thanks to Professors Tosihusa Kimura, Kenji Yajima, Tatsuo Itoh, Katsunori Iwasaki and Shu Nakamura, who give me opportunities for talks in their seminars and offer me many suggestions. Also I heartily thank to Professor Kazuo Okamoto, who inspires me by appropriate advice. I sincerely thank to Mr. Kazuo Narita for his good bookbinding.

References

[1] Ambarzumian,V., Über eine Frage der Eigenwerttheorie, Z.Physik 53 (1929), 690 - 695.

[2] Borg,G., Eine Umkehrung der Sturm-Liouvilleschen Eigenwertfrage, Acta Math. 78 (1946), 1 - 96.

[3] Coddington,E.A., and Levinson,N., Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[4] Gasymov,M.G., and Levitan,B.M., The inverse problem for a Dirac system (English translation), Soviet Math. Dokl. 7 (1966), 495 - 499.

[5] Gel'fand,I.M., and Levitan,B.M., On the determination of a differential equation from its spectral function (English translation), Amer. Math. Soc. Transl. Ser.2, 1. (1955), 253 - 304.

[6] Gohberg,I.C., and Kreĭn,M.G., Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monographs Vol.18, Amer. Math. Soc., Providence, R.I., 1969.

[7] Hald,O.H., Inverse eigenvalue problems for layered media, Comm. Pure Appl. Math. 30 (1977), 69 - 94.

[8] Hald,O.H., The inverse Sturm-Liouville problem with symmetric potentials, Acta Math. 141 (1978), 263 - 291.

[9] Hald,O.H., Inverse eigenvalue problems for the mantle, Geophys. J.R. astr. Soc. 62 (1980), 41 - 48.

[10] Hald,O.H., Discontinuous inverse eigenvalue problems, Comm. Pure Appl. Math. 37 (1984), 539 - 577.

[11] Heim,D.S., and Sharpe,C.B., The synthesis of nonuniform lines of finite length - Part I, IEEE Trans. Circuit Theory 14 (1967), 394 - 403.

[12] Hochstadt,H., On inverse problems associated with

second-order differential operators, Acta Math. 119 (1967), 173 - 192.

[13] Hochstadt,H., The inverse Sturm-Liouville problem, Comm. Pure Appl. Math. 26 (1973), 715 - 729.

[14] Hochstadt,H., On inverse problems associated with Sturm-Liouville operators, J. Differential Equations 17 (1975), 220 - 235.

[15] Hochstadt,H., On the determination of the density of a vibrating string from spectral data, J. Math. Anal. Appl. 55 (1976), 673 - 685.

[16] Hochstadt,H., On the well-posedness of the inverse Sturm-Liouville problem, J. Differential Equations 23 (1977), 402 - 413.

[17] Hochstadt,H., and Lieberman,B., An inverse Sturm-Liouville problem with mixed given data, SIAM J. Appl. Math. 34 (1978), 676 - 680.

[18] Isaacson,E.L., McKean,H.P., and Trubowitz,E., The inverse Sturm-Liouville problem II., Comm. Pure Appl. Math. 37 (1984), 1 - 11.

[19] Isaacson,E.L., and Trubowitz,E., The inverse Sturm-Liouville problem I., Comm. Pure Appl. Math. 36 (1983), 767 - 783.

[20] Iwasaki,K., On the inverse Sturm-Liouville problem with spatial symmetry, to appear in Funkcial. Ekvac.

[21] Iwasaki,K., Inverse problem for Sturm-Liouville and Hill equations, to appear in Ann. Mat. Pura Appl.

[22] Johnson,W.C., Transmission Lines and Networks, McGraw-Hill, New York, 1950.

[23] Kato,T., Perturbation Theory for Linear Operators, Second ed., Springer, Berlin-Heidelberg-New York, 1976.

[24] Kitamura,S., and Nakagiri,S., Identifiability of spatially

varying and constant parameters in distributed systems of parabolic type, SIAM J. Control Optim. 15 (1977), 785 - 802.

[25] Kolmogorov,A.N., and Fomin,S.V., Introductory Real Analysis (English translation), Prentice-Hall,INC., Englewood Cliffs, New Jersey.

[26] Leĭbenzon,Z.L., An inverse problem of the spectral analysis of ordinary differential operators of higher order (English translation), Trans. Moscow Math. Soc. 15 (1966), 78 - 163.

[27] Levinson,N., The inverse Sturm-Liouville problem, Mat. Tidsskr, B. (1949), 25 - 30.

[28] Levitan,B.M., and Gasymov,M.G., Determination of a differential equation by two of its spectra (English translation), Russ. Math. Surveys 19 - 2 (1964), 1 - 63.

[29] Levitan,B.M., and Sargsjan,I.S., Introduction to Spectral Theory, Transl. Math. Monographs Vol.39, Amer. Math. Soc., Providence, R.I., 1975.

[30] Marchenko,V.A., Some problems in the theory of second-order differential operators, Dokl. Akad. Nauk SSSR 72 (1950), 457 - 460, (in Russian).

[31] McLaughlin,J.R., An inverse eigenvalue problem of order four, SIAM J. Math. Anal. 7 (1976), 646 - 661.

[32] McLaughlin,J.R., An inverse eigenvalue problem of order four - an infinite case, SIAM J. Math. Anal. 9 (1978), 395 - 413.

[33] McLaughlin,J.R., On uniqueness theorems for second order inverse eigenvalue problems, J. Math. Anal. Appl. 118 (1986), 38 - 41.

[34] Mizutani,A., On the inverse Sturm-Liouville problem, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. 31 (1984), 319 -

350.

[35] Murayama,R., The Gel'fand-Levitan theory and certain inverse problems for the parabolic equation, J. Fac. Sci. Univ. Tokyo 28 (1981), 317 - 330.

[36] Nagumo,M., Partial Differential Equations I, (in Japanese), Iwanami, Tokyo, 1957.

[37] Nakagiri,S., Identifiability of linear systems in Hilbert spaces, SIAM J. Control Optim. 21 (1983), 501 - 530.

[38] Nakagiri,S., Kitamura,S., and Murakami,H., Mathematical treatment of the constant parameter identifiability of distributed systems of parabolic type, Math. Sem. Notes Kobe Univ., 5 (1977), 97 - 105.

[39] Nakagiri,S., and Yamamoto,M., Identifiability of linear retarded systems in Banach spaces, submitted to Funkcial. Ekvac.

[40] Petrovsky,I.G., Lectures on Partial Differential Equations (English translation), Intersciences, New York, 1954.

[41] Picard,E., Leçons sur quelques types simples d'équations aux dérivées partielles avec des applications à la physique mathématique, Gauthier-Villars, Paris, 1950.

[42] Russell,D.L., Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory, J. Math. Anal. Appl. 40 (1972), 336 - 368.

[43] Russell,D.L., Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems, J. Math. Anal. Appl. 62 (1978), 186 - 225.

[44] Sahnovič, L.A., Inverse problem for differential operators of order $n > 2$ with analytic coefficients, Mat. Sb. 46(88) (1958), 61 - 76, (in Russian).

[45] Sahnovič, L.A., The inverse problem for fourth-order

equations, Mat. Sb. 56(98) (1962), 137 - 146, (in Russian).

[46] Suzuki,T., Deformation formulas and their applications to spectral and evolutional inverse problems, in "U.S.-Japan Seminar on Nonlinear Partial Differential Equations in Applied Sciences" (Fujita,H., Lax,P.D., and Strang,G., Eds.), 289 - 311, Kinokuniya & North-Holland, Japan/Amsterdam, 1983.

[47] Suzuki,T., Gel'fand-Levitan's theory, deformation formulas, and inverse problems, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. 32 (1985), 231 - 271.

[48] Suzuki,T., On the inverse Sturm-Liouville problem for spatially symmetric operators, I., J. Differential Equations, 56 (1985), 165 - 194.

[49] Suzuki,T., ditto, II., J. Differential Equations 58 (1985), 243 - 256.

[50] Suzuki,T., ditto, III., J. Differential Equations 58 (1985), 267 - 281.

[51] Suzuki,T., Inverse problems for heat equations on compact intervals and on circles, I., J. Math. Soc. Japan 38 (1986), 39 - 65.

[52] Suzuki,T., and Murayama,R., A uniqueness theorem in an identification problem for coefficients of parabolic equations, Proc. Japan Acad. 56 (1980), 259 - 263.

[53] Timoshenko,S., Vibration Problems in Engineering, Van Nostrand, New York, 1955.

[54] Uchiyama,K., On the uniqueness of the inverse eigenvalue problem on a finite interval, Master's thesis (in Japanese), Univ. Tokyo, Tokyo, 1986.

[55] Willis,C., Inverse Sturm-Liouville problems with two discontinuities, Inverse Problems 1 (1985), 263 - 289.

[56] Yamamoto,M., Some identification problem of vibrating systems, Kôkyûroku, RIMS, Kyoto Univ., 562 (1985), 132 - 166.

[57] Yosida,K., Lectures on Differential and Integral Equations, Intersciences, New York, 1960.

[58] Yosida,K., Functional Analysis, Third ed., Springer, Berlin-Göttingen-Heidelberg, 1971.

[59] Žikov,V.V., On inverse Sturm-Liouville problems on a finite segment (English translation), Math. USSR-Izvestija 1 (1967), 923 - 934.