Micro Local Theory of Boundary Value Problems
(境界値問題の超局所理論)

片岡清臣

Micro Local Theory of Boundary Value Problems

The subject of this paper is to study non-characteristic boundary value problems for partial differential equations in a micro local situation. Throughout this paper, we chiefly use the method of algebraic analysis, not of functional analysis, which will be very useful to understand the essence of boundary value problems or mixed problems intuitively. As the most important application in this paper, we obtained the theorem on propagation of micro-analyticity of solutions up to the boundary for partially micro-hyperbolic operators in one side and for diffractive operators.

This paper consists of two parts,

" Micro Local Theory of Boundary Value Problems I "

and

" Micro Local Theory of Boundary Value Problems II and a Theorem on Regularity for Diffractive Operators ".

In the former part, we introduce the notion of " Mildness " and micro-local Green's formula, which are indispensable for the systematic study of boundary value problems. In the latter part, we formulate non-characteristic boundary value problems micro-locally and apply these methods to solvability or regularity of boundary value problems for some kinds of pseudo--differential operators.

As for more detailed explanation, the reader may be referred to see the introduction of each part.

Micro Local Theory of Boundary Value Problems I

By Kiyômi Kataoka

Abstract

In [9] and [11], H.Komatsu-T.Kawai and P.Schapira defined boundary values of hyperfunction solutions to non-characteristic boundary value problems. Since then, many works have been published. Above all the Green's formula used by A.Kaneko is seemed to be very fundamental for boundary value problems. The auther introduces the concept of mildness for hyperfunctions, which expresses a wide class of hyperfunctions having boundary values, and shows that mildness is kept under several operations, integration along fiber, product, ..., and so on. Simultaneously mildness and these operations are micro-localized and the relationship with boundary value problems for pseudo-differential operators is clarified. The Green's formula, in particular, is also micro-localized, too. Lastly the consistency with topological boundary values is stated.

Contents

Introduction

Let $P(x,D)$ be a differential operator of order m defined in $M=\{x\in\mathbb{R}^n;\ |x|<r\}$. Suppose that $N=\{x\in M;\ x_1=0\}$ is non-characteristic with respect to P. Then, Komatsu-Kawai and Schapira independently proved that any hyperfunction solution to $P(x,D)u=0$ in $\{x\in M; x_1>0\}$ has a unique extension $\tilde{u}(x)\in\Gamma_{M+}(M,B_M)$ and "boundary values" $(f_0,\ldots,f_{m-1})\in\Gamma(N,B_N)^m$ such that $\tilde{u}=u$ in $\{x\in M; x_1>0\}$ and $P\tilde{u}=\sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1)$. As for these hyperfunctions $u(x)$, we want to stress that every normal derivative up to the infinite order has a boundary value on $x_1=+0$. More precisely for every differential operator $J(x,D)$ (of finite or infinite order) defined in a neighborhood of the boundary $J(x,D)u(x)$ has a boundary value $J(x,D)u|_{x_1=+0}$ as a section of B_N (such an approach is seen in the treatment of hyperfunctions with real analytic parameters by A.Kaneko [1]). In order to generalize this property apart from solutions to differential equations, the auther introduced the concept of mildness. Let $f(x)$ be a hyperfunction defined in $\{x\in M;\ \delta>x_1>0,\ |x'-x_0'|<\delta\}$. Then $f(x)$ is said to be mild at $(0,x_0')\in N$ from the positive side of N if and only if $f(x)$ has an extension $\tilde{f}(x)$ to $\Gamma_{M+}(\{|x_1|<\delta,\ |x'-x_0'|<\delta\},B_M)$ such that $\tilde{f}(x)$ considered as a section of $C_{M+|X}$ is continued as a section of $C_{N|X}$ to $S_Y^*X\cap\pi^{-1}((0,x_0'))=\{(0,x';\zeta_1,i\eta')\in S_N^*X;\ x'=x_0',\ \eta'=0\}$. We denote by $\hat{B}_{N|M+}$ the sheaf of germs of mild hyperfunctions from the positive side of N.

In 2.1 of §2, we begin with the micro localization of $\hat{B}_{N|M+}$. That is, a sheaf $\hat{C}_{N|M+}$ on iS^*N is introduced and the following exact sequence is obtained.

$$0\longrightarrow \mathcal{A}_M|_N\longrightarrow \hat{B}_{N|M+}\longrightarrow \pi_{N*}\hat{C}_{N|M+}\longrightarrow 0$$

The softness of $\hat{C}_{N|M+}$ and $\hat{B}_{N|M+}$ are proved there. Further, to analyze mild hyperfunctions more minutely, we characterize mildness by using defining functions, that is, a mild hyperfunction is written as a sum of boundary values of holomorphic functions defined in such a domain as $D=\{z\in\mathbb{C}^n;\ |z_1|<\delta,\ |z'-x_0'|<\delta\ ,\langle \mathrm{Im}z',\xi_0\rangle>\delta|\mathrm{Im}z'|+\frac{1}{\delta}(|\mathrm{Im}z_1|+(-\mathrm{Re}z_1)_+)\}$, where $(x)_+=x$ if $x\geq 0$ and $(x)_+=0$ if $x<0$. Here we remark that the intersection of D with the complex boundary $\{z_1=0\}$ constitutes a wedge in $\mathbb{C}^{n-1}$, $\{z'\in\mathbb{C}^{n-1};\langle \mathrm{Im}z',\xi_0\rangle>\delta|\mathrm{Im}z'|,\ |z'-x_0'|<\delta\}$. Using these facts, the trace operation and the canonical extension are defined as the following sheaf homomorphisms.

$$\mathrm{Trace}\ :\ \hat{B}_{N|M+}\ni f(x)\longrightarrow f(+0,x')\in B_N \quad \text{or} \quad \hat{C}_{N|M+}\longrightarrow C_N\ ,$$

$$\mathrm{ext}\ \ :\ \hat{B}_{N|M+}\ni f(x)\longrightarrow f(x)Y(x_1)\in \mathcal{H}^0_{M+}(B_M) \quad \text{and so on.}$$

Especially ext is injective. The subject of 2.2 is the micro local Green's formula (Theorem 2.2.8). To obtain it we define product operations ; $\hat{B}_{N|M+}\underset{\mathbb{C}}{\otimes}\cdots\underset{\mathbb{C}}{\otimes}\hat{B}_{N|M+}\longrightarrow\hat{B}_{N|M+}$, $\hat{B}_{N|M+}\otimes\mathcal{H}^0_{M+}(B_M)\longrightarrow\mathcal{H}^0_{M+}(B_M)$, integration along fiber , and so on under suitable conditions on singular supports. In 2.3, we treat topological properties of mild hyperfunctions. It is proven there that for any mild hyperfunction $f(x_1,x')$ with proper support in x', there exists the limit of $f(x_1,x')$ as $x_1\to+0$ in the space of analytic functionals and that it coincides with Trace(f).

The subject of §1 is the division theorem for $C_{N|X}$ which has been proved by Kashiwara-Kawai (local version) and Schapira, Kataoka (semi global version). Here we give the projection operators concretely using symbols of pseudo-differential operators.

§1. The division problem for the sheaf $C_{N|X}$.

In this section we introduce the division theorem of the sheaf $C_{N|X}$ whose local form was proved by Kashiwara-Kawai [4] [5] and whose global form was proved by Schapira [12] (and also see Kataoka [7]), in particular we will give concretely the projection operators.

1.1. Notations.

Let M be a n-dimensional real analytic manifold, N be its submanifold of codimension 1 and X,Y be their complex neighborhoods respectively. For the sake of simplicity, we assume $M=\mathbb{R}^n \ni x=(x_1, x')$ and $N=\{x\in\mathbb{R}^n; x_1=0\}$. Further we set $M+=\{x\in\mathbb{R}^n; x_1\geqq 0\}$.

$$\begin{array}{ccc} M=\mathbb{R}^n \ni (x_1, x') & \hookleftarrow & N=\{0\}\times\mathbb{R}^{n-1}\ni(0,x') \\ \cap & & \cap \\ X=\mathbb{C}^n \ni (z_1, z') & \hookleftarrow & Y=\{0\}\times\mathbb{C}^{n-1}\ni(0,z') \end{array}$$

And we define the pairing of $TX=\mathbb{C}^n\times\mathbb{C}^n \ni (z;\sum_{j=1}^{n}(w_j\partial/\partial z_j+\overline{w}_j\partial/\partial\overline{z}_j))$ and $T^*X=\mathbb{C}^n\times\mathbb{C}^n\ni(z;\zeta)$ by $-\mathrm{Re}\langle w,\zeta\rangle$. In this paper, we denote $\sum_{j=1}^{n} w_j\zeta_j$ by $\langle w,\zeta\rangle$ and $w=u+iv$, $\zeta=\xi+i\eta$. Then S_N^*X, S_{M+}^*X, G+, H+, I are written as follows.

$$S_N^*X = \{(0,x';\zeta_1, i\eta')\in S^*X;\ x', \eta' \text{ real}\}$$

$$G+ = \{(0,x';\zeta_1, i\eta')\in S_N^*X;\ \mathrm{Re}\zeta_1>0\}$$

$$H+ = \{(x_1,x';i\eta_1, i\eta')\in S^*X;\ x', \eta=(\eta_1,\eta') \text{ real and } x_1>0.\}$$

$$I = \{(0,x';i\eta)\in S_N^*X;\ x', \eta \text{ real}\}$$

$$S_{M+}^*X = G+\sqcup H+\sqcup I.$$

$C_{M+|X}$ and $C_{N|X}$ are canonically defined as sheaves on S_{M+}^*X and S_N^*X respectively (see Kataoka [8] §4 and Kashiwara-Kawai [4], [5]).

For the sake of reader's convenience, we pick up some important properties of these sheaves (see Kataoka [8] §4 and Kashiwara-Kawai [4], [5]).

Proposition 1.1.1. Let $\pi_{M+/X}$; $S^*_{M+}X \longrightarrow X$ and $\pi_{N/X}$; $S^*_N X \rightarrow X$ be canonical projections. Then we have,

$$\mathcal{H}^0_{M+}(\mathcal{B}_M)\big|_N \overset{\sim}{\Rightarrow} \pi_{M+/X*}\mathcal{C}_{M+|X}\big|_N$$

$$0 \longrightarrow \mathcal{H}^0_N(\mathcal{B}_M) \longrightarrow \pi_{N/X*}\mathcal{C}_{N|X} \longrightarrow \mathcal{O}_X\big|_N \longrightarrow 0 \quad \text{exact.}$$

Further, $\mathcal{C}_{M+|X}\big|_{G+} = \mathcal{C}_{N|X}\big|_{G+}$, $\mathcal{C}_{M+|X}\big|_{H+} = \mathcal{C}_M\big|_{H+}$ and the following canonical injections are obtained.

$$\mathcal{C}_{N|X}\big|_I \overset{i'}{\hookrightarrow} \mathcal{C}_{M+|X}\big|_I \overset{i}{\hookrightarrow} \mathcal{C}_M\big|_I$$

Proposition 1.1.2. $\mathcal{C}_{M+|X}$ (or $\mathcal{C}_{N|X}$) is a $\mathcal{P}_X\big|_{S^*_{M+}X}$ -module ($\mathcal{P}_X\big|_{S^*_N X}$ -module), where $\mathcal{P}_X$ is the sheaf of germs of pseudo-differential operators.

Proposition 1.1.3. We denote the closure of H+ and G+ by $\overline{H}+=H+\cup I$ and $\overline{G}+=G+\cup I$ respectively. Then the sections of $\mathcal{C}_{M+|X}$ (or $\mathcal{C}_{N|X}$) have the unique continuation property along the fiber of ι^+ ; $\overline{G}+\setminus S^*_Y X \ni (0,x';\zeta_1,i\eta') \longrightarrow (x';i\eta') \in iS^*N$ (or ι; $S^*_N X \setminus S^*_Y X \longrightarrow iS^*N$).

Proposition 1.1.4. Setting $U^{\pm}_k = \{(0,x';\zeta_1,i\eta') \in S^*_N X;\ \eta_k = \pm 1\}$ for each k=2,,,n, we have the following sheaf isomorphism.

$$\beta^{\pm}_k \ ; \ \mathcal{C}_{N|X}\big|_{U^{\pm}_k} \xrightarrow{\sim} \mathcal{C}\mathcal{O}\big|_{\widetilde{U}^{\pm}_k} \ ,$$

where $\widetilde{U}^{\pm}_k = \{(\zeta_1,x';i\eta') \in \mathbb{C}\times iS^*N \ ;\ \eta_k = \pm 1\}$ and $\mathcal{C}\mathcal{O}$ is the sheaf of germs of microfunctions in (ζ_1,x') with holomorphic parameter ζ_1.

Then the induced isomorphism of pseudo-differential operators are given as the following quantized contact transformation.

i) $\beta_k^{\pm}\frac{\partial}{\partial x_j}(\beta_k^{\pm})^{-1}=\frac{\partial}{\partial x_j}$ for $j\neq 1$, $\beta_k^{\pm}\frac{\partial}{\partial x_1}(\beta_k^{\pm})^{-1}=\mp i\zeta_1\frac{\partial}{\partial x_k}$.

ii) $\beta_k^{\pm}\, x_j(\beta_k^{\pm})^{-1}=x_j$ for $j\neq 1,k$, $\beta_k^{\pm}\, x_1(\beta_k^{\pm})^{-1}=\mp i\frac{\partial}{\partial\zeta_1}(\frac{\partial}{\partial x_k})^{-1}$,

$\beta_k^{\pm}\, x_k(\beta_k^{\pm})^{-1}=x_k+\frac{\partial}{\partial\zeta_1}\zeta_1(\frac{\partial}{\partial x_k})^{-1}$.

iii) $\beta_k^{\pm}(\sum_{j=0}^{m-1}f_j(x')\delta^{(j)}(x_1))=-\frac{1}{2\pi}\sum_{j=0}^{m-1}(i\zeta_1)^j(\mp\frac{\partial}{\partial x_k})^{j+1}f_j(x')$.

Proof Remember the sheaf isomorphism $\beta_k^{\pm}$ (Kataoka [8] §4.2 Theorem 4.2.3) ;

$$\beta_k^{\pm}\ ;\ F(x)=\sigma(f(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,p,\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_1\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n)$$

$$\longrightarrow G(\zeta_1,x')=\sigma(f(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,\tilde{p},\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_2\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n),$$

where $p=\langle z,\zeta\rangle=\sum_{j\neq k}z_j\zeta_j\pm z_k i$, $\tilde{p}=\sum_{j\neq 1,k}z_j\zeta_j\pm z_k i$ and

$f(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,p,\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_1\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n$ is a normalized Radon transform of a germ $F(x)$ of $C_{N|X}$. From this expression, i) and ii) are concluded, for example,

$$\beta_k^{\pm}(\frac{\partial}{\partial x_1}F(x))=\beta_k^{\pm}(\sigma(\zeta_1\frac{\partial f}{\partial p}(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,p,\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_1\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n))$$

$$=\sigma(\zeta_1\frac{\partial f}{\partial\tilde{p}}(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,\tilde{p},\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_2\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n)$$

$$=\zeta_1(\mp i)\frac{\partial}{\partial x_k}G(\zeta_1,x')=\mp i\zeta_1\frac{\partial}{\partial x_k}\beta_k^{\pm}(F(x)),$$

$$\beta_k^{\pm}(x_1F(x))=\beta_k^{\pm}(\sigma(z_1\cdot f(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,p,\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_1\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n))$$

by normalization

$$=\beta_k^{\pm}(\sigma(-(\frac{\partial}{\partial p})^{-1}\frac{\partial}{\partial\zeta_1}f(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,p,\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_1\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n))$$

$$=-\sigma((\frac{\partial}{\partial\tilde{p}})^{-1}\frac{\partial}{\partial\zeta_1}f(z_2,\overset{\vee}{\underset{k}{\cdot}},,z_n,\tilde{p},\zeta_1,,\overset{\vee}{\underset{k}{\cdot}},,\zeta_n)d\zeta_2\wedge\cdots\overset{\vee}{\underset{k}{\cdot}}\cdots\wedge d\zeta_n)$$

$$= -(\pm i)\left(\frac{\partial}{\partial x_k}\right)^{-1}\frac{\partial}{\partial \zeta_1}G(\zeta_1, x') = \mp i\frac{\partial}{\partial\zeta_1}\left(\frac{\partial}{\partial x_k}\right)^{-1}\beta_k^{\pm}(F(x)),$$

$$\beta_k^{\pm}(x_k F(x)) = \beta_k^{\pm}(\sigma(z_k f(z_2,\overset{\vee}{\underset{k}{\ldots}},z_n,p,\zeta_1,\overset{\vee}{\underset{k}{\ldots}},\zeta_n)d\zeta_1\wedge\overset{\vee}{\underset{k}{\cdots}}\wedge d\zeta_n))$$

$$= \mp i\,\beta_k^{\pm}(\sigma((p-z_1\zeta_1-\overset{\vee}{\underset{k}{\cdots}}-z_n\zeta_n)f(z_2,\overset{\vee}{\underset{k}{\ldots}},z_n,p,\zeta_1,\overset{\vee}{\underset{k}{\ldots}},\zeta_n)d\zeta_1\wedge\overset{\vee}{\underset{k}{\cdots}}\wedge d\zeta_n))$$

$$= \mp i\,\beta_k^{\pm}(\sigma(\{(p-z_2\zeta_2-\overset{\vee}{\underset{k}{\cdots}}-z_n\zeta_n)f(z_2,\overset{\vee}{\underset{k}{\ldots}},z_n,p,\zeta_1,\overset{\vee}{\underset{k}{\ldots}},\zeta_n)$$

$$+ \left(\frac{\partial}{\partial p}\right)^{-1}\frac{\partial}{\partial\zeta_1}\zeta_1 f(z_2,\overset{\vee}{\underset{k}{\ldots}},z_n,p,\zeta_1,\overset{\vee}{\underset{k}{\ldots}},\zeta_n)\}d\zeta_1\wedge\overset{\vee}{\underset{k}{\cdots}}\wedge d\zeta_n))$$

$$= \mp i\sigma(\{(\tilde{p}-z_2\zeta_2-\overset{\vee}{\underset{k}{\cdots}}-z_n\zeta_n)f(z_2,\overset{\vee}{\underset{k}{\ldots}},z_n,\tilde{p},\zeta_1,\overset{\vee}{\underset{k}{\ldots}},\zeta_n)$$

$$+ \left(\frac{\partial}{\partial\tilde{p}}\right)^{-1}\frac{\partial}{\partial\zeta_1}\zeta_1 f(z_2,\overset{\vee}{\underset{k}{\ldots}},z_n,\tilde{p},\zeta_1,\overset{\vee}{\underset{k}{\ldots}},\zeta_n)\}d\zeta_2\wedge\overset{\vee}{\underset{k}{\cdots}}\wedge d\zeta_n)$$

$$= x_k G(\zeta_1, x') + \left(\frac{\partial}{\partial x_k}\right)^{-1}\frac{\partial}{\partial\zeta_1}\zeta_1 G(\zeta_1, x')$$

$$= \left\{x_k + \frac{\partial}{\partial\zeta_1}\zeta_1\left(\frac{\partial}{\partial x_k}\right)^{-1}\right\}\beta_k^{\pm}(F(x)).$$

Furthermore the Radon transform of $F(x) = \sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1)$ is given by (say, $f_j(x')\in\Gamma_c(\mathbb{R}^{n-1}, B_N)$ for every j);

$$f(p,\zeta)d\sigma(\zeta) = \left\{\frac{(n-1)!}{(-2\pi i)^n}\int_{\mathbb{R}^n}\frac{F(y)}{(p-\langle y,\zeta\rangle)^n}dy\right\}d\sigma(\zeta)$$

$$= \left\{\frac{(n-1)!}{(-2\pi i)^n}\sum_{j=0}^{m-1}\left(\zeta_1\frac{\partial}{\partial p}\right)^j\int_{\mathbb{R}^{n-1}}\frac{f_j(y')}{(p-\langle y',\zeta'\rangle)^n}dy'\right\}d\sigma(\zeta),$$

where $d\sigma(\zeta) = \sum_{l=1}^{n}(-1)^{l-1}\zeta_l d\zeta_1\wedge\overset{\vee}{\underset{l}{\cdots}}\wedge d\zeta_n$. Since $\zeta_k = \pm i$ on $U_k^{\pm}$,

$$\beta_k^{\pm}(F(x)) = (-1)^{k-1}(\pm i)\sigma\left(\left\{\frac{(n-1)!}{(-2\pi i)^n}\sum_{j=0}^{m-1}\left(\zeta_1\frac{\partial}{\partial\tilde{p}}\right)^j\int_{\mathbb{R}^{n-1}}\frac{f_j(y')}{(\tilde{p}-\langle y',\zeta'\rangle)^n}dy'\right\}\right.$$

$$\left.\times\, d\zeta_2\wedge\overset{\vee}{\underset{k}{\cdots}}\wedge d\zeta_n\right)$$

$$= -\frac{(n-1)!}{(-2\pi i)^n}\sum_{j=0}^{m-1}\zeta_1{}^j\cdot\frac{1}{1-n}\cdot\left(\mp i\frac{\partial}{\partial x_k}\right)^{j+1}\sigma\left(\left\{\int_{\mathbb{R}^{n-1}}\frac{f_j(y')dy'}{(\tilde{p}-\langle y',\zeta'\rangle)^{n-1}}\right\}d\sigma(\zeta')\right)$$

$$= -\frac{1}{2\pi}\sum_{j=0}^{m-1}(i\zeta_1)^j\left(\mp\frac{\partial}{\partial x_k}\right)^{j+1}f_j(x').$$

1.2 The division theorem for $C_{N|X}$.

Let $P(x,D_x)$ be a pseudo-differential operator defined in a neighborhood of $\tau^{-1}(p'_0)=\{(0,x'_0;\zeta_1,i\eta'_0)\in S^*_N X;\ \zeta_1\in\mathbb{C}\}$ of the form

$$P(x,D_x)=D_{x_1}^{\ m}+P_1(x,D_{x'})D_{x_1}^{\ m-1}+\cdots\cdots+P_m(x,D_{x'}),$$

where $D_{x_j}=\partial/\partial x_j$, $D_{x'}=(D_{x_2},,,D_{x_n})$ and order of $P_j\leq j$, $[P_j(x,D_{x'}),x_1]=0$ for every j. We denote the principal symbol of $P(x,D_x)$ by $\sigma(P)$ $=\zeta_1^{\ m}+p_1(z,\zeta')\zeta_1^{\ m-1}+\cdots+p_m(z,\zeta')$.

By Proposition 1.1.2, $P(x,D_x)$ induces the sheaf isomorphism

$$P(x,D_x)\ ;\ C_{N|X}\xrightarrow{\ \sim\ }C_{N|X}$$

on $S^*_N X\cap\{(z;\zeta)\in S^*X;\ \sigma(P)\neq 0\}$. On the other hand $\tau^{-1}(p'_0)\cap\{\sigma(P)=0\}$ $=\{(0,x'_0;\zeta_1,i\eta'_0);\ \zeta_1^{\ m}+p_1(0,x'_0,i\eta'_0)\zeta_1^{\ m-1}+\cdots+p_m(0,x'_0,i\eta'_0)=0\}$ is a finite set of number m counting multiplicities, in particular $P(x,D_x)$ is invertible on an open dense subset of each fiber of τ. So we have the following statement.

Proposition 1.2.1. Let $P(x,D_x)$ be a pseudo-differential operator stated above, then

$$P(x,D_x)\ ;\quad C_{N|X}\big|_{S^*_N X\setminus S^*_Y X}\longrightarrow C_{N|X}\big|_{S^*_N X\setminus S^*_Y X}$$

and

$$P(x,D_x)\ ;\quad C_{M+|X}\big|_{\overline{G}+\setminus S^*_Y X}\longrightarrow C_{M+|X}\big|_{\overline{G}+\setminus S^*_Y X}$$

are injective sheaf homomorphisms on the domain of definition of P.

Proof These are direct corollaries of Proposition 1.1.3.

Thus our interest concentrates upon calculating the cokernel of $P;\ C_{N|X}\to C_{N|X}$. By Proposition 1.1.4 this is equivalent to calculating $\mathcal{CO}/Q(\zeta_1,x',D_{\zeta_1},D_{x'})\mathcal{CO}$, where $Q(\zeta_1,x',D_{\zeta_1},D_{x'})=\beta^{\pm}_k\circ P\circ(\beta^{\pm}_k)^{-1}$.

<u>Lemma 1.2.2</u> Let $P(z,D_z)$ be a pseudo differential operator of the form

$$P(z,D_z)=D_{z_1}^{m}+P_1(z,D_{z'})D_{z_1}^{m-1}+\cdots+P_m(z,D_{z'}),$$

where order of $P_j(z,D_{z'})\leq j$ and $P_j(z,D_{z'})$ commutes with z_1 for every j. Then

$$Q(\zeta_1,z',D_{\zeta_1},D_{z'})=\beta_k^{\pm}\cdot P(z,D_z)\cdot(\beta_k^{\pm})^{-1}=\sum_{l=-\infty}^{m}Q_l(\zeta_1,z',D_{\zeta_1},D_{z'}),$$

where Q_l is the l-th homogeneous part of Q, satisfies the following relations.

$$\left.\frac{\partial^{r+s}Q_l(\zeta_1,z';\tau,\zeta')}{\partial\zeta_1^{\,r}\partial\tau^{s}}\right|_{\tau=0}=0$$

for every $r,s\geq 0$ and l satisfying $r>s+m$, where τ and ζ' are variables corresponding to D_{ζ_1} and $D_{z'}=(D_{z_2},\ldots,D_{z_n})$ respectively.

<u>Proof</u> We may assume that each $P_j(z,D_{z'})$ is defined at $(0;\zeta_0')$, so P has the Taylor expansion in z as follows.

$$P(z,D_z)=\sum_{l=0}^{m}\sum_{J=(j_1,,j_n)\geq 0}D_{z_1}^{\,l}\,z^{J}\,R_J^{l}(D_{z'}),$$

where each $R_J^l(D_{z'})=R_J^l(D_{z_2},,D_{z_n})$ is a pseudo-differential operator with constant coefficient defined in some neighborhood of $\{\zeta'=\zeta_0'\}$, and this series converges as a pseudo-differential operator at $(0;\zeta_0')$. Therefore from Proposition 1.1.4 $Q=\beta_k^{\pm}\cdot P\cdot(\beta_k^{\pm})^{-1}$ is written in the form

$$Q(\zeta_1,z',D_{\zeta_1},D_{z'})=\sum_{l=0}^{m}\sum_{J\geq 0}(\mp i\zeta_1D_{z_k})^{l}z_2^{j_2}\overset{\vee}{\underset{k}{\cdots}}z_n^{j_n}(z_k+D_{\zeta_1}D_{z_k}^{-1}\zeta_1)^{j_k}$$
$$\times(\mp iD_{\zeta_1}D_{z_k}^{-1})^{j_1}R_J^l(D_{z'})$$

$$=\sum_{l=0}^{m}\sum_{J\geq 0}(\mp i)^{l+j_1}z_2^{j_2}\overset{\vee}{\underset{k}{\cdots}}z_n^{j_n}\zeta_1^{\,l}D_{z_k}^{\,l}(z_k+D_{\zeta_1}\zeta_1D_{z_k}^{-1})^{j_k}D_{\zeta_1}^{j_1}D_{z_k}^{-j_1}R_J^l(D_{z'}).$$

Consequently it suffices to prove this theorem in the case of

$Q=\zeta_1{}^l D_{z_k}{}^l (z_k + D_{\zeta_1}\zeta_1 D_{z_k}{}^{-1})^j D_{\zeta_1}{}^t$, where $0\le l\le m$ and $j,t\ge 0$. We claim that $D_{z_k}{}^l (z_k + D_{\zeta_1}\zeta_1 D_{z_k}{}^{-1})^j$ has the following expansion.

$$D_{z_k}{}^l (z_k + D_{\zeta_1}\zeta_1 D_{z_k}{}^{-1})^j = \sum_{\text{finite sum}} A^{l,j}_{a,b,c,d}\, \zeta_1{}^a z_k{}^b D_{\zeta_1}{}^c D_{z_k}{}^d ,$$

where $\{A^{l,j}_{a,b,c,d}\}$ are constants and the summation is taken over all index satisfying $a\ge 0, b\ge 0, c\ge 0$, $l\ge d\ge l-j$ and $a\le c$. This claim is easily proved by induction on j. Thus Q has the following expansion;

$$Q = \sum A^{l,j}_{a,b,c,d}\, \zeta_1{}^{a+l} z_k{}^b D_{\zeta_1}{}^{c+t} D_{z_k}{}^d .$$

By remarking that $a+l\le c+l\le c+t+l\le (c+t)+m$, we obtain the desired result,

$$\left.\frac{\partial^{r+s} Q_i(\zeta_1, z_k; \tau, \zeta_k)}{\partial \zeta_1{}^r \partial \tau^s}\right|_{\tau=0} = 0 \quad \text{for every } r,s \text{ such that } r>s+m,$$

where Q_i is the i-th homogeneous part of Q.

Theorem 1.2.3. We inherit the notations from the previous lemma. Let K be a compact set in ζ_1-plane with real analytic boundary γ. Recall that $Q(\zeta_1, x', D_{\zeta_1}, D_{x'})$ is defined in a neighborhood of $H=\{(\zeta_1, x'; 0, i\eta')\in U_k^{\pm}\ ;\ x'=x_0',\ \eta'=\eta_0'\}\cong \mathbb{C}$, and that the restriction of its principal symbol $Q_m(\zeta_1, x'; \tau, \zeta')$ to H is a polynomial in ζ_1 of degree m. We assume that $Q_m(\zeta_1, x_0'; 0, i\eta_0')$ never vanishes on γ. Then setting $\widetilde{K}=\{(\zeta_1, x_0'; 0, i\eta_0')\in H;\ \zeta_1\in K\}$, we have the following direct decomposition of sections of $C\mathcal{O}$.

$$\Gamma(\widetilde{K}, C\mathcal{O}) = Q(\zeta_1, x', D_{\zeta_1}, D_{x'})\Gamma(\widetilde{K}, C\mathcal{O}) \oplus \sum_{l=0}^{s-1} C_N\big|_{(x_0'; i\eta_0')}\cdot \zeta_1{}^l,$$

where s is the number of zeros of Q_m in $\widetilde{K}$ counting multiplicities.

Proof By the Weierstrass's division theorem for pseudo-

-differential operators, $P(x,D_x)$ is decomposed into the product $P=P'P''$ of pseudo-differential operators defined on $\{(0,x_0';\zeta_1,i\eta_0');\zeta_1\in\mathbb{C}\}$, where P'' is invertible on $\{(0,x_0';\zeta_1,i\eta_0')\in S_N^*X;\ \zeta_1\in K,\ (\eta_{0,k}'=\pm 1)\}$ and P' is written in the form

$$P'(x,D_x)= D_{x_1}^{\ s} + P_1'(x,D_{x'})D_{x_1}^{\ s-1}+\cdots+ P_s'(x,D_{x'})$$

with order of $P_j'\leqq j$ and $[P_j',\ x_1]=0$ for every j. So putting $Q'(\zeta_1,x',D_{\zeta_1},D_{x'})=\beta_k^{\pm}\cdot P'(x,D_x)\cdot(\beta_k^{\pm})^{-1}$, we prove this theorem for Q' in place of Q. In other words we may assume s=m.

Let $h(\zeta_1,x')$ be a section of $C\mathcal{O}$ on $\widetilde{K}$, then the following integral is well-defined as a section of $C\mathcal{O}$ on $\widetilde{K}$;

(1.1) $$h_1(\zeta_1,x') = \frac{1}{2\pi i}\oint_{\gamma} \frac{1}{w-\zeta_1}\ (Q^{-1}h)(w,x')dw.$$

In fact, due to the Cauchy's integral theorem for microfunctions with holomorphic parameters this integral is invariant under the change of path of integration. Now we calculate the difference

$$h_2(\zeta_1,x') = h(\zeta_1,x') - Q(\zeta_1,x',D_{\zeta_1},D_{x'})h_1(\zeta_1,x').$$

(1.2) $$Q(\zeta_1,x',D_{\zeta_1},D_{x'})h_1(\zeta_1,x')$$

$$= \frac{1}{2\pi i}\oint_{\gamma} Q(\zeta_1,x',D_{\zeta_1},D_{x'})(w-\zeta_1)^{-1}(Q^{-1}h)(w,x')dw$$

$$= \frac{1}{2\pi i}\oint_{\gamma}\Big\{\sum_{l=-\infty}^{m}\ \sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial^j(w-\zeta_1)^{-1}}{\partial\zeta_1^{\ j}}\frac{\partial^j Q_l}{\partial\tau^j}(\zeta_1,x',D_{\zeta_1},D_{x'})\Big\}\ (Q^{-1}h)(w,x')dw$$

$$= \frac{1}{2\pi i}\oint_{\gamma}\Big\{\sum_{l=-\infty}^{m}\ \sum_{j=0}^{\infty}(w-\zeta_1)^{-j-1}\frac{\partial^j Q_l}{\partial\tau^j}(\zeta_1,x',0,D_{x'})\Big\}\ (Q^{-1}h)(w,x')dw,$$

where $Q_l(\zeta_1,x',D_{\zeta_1},D_{x'})$ is the homogeneous part of degree l of Q. On the other hand, since Q has a Taylor expansion in D_{ζ_1} as follows;

$$Q(\zeta_1,x',D_{\zeta_1},D_{x'}) = \sum_{l=-\infty}^{m} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j Q_l}{\partial \tau^j}(\zeta_1,x',0,D_{x'})D_{\zeta_1}^j .$$

So

$$(1.3) \qquad h(\zeta_1,x') = \frac{1}{2\pi i}\oint_\gamma (w-\zeta_1)^{-1}h(w,x')dw$$

$$= \frac{1}{2\pi i}\oint_\gamma (w-\zeta_1)^{-1}Q(w,x',D_w,D_{x'})(Q^{-1}h)(w,x')dw$$

$$= \frac{1}{2\pi i}\oint_\gamma \left\{\sum_{l=-\infty}^{m} \sum_{j=0}^{\infty} \frac{1}{j!}(w-\zeta_1)^{-1} \frac{\partial^j Q_l}{\partial \tau^j}(w,x',0,D_{x'})D_w{}^j\right\}(Q^{-1}h)(w,x')dw$$

Using the Leibniz's rule

$$(w-\zeta_1)^{-1} \frac{\partial^j Q_l}{\partial \tau^j}(w,x',0,D_{x'})D_w{}^j$$

$$= \sum_{r=0}^{j} (-1)^r \binom{j}{r} D_w{}^{j-r}\left\{\frac{\partial^r}{\partial w^r}\left((w-\zeta_1)^{-1}\frac{\partial^j Q_l}{\partial \tau^j}(w,x',0,D_{x'})\right)\right\} ,$$

$(w-\zeta_1)^{-1}Q(w,x',D_w,D_{x'})$ is decomposed into the sum of two pseudo-differential operators as follows.

$$(w-\zeta_1)^{-1}Q(w,x',D_w,D_{x'})$$

$$= \sum_{l=-\infty}^{m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\left\{\frac{\partial^j}{\partial w^j}\left((w-\zeta_1)^{-1}\frac{\partial^j Q_l}{\partial \tau^j}(w,x',0,D_{x'})\right)\right\}$$

$$+ D_w\left[\sum_{l=-\infty}^{m} \sum_{j=0}^{\infty} \sum_{r=0}^{j-1} \frac{(-1)^r}{r!(j-r)!} D_w{}^{j-r-1}\left\{\frac{\partial^r}{\partial w^r}\left((w-\zeta_1)^{-1}\frac{\partial^j Q_l}{\partial \tau^j}(w,x',0,D_{x'})\right)\right\}\right]$$

Here, the second term does not contribute the integral in (1.3). Therefore it follows that;

$$(1.4) \qquad h(\zeta_1,x') =$$

$$\frac{1}{2\pi i}\oint_\gamma \left[\sum_{l=-\infty}^{m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!}\left\{\frac{\partial^j}{\partial w^j}\left((w-\zeta_1)^{-1}\frac{\partial^j Q_l}{\partial \tau^j}(w,x',0,D_{x'})\right)\right\}\right](Q^{-1}h)(w,x')dw$$

$$= \frac{1}{2\pi i}\oint_\gamma \Big\{\sum_{l=-\infty}^{m}\sum_{j=0}^{\infty}\sum_{r=0}^{j}\frac{(-1)^r}{r!}(w-\zeta_1)^{-j+r-1}\frac{\partial^{r+j}Q_l}{\partial w^r\partial\tau^j}(w,x',0,D_{x'})\Big\}$$

$$\times(Q^{-1}h)(w,x')dw.$$

From (1.2) and (1.4), we obtain the following.

$$h_2(\zeta_1,x') =$$

$$\frac{1}{2\pi i}\oint_\gamma \Big[\sum_{l=-\infty}^{m}\sum_{j=0}^{\infty}(w-\zeta_1)^{-j-1}\Big\{\sum_{r=0}^{j}\frac{(\zeta_1-w)^r}{r!}\frac{\partial^{r+j}Q_l}{\partial w^r\partial\tau^j}(w,x',0,D_{x'})$$

$$-\frac{\partial^j Q_l}{\partial\tau^j}(\zeta_1,x',0,D_{x'})\Big\}\Big](Q^{-1}h)(w,x')dw$$

$$= \frac{-1}{2\pi i}\oint_\gamma\Big\{\sum_{l=-\infty}^{m}\sum_{j=0}^{\infty}(w-\zeta_1)^{-j-1}\Big(\sum_{r=j+1}^{\infty}\frac{(\zeta_1-w)^r}{r!}\frac{\partial^{r+j}Q_l}{\partial w^r\partial\tau^j}(w,x',0,D_{x'})\Big)\Big\}$$

$$\times(Q^{-1}h)(w,x')dw.$$

Further by Lemma 1.2.2,

$$= -\frac{1}{2\pi i}\oint\Big\{\sum_{l=-\infty}^{m}\sum_{j=0}^{\infty}(w-\zeta_1)^{-j-1}\Big(\sum_{r=j+1}^{j+m}\frac{(\zeta_1-w)^r}{r!}\frac{\partial^{r+j}Q_l}{\partial w^r\partial\tau^j}(w,x',0,D_{x'})\Big)\Big\}$$

$$\times(Q^{-1}h)(w,x')dw.$$

Consequently

(1.5) $\quad h_2(\zeta_1,x') =$

$$\frac{1}{2\pi i}\oint_\gamma\sum_{r=0}^{m-1}(\zeta_1-w)^r\Big(\sum_{l=-\infty}^{m}\sum_{j=0}^{\infty}\frac{(-1)^j}{(r+j+1)!}\frac{\partial^{2j+r+1}Q_l}{\partial w^{j+r+1}\partial\tau^j}(w,x',0,D_{x'})\Big)(Q^{-1}h)(w,x')dw.$$

From this expression, $h_2(\zeta_1,x')$ is a polynomial in ζ_1 of degree less than m-1 with coefficients in germs of C_N at $(x_0';i\eta_0')$.

Next we will prove the uniqueness of this decomposition. To do so, it is sufficient to prove that

(1.6) $$\oint_\gamma(w-\zeta_1)^{-1}(Q^{-1}h)(w,x')dw = 0$$

as a section of $C\mathcal{O}$ on $\widetilde{K}$ for every $h \in \sum_{r=0}^{m-1} C_N|_{(x_0';i\eta_0')} \cdot \zeta_1^r$. Since h is a section defined on H and Q is invertible on $H \setminus \mathrm{int}(\widetilde{K})$ by the assumption, the above integral is equal to

$$\oint_{|w|=1/R} (w-\zeta_1)^{-1}(Q^{-1}h)(w,x')dw$$

$$= \oint_{|v|=R} v^{-1}(1-\zeta_1 v)^{-1}(Q^{-1}h)(v^{-1},x')dv$$

for sufficiently small R>0 such that $\{|\zeta_1|< R^{-1}\} \supset \widetilde{K}$. Write that

$$P(x,D_x) = D_{x_1}^{m} + \sum_{j=1}^{m} D_{x_1}^{m-j} P_j'(x,D_{x'}),$$

where order of $P_j' \leq j$ and $[P_j', x_1]=0$. Then by Proposition 1.1.4 $Q=\beta_k^{\pm} \cdot P \cdot (\beta_k^{\pm})^{-1}$ is written as follows.

$$(\mp iwD_{x_k})^m + \sum_{j=1}^{m} (\mp iwD_{x_k})^{m-j} P_j'(\mp iD_w D_{x_k}^{-1}, x_2, \ldots, x_{k-1}, x_k + D_w w D_{x_k}^{-1}, x_{k+1}, \ldots$$

$$\ldots, x_n; D_{x'})$$

In the coordinate system $(v=w^{-1}, x_2, \ldots, x_n)$,

$$Q = (\mp iv^{-1}D_{x_k})^m + \sum_{j=1}^{m} (\mp iv^{-1}D_{x_k})^{m-j} P_j'(\pm iv^2 D_v D_{x_k}^{-1}, x_2, \ldots, x_{k-1},$$

$$x_k - vD_v D_{x_k}^{-1} + D_{x_k}^{-1}, x_{k+1}, \ldots, x_n; D_{x'})$$

$$= v^{1-m}\{(\mp iD_{x_k})^m + \sum_{j=1}^{m} (\mp iD_{x_k})^{m-j} v^{j-1} P_j'(\pm iv^2 D_v D_{x_k}^{-1}, x_2, \ldots, x_{k-1},$$

$$x_k - vD_v D_{x_k}^{-1} + D_{x_k}^{-1}, x_{k+1}, \ldots, x_n; D_{x'})v\}v^{-1}.$$

Therefore for $h(w,x') = \sum_{r=0}^{m-1} f_r(x')v^{-r} \in \sum_{r=0}^{m-1} C_N|_{(x_0';i\eta_0')} \cdot w^r$,

$$v^{-1}(1-\zeta_1 v)^{-1}(Q^{-1}h)(v^{-1},x') = (1-\zeta_1 v)^{-1}(R(v,x',D_v,D_{x'}))^{-1}\left(\sum_{r=0}^{m-1} f_r(x')v^{m-r-1}\right),$$

where $R(v,x',D_v,D_{x'}) = (\mp iD_{x_k})^m + \sum_{j=1}^{m} (\mp iD_{x_k})^{m-j} v^{j-1} P_j'(\pm iv^2 D_v D_{x_k}^{-1},$

$x_2, \ldots, x_{k-1}, x_k - vD_v D_{x_k}^{-1} + D_{x_k}^{-1}, x_{k+1}, \ldots, x_n; D_{x'})v$ is an invertible pseudo-differential operator in (v,x') defined in the neighborhood of $\{(v=0,x'=x_0';i\eta_0'dx')\}$. So the residue of $v^{-1}(1-\zeta_1 v)^{-1}(Q^{-1}h)(v^{-1},x')$ at $v=0$ is equal to zero. That is to say, the equation (1.6) holds. Thus the proof is completed.

Remark From the proof it is easily understood that the hypothesis of the analyticity of $\gamma=\partial K$ is unnecessary.

Corollary 1.2.4. (The division theorem for $C_{N|X}$)

We fix a point $(x_0';i\eta_0')\in iS^*N$. Let K be a compact subset of $\tau^{-1}((x_0';i\eta_0'))=\{(0,x';\zeta_1,i\eta')\in S_N^*X;\ x'=x_0',\ \eta'=\eta_0'\}$ and $P(x,D_x)$ be a pseudo-differential operator of finite order defined in the neighborhood of $\tau^{-1}((x_0';i\eta_0'))$. Assume that the principal symbol $\sigma(P)$ of P never vanishes on $\partial K=K\setminus\mathrm{int}(K)$. Then, putting $s=$(the number of zeros of $\sigma(P)$ in $\mathrm{int}(K)$), the following direct decomposition of sections of $C_{N|X}$ holds.

$$\Gamma(K,\ C_{N|X}) = P(x,D_x)\Gamma(K,\ C_{N|X}) \oplus \sum_{r=0}^{s-1} C_N\Big|_{(x_0';i\eta_0')}\cdot\delta^{(r)}(x_1)$$

Proof Remarking that P is decomposed into the product $P=P'P''$ of pseudo-differential operators defined on $\tau^{-1}((x_0';i\eta_0'))$, where P'' is invertible on K and P' is of the form

$$D_{x_1}^{\,s} + P_1(x,D_{x'})D_{x_1}^{\,s-1} + \cdots + P_s(x,D_{x'}),$$

this decomposition is a direct consequence of Theorem 1.2.3.

§2. Micro local boundary value problems.

Through this section we formulate boundary value problems in micro local sense and introduce some general tools to treat them. Especially the micro local Green's formula will be indispensable for the study of mixed boundary value problems.

2.1. The concept of mildness at the boundary.

Let $u(x)$ be a hyperfunction defined on $U=\{x\in\mathbb{R}^n;\ x_1>0,\ |x|<R\}$. Then the boundary value $u(+0,x')$ is not well-defined in general as a hyperfunction in x', even if $u(x)$ depends on the variable x_1 analytically in U. But if $u(x)$ is extensible to the neighborhood of $\{x_1=0,\ |x'|<R\}$ as a hyperfunction with real analytic parameter x_1, the boundary value $u(+0,x')$ is defined as the substitution $u(0,x')$ (see S-K-K [10] CH I). In this section we extend this trace operation to an operation on a wider class of hyperfunctions. In the sequel a section of B_M defined on $\{x\in\mathbb{R}^n; x_1>0,\ |x|<\varepsilon\}$ with some $\varepsilon>0$ is identified with a germ of $\mathcal{H}^0_{M+}(B_M)/\mathcal{H}^0_N(B_M)$ at $x=0$. We inherit the notations $M, N, M+, X, Y, ..$etc from §1.

<u>Definition 2.1.1.</u> Let $u(x)$ be a section of $\mathcal{H}^0_{M+}(B_M)$ defined in a neighborhood of $x_0\in N$. Then <u>$u(x)$ is said to be mild at</u> x_0 <u>from the positive side of N</u>, if and only if $u(x)$ is extensible to $S^*_YX\cap S^*_NX\cap\pi_{N/X}^{-1}(x_0)$ as a section of $C_{N|X}$. In fact $u(x)$ is identified with a section of $C_{M+|X}$ on $\overline{G}+$, so it always defines a section of $C_{N|X}$ on a neighborhood of $S^*_YX\cap S^*_NX\cap G+\cap\pi_{N/X}^{-1}(x_0)$. Furthermore by Proposition 1.1.3 and the fact that $\mathcal{H}^0_{S^*_YX\cap S^*_NX}(C_{N|X})$ has the unique continuation property along fibers of $\pi: S^*_YX\cap S^*_NX\to N$ (see [8] 4.3), the extension of $u(x)$ to $S^*_YX\cap S^*_NX\cap\pi_{N/X}^{-1}(x_0)$ is unique if it exists.

Let $v(x)=[u(x)]$ be a germ of $\mathcal{H}^0_{M+}(B_M)/\mathcal{H}^0_N(B_M)$ at $x_0\in N$, where $u(x) \in\mathcal{H}^0_{M+}(B_M)_{x_0}$. Then <u>$v(x)$ is said to be mild at</u> x_0 <u>from the positive side of N</u> if and only if $u(x)$ is mild at x_0 from the positive side of N. Since $\mathcal{H}^0_N(B_M)\subset\pi_{N/X*}C_{N|X}$, this definition does not depend on the choice of $u(x)$.

<u>Remark 2.1.2.</u> From the definition, if $u(x)\in\mathcal{H}^0_{M+}(B_M)/\mathcal{H}^0_N(B_M)$ is mild at $x_0\in N$, u is mild at x for sufficiently close x to x_0 in N. So mildness is a sheaf-theoretic concept. Hereafter by $B_{N|M\pm}$ $[\mathcal{H}^0_{M\pm}(B_M)/\mathcal{H}^0_N(B_M)]_N$ is denoted and the sheaf of germs of mild sections of $B_{N|M\pm}$ from the positive (or negative) side of N is denoted by $\hat{B}_{N|M\pm}$. Then $\hat{B}_{N|M\pm}$ is $\mathcal{D}_X$-subsheaf of $B_{N|M\pm}$.

<u>Example 2.1.3.</u> Let $P(x,D_x)$ be a differential operator of finite order defined on the neighborhood of $x_0\in N$ in M. Suppose that N is non-characteristic with respect to P. Then every solution of P in $B_{N|M\pm}|_{x_0}$ is mild at x_0 from the positive (negative) side of N. In fact, let $u(x)$ be a germ of $\mathcal{H}^0_{M+}(B_M)$ at x_0 with $Pu=f\in\mathcal{H}^0_N(B_M)$. Then, since P is invertible on S^*_YX as a pseudo-differential operator by the non-charactericity of N and f is identified with a section of $C_{N|X}$ on $\pi_{N/X}^{-1}(x_0)$, $P^{-1}f$ gives the extension of $u(x)$ to $S^*_YX\cap S^*_NX\cap\pi_{N/X}^{-1}(x_0)$ as a section of $C_{N|X}$.

<u>Example 2.1.4.</u> Let $u(x)$ be a germ of B_M at $x_0\in N$. Suppose that $SS(u(x))\cap iS^*_NM\cap\pi^{-1}(x_0)=\phi$ (that is, u is real analytic in the direction transversal to N. See S-K-K [10] CH I). Then $u(x)\big|_{\mathrm{int}(M\pm)}$

is mild at x_0 from the positive side (negative side) of N. We leave the proof of mildness of u in the later part of this section, where the more intuitive explanation of mildness will be given.

To proceed to the micro-localization of mildness, we introduce the sheaf $\hat{C}_{N|M+}$ on iS^*N.

<u>Definition 2.1.5.</u> We define $S_N^*X^\infty$ and τ^∞ canonically as follows.

$$\begin{array}{ccc} S_N^*X^\infty = (S_N^*X \setminus S_Y^*X) \cup iS^*N \times \{\infty\} & \xrightarrow{\tau^\infty} & iS^*N \\ \quad\quad \psi \quad\quad\quad\quad\quad \psi & & \psi \\ (0,x';\zeta_1,i\eta') \quad (0,x';\infty,i\eta') & & (x';i\eta') \end{array}$$

Clearly $S_N^*X^\infty$ is a real analytic manifold and τ^∞ is a real analytic projection with fiber $\mathbb{P}^1=\mathbb{C}\cup\{\infty\}$. Putting i=the injection : $S_N^*X \setminus S_Y^*X \longrightarrow S_N^*X^\infty$, we define

$$C_{N|X}^\infty = i_*(C_{N|X}|_{S_N^*X \setminus S_Y^*X}).$$

as a sheaf on $S_N^*X^\infty$. Set the projection

$$\tau^+ \ ; \quad \overline{G+} \setminus S_Y^*X = \{(0,x';\zeta_1,i\eta') \in S_N^*X;\ \mathrm{Re}\,\zeta_1 \geq 0,\ \eta' \neq 0\} \longrightarrow (x';i\eta') \in iS^*N,$$

then the sheaf $\hat{C}_{N|M+}$ on iS^*N is defined as

$$\hat{C}_{N|M+} = \tau_*^+ C_{M+|X} \cap C_{N|X}^\infty |_{iS^*N \times \{\infty\}} \,/\, \tau_* C_{N|X} \,.$$

Here we remark that $\tau_* C_{N|X} = \tau_*^\infty C_{N|X}^\infty$. Obviously $\hat{C}_{N|M+}$ is a $\tau_*\mathcal{P}_X$-module.

Furthermore, to analyse the relation of $\hat{B}_{N|M+}$ and $\mathcal{H}_{M+}^0(B_M)$ we introduce the sheaf $C_{N|M+}$ on iS^*N.

Definition 2.1.6. $C_{N|M+}$ is defined as the image sheaf under the sheaf homomorphism

$$\pi_N^{-1}\mathcal{H}^0_{M+}(B_M)\big|_N \longrightarrow \iota_*^+(C_{M+|X}\big|_{\overline{G}+\setminus S_Y^*X}).$$

This is, of course, isomorphic to the image sheaf under

$$\pi_N^{-1}\mathcal{H}^0_{M+}(B_M)\big|_N \longrightarrow \iota\big|_{I*}(C_M\big|_{I\setminus S_Y^*X}),$$

since $C_{M+|X}\big|_I \longrightarrow C_M\big|_I$ is injective. $C_{N|M+}$ is a $\pi_N^{-1}\mathcal{O}_X$-module, but not a $\iota_*\mathcal{P}_X$-module.

Fix a coordinate (x_1,x') of M and the induced sheaf isomorphism β_k^ε ; $C_{N|X}\big|_{U_k^\varepsilon} \longrightarrow C\mathcal{O}\big|_{U_k^\varepsilon}$ ($\varepsilon=\pm1$). Then the sheaf homomorphism on $V_k^\varepsilon=\{(x';i\eta')\in\mathbb{R}^{n-1}\times i\mathbb{R}^{n-1};\eta_k=\varepsilon\}$

(2.1) $$q_k : C_{N|M+} \longrightarrow \gamma_*(C\mathcal{O}\big|_{G+})$$

is obtained, where γ is a projection given by

$$U_k^\varepsilon\cap G+ =\{(0,x';\zeta_1,i\eta')\in\mathbb{R}^{n-1}\times\mathbb{C}\times i\mathbb{R}^{n-1};\mathrm{Re}\,\zeta_1>0,\eta_k=\varepsilon\}\longrightarrow V_k^\varepsilon.$$

Surely this is injective by Proposition 1.1.3. So we can study Image(q_k) instead of $C_{N|M+}$.

Definition 2.1.7. Set $L, L+, \widetilde{L}, \widetilde{L}+$ and π, λ as follows.

$$L = \mathbb{P}^1\times\mathbb{R}^{n-1}=\{(\zeta_1,x')=(\zeta_1,x_2,\ldots,x_n)\in(\mathbb{C}\cup\{\infty\})\times\mathbb{R}^{n-1}\}$$
$$\cup$$
$$L+=\tfrac{1}{2}\mathbb{P}^1\times\mathbb{R}^{n-1}=\{(\zeta_1,x')\in L;\ \mathrm{Re}\,\zeta_1\geq 0 \text{ or } \zeta_1=\infty\}$$
$$\widetilde{L} = \mathbb{P}^1\times iS^*\mathbb{R}^{n-1}=\{(\zeta_1,x';i\eta')\in(\mathbb{C}\cup\{\infty\})\times\mathbb{R}^{n-1}\times(i\mathbb{R}^{n-1}/\mathbb{R}+)\}$$
$$\cup$$
$$\widetilde{L}+=\tfrac{1}{2}\mathbb{P}^1\times iS^*\mathbb{R}^{n-1}=\{(\zeta_1,x';i\eta')\in\widetilde{L};\ \mathrm{Re}\,\zeta_1\geq 0 \text{ or } \zeta_1=\infty\}$$

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\pi} & L \\ \cup & & \cup \\ \tilde{L}+ & \xrightarrow{\pi} & L+ \end{array} \qquad \begin{array}{ccc} \mathrm{int}(\tilde{L}+) & \xhookrightarrow{\lambda} & \tilde{L}+ \\ \downarrow \pi & & \downarrow \pi \\ \mathrm{int}(L+) & \xhookrightarrow{\lambda} & L+ \end{array}$$

where $\mathrm{int}(\tilde{L}+)=\tilde{L}+\cap\{\mathrm{Re}\,\zeta_1>0,\ \zeta_1\neq\infty\}$, $\mathrm{int}(L+)=L+\cap\{\mathrm{Re}\,\zeta_1>0,\ \zeta_1\neq\infty\}$. We denote by $B\mathcal{O}$ the sheaf of germs of hyperfunctions on $L=\mathbb{P}^1\times\mathbb{R}^{n-1}$ depending holomorphically on ζ_1. Then the sheaf $C\mathcal{O}_+^{\infty}$ is defined on $\tilde{L}+$ as follows.

$$C\mathcal{O}_+^{\infty} = \text{Image of } (\pi^{-1}\lambda_*(B\mathcal{O}|_{\mathrm{int}(L+)}) \longrightarrow \lambda_*(C\mathcal{O}|_{\mathrm{int}(\tilde{L}+)}))$$

$C\mathcal{O}_+^{\infty}$ coincides with $C\mathcal{O}$ on $\mathrm{int}(\tilde{L}+)$, but $C\mathcal{O}_+^{\infty}\subsetneqq\lambda_*(C\mathcal{O}|_{\mathrm{int}(\tilde{L}+)})$ on $\partial\tilde{L}+$. In fact sections of $C\mathcal{O}_+^{\infty}$ have boundary values.

<u>Proposition 2.1.8.</u> The following injective sheaf homomorphism is well-defined.

$$(2.2)\qquad \begin{array}{ccc} C\mathcal{O}_+^{\infty}|_{S^1\times iS^*\mathbb{R}^{n-1}} & \xrightarrow{b} & \mathcal{H}_F^0(C_{S^1\times\mathbb{R}^{n-1}})|_{S^1\times iS^*\mathbb{R}^{n-1}}, \\ \Psi & & \Psi \\ f(\zeta_1,x') & & f(it+0,x') \end{array}$$

where $S^1\times iS^*\mathbb{R}^{n-1}=\{(\zeta_1,x';i\eta')\in\tilde{L}\ ;\ \mathrm{Re}\,\zeta_1=0 \text{ or } \zeta_1=\infty\}$, $S^1\times\mathbb{R}^{n-1}=\{(\zeta_1,x')\in\mathbb{P}^1\times\mathbb{R}^{n-1};\ \mathrm{Re}\,\zeta_1=0 \text{ or } \zeta_1=\infty\}$, and $S^1\times iS^*\mathbb{R}^{n-1}\cong\{(t,x';i\tau,i\eta')\in(\mathbb{R}\cup\{\infty\})\times\mathbb{R}^{n-1}\times(i\mathbb{R}^n/\mathbb{R}+);\ \tau=0\}$, $F=\{(t,x';i\tau,i\eta');\ \tau\leq 0\}$.

Conversely the Cauchy integral

$$(2.3)\qquad h(f(t,x'))=\int_{\mathbb{R}\cup\{\infty\}}\frac{-1}{2\pi i}\cdot\frac{f(t,x')}{it-\zeta_1}i\,dt=\int_{\mathbb{R}\cup\{\infty\}}\frac{-1}{2\pi i}\,\frac{f(u^{-1},x')}{(i-u\zeta_1)(u+i0)}i\,du$$

defined as a sheaf homomorphism

$$h\ ;\quad \alpha_*(\mathcal{H}_F^0(C_{S^1\times\mathbb{R}^{n-1}})|_{S^1\times iS^*\mathbb{R}^{n-1}}) \longrightarrow \alpha_*^+C\mathcal{O}_+^{\infty},$$

where $\alpha:\ S^1\times iS^*\mathbb{R}^{n-1}\longrightarrow iS^*\mathbb{R}^{n-1}$, $\alpha^+\ :\ \tfrac{1}{2}\mathbb{P}^1\times iS^*\mathbb{R}^{n-1}\longrightarrow iS^*\mathbb{R}^{n-1}$ are projections, gives the left inverse of

$$b\ ;\quad \alpha_*^+ C\mathcal{O}_+^\infty \longrightarrow \alpha_*(C_{S^1\times\mathbb{R}^{n-1}}\Big|_{S^1\times iS^*\mathbb{R}^{n-1}}),$$

that is, $h\circ b$=identity.

Proof The well-definedness of (2.2) is reduced to the N-regularity of the partial Cauchy-Riemann equation by P.Schapira [12]. Here we prove this only by properties of $C_{M+|X}$. Without loss of generality we may prove the well-definedness of b in (2.2) only on $(S^1-\{\infty\})\times V_n^+=\{(t,x';i\eta')\in\mathbb{R}\times\mathbb{R}^{n-1}\times i\mathbb{R}^{n-1};\ \eta_n=1\}\simeq U_n^+\cap I=\{(0,x';it,i\eta');\eta_n=1\}$. Then, according to Theorem 4.2.12 §4 [8], $C\mathcal{O}_+^\infty|_{\mathbb{R}\times V_n^+}$ is isomorphic to $C_{M+|X}|_{U_n^+\cap I}$ by the quantized Legendre transform β_n^+. Therefore, using the sheaf imbedding $i : C_{M+|X}|_I \hookrightarrow \mathcal{H}^0_{H+}(C_M)|_I$ (Theorem 4.2.8 §4 [8]), b is written as the composite homomorphism

$$\tilde{\beta}_n^+\circ i\circ(\beta_n^+)^{-1}\ :\ C\mathcal{O}_+^\infty\Big|_{\mathbb{R}\times V_n^+} \longrightarrow \mathcal{H}_F^0(C_{\mathbb{R}\times\mathbb{R}^{n-1}})\Big|_{\mathbb{R}\times V_n^+},$$

where $\tilde{\beta}_n^+$ is the quantized Legendre transform for microfunctions given by

$$\frac{i(n-1)!}{(-2\pi i)^n}\int_{\eta_n=1}\frac{g(y)dyd\eta_2\cdots d\eta_{n-1}}{(\langle x'-y',\eta'\rangle-y_1t+i0)^n}. \tag{2.4}$$

In fact this homomorphism is injective and we see that it is compatible with the trace homomorphism

$$\begin{array}{ccc} \lambda_*(B\mathcal{O}|_{int(L+)})|_{\partial L+} & \longrightarrow & B_{\mathbb{R}\times\mathbb{R}^{n-1}} \\ \Psi & & \Psi \\ f(\zeta_1,x') & & f(it+0,x') \end{array}$$

as follows. Let $f(\zeta_1,x')$ be a germ of $\lambda_*(B\mathcal{O}|_{int(L+)})$ at $(it_0,x_0')\in\partial L+$. By the partial flabbiness of $B\mathcal{O}$ with respect to the real parameter x' (see Lemma 4.2.16 §4 [8]), we may assume $f(\zeta_1,x')$

is a section of $\mathcal{BO}$ defined on $\{\zeta_1\in\mathbb{C};\ |\zeta_1-it_0|<\delta,\ \mathrm{Re}\,\zeta_1>0\}\times\mathbb{R}^{n-1}$ with support in $\{(\zeta_1,x')\in\mathbb{C}\times\mathbb{R}^{n-1};\ |x'-x_0'|\le 1\}$. So we have

$$(\beta_n^+)^{-1}([f]) = \sigma\Big(\frac{-(n-2)!}{(-2\pi i)^{n-1}}\Big\{\int_{\mathbb{R}^{n-1}}\frac{f(\zeta_1,y')dy'}{(\langle z,\zeta\rangle-\langle y',\zeta'\rangle)^{n-1}}\Big\}_{\zeta_n=i}\, d\sigma(\zeta)\Big).$$

Therefore $i\circ(\beta_n^+)^{-1}([f]) =$

$$-\frac{i(n-2)!}{(-2\pi i)^{n-1}}\int d\eta_1\cdots d\eta_{n-1}\Big(\int_{\mathbb{R}^{n-1}}\frac{f(i\eta_1+0,y')dy'}{(x_1\eta_1+\langle x'-y',\eta'\rangle+i0)^{n-1}}\Big|_{\eta_n=1}\Big).$$

This integral is well-defined as an integral transformation for micro-functions, in fact it gives the inverse transformation of (2.4). (cf. Theorem 4.2.8 §4 [8]) So $\tilde{\beta}_n^+\circ i\circ(\beta_n^+)^{-1}([f]) = [f(it+0,x')]\big|_{\mathbb{R}\times V_n^+}$.

Next, let $g(t,x')$ be a germ of $\alpha_*(\mathcal{H}_F^0(\mathcal{C}_{S^1\times\mathbb{R}^{n-1}})\big|_{S^1\times iS^*\mathbb{R}^{n-1}})$ at $p_0\in iS^*\mathbb{R}^{n-1}$. By the flabbiness of $\mathcal{C}_{S^1\times\mathbb{R}^{n-1}}$ and the fundamental exact sequence, there exists a hyperfunction $f(t,x')$ defined on $S^1\times\mathbb{R}^{n-1}$ such that $[f(t,x')]=g(t,x')$ on $\alpha^{-1}(p_0)$ and $SS(f)\subset F$. So

$$A(\zeta_1,x') = \int_{\mathbb{R}\cup\{\infty\}} -\frac{1}{2\pi i}\,\frac{f(t,x')}{it-\zeta_1}\,i\,dt$$

$$\Big(= \int_{\mathbb{R}\cup\{\infty\}} -\frac{1}{2\pi i}\,\frac{f(u^{-1},x')}{(i-u\zeta_1)(u+i0)}\,i\,du\ \Big)$$

is a hyperfunction with holomorphic parameter defined on $\{(\zeta_1,x')\in\mathbb{C}\times\mathbb{R}^{n-1};\ \mathrm{Re}\,\zeta_1>0\}$. On the other hand, by the theory of integration for microfunctions, the spectre of $A(\zeta_1,x')$ on $\{(\zeta_1,x';i\eta'dx');\mathrm{Re}\,\zeta_1>0,\ (x';i\eta')=p_0\}$ is uniquely determined as a section of microfunctions with holomorphic parameter only by $g(t,x')$. So h defines a sheaf homomorphism $\alpha_*(\mathcal{H}_F^0(\mathcal{C}_{S^1\times\mathbb{R}^{n-1}})\big|_{S^1\times iS^*\mathbb{R}^{n-1}}) \longrightarrow \alpha_*^+\mathcal{C}\mathcal{O}_+^\infty$.

Let $f(\zeta_1,x')$ be a germ of $\alpha_*^+\mathcal{C}\mathcal{O}_+^\infty$ at $p_0\in iS^*\mathbb{R}^{n-1}$. So $(\zeta_1+1)f(\zeta_1,x')$

is also a germ of $\alpha^{+}_{*}C\mathcal{O}^{\infty}_{+}$ at the same point. Then by the theory of boundary value problems for hyperfunctions (Komatsu-Kawai [9], Schapira [11]) there exists a unique section $g(\zeta_1,x')$ of $C_{\mathbb{P}^1\times\mathbb{R}^{n-1}}$ (we write $\zeta_1=s+it$) on a neighborhood of $\mathbb{P}^1\times\{p_0\}=\{(s,t,x';i\sigma,i\tau,i\eta')\in(\mathbb{R}\times\mathbb{R}\cup\{\infty\})\times\mathbb{R}^{n-1}\times(i\mathbb{R}^{1+n}/\mathbb{R}+);\ \sigma=\tau=0,\ (x';i\eta')=p_0\}$ with support in $\{(s,t,x';i\sigma,i\tau,i\eta');\ s\geq 0 \text{ or } s+it=\infty\}$ satisfying

$$g(\zeta_1,x') = (\zeta_1+1)f(\zeta_1,x') \quad \text{on } \mathbb{P}^1\times\{p_0\}\cap\{s>0\}$$

and

$$\bar{\partial}_{\zeta_1}g(\zeta_1,x') = \tfrac{1}{2}(1+it)f(it+0,x')\delta(s)d\bar{\zeta}_1 \qquad \text{on } \mathbb{P}^1\times\{p_0\},$$

where the last condition for $\zeta_1=\infty$ means that, putting $w=u+iv=\zeta_1^{-1}$,

$$\bar{\partial}_w g(w^{-1},x') = \tfrac{1}{2}\cdot\frac{1+iv}{iv+0}\, f((iv+0)^{-1},x')\delta(u)d\bar{w}\ .$$

So $(\zeta_1+1)^{-1}(\zeta_1-\tilde{\zeta}_1)^{-1}g(\zeta_1,x')d\zeta_1$ is well-defined as a differential form defined on $\{(\tilde{\zeta}_1,\zeta_1,x';i\eta'dx')\in\mathbb{C}\times\mathbb{P}^1\times iS^*\mathbb{R}^{n-1};\ \mathrm{Re}\,\tilde{\zeta}_1>0,\ \zeta_1\neq\tilde{\zeta}_1,\ (x';i\eta')=p_0\}$, in addition,

$$d_{\zeta_1}\{(\zeta_1+1)^{-1}(\zeta_1-\tilde{\zeta}_1)^{-1}g(\zeta_1,x')d\zeta_1\} = i(it-\tilde{\zeta}_1)^{-1}f(it+0,x')\delta(s)ds\wedge dt$$

$$(= -d_w\{(1+w)^{-1}(1-w\tilde{\zeta}_1)^{-1}g(w^{-1},x')dw\} = -i(1-iv\tilde{\zeta}_1)^{-1}(iv+0)^{-1}f((iv+0)^{-1},x')$$
$$\times\,\delta(u)du\wedge dv\ ).$$

Therefore $\quad h(f(it+0,x')) = \int_{\mathbb{R}\cup\{\infty\}} \frac{i}{2\pi}(it-\tilde{\zeta}_1)^{-1}f(it+0,x')i\,dt$

$$= \int_{\mathbb{P}^1-\{\tilde{\zeta}_1\}} \frac{i}{2\pi}\, d_{\zeta_1}\{(\zeta_1+1)^{-1}(\zeta_1-\tilde{\zeta}_1)^{-1}g(\zeta_1,x')d\zeta_1\}\ .$$

Now we confine $\tilde{\zeta}_1$ in $E=\{\tilde{\zeta}_1\in\mathbb{C};\ |\tilde{\zeta}_1-2|<1\}$, so

$$= \int_{\mathbb{P}^1} \frac{i}{2\pi}\, d_{\zeta_1}\{(1-\chi_E(\zeta_1))(\zeta_1+1)^{-1}(\zeta_1-\tilde{\zeta}_1)^{-1}g(\zeta_1,x')d\zeta_1\}$$

$$+ \int_{\mathbb{P}^1-\{\tilde{\zeta}_1\}} \frac{i}{2\pi} d_{\zeta_1}\{ \chi_E(\zeta_1)(\zeta_1-\tilde{\zeta}_1)^{-1} f(\zeta_1,x')d\zeta_1\},$$

where $\chi_E(\zeta_1)$ is the characteristic function of $E\subset\mathbb{P}^1$, and by the formula of Stokes the first term vanishes. Consequently

$$h(f(it+0,x')) = \int_{\mathbb{P}^1-\{\tilde{\zeta}_1\}} \frac{i}{2\pi} d_{\zeta_1}\{\chi_E(\zeta_1)(\zeta_1-\tilde{\zeta}_1)^{-1} f(\zeta_1,x')d\zeta_1\}$$

$$= \int_{|\zeta_1-2|=1} \frac{1}{2\pi i} (\zeta_1-\tilde{\zeta}_1)^{-1} f(\zeta_1,x')d\zeta_1 = f(\tilde{\zeta}_1,x')$$

Considering the identity theorem, this shows $h\circ b(f)=f$.

<u>Corollary 2.1.9.</u> The sheaf $\alpha_*^+ C\mathcal{O}_+^\infty$ is flabby.

<u>Proof</u> Consider the following decomposition of identity.

$$\alpha_*^+ C\mathcal{O}_+^\infty \xrightarrow{\ b\ } \alpha_*(\mathcal{H}_F^0(C_{S^1\times\mathbb{R}^{n-1}})\big|_{S^1\times iS^*\mathbb{R}^{n-1}}) \xrightarrow{\ h\ } \alpha_*^+ C\mathcal{O}_+^\infty$$

$$h\circ b = \text{identity}.$$

So the flabbiness of $\alpha_*^+ C\mathcal{O}_+^\infty$ follows from the flabbiness of the sheaf of microfunctions $C_{S^1\times\mathbb{R}^{n-1}}$.

<u>Proposition 2.1.10.</u> In the sheaf homomorphism q_k of (2.1), Image(q_k) coincides with the sheaf $\alpha_*^+ C\mathcal{O}_+^\infty$. Especially, since flabbiness is a local property, $C_{N|M+}$ is flabby.

<u>Proof</u> We inherit the notations from the preceding propositions. The proof of Image$(q_k)\subset\alpha_*^+ C\mathcal{O}_+^\infty$ is included in the proof of Theorem 4.2.12 §4 [8]("$G(\zeta_1,x')$" represents a section of $\alpha_*^+ C\mathcal{O}_+^\infty$), so we omit it here. Let $g(\zeta_1,x')$ be a germ of $\alpha_*^+ C\mathcal{O}_+^\infty$ at $p_0=(x_0';i\eta_0')\in iS^*\mathbb{R}^{n-1}$. Then, by the proof of Proposition 2.1.8, $g(\zeta_1,x')$ is represented by a section $A(\zeta_1,x')$ of $B\mathcal{O}$ defined on $\{(\zeta_1,x')\in\mathbb{C}\times\mathbb{R}^{n-1};\mathrm{Re}\,\zeta_1>0\}$. Further

by the partial flabbiness of $B\mathcal{O}$ (Lemma 4.2.16 [8]) we may assume that the support of $A(\zeta_1, x')$ is contained in $\{(\zeta_1, x');\ |x'-x_0'| \leq 1\}$. Therefore we have

$$(\beta_k^{\varepsilon})^{-1}(g) = \sigma\Big(-\frac{(n-2)!}{(-2\pi i)^{n-1}}\Big\{\int_{\mathbb{R}^{n-1}} \frac{A(\zeta_1, y')dy'}{(\langle z,\zeta\rangle - \langle y', \zeta'\rangle)^{n-1}}\Big\}_{\zeta_k = i\varepsilon} d\sigma(\zeta)\Big).$$

On the other hand $G(z, \zeta_1, \ldots \overset{\vee}{\underset{k}{\cdot}} \ldots, \zeta_n) =$

$$-\frac{(n-2)!}{(-2\pi i)^{n-1}}\Big\{\int_{\mathbb{R}^{n-1}} \frac{A(\zeta_1, y')dy'}{(\langle z,\zeta\rangle - \langle y', \zeta'\rangle)^{n-1}}\Big\}_{\zeta_k = i\varepsilon}$$

is holomorphic on $D = \{(z, \zeta_1, \ldots \overset{\vee}{\underset{k}{\cdot}} \ldots, \zeta_n) \in \mathbb{C}^n \times \mathbb{C}^{n-1};\ -\mathrm{Re}\langle z,\zeta\rangle > (|x_0'|+1)\cdot|\mathrm{Re}\,\zeta'|,\ \mathrm{Re}\,\zeta_1 > 0\}$, and so, by the theory of cohomological Radon transformations (see §2.1 and 2.2 [8]), $G(z, \zeta_1, \ldots \overset{\vee}{\underset{k}{\cdot}} \ldots, \zeta_n)d\sigma(\zeta)$ defines an element of $H^{n-1}(\Omega, \mathcal{O}_{\mathbb{C}^n})$, where

$$\Omega = \bigcup_{\zeta \in \{\zeta \in \mathbb{C}^n;\ \mathrm{Re}\,\zeta_1 > 0,\ \mathrm{Re}\,\zeta' = 0,\ \zeta_k = i\varepsilon\}} \{z \in \mathbb{C}^n;\ -\mathrm{Re}\langle z,\zeta\rangle > 0\}.$$

Note that

$$S = \mathbb{C}^n - \Omega = \bigcap_{\{\xi_1 > 0,\ \eta \in \mathbb{R}^n,\ \eta_k = \varepsilon\}} \{z \in \mathbb{C}^n;\ -x_1\xi_1 + \langle y, \eta\rangle \leq 0\}$$

$$= \{z \in \mathbb{C}^n;\ x_1 \geq 0,\ y_1 = \cdots \overset{\vee}{\underset{k}{\cdot}} \cdots = y_n = 0,\ \varepsilon y_k \leq 0\}$$

and $H^{n-1}(\Omega, \mathcal{O}_{\mathbb{C}^n}) = H^n_S(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. Hence $G d\sigma(\zeta)$ defines an element c of $H^n_S(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. Further, using the exact sequence

$$H^n_{M+}(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \longrightarrow H^n_S(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \longrightarrow H^n_{S-M+}(\{z \in \mathbb{C}^n;\ \varepsilon y_k < 0\}, \mathcal{O}_{\mathbb{C}^n})$$

where $M+ = \{z \in \mathbb{C}^n; \mathrm{Im}\,z = 0,\ \mathrm{Re}\,z_1 \geq 0\}$, c is represented by an element of $H^n_{M+}(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = \Gamma_{M+}(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ because $H^n_{S-M+}(\{z \in \mathbb{C}^n;\ \varepsilon y_k < 0\}, \mathcal{O}_{\mathbb{C}^n})$ is isomorphic to $H^{n-1}(\{z\in\mathbb{C}^n; \varepsilon y_k<0\}\setminus\{x_1\geq 0, y_1=\cdots\overset{\vee}{k}\cdots=y_n=0\}, \mathcal{O}_{\mathbb{C}^n})$

$$\simeq H^{n-1}((\mathbb{C}^{n-1}-\{x_1\geq 0, y_1=\cdots=y_{n-1}=0\})\times\{z\in\mathbb{C}; \mathrm{Re}\, z>0\}, \mathcal{O}_{\mathbb{C}^n})$$

and this is zero by the Malgrange's vanishing theorem with Stein holomorphic parameter (cf. Lemma 4.2.16 [8]). Considering the canonical homomorphism

$$H^n_S(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) \longrightarrow \Gamma(V^{\varepsilon}_k, \tau^+_*(C_{M+|X}|_{\overline{G}+\setminus S^*_Y X})),$$

this leads to $(\beta^{\varepsilon}_k)^{-1}(g)\in C_{N|M+}|_{p_0}$. So the proof is completed.

<u>Corollary 2.1.11.</u> A germ $f(x)$ of $\tau^+_*(C_{M+|X}|_{\overline{G}+\setminus S^*_Y X})$ at $p_0\in iS^*N$. belongs to $C_{N|M+}$, in other words f is represented by a germ of $\mathcal{H}^0_{M+}(B_M)$ at $\pi(p_0)$, if and only if the following condition holds.

Choosing a local coordinate (x_1, x') of M with $p_0=(x'_0; i\eta'_0)\in V^{\varepsilon}_k$, then $\beta^{\varepsilon}_k(f)(w^{-1}, x')$ is represented by a section of $B\mathcal{O}$ on $\{(w,x')\in \mathbb{C}\times\mathbb{R}^{n-1}; \mathrm{Re}\, w>0, |w|<\delta, |x'-x'_0|<\delta\}$ as a section of $C\mathcal{O}$ near $\{w=0, x'=x'_0, \eta'=\eta'_0\}$.

In particular, the following pseudo-differential operators operate on $C_{N|M+}$ as sheaf homomorphisms.

$$P = \sum_{j=0}^{\text{finite}} p_j(x, D_{x'})D^j_{x_1}$$

where $\{p_j(x, D_{x'})\}$ are pseudo-differential operators commuting with x_1.

<u>Proof</u> These are direct consequences from Theorem 4.2.12 [8] and Proposition 2.1.10. And the last assertion follows from the explicit form of $\beta^{\varepsilon}_k\circ P\circ(\beta^{\varepsilon}_k)^{-1}$ (cf. Theorem 1.2.3) :

$$\sum_{j=0}^{\text{finite}} p_j(i\varepsilon w^2 D_w D^{-1}_{x_k}, x_2, \ldots, x_{k-1}, x_k - wD_w D^{-1}_{x_k} + D^{-1}_{x_k}, x_{k+1}, \ldots, x_n; D_{x'})(-i\varepsilon w^{-1}D_{x_k})^j.$$

(Recall that $C\mathcal{O}^{\infty}_{+}$ is a $\mathcal{P}_X$-module).

As for $\hat{C}_{N|M+}$, we have the following theorem.

Theorem 2.1.12. $\hat{C}_{N|M+}$ is a soft sheaf on iS^*N.

Proof To simplify the proof, we assume $M=\mathbb{R}^n \ni (x_1,x')$ and $N=\{x_1=0\}$. Then $iS^*\mathbb{R}^{n-1}$ is covered by open sets $\{V_k^\varepsilon\}_{k,\varepsilon}$ ($V_k^\varepsilon=\{(x';i\eta')\in \mathbb{R}^{n-1}\times i\mathbb{R}^{n-1};\ \eta_k=\varepsilon\}$), so we have only to show the following statement.
" For any open subset V of V_k^ε and any section $c\in\Gamma(\overline{V}\cap V_k^\varepsilon,\hat{C}_{N|M+})$ such that the closure of $\mathrm{support}(c)\cap\overline{V}\cap V_k^\varepsilon$ in $iS^*\mathbb{R}^{n-1}$ is contained in V_k^ε, there exists a section $c'\in\Gamma_{(\pi_N|_{V_k^\varepsilon})\text{-proper}}(V_k^\varepsilon,\ \hat{C}_{N|M+})$ satisfying $c'=c$ on V."

We prove this for the case of $k=n$ $\varepsilon=+1$. By the quantized Legendre transform β_n^+, $C^\infty_{N|X}|_{V_n^+\times\{\infty\}}$ is isomorphic to $\mu_*\mathbb{C}\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}|_{\{\infty\}\times V_n^+}$, where μ: $\mathbb{C}\times iS^*\mathbb{R}^{n-1}\hookrightarrow\mathbb{P}^1\times iS^*\mathbb{R}^{n-1}$ is the imbedding. On the other hand, the integral transformation

$$(2.5)\qquad E(f)(\zeta_1,x') = -\frac{1}{2\pi i}\int_{|s|=R}\frac{f(s,x')}{s-\zeta_1}ds$$

with $|\zeta_1|>R$ for $f\in\mu_*\mathbb{C}\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}|_{\{\infty\}\times iS^*\mathbb{R}^{n-1}}$ and sufficiently large $R>0$ induces an injection on $\{\infty\}\times iS^*\mathbb{R}^{n-1}$

$$E\ :\ \mu_*\mathbb{C}\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}|_{\{\infty\}}\,/\,\alpha_*\mathbb{C}\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}\hookrightarrow\mathbb{C}\mathcal{O}_{\mathbb{P}^1\times\mathbb{R}^{n-1}}|_{\{\infty\}},$$

where $\{\infty\}$ means $\{\infty\}\times iS^*\mathbb{R}^{n-1}$ and α: $\mathbb{C}\times iS^*\mathbb{R}^{n-1}\to iS^*\mathbb{R}^{n-1}$ is the projection. In fact $E\circ E=E$ and $E(f)-f\in\alpha_*\mathbb{C}\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}$ for any $f\in\mu_*\mathbb{C}\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}|_{\{\infty\}}$. Hence $E\circ\beta_n^+$ gives the injective sheaf homomorphism on V_n^+

$$(2.6)\quad T\ ;\ \hat{C}_{N|M+} = \iota_*^+C_{M+|X}\cap C^\infty_{N|X}|_{\{\infty\}\times iS^*N}/\iota_*C_{N|X}\longrightarrow\mathbb{C}\mathcal{O}_{\mathbb{P}^1\times\mathbb{R}^{n-1}}|_{\{\infty\}}.$$

Let V be an open subset of V_n^+ and c be a section of $\Gamma(\overline{V}\cap V_n^+,\hat{C}_{N|M+})$

such that the closure of support$(c)\cap\overline{V}\cap V_n^+$ is contained in V_n^+. Then $T(c)$ is a section of $C\mathcal{O}_{\mathbb{P}^1\times R^{n-1}}$ defined on a neighborhood of $\{\infty\}\times(\overline{V}\cap V_n^+)$. By the assumption for c, $T(c)$ is a section of $C\mathcal{O}_{\mathbb{P}^1\times\mathbb{R}^{n-1}}$ on $\{(\zeta_1,x';i\eta')\in\mathbb{P}^1\times iS^*\mathbb{R}^{n-1};(x';i\eta')\in V$ and $R(x')\leqq|\zeta_1|\leqq\infty\}$ for some positive valued real analytic function $R(x')$ defined in $\mathbb{R}^{n-1}$. Setting the real submanifold $L=\{(\zeta_1,x')\in\mathbb{P}^1\times\mathbb{R}^{n-1};\ |\zeta_1|=R(x')\}$, the substitution of $T(c)(\zeta_1,x')$ on L is a section of C_L on $\{(R(x')e^{i\theta},x';i\tau d\theta+i\eta'dx');$ $(x';i\eta')\in V,\ \tau=0\}$. So by the flabbiness of C_L and the assumption for c, there exists a section $g(\theta,x')$ of C_L on $\{(R(x')e^{i\theta},x';i\eta'dx');$ $(x';i\eta')\in V_n^+\}$ with ν-proper support, where

$$\nu:\ \{(R(x')e^{i\theta},x';i\eta'dx');(x';i\eta')\in V_n^+\}\longrightarrow x'\in\mathbb{R}^{n-1}=N$$

is the projection. Using $g(\theta,x')$, $T(c)(\zeta_1,x')$ is extended to $\{(\zeta_1,x';i\eta')\in\mathbb{P}^1\times iS^*\mathbb{R}^{n-1};\ R(x')<|\zeta_1|\leqq\infty,(x';i\eta')\in V_n^+\}$ as a section of $C\mathcal{O}_{P^1\times i S^* R^{n-1}}$ given by

$$f(\zeta_1,x') = -\frac{1}{2\pi}\int_0^{2\pi}\frac{g(\theta,x')}{R(x')e^{i\theta}-\zeta_1}R(x')e^{i\theta}\,d\theta.$$

Next, recall that $\beta_n^+(c)$ defines a section g of $C\mathcal{O}_+^\infty$ on $\{\zeta_1\in\frac{1}{2}\mathbb{P}^1;\zeta_1\neq\infty\}$ modulo $\alpha_*C\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}$ at every point of V and $T(c)-g\in\alpha_*C\mathcal{O}_{\mathbb{C}\times\mathbb{R}^{n-1}}$. So $T(c)\in\Gamma(V,\ \alpha_*^+C\mathcal{O}_+^\infty)$ and $f(\zeta_1,x')\in\Gamma((\alpha^+)^{-1}(V),C\mathcal{O}_+^\infty)\cap\Gamma(\{\infty\}\times V_n^+,C\mathcal{O}_{\mathbb{P}^1\times\mathbb{R}^{n-1}})$ $\subset\Gamma(\frac{1}{2}\mathbb{P}^1\times V\cup\{\infty\}\times V_n^+,C\mathcal{O}_+^\infty)$. Now we consider the boundary value $b(f)=$ $f(it+0,x')$ defined in (2.2) Proposition 2.1.8, which is a section of $\mathcal{H}_F^0(C_{S^1\times\mathbb{R}^{n-1}})$ on $\{(t,x';i\eta'dx')\in(\mathbb{R}\cup\{\infty\})\times iS^*\mathbb{R}^{n-1};(x';i\eta')\in V$, or $t=\infty$ and $(x';i\eta')\in V_n^+\}$. Since $f\in\Gamma_{(\pi_N|V_n^+)\text{-proper}}(V_n^+,C\mathcal{O}_{\mathbb{P}^1\times\mathbb{R}^{n-1}}|_{\{\infty\}\times V_n^+})$, by the flabbiness of $\mathcal{H}_F^0(C_{S^1\times\mathbb{R}^{n-1}})$, there exists a section $e(t,x')$ of $\mathcal{H}_F^0(C_{S^1\times\mathbb{R}^{n-1}})$ on $S^1\times V_n^+$ such that $e(t,x')=f(it+0,x')$ on

$\{(t,x';i\eta'dx')\in S^1\times V_n^+;\ (x';i\eta')\in V$ or $t=\infty\}$ and the support of $e(t,x')$ is Λ-proper with Λ: $S^1\times V_n^+\ni(t,x';i\eta')\to x'\in\mathbb{R}^{n-1}$. Then $h(e(t,x'))$ (see (2,3) Proposition 2.1.8) is a section of $\Gamma(\pi|_{V_n^+})$-proper$(V_n^+,\ \alpha_*^+C\mathcal{O}_+^\infty)$ which coincides with $f=T(c)$ on V by Proposition 2.1.8. Furthermore, by the change of the path of integration in (2.3), we see that $h(e)(\zeta_1,x')$ is also holomorphic on $\{(\zeta_1,x';i\eta');\ \zeta_1=\infty,(x';i\eta')\in V_n^+\}$ (In this case $t\cdot e(t,x')=t\cdot f(it+0,x')$ is holomorphic on $\{\infty\}\times V_n^+$.). Therefore $(\beta_n^+)^{-1}(h(e))$ is a section of $\Gamma(\pi|_{V_n^+})$-proper$(V_n^+,\ \imath_*^+C_{M+|X}\cap C^\infty_{N|X}|_{\{\infty\}\times iS^*N})$ which coincides with c modulo $\imath_*C_{N|X}$ on V. This completes the proof.

To obtain the explicit representations of sections of $C_{N|M+}$ and $\hat{C}_{N|M+}$, we consider the monoidal transform of X with center M+.

<u>Definition 2.1.13</u>. We inherit the notations from 1.1. The normal spherical bundle $S_{M+}X$ of M+ in X and the projection $\tau_{M+/X}$ are defined as follows.

$$\begin{array}{ccc} S_{M+}X = (S_NX-S_NM+)\sqcup iSM|_{M+} & \xrightarrow{\tau_{M+/X}} & M+ \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (0,x';w_1=u_1+iv_1,iv') & & x \end{array},$$

where $S_NM+=\{(0,x';+1,0)\}$ and $iSM|_N=\{(0,x';iv)\}$ is identified with $\{(0,x';+\infty+iv_1,iv')\in\overline{S_NX-S_NM+}\}$. Then the monoidal transform of X with center M+ is canonically defined as

$$\widetilde{{}^{M+}X}=(X-M+)\sqcup S_{M+}X,$$

which is equipped with the natural topology (see Proposition 4.1.4 [8]). Further we define the closed subset D+ of $S_{M+}X\underset{M+}{\times}S^*_{M+}X$ as

$$D+ = \tfrac{1}{2}S_{M+}X \underset{M+}{\times} S^*_{M+}X = \{(x_1, x'; iv; i\eta);\ x_1 > 0,\ \langle v, \eta\rangle \geq 0\}$$
$$\sqcup\{(0, x'; w_1, iv'; \zeta_1, i\eta');\ -\mathrm{Re}(w_1\zeta_1) + \langle v', \eta'\rangle \geq 0,\ \mathrm{Re}\zeta_1 \geq 0, (w_1, iv') \neq (1, 0)\}$$
$$\sqcup\{(0, x'; +\infty + iv_1, iv'; i\eta);\ \langle v, \eta\rangle \geq 0\}$$
and
$$\widetilde{D+_X^+} = (X - M+) \sqcup D+.$$

So we have the following commutative diagram, where π, τ, π', τ' are the canonical projections. $\widetilde{D+_X^+}$ is equipped with the weakest topology under which π' is continuous and D+ has the original topology as a closed subset of $\widetilde{D+_X^+}$. Then the topology of $\widetilde{M+_{X^*}}$ defined in Definition 4.1.1 [8] coincides with the strongest topology under which τ' is continuous (cf. Proposition 4.1.4 [8]). In particular τ' is a continuous, proper, separated and surjective mapping with contractible fibers. So we have the following isomorphism.

$$\begin{tikzcd} & \widetilde{D+_X^+} \arrow[dl, "\pi'"'] \arrow[dr, "\tau'"] & \\ \widetilde{M+_X} \arrow[dr, "\tau"'] \arrow[rr, phantom, "\circlearrowleft"] & & \widetilde{M+_{X^*}} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Lemma 2.1.14. $C_{M+|X} \xleftarrow{\sim} \mathbb{R}\tau'_{!}\, \pi'^{-1}\mathbb{R}\Gamma_{S_{M+}X}\, \tau^{-1}\mathcal{O}_X\, [n] \otimes \omega_{M+/X}$

Proof This is a direct consequence from the definition of $C_{M+|X}$ (see Definition 4.1.2 [8]) and Proposition 1.2.2 S-K-K [10].

Definition 2.1.15. We denote by L $S_{M+}X \cap \tau^{-1}(N)$. Then the closed subset F+ of L and the projection θ are introduced as follows.

$$F+ = \{(0, x'; w_1, iv') \in L;\ v_1 = 0, 0 \leq u_1 \leq +\infty\} \xrightarrow{\ \theta\ } (x'; iv') \in iSN$$

Indeed these are coordinate-invariant. Putting $j: X - M+ \hookrightarrow \widetilde{M+_X}$, we define

$$\widetilde{\mathcal{A}}_{M+} = j_*(\mathcal{O}_X|_{X-M+})|_{S_{M+}X}, \quad q_{M+} = \widetilde{\mathcal{A}}_{M+} / \tau^{-1}\mathcal{A}_M$$

and the sheaves $\widetilde{A}_{M+}$, $\widetilde{B}_{M+}$ on iSN as follows.

$$\widetilde{A}_{M+} = \theta_*(\widetilde{\mathcal{A}}_{M+}|_{F+}), \qquad \widetilde{B}_{M+} = R^1\theta_*\mathbb{R}\Gamma_{F+}(\widetilde{\mathcal{A}}_{M+}|_L)$$

Lemma 2.1.16. We have the following quasi-isomorphisms.

$$\mathbb{R}\Gamma_{S_{M+}X}\tau^{-1}\mathcal{O}_X[1] \xleftarrow{\sim} q_{M+}, \qquad \mathbb{R}\theta_*(\widetilde{\mathcal{A}}_{M+}|_{F+}) \xleftarrow{\sim} \widetilde{A}_{M+}$$

And therefore $C_{M+|X} \xleftarrow{\sim} \mathbb{R}\tau_*^! \pi'^{-1} q_{M+}[n-1]\otimes\omega_{M+/X}$.

Proof These follow from the vanishing theorem of cohomology groups of open convex sets in $\mathbb{C}^n$ with coefficients in holomorphic functions.

Now, to obtain the intuitive explanations of $\widetilde{A}_{M+}$ and $\widetilde{B}_{M+}$, we will express germs of these sheaves as holomorphic functions.

Set $M=\mathbb{R}^n\ni(x_1,x')$ $N=\{x_1=0\}$ $M+=\{x_1\geq 0\}$ and a directed set $\mathcal{H}$ of positive continuous functions on $\mathbb{R}^2_+=\{(t,s)\in\mathbb{R}^2;\ t\geq 0, s\geq 0\}$ as

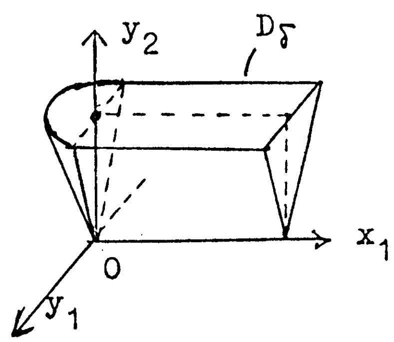
$$(2.7)\qquad \mathcal{H}=\{h\in C^0(\mathbb{R}^2_+)\ ;\ h\geq 0, h(t,0)\equiv 0, \partial h/\partial s\in C^0(\mathbb{R}^2_+), \partial h/\partial s\geq 0, \partial h/\partial s(0,0)=0\}$$

with the natural order ($h\leq h' \Leftrightarrow h(t,s)\leq h'(t,s)$ for every (t,s)).
Then the stalk of $\widetilde{A}_{M+}$ at $(0;i\partial/\partial x_n)\in iS\mathbb{R}^{n-1}$ is given by

$$(2.8)\qquad \widetilde{A}_{M+}|(0;i\partial/\partial x_n) = \varinjlim_{\delta\to+0}\Gamma(D_\delta,\ \mathcal{O}_{\mathbb{C}^n}),$$

where $D_\delta=\{z\in\mathbb{C}^n;\ |z|<\delta$,
$y_n>\frac{1}{\delta}(|y_1|+\cdots+|y_{n-1}|+|x_1|\cdot Y(-x_1))\}$ ($Y(t)$ is the Heaviside function) and the stalk of $\widetilde{B}_{M+}$ at $(0;i\partial/\partial x_n)$ is given by

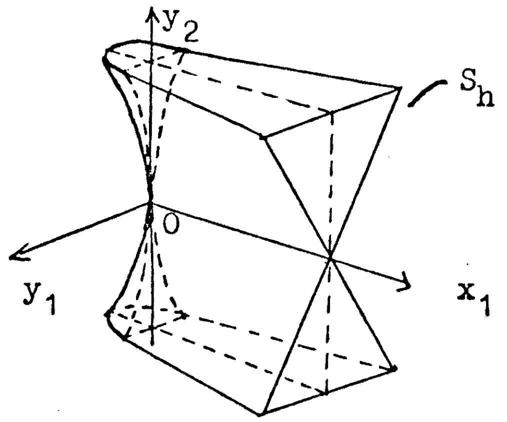
Fig.2.1 (n=2)



(2.9) $$\widetilde{B}_{M+}\big|(0;i\partial/\partial x_n) = \varinjlim_{\substack{\delta\to+0 \\ h\in\mathcal{X}\uparrow}} H^1_{S_h}(D_\delta, \mathcal{O}_{\mathbb{C}^n})$$

$$= \varinjlim \Gamma(D_\delta\setminus S_h, \mathcal{O}_{\mathbb{C}^n}) \,/\, \Gamma(D_\delta, \mathcal{O}_{\mathbb{C}^n}),$$

Fig.2.2 (n=2)



where S_h is a closed subset of $\mathbb{C}^n$ of the form : $S_h=\{z\in\mathbb{C}^n;\ |z_1|\leq h(+0,|y'|)\}$ $\cup\{\ z\in\mathbb{C}^n;\ |y_1|\leq h(x_1,|y'|),\ x_1\geq 0\}$ for $h\in\mathcal{X}$.

Remark 2.1.17 The domain of defining functions of germs of $\widetilde{B}_{M+}$, $D_\delta\setminus S_h$ in (2.9), is not Stein in general. The auther does not know the envelope of holomorphy of such a domain.

Proposition 2.1.18 The following injective sheaf homomorphisms are canonically defined .

$$\widetilde{B}_{M+}\otimes\omega_N \xhookrightarrow{\ b\ } \tau_N^{-1}\mathcal{H}^0_{M+}(B_M)$$

and

$$\begin{array}{ccccc} \widetilde{A}_{M+} & \xhookrightarrow{\ c\ } & \widetilde{B}_{M+} & \longrightarrow & \tau_N^{-1}\mathcal{H}^0_{M+}(B_M) \\ \Psi & & \Psi & & \Psi \\ f(z) & \longmapsto & \left[-\frac{1}{2\pi i}\, f(z)\log z_1\right] & \longmapsto & b(f(x_1, z')Y(x_1)\otimes\omega_N) \end{array}$$

Here we remark that c is not a $\tau_N^{-1}\mathcal{O}_X$-homomorphism, in fact, the equation

$$c\Big(\frac{\partial f}{\partial z_1}\Big) = \frac{\partial}{\partial z_1}c(f) - \frac{1}{2\pi i}\big[f(0,z')z_1^{-1}\big]$$

holds.

Proof To explain b and c we use the expressions in (2.8) and (2.9). Note that, setting $E_\delta=\{z\in\mathbb{C}^n;\ |z|<\delta,\ y_n>\frac{1}{\delta}(|y_1|+\cdots+|y_{n-1}|)\}$, the isomorphism

$$H^1_{S_h}(D_\delta,\ \mathcal{O}_{\mathbb{C}^n})\cong H^1_{S_h}(E_\delta,\ \mathcal{O}_{\mathbb{C}^n})=\Gamma(E_\delta\setminus S_h,\mathcal{O}_{\mathbb{C}^n})\ /\ \Gamma(E_\delta,\mathcal{O}_{\mathbb{C}^n})$$

holds for sufficiently small $\delta>0$. Taking a holomorphic function $f(z)\in\Gamma(E_\delta\setminus S_h,\mathcal{O}_{\mathbb{C}^n})$, we define a section of $\Gamma(\{|x|<\delta\},\mathcal{B}_{\mathbb{R}^n})$ as

$$b(f)=b(f(z)\big|_{(E_\delta\setminus S_h)\cap\{y_1>0\}})-b(f(z)\big|_{(E_\delta\setminus S_h)\cap\{y_1<0\}}).$$

In fact the boundary values in the right hand side are determined if the orientation of N is given, and clearly the support of b(f) is contained in M+. Further by the theorem of the edge of the wedge this induces the injective homomorphism

$$b\ :\ \tilde{\mathcal{B}}_{M+}\big|_{(0;i\partial/\partial x_n)}\longrightarrow\mathcal{H}^0_{M+}(\mathcal{B}_M)\big|_0$$

It is easy to see that b defines canonically an injective sheaf homomorphism : $\tilde{\mathcal{B}}_{M+}\otimes\omega_N\longrightarrow\tau_N^{-1}\mathcal{H}^0_{M+}(\mathcal{B}_M)$. Next, for a holomorphic function on D_δ, $-\frac{1}{2\pi i}f(z)\log z_1$ is considered as a holomorphic function on $D_\delta\setminus\{z\in\mathbb{C}^n;\ \mathrm{Re}z_1\geq 0,\ \mathrm{Im}z_1=0\}$. So the sheaf homomorphism

$$\begin{array}{cccc} c\ : & \tilde{\mathcal{A}}_{M+} & \longrightarrow & \tilde{\mathcal{B}}_{M+}\\ & \Psi & & \Psi\\ & f(z) & & [-\frac{1}{2\pi i}f(z)\log z_1]\end{array}$$

is well-defined and injective. Further this is independent of the coordinate system.

Example 2.1.19. By the preceding proposition we know that $\tilde{\mathcal{B}}_{M+}$ defines a germ of $\mathcal{H}^0_{M+}(\mathcal{B}_M)$. Conversely any germ of $\mathcal{H}^0_{M+}(\mathcal{B}_M)$

is represented as a sum of boundary values of germs of $\widetilde{\hat{B}}_{M+}$. Indeed the cohomology group $H^n_{M+}(\{z\in\mathbb{C}^n;\ |z|<\delta\},\mathcal{O}_{\mathbb{C}^n}) = H^{n-1}(\{|z|<\delta\}\setminus M+,\mathcal{O}_{\mathbb{C}^n})$ is calculated by the Stein covering $\{W_j^{\pm}\}_{j=2,\ldots,n}$ with $W_j^{\pm} = \{|z|<\delta\}\cap (\{z_1\in\mathbb{C};\ z_1\notin[0,+\infty)\}\times\{z'\in\mathbb{C}^{n-1};\ \operatorname{Im} z_j \gtrless 0\})$. So any section of $\mathcal{H}^0_{M+}(B_M)$ on $\{|x|<\delta\}$ is written as a sum of boundary values of sections of $\Gamma(\{z\in\mathbb{C}^n;\ z_1\notin[0,+\infty),\ |z|<\delta,\ \operatorname{Im} z_2>0,\ldots,\operatorname{Im} z_n>0\},\mathcal{O}_{\mathbb{C}^n})$ ...etc.

The typical example of sections of $\widetilde{\hat{B}}_{M+}$ which does not belong to $\widetilde{\hat{A}}_{M+}$ is $f(z')z_1^{\lambda(z')}$, where $f(z')$ and $\lambda(z')$ are holomorphic functions in z' at $z'=0$ with $\lambda(z')\neq 0,1,2,\ldots$ Such a function is important in the theory of boundary value problems with regular singularities.

In the sequel, we prove that the mild hyperfunctions $\hat{B}_{N|M+}$ coincide with sums of boundary values of sections of $\widetilde{\hat{A}}_{M+}$.

<u>Lemma 2.1.20.</u> The image of the imbedding $b\circ c$; $\widetilde{\hat{A}}_{M+}\otimes\omega_N \hookrightarrow \tau_N^{-1}\mathcal{H}^0_{M+}(B_M)$ defined in Proposition 2.1.18 consists of mild hyperfunctions. Particularly the induced sheaf homomorphism

$$\widetilde{\hat{A}}_{M+}\otimes\omega_N \longrightarrow \tau_N^{-1}\hat{B}_{N|M+}$$

is injective and $\tau_N^{-1}\mathcal{D}_X$ - homomorphism. Indeed this coincides with the homomorphism induced by

$$\begin{array}{ccc} \widetilde{\hat{A}}_{M+}\otimes\omega_N & \longrightarrow & \tau_N^{-1}B_{N|M+}\,. \\ \Psi & & \Psi \\ f(z) & & b(f(z)|_{\{\operatorname{Re} z_1>0\}}) \end{array}$$

<u>Proof</u> Fix a germ $f(z)$ of $\widetilde{\hat{A}}_{M+}$ at $(0;i\partial/\partial x_n)$. Then $c(f) = [-\frac{1}{2\pi i}f(z)\log z_1]$ defines not only a germ of $\widetilde{\hat{B}}_{M+}\subset\tau_N^{-1}\mathcal{H}^0_{M+}(B_M)$ but

also a germ of $\pi_{N/X*}(C_{N|X}|_{S_Y^*X\cap S_N^*X})$. In fact by moving the cut-off hyperplane, $[-\frac{1}{2\pi i}f(z)\log z_1]$ defines an element of

$$H^1_{K_\theta}(\{z\in\mathbb{C}^n;\ |z|<\delta,\ y_n>\frac{1}{\delta}(|z_1|+|y_2|+\cdots+|y_{n-1}|)\},\ \mathcal{O}_{\mathbb{C}^n})$$

with $K_\theta=\{z\in\mathbb{C}^n;\ z_1e^{i\theta}\in[0,+\infty)\}$ for every $\theta\in[0,2\pi]$. Further the trace operations and the relative cohomology exact sequences induce the natural homomorphisms as follows.

$$H^1_{K_\theta}(\{|z|<\delta,\ y_n>\frac{1}{\delta}(|z_1|+|y_2|+\cdots+|y_{n-1}|)\},\mathcal{O}_{\mathbb{C}^n})\xrightarrow{y_2=\cdots=y_{n-1}=+0}$$

$$\longrightarrow H^{n-1}_{K_\theta\cap\{y_2=\cdots=y_{n-1}=0\}}(\{|z|<\delta,\ y_n>\frac{1}{\delta}|z_1|\},\ \mathcal{O}_{\mathbb{C}^n})\longrightarrow$$

$$\longrightarrow H^n_{K_\theta\cap\{y_2=\cdots=y_{n-1}=0,\,y_n=\frac{1}{\delta}|z_1|\}}(\{|z|<\delta\},\ \mathcal{O}_{\mathbb{C}^n}).$$

On the other hand the stalk of $C_{N|X}$ at $(0;e^{i\theta},0)\in S_Y^*X\cap S_N^*X$ is isomorphic to

$$\varinjlim_{\delta\to+0} H^n_{\Lambda_\delta}(\{|z|<\delta\},\ \mathcal{O}_{\mathbb{C}^n})$$

with $\Lambda_\delta=\{z\in\mathbb{C}^n;\ \mathrm{Re}(z_1e^{i\theta})\geq\delta(|y'|+|\mathrm{Im}(z_1e^{i\theta})|)\}$. Combination of these facts assures the existence of the homomorphism

$$\widetilde{\hat{A}}_{M+}\longrightarrow\tau_N^{-1}\pi_{N/X*}(C_{N|X}|_{S_Y^*X\cap S_N^*X}).$$

Thus the mildness of $b\circ c(\widetilde{\hat{A}}_{M+})$ is proved. The other claims are easily verified.

<u>Proposition 2.1.21</u>. For any open subset U of iS^*N with proper convex fibers, every section of $\pi_{N*}\hat{C}_{N|M+}$ with $\pi_N|_U$-proper support is identified with a section of $\widetilde{\hat{A}}_{M+}/\tau_N^{-1}\mathcal{A}_M$ on U° (the polar set of U), that is,

$$(\pi_N|_U)_!\hat{C}_{N|M+} \overset{\sim}{\Leftarrow} (\tau_N|_{U^o})_*\widetilde{A}_{M+} / \mathcal{A}_{M|N}.$$

<u>Proof</u> Let us calculate $\mathcal{F} = \mathbb{R}(\pi_N|_U)_!\mathbb{R}\iota^+_! C_{M+|X}$. By Lemma 2.1.16, $\mathcal{F} \overset{\sim}{\Leftarrow} \mathbb{R}(\pi_N|_U)_!\mathbb{R}\iota^+_!(\mathbb{R}\tau'^!_* \pi'^{-1}q_{M+}|_{\overline{G}+\setminus S^*_Y X})[n-1]$. Consider the following commutative diagram.

$$\begin{tikzcd}
& \tfrac{1}{2}L \underset{N}{\times} \overline{G}+ \arrow[dl, "\pi'"'] \arrow[dr, "\tau'"] & & \tfrac{1}{2}L \underset{N}{\times} (\overline{G}+\setminus S^*_Y X) \arrow[ll, hook', "i"'] \arrow[dr, "\tau'"] & & \tfrac{1}{2}L \underset{N}{\times} (\iota^+)^{-1}(U) \arrow[ll, hook', "f''"'] \arrow[dr, "\tau'"] & \\
L \arrow[dr, "\tau"'] & & \overline{G}+ \arrow[dl, "\pi"] & & \overline{G}+\setminus S^*_Y X \arrow[ll, hook', "i"'] \arrow[dl, "\iota^+"] & & (\iota^+)^{-1}(U) \arrow[ll, hook', "f'"'] \arrow[dl, "\iota^+"] \\
& N & & iS^*N \arrow[ll, "\pi_N"'] & & U \arrow[ll, hook', "f"'] &
\end{tikzcd} \tag{2.10}$$

Therefore $\mathcal{F} \overset{\sim}{\Leftarrow} \mathbb{R}\pi_{N*}\mathbb{R}f_!f^{-1}\mathbb{R}\iota^+_!i^{-1}\mathbb{R}\tau'^!_* \pi'^{-1}q_{M+}[n-1]$, by the theorems in CH I S-K-K [10],

$$= \mathbb{R}\pi_{N*}\mathbb{R}f_!(\mathbb{R}\iota^+_!f'^{-1})(\mathbb{R}\tau'^!_*i^{-1})\,\pi'^{-1}q_{M+}[n-1]$$

$$= \mathbb{R}\pi_{N*}\mathbb{R}f_!\mathbb{R}\iota^+_!(\mathbb{R}\tau'^!_*f''^{-1})i^{-1}\,\pi'^{-1}q_{M+}[n-1]$$

$$= \mathbb{R}(\pi_N\circ f\circ\iota^+\circ\tau')_!(\pi'\circ i\circ f'')^{-1}q_{M+}[n-1]$$

$$= \mathbb{R}(\tau\circ\pi'\circ i\circ f'')_!(\pi'\circ i\circ f'')^{-1}q_{M+}[n-1]$$

$$= \mathbb{R}\tau_*\mathbb{R}(\pi'\circ i\circ f'')_!(\pi'\circ i\circ f'')^{-1}q_{M+}[n-1].$$

Set the continuous maps g_1, g_2 as follows.

$$\begin{array}{ccccc}
\tfrac{1}{2}L \underset{N}{\times} (\iota^+)^{-1}(U) & \xrightarrow{\quad g_1 \quad} & L \underset{N}{\times} U & \xrightarrow{\quad g_2 \quad} & L \\
\cup & & \cup & & \cup \\
(0,x';w_1,iv';\zeta_1,i\eta') & & (0,x';w_1,iv';i\eta') & & (0,x';w_1,iv')
\end{array}$$

Note that $g_2g_1=\pi' i f''$. Therefore $\mathcal{F} \overset{\sim}{\Leftarrow} \mathbb{R}\tau_*\mathbb{R}g_{2!}\mathbb{R}g_{1!}g_1^{-1}g_2^{-1}q_{M+}[n-1]$.
The fiber of g_1 is given by $\{(\xi_1,\eta_1)\in\mathbb{R}^2;\ \xi_1\geqq 0, -\xi_1u_1+\eta_1v_1+\langle v',\eta'\rangle\geqq 0\}$ and so this is topologically isomorphic to

$$\begin{cases} \mathbb{R}\times[0,+\infty) & \text{on } (\{v_1\neq 0\}\cup\{v_1=0,u_1<0\}\cup\{u_1=v_1=0,\ \langle v',\eta'\rangle\geqq 0\})\cap\{u_1\neq+\infty\} \\ \mathbb{R}\times[0,1] & \text{on } \{v_1=0,u_1>0,\ \langle v',\eta'\rangle>0\}\cap\{u_1\neq+\infty\} \\ \mathbb{R} & \text{on } \{v_1=0,u_1>0,\langle v',\eta'\rangle=0\}\cup\{v_1=0,u_1=+\infty,\ \langle v',\eta'\rangle\geqq 0\} \\ [0,+\infty) & \text{on } \{u_1=+\infty,\ v_1\neq 0\} \\ \phi & \text{on } \{v_1=0,\ 0\leqq u_1\leqq+\infty,\ \langle v',\eta'\rangle<0\}. \end{cases}$$

Using the isomorphism $R^j g_{1!}g_1^{-1}(g_2^{-1}q_{M+})|_{p_0} = H_c^j(g_1^{-1}(p_0),\mathbb{C})\otimes_{\mathbb{C}} g_2^{-1}q_{M+}|_{p_0}$

$$= \begin{cases} g_2^{-1}q_{M+}|_{p_0} & \text{for } j=1 \text{ and } p_0\in\{v_1=0,0<u_1\leqq+\infty,\langle v',\eta'\rangle\geqq 0\} \\ 0 & \text{otherwise,} \end{cases}$$

we have $\mathbb{R}g_{1!}g_1^{-1}g_2^{-1}q_{M+}[n-1]\xleftarrow{\sim} g_2^{-1}\Gamma_{F+\setminus F-}(q_{M+}|_{F+})|_{\{\langle v',\eta'\rangle\geqq 0\}}[n-2]$, where $F- = \{(0,x';w_1,iv')\in S_{M-}X|_N;\ v_1=0, 0\geqq u_1\geqq-\infty\}$, so

$$\mathcal{F}\xleftarrow{\sim}\mathbb{R}\tau_*\mathbb{R}g_{2!}(g_2^{-1}\Gamma_{F+\setminus F-}(q_{M+}|_{F+})|_{\{\langle v',\eta'\rangle\geqq 0\}})[n-2].$$

The fiber of $g_2|_{\{\langle v',\eta'\rangle\geqq 0\}}$ is given by $\{\eta'\in S^{n-2}; i\eta'\in U\cap\pi_N^{-1}(x'), \langle v',\eta'\rangle\geqq 0\}$ and so this is topologically isomorphic to

$$\begin{cases} \mathbb{R}^{n-2} & \text{on } (x';iv')\in U^o=\{(x';iv')\in iSN;\ \langle v',\eta'\rangle\geqq 0 \text{ for } \forall(x';i\eta')\in U\} \\ [0,+\infty)\times\mathbb{R}^{n-3} \text{ or } \phi & \text{otherwise.} \end{cases}$$

Therefore $\mathbb{R}g_{2!}(g_2^{-1}\Gamma_{F+\setminus F-}(q_{M+}|_{F+})|_{\{\langle v',\eta'\rangle\geqq 0\}})[n-2]\xleftarrow{\sim}$ $\Gamma_{F+\setminus F-}(q_{M+}|_{F+})|_{\theta^{-1}(U^o)}$ and so $\mathcal{F}\xleftarrow{\sim}\mathbb{R}\tau_*(\Gamma_{F+\setminus F-}(q_{M+}|_{F+})|_{\theta^{-1}(U^o)})$ $=\mathbb{R}(\tau_N|_{U^o})_*\mathbb{R}\theta_*\Gamma_{F+\setminus F-}(q_{M+}|_{F+})$. Consequently we have the isomorphism

$$R^1(\pi_N|_U)_!\mathbb{R}\iota_!^+C_{M+|X}\xleftarrow{\sim}R^1(\tau_N|_{U^o})_*\mathbb{R}\theta_*\Gamma_{F+\setminus F-}(q_{M+}|_{F+}).$$

Define the sheaf $q_{M+\cup M-}$ on $(S_{M+}X-S_NM-)\cup(S_{M-}X-S_NM+)$ as

$$q_{M+\cup M-} = \begin{cases} q_{M+} & \text{on } S_{M+}X-S_NM- \\ q_{M-} & \text{on } S_{M-}X-S_NM+ \end{cases}$$

because $q_{M+} = q_{M-}$ holds on S_NX-S_NM. Setting $F= F+\cup F- =$ $\{(0,x';u_1,iv')\in\mathbb{R}^{n-1}\times(\{\pm\infty\}\cup\mathbb{R})\times iS^{n-2}\}\subset(S_{M+}X-S_NM-)\cup(S_{M-}X-S_NM+)$ and the projection $\tilde{\theta}: F \longrightarrow iSN$, the following exact sequence on F holds.

$$0 \longrightarrow \Gamma_{F+\setminus F-}(q_{M+}|_{F+}) \longrightarrow q_{M+\cup M-}|_F \longrightarrow q_{M-}|_{F-} \longrightarrow 0$$

So $R^1(\tau_N|_{U^o})_*\mathbb{R}\theta_*\Gamma_{F+\setminus F-}(q_{M+}|_{F+}) \tilde{\Leftarrow}$

$$(\tau_N|_{U^o})_*\tilde{\theta}_*(q_{M-}|_{F-}) \;/\; (\tau_N|_{U^o})_*\tilde{\theta}_*(q_{M+\cup M-}|_F),$$

since $(\tau_N|_{U^o})_*\theta_*\Gamma_{F+\setminus F-}(q_{M+}|_{F+}) = 0$ and $R^1(\tau_N|_{U^o})_*\mathbb{R}\tilde{\theta}_*(q_{M+\cup M-}|_F)=0$ by the proper convexity of $\tilde{\theta}^{-1}(U^o)$. Thus, because of $\iota_!^+C_{M+|X}=0$,

$$(2.11)\quad \{(\tau_N|_{U^o})_*\tilde{A}_{M-}/\mathcal{A}_M|_N\} \;/\; (\tau_N|_{U^o})_*\tilde{\theta}_*(q_{M+\cup M-}|_F)$$
$$\tilde{\Rightarrow} R^1(\pi_N|_U)_!\mathbb{R}\iota_!^+C_{M+|X} = (\pi_N|_U)_!R^1\iota_!^+C_{M+|X}.$$

Put the imbedding $k : \overline{G+\setminus S_Y^*X} \hookrightarrow S_N^*X^\infty$. Then we have a relative cohomology exact sequence;

$$0 \longrightarrow \iota_*^+C_{M+|X} \longrightarrow k_*C_{M+|X}|_{\{\infty\}\times iS^*N} \longrightarrow R^1\iota_!^+C_{M+|X} \longrightarrow .$$

Recalling the natural homomorphism $(\tau_N|_{U^o})_*\tilde{A}_{M-} \longrightarrow \pi_{N*}\hat{C}_{N|M-}$, especially $(\tau_N|_{U^o})_*\tilde{A}_{M-} \longrightarrow (\pi_N|_U)_!\hat{C}_{N|M-}$ (see the preceding lemma), we obtain the following natural homomorphisms.

$$(\tau_N|_{U^o})_*\tilde{A}_{M-} \xrightarrow{g} (\pi_N|_U)_!\hat{C}_{N|M-} \xrightarrow{g'} (\pi_N|_U)_!(k_*C_{M+|X}|_{\{\infty\}\times iS^*N}/\iota_*^+C_{M+|X})$$
$$\xrightarrow{g''} (\pi_N|_U)_!R^1\iota_!^+C_{M+|X}$$

So, since $g''g'g$ is surjective by (2.11) and g'' is injective, for every germ e_1 of $(\pi_N|_U)_!\hat{C}_{N|M-}$ there exists a germ $e_2 \in \mathrm{Image}(g)$ satisfying $e_1 - e_2 \in \mathrm{Kernel}(g') = (\pi_N|_U)_!(\iota_*^+C_{M+|X} \cap C_{N|X}^\infty|_{\{\infty\}\times iS^*N} \cap \iota_*^-C_{M-|X}/\iota_*C_{N|X})$, where we mean by $\iota_*^+C_{M+|X} \cap (C_{N|X}^\infty|_{\{\infty\}\times iS^*N}) \cap \iota_*^-C_{M-|X}$ the sections of $C_{N|X}$ on $\{(0,x';\zeta_1,i\eta')\in S_N^*X\setminus S_Y^*X;\ \mathrm{Re}\,\zeta_1 \neq 0 \text{ or } R < |\zeta_1| < +\infty\}$ for sufficiently large $R > 0$ which have boundary values on $I\setminus S_Y^*X$ as $C_M|_I$ from the both sides $\{\mathrm{Re}\,\zeta_1 \gtrless 0\}$. Hence considering the boundary value operation induced by $C_{M\pm|X}|_I \xrightarrow{i_\pm} C_M|_I$;

$$(\pi_N|_U)_!(\iota_*^+ C_{M+|X} \cap C^\infty_{N|X}|_{\{\infty\}\times iS^*N} \cap \iota_*^- C_{M-|X} / \iota_* C_{N|X}) \ni [f(x)]$$

$$\xrightarrow[\text{injective}]{} i_+(f(x)|_{\{Re\zeta_1>0\}}) - i_-(f(x)|_{\{Re\zeta_1<0\}}) \in (\pi_N|_U)_! \iota_!(C_M|_I),$$

e_1-e_2 defines a germ of $(\pi_N|_U)_!\iota_!(C_M|_{I\setminus S^*_Y X}) = \{[f(x)] \in B_M|_N / \mathcal{A}_M|_N$; $SS(f)\cap \pi^{-1}(N) \subset \iota^{-1}(U)\} \cong \tau_{N*}(\widetilde{\theta}_*(q_{M+}\cup_{M-}|_F)|_{U^o}) \hookrightarrow (\tau_N|_{U^o})_* \widetilde{A}_{M-}/\mathcal{A}_M|_N$.
Therefore e_1 also belongs to Image(g), that is,

$$(\tau_N|_{U^o})_*\widetilde{A}_{M-}/\mathcal{A}_M|_N \longrightarrow (\pi_N|_U)_!\hat{C}_{N|M-}$$

is surjective. On the other hand $0 \longrightarrow \mathcal{A}_M|_N \longrightarrow \hat{B}_{N|M-} \longrightarrow \pi_{N*}\hat{C}_{N|M-}$ is exact because of $\pi_{N/X*}C_{N|X} = \mathcal{A}_M|_N \oplus \mathcal{H}^0_N(B_M)$ (see Proposition 1.1.1). Combination of this and the preceding lemma assures the injectivity of $(\tau_N|_{U^o})_*\widetilde{A}_{M-}/\mathcal{A}_M|_N \longrightarrow (\pi_N|_U)_!\hat{C}_{N|M-}$. Changing the signature - into +, the proof is completed.

<u>Corollary 2.1.22</u>. We have the exact sequence of $\mathcal{D}_X|_N$-modules;

$$0 \longrightarrow \mathcal{A}_M|_N \longrightarrow \hat{B}_{N|M+} \longrightarrow \pi_{N*}\hat{C}_{N|M+} \longrightarrow 0.$$

From this we obtain that $\hat{B}_{N|M+}$ is a soft sheaf on N.

<u>Proof</u> We have only to show the surjectivity of $\hat{B}_{N|M+} \longrightarrow \pi_{N*}\hat{C}_{N|M+}$. By the softness of $\hat{C}_{N|M+}$ and Proposition 2.1.21, any germ of $\pi_{N*}\hat{C}_{N|M+}$ is represented by a sum of boundary values of sections of $\widetilde{A}_{M+}$. Recalling that $\widetilde{A}_{M+} \hookrightarrow \tau_N^{-1}\hat{B}_{N|M+}$, this assures the surjectivity. Moreover let S be a closed set in N and f be a section of $\hat{B}_{N|M+}$ on S. Remark that this sequence is globally exact on any open subset of N due to the cohomological triviality of $\mathcal{A}_M$. So combination of this fact and the softness of $\hat{C}_{N|M+}$ admits the existence of a section $f' \in \Gamma(N, \hat{B}_{N|M+})$ satisfying $f'|_S - f \in \Gamma(S, \mathcal{A}_M|_N)$. So we

may assume that $f \in \Gamma(U, \hat{B}_{N|M+})$ and $f'-f \in \Gamma(U, \mathcal{A}_M|_N)$ for a suitable open neighborhood U of S. Choose a real analytic vector field Z defined in a neighborhood of N in M which is transversal to N, and a continuous function $\varphi(x')$ defined on N with support in U such that $\varphi \equiv 1$ on a neighborhood of S. Then $\varphi(x')$ is extended by the vector field Z to a continuous function $\psi(x_1, x')$ defined on a neighborhood of N in M such that $\psi|_N = \varphi$. Note that the product $(f'-f)\psi$ is a well-defined continuous function defined on a neighborhood of N in M satisfying $(f'-f)\psi = f'-f$ on a neighborhood of S in M. Moreover $F=(f'-f)\psi$ depends real analytically on the transversal parameter x_1 to N. Consequently $f'-F$ is a mild hyperfunction from the positive side of N satisfying $f'-F=f$ on S. This completes the proof.

From Proposition 2.1.21 and its corollary we know that any germ of $\hat{B}_{N|M+}$ is written as a sum of boundary values of sections of $\widetilde{A}_{M+}$. On the other hand as for the uniqueness of such a representation, we have the following theorem.

<u>Theorem 2.1.23.</u> (The edge of the wedge theorem for $\widetilde{A}_{M+}$)

i) Let K be a closed subset of iSN with connected fibers. We denote by $\gamma(K)$ the convex hull of K in each fiber. Then we have,

$$(\tau_N|_{\gamma(K)})_* \widetilde{A}_{M+} \xrightarrow{\sim} (\tau_N|_K)_* \widetilde{A}_{M+} \tag{2.12}$$

and when K=iSN,

$$\mathcal{A}_M|_N \xrightarrow{\sim} \tau_{N*}\widetilde{A}_{M+}.$$

ii) Let $K_1, \ldots, K_m$ be closed subsets of iSN such that each $\tau_N|_{K_j}$ is a surjective mapping with proper convex fibers. Then the following exact sequence holds for every open subset U of N.

$$(2.13) \quad \bigoplus_{j,k=1}^{m} \Gamma(\tau_N^{-1}(U)\cap\gamma(K_j\cup K_k),\widetilde{A}_{M+}) \xrightarrow{F_1} \bigoplus_{j=1}^{m} \Gamma(\tau_N^{-1}(U)\cap K_j,\widetilde{A}_{M+})$$

$$\xrightarrow{F_2} \Gamma(U,\ \hat{B}_{N|M+})$$

where F_1 and F_2 are given as follows.

$$F_1(\bigoplus_{j,k=1}^{m} f_{jk}(z)) = \bigoplus_{j=1}^{m} \sum_{k=1}^{m}(f_{jk}(z)-f_{kj}(z))$$

$$F_2(\bigoplus_{j=1}^{m} f_j(z)) = \sum_{j=1}^{m} b_{K_j}(f_j\big|\{\mathrm{Re}\,z_1>0\})$$

Further, if $\{\mathrm{int}(K_j^o)\}_{j=1,2,,m}$ cover iS^*N, the homomorphism F_2 is surjective.

<u>Proof</u> The results in i) are derived directly from Proposition 2.1.21. Let $f=(f_1,\dots,f_m)$ be a section of Kernel(F_2). Since $\sum_{j=1}^{m} b_{K_j}(f_j)=0$, by Proposition 2.1.21 the support of $[f_m]$ in $\hat{C}_{N|M+}$ is contained in $\mathrm{int}(K_m^o)\cap\{\bigcup_{j=1}^{m-1}\mathrm{int}(K_j^o)\}\subset\bigcup_{j=1}^{m-1}\mathrm{int}(K_m^o\cap K_j^o)=\bigcup_{j=1}^{m-1}\mathrm{int}(\gamma(K_m\cup K_j)^o)$. Therefore by the softness of $\hat{C}_{N|M+}$ on $\pi_N^{-1}(U)$ there exist sections $c_1,\dots,c_{m-1}$ of $\hat{C}_{N|M+}$ on $\pi_N^{-1}(U)$ such that $c_1+\cdots+c_{m-1}=[f_m]$ on $\pi_N^{-1}(U)$ and the support of c_j in $\hat{C}_{N|M+}$ is contained in $\mathrm{int}(\gamma(K_m\cup K_j)^o)$ for every j. On the other hand by Proposition 2.1.21 we can take sections $g_1,\dots,g_{m-1}$ of $\widetilde{A}_{M+}$ on $\tau_N^{-1}(U)\cap\gamma(K_m\cup K_1),\dots,\tau_N^{-1}(U)\cap\gamma(K_m\cup K_{m-1})$ respectively such that $c_1=[g_1],\dots,c_{m-1}=[g_{m-1}]$. Set $g=(g_{jk})_{j,k}\in \bigoplus_{j,k}\Gamma(\tau_N^{-1}(U)\cap\gamma(K_j\cup K_k),\widetilde{A}_{M+})$ as : $g_{mj}=-g_{jm}=\frac{1}{2}g_j$ for every j and the other components are equal to zero. So the m-th component of $f-F_1(g)$ belongs to $\Gamma(U,\mathcal{A}_{M|N})$ and hence it may be assumed to be zero. Consequently the induction step on m completes the proof.

<u>Corollary 2.1.24</u> (Canonical flabby extensions)

We have a natural imbedding of $\hat{B}_{N|M+}$ into $\mathcal{H}^0_{M+}(B_M)_N$ as a sheaf homomorphism on N.

$$\mathrm{ext} \quad ; \quad \hat{B}_{N|M+} \longrightarrow \mathcal{H}^0_{M+}(B_M)_N \ ,$$

which gives a right inverse of the natural homomorphism $\mathcal{H}^0_{M+}(B_M)_N \longrightarrow B_{N|M+}$. Here we remark that ext is not a $\mathcal{D}_X$-homomorphism as seen in Proposition 2.1.18. Moreover the micro local version of ext is also defined as the following sheaf imbedding.

$$\mathrm{ext} \quad ; \quad \hat{C}_{N|M+} \longrightarrow C_{N|M+} \subset \iota^+_* C_{M+|X},$$

which gives a right inverse of the homomorphism : $C_{N|M+} \longrightarrow \iota^+_* C_{M+|X}/\iota_* C_{N|X}$. In fact the diagram

$$\begin{tikzcd}
\pi_N^{-1}\hat{B}_{N|M+} \arrow[r] \arrow[d, "\mathrm{ext}"] \arrow[dr, phantom, "\circlearrowleft"] & \hat{C}_{N|M+} \arrow[d, "\mathrm{ext}"] \\
\pi_N^{-1}\mathcal{H}^0_{M+}(B_M) \arrow[r] & C_{N|M+}
\end{tikzcd}$$

is commuting. Particularly for every $f \in \hat{B}_{N|M+}$, ext(f) is one of the extensions which have the smallest singular support in extensions to $\mathcal{H}^0_{M+}(B_M)$.

From now on, we call ext(f) "the canonical flabby extension of $f \in \hat{B}_{N|M+}$".

<u>Proof</u> These are direct consequences from Proposition 2.1.18 and the preceding theorem. The micro local version of ext is also well-defined by using the softness of $\hat{C}_{N|M+}$ and Proposition 2.1.21. Indeed the micro local version of ext coincides with the integral transformation $(\beta_k)^{-1} \circ E \circ \beta_k$ introduced in (2.5) in Theorem 2.1.12.

<u>Corollary 2.1.25</u> (Trace operator)

By the trace homomorphism on iSN ;

$$\widetilde{\hat{A}}_{M+} \ni f(z) \longrightarrow f(0,z') \in \widetilde{\hat{\mathcal{A}}}_N ,$$

the trace homomorphisms for $\hat{B}_{N|M+}$ and $\hat{C}_{N|M+}$ are induced.

$$\text{Trace} \quad : \quad \hat{B}_{N|M+} \ni f(x) \longrightarrow f(+0,x') \in B_N \quad \text{on } N$$

$$\text{Trace} \quad : \quad \hat{C}_{N|M+} \ni f(x) \longrightarrow f(+0,x') \in C_N \quad \text{on } iS^*N.$$

Certainly this is the natural extension of the substitution for hyperfunctions with real analytic parameter.

<u>Lemma 2.1.26.</u> Let X,Y be topological spaces and f ; $X \to Y$ be a continuous map. Suppose that any open subset of X and Y is para-compact and there exists an increasing sequence $\{E_j\}_{j=1,2,..}$ of closed subsets of X satisfying that ;

$$E_1 \subset \mathrm{int}(E_2) \subset E_2 \subset \mathrm{int}(E_3) \subset E_3 \subset \cdots\cdots, \quad \bigcup_{j=1}^{\infty} E_j = X \quad \text{and}$$

$f|_{E_j}$; $E_j \longrightarrow Y$ is a proper map with void or contractible fibers for every j.

Then for any complex $\mathcal{F}$ of sheaves on Y,

$$Rf_* f^{-1}\mathcal{F} \xleftarrow{\sim} Rg_* g^{-1}\mathcal{F}$$

holds, where g ; $f(X) \hookrightarrow Y$ is the imbedding.

The relation between $\mathcal{H}^0_{M+}(B_M)_N$ and $\widetilde{\hat{B}}_{M+}$ is clarified in the following proposition.

<u>Proposition 2.1.27.</u> Let K be a closed subset of iS^*N. Set $S = (\tau^+)^{-1}(K) \cup (S^*_Y X \cap \bar{G}+)$. Then,

i) if each fiber of K is non-void and proper convex,

$$(\tau_N|_{\mathrm{int}(K^o)})_*\widetilde{\mathcal{B}}_{M+} \xrightarrow{\sim} \pi_{M+/X*}\Gamma_S(\mathcal{C}_{M+|X}|_{\overline{G}+}),$$

that is , a section of $\mathcal{H}^0_{M+}(\mathcal{B}_M)_N$ whose singular support is contained in $\tau^{-1}(K)\cup\{\pm idx_1\}$ is uniquely written as a boundary value of a section of $\widetilde{\mathcal{B}}_{M+}$ on $\mathrm{int}(K^o)$.

ii) if each fiber of $V=iS^*N-K$ is non-void and proper convex,

$$R^{n-1}\tau_{N*}\mathbb{R}\Gamma_{(V^o)^a}\mathbb{R}\theta_*\mathbb{R}\Gamma_{F+}(\mathcal{Q}_{M+}|_L) \xrightarrow{\sim} \pi_{M+/X*}\Gamma_S(\mathcal{C}_{M+|X}|_{\overline{G}+}),$$

where a is the antipodal map.

<u>Proof</u> Recalling Lemma 2.1.16, we have

$$(2.14)\qquad \mathbb{R}\pi_{M+/X*}\mathbb{R}\Gamma_S(\mathcal{C}_{M+|X}|_{\overline{G}+}) \xleftarrow{\sim} \mathbb{R}\pi_{M+/X*}\mathbb{R}\Gamma_S\mathbb{R}\tau^!_*\pi'^{-1}(\mathcal{Q}_{M+}|_L)[n-1]$$
$$= \mathbb{R}\pi_{M+/X*}\mathbb{R}\tau^!_*\mathbb{R}\Gamma_{\tau'^{-1}(S)}\pi'^{-1}(\mathcal{Q}_{M+}|_L)[n-1]$$
$$= \mathbb{R}\tau_*\mathbb{R}\pi^!_*\mathbb{R}\Gamma_{\tau'^{-1}(S)}\pi'^{-1}(\mathcal{Q}_{M+}|_L)[n-1].$$

We denote by j the imbedding $D^+-\tau'^{-1}(S)\hookrightarrow D^+$. Then the following triple is a triangle.

$$(2.15)\qquad \begin{tikzcd}[column sep=small] & \mathbb{R}\pi^!_*\mathbb{R}\Gamma_{\tau'^{-1}(S)}\pi'^{-1}(\mathcal{Q}_{M+}|_L) \arrow[dl] & \\ \mathcal{Q}_{M+}|_L \arrow[rr] & & \mathbb{R}\pi^!_*\mathbb{R}j_*j^{-1}\pi'^{-1}(\mathcal{Q}_{M+}|_L) \arrow[ul, "+1"'] \end{tikzcd}$$

Note that the map $\pi'\circ j$ is written as the composite of g_2 and g_1, where g_1 and g_2 are defined as follows.

$$\begin{array}{ccccc} D^+-\tau'^{-1}(S) & \xrightarrow{\quad g_1\quad} & L\underset{N}{\times}(iS^*N-K) & \xrightarrow{\quad g_2\quad} & L \\ \Psi & & \Psi & & \Psi \\ (0,x';w_1,iv';\zeta_1,i\eta') & & (0,x';w_1,iv';i\eta') & & (0,x';w_1,iv') \end{array}$$

As seen in the proof of Proposition 2.1.21 the fiber of g_1 is a non void convex set on $\{(0,x';w_1,iv')\notin F+ \text{ or } \langle v',\eta'\rangle \geq 0\}$ and is void on the other point. Therefore by the preceding lemma we have

$$\mathbb{R}\pi'_*\mathbb{R}j_*j^{-1}\pi'^{-1}(q_{M+}|_L) \xleftarrow{\sim} \mathbb{R}g_{2*}\mathbb{R}k_*k^{-1}g_2^{-1}(q_{M+}|_L) = \mathbb{R}(g_2|_Y)_*(g_2|_Y)^{-1}(q_{M+}|_L),$$

where k ; $Y=\{(0,x';w_1,iv';i\eta')\in L \underset{N}{\times} (iS^*N-K);\ (0,x';w_1,iv')\notin F+ \text{ or } \langle v',\eta'\rangle \geq 0\} \hookrightarrow L \underset{N}{\times} (iS^*N-K)$ is the imbedding. Hence from the triangle in (2.15), it follows that,

$$\mathbb{R}\pi'_*\mathbb{R}\Gamma_{\tau'^{-1}(S)}\pi'^{-1}(q_{M+}|_L) \xleftarrow{\sim} \mathbb{R}\mathrm{dist}_{g_2|Y}(q_{M+}|_L). \tag{2.16}$$

Consider the following triangle

$$\begin{tikzcd} & \mathbb{R}\mathrm{dist}_{h_2}(q_{M+}|_L) \arrow[dl] & \\ \mathbb{R}\mathrm{dist}_{g_2|Y}(q_{M+}|_L) \arrow[rr] & & \mathbb{R}h_{2*}\mathbb{R}\mathrm{dist}_{h_1}h_2^{-1}(q_{M+}|_L) \arrow[ul, "+1"'] \end{tikzcd} \tag{2.17}$$

attached to the triple of natural continuous maps as below.

$$\begin{tikzcd} Y \arrow[rr, "g_2|Y"] \arrow[dr, "h_1"'] & & L \\ & L \underset{N}{\times} (iS^*N-K) \arrow[ur, "h_2"'] & \end{tikzcd}$$

Let us calculate the other two complexes in (2.17). Firstly, by the preceding lemma and the assumption for K we see $\mathbb{R}\mathrm{dist}_{h_2}(q_{M+}|_L)=0$. Secondly, $\mathbb{R}h_{2*}\mathbb{R}\mathrm{dist}_{h_1}h_2^{-1}(q_{M+}|_L) = \mathbb{R}h_{2*}\mathbb{R}\Gamma_P h_2^{-1}(q_{M+}|_L)$, where $P=\{(0,x';u_1,iv';i\eta')\in F+ \underset{N}{\times} (iS^*N-K);\ \langle v',\eta'\rangle < 0\}$ is a locally closed subset of $L \underset{N}{\times} (iS^*N-K)$,

$$= \mathbb{R}h_{2*}\Gamma_P\mathbb{R}\Gamma_{h_2^{-1}(F+)}h_2^{-1}(q_{M+}|_L) \xleftarrow{\sim} \mathbb{R}h_{2*}\Gamma_P h_2^{-1}\mathbb{R}\Gamma_{F+}(q_{M+}|_L)$$

since every point of iS^*N-K has a neighborhood base consisting of open convex sets. Therefore, (2.18) $\mathbb{R}\mathrm{dist}_{g_2|Y}(q_{M+}|_L) \xrightarrow{\sim} \mathbb{R}h_{2*}\Gamma_P h_2^{-1}\mathbb{R}\Gamma_{F+}(q_{M+}|_L)$.

Taking account of the triangle

$$\begin{tikzcd}[column sep=small] & \mathbb{R}h_{2*}\Gamma_P h_2^{-1}\mathbb{R}\Gamma_{F+}(q_{M+}|_L) \arrow[dl] & \\ \mathbb{R}h_{2*}h_2^{-1}\mathbb{R}\Gamma_{F+}(q_{M+}|_L) \arrow[rr] & & \mathbb{R}h_{2*}(h_2^{-1}\mathbb{R}\Gamma_{F+}(q_{M+}|_L)|_{h_2^{-1}(F+)-P}) \arrow[ul, "+1"'] \end{tikzcd}$$

we have,

(2.19) $\mathbb{R}h_{2*}\Gamma_P h_2^{-1}\mathbb{R}\Gamma_{F+}(q_{M+}|_L) \simeq \mathbb{R}\mathrm{dist}_{h_2^!}\mathbb{R}\Gamma_{F+}(q_{M+}|_L)$

where $h_2^!$; $\{(0,x';u_1,iv';i\eta')\in F+ \underset{N}{\times} (iS^*N-K);\ \langle v',\eta'\rangle \geq 0\} \longrightarrow F+$.

From an easy application of the preceding lemma, we obtain that ;

(2.20) $\mathbb{R}\mathrm{dist}_{h_2^!}\mathbb{R}\Gamma_{F+}(q_{M+}|_L) \simeq \begin{cases} \mathbb{R}i_*i^{-1}\mathbb{R}\Gamma_{F+}(q_{M+}|_L)[2-n] \\ \qquad\qquad \text{in the case i).} \\ \mathbb{R}\Gamma_{\theta^{-1}((V^o)^a)}\mathbb{R}\Gamma_{F+}(q_{M+}|_L) \\ \qquad\qquad \text{in the case ii).} \end{cases}$

(i : $\theta^{-1}(\mathrm{int}(K^o)) \hookrightarrow F+$)

Combination of these facts (2.14)$\sim$(2.20) completes the proof.

As the final of § 2.1, we want to state a conjecture on the fundamental relationship between $\mathcal{H}^0_{M+}(B_M)$ and $C_{N|M+}$. Together with the flabbiness of $C_{N|M+}$, this means the decomposability of singularities of hyperfunctions with support in M+. This is true when n=2 (see the proof of Proposition 2.1.10).

Conjecture The following sequence is exact.

$$0 \longrightarrow \pi_{M+/X*}\Gamma_{\bar{G}+\cap S_Y^*X}(C_{M+|X}|_{\bar{G}+}) \longrightarrow \mathcal{H}^0_{M+}(B_M)_N \longrightarrow \pi_{N*}C_{N|M+} \longrightarrow 0$$

2.2 Product and Green's formula.

A.Kaneko used effectively Green's formula to study singularities of boundary values of hyperfunction solutions of boundary value problems ([2],[3]). He constructed a defining function of a fundamental solution of the adjoint system with small singularity in the complex domain and made the product of this and the given solution of the original system as the trace of the product of two defining functions in the complex domain. Here we inherit this idea and generalize it in the micro local situation.

Definition 2.2.1 For a hyperfunction f(x) which is mild from the positive side of N, we define the singular support of f near N as,

$$SS(f) \overset{\text{def}}{=} SS(ext(f)) \subset iS^*M|_{M+},$$

where ext(f) is the canonical flabby extension of $f \in \hat{B}_{N|M+}$.

The singular support of a section of $\mathcal{H}^0_{M+}(B_M)$ has a fibrated structure on N, that is,

$$SS(f) \cap \pi^{-1}(N) - iS^*_N M = \tau^{-1} \circ \tau(SS(f) \cap \pi^{-1}(N) - iS^*_N M)$$

for every $f \in \mathcal{H}^0_{M+}(B_M)$, where $\tau: iS^*M|_N - iS^*_N M \longrightarrow iS^*N$ is the projection (see Theorem 4.3.3 [8], which is a direct consequence from the unique cintinuation property of $C_{M+|X}$). So we define <u>the reduced singular support</u> τ-SS(f) of $f \in \mathcal{H}^0_{M+}(B_M)$ on N as,

$$\tau\text{-}SS(f) \overset{\text{def}}{=} \tau(SS(f) \cap \pi^{-1}(N) - iS^*_N M) \subset iS^*N,$$

and furthermore

$$\tau\text{-}SS(f) = \tau\text{-}SS(ext(f)) \quad \text{for every } f \in \hat{B}_{N|M+}.$$

Then, easily to see, the following relations hold.

$\ell\text{-}SS(f)$ = (the support of $[f]$ in $C_{N|M+}$) for every $f \in \mathcal{H}^0_{M+}(B_M)_N$

$\ell\text{-}SS(f)$ = (the support of $[f]$ in $\hat{C}_{N|M+}$) for every $f \in \hat{B}_{N|M+}$

Theorem 2.2.2 (Product : $\hat{B}_{N|M+} \underset{\mathbb{C}}{\otimes} \cdots \underset{\mathbb{C}}{\otimes} \hat{B}_{N|M+} \longrightarrow \hat{B}_{N|M+}$)

Let $f_1, \ldots, f_m$ be sections of $\hat{B}_{N|M+}$. If $t_1\eta_1' + \cdots + t_m\eta_m'$ for $t_1 \geq 0, \ldots, t_m \geq 0$ with $t_1 + \cdots + t_m > 0$ and $(x'; i\eta_1') \in \ell\text{-}SS(f_1), \ldots, (x'; i\eta_m') \in \ell\text{-}SS(f_m)$ is not equal to zero in each fiber of iT^*N, the commutative product of $f_1, \ldots, f_m$

$$f_1(x) \otimes \cdots \otimes f_m(x) \longrightarrow f_1(x) \cdot \cdots \cdot f_m(x)$$

is well defined as a section of $B_{N|M+} = \mathcal{H}^0_{M+}(B_M)/\mathcal{H}^0_N(B_M)\big|_N$. We assert that the product $f_1(x) \cdot \cdots \cdot f_m(x)$ is also mild and

$$\ell\text{-}SS(f_1 \cdot \cdots \cdot f_m) \subset \ell\text{-}SS(f_1)^\vee \cdots {}^\vee \ell\text{-}SS(f_m),$$

where $K_1^\vee \cdots {}^\vee K_m = \{(x'; i\eta') \in iS^*N \ ; \ \eta' = t_1\eta_1' + \cdots + t_m\eta_m'$ for some $t_1 \geq 0, \ldots, t_m \geq 0$ with $t_1 + \cdots + t_m > 0$ and some $(x'; i\eta_1') \in K_1, \ldots, (x'; i\eta_m') \in K_m\}$ for subsets $K_1, \ldots, K_m$ of iS^*N.

Proof Through Proposition 2.1.21 Theorem 2.1.23 and the softness of $\hat{C}_{N|M+}$ the product operation on $\tilde{A}_{M+}$;

$$\tilde{A}_{M+} \underset{\mathbb{C}}{\otimes} \tilde{A}_{M+} \ni f(z) \otimes g(z) \longrightarrow f(z)g(z) \in \tilde{A}_{M+}$$

induces an operation on sections of $\hat{B}_{N|M+}$. Obviously this coincides with the ordinary product operation on $B_{N|M+}$. So it follows that $f_1(x)f_2(x)$ is mild and $\ell\text{-}SS(f_1f_2) \subset \ell\text{-}SS(f_1)^\vee \ell\text{-}SS(f_2)$. Repeating this process, we reach the assertion.

Theorem 2.2.3 (Product : $\hat{B}_{N|M+} \underset{\mathbb{C}}{\otimes} \mathcal{H}^0_{M+}(B_M)_N \longrightarrow \mathcal{H}^0_{M+}(B_M)_N$)

Let $f(x)$ be a section of $\hat{B}_{N|M+}$ and $g(x)$ be a section of $\mathcal{H}^0_{M+}(B_M)_N$. Suppose that $\ell\text{-SS}(f) \cap \ell\text{-SS}(g)^a = \phi$. Then the product ;

$$f(x) \otimes g(x) \longrightarrow f(x)g(x)$$

is well-defined as a section of $\mathcal{H}^0_{M+}(B_M)$, which coincides with the ordinary product of two hyperfunctions in $\mathrm{int}(M+)$. And the following estimate holds.

$$\ell\text{-SS}(fg) \subset \ell\text{-SS}(f) \vee \ell\text{-SS}(g)$$

Furthermore this definition satisfies the associatve law in the following sense.

$$f_1(x)(f_2(x)g(x)) = f_2(x)(f_1(x)g(x)) = (f_1(x)f_2(x))g(x)$$

for all $f_1, f_2 \in \hat{B}_{N|M+}$ and $g \in \mathcal{H}^0_{M+}(B_M)_N$ such that $t_1\eta_1' + t_2\eta_2' + t_3\eta_3'$ is not zero in each fiber for $t_1 \geq 0$ $t_2 \geq 0$ $t_3 \geq 0$ with $t_1+t_2+t_3 > 0$ and $(x'; i\eta_1') \in \ell\text{-SS}(f_1)$ $(x'; i\eta_2') \in \ell\text{-SS}(f_2)$ $(x'; i\eta_3') \in \ell\text{-SS}(g)$.

We remark that , if $g(x)$ is a section of $\mathcal{H}^0_N(B_M)$, fg is also a section of $\mathcal{H}^0_N(B_M)$.

<u>Proof</u> We fix an open subset U of N. Let $f(x)$ be a section of $\hat{B}_{N|M+}$ on U and $g(x)$ be a section of $\mathcal{H}^0_{M+}(B_M)_N$ on U satisfying that $\ell\text{-SS}(f) \cap \ell\text{-SS}(g)^a = \phi$. First suppose that $\ell\text{-SS}(f)$ is contained in an open subset V of $iS^*N|_U$ with $V \cap \ell\text{-SS}(g)^a = \phi$ and each fiber of V is non void and proper convex. By Proposition 2.1.21 f is written as the boundary value of a section $F(z) \in \Gamma(V^o|_U, \tilde{A}_{M+}) = \Gamma(\theta^{-1}(V^o)|_U, \tilde{\mathcal{A}}_{M+}|_{F+})$. Since $\ell\text{-SS}(g) \subset iS^*N - V^a$, by Proposition 2.1.27 g is identified with a section of $\Gamma(U, R^{n-1}\tau_{N*}\mathbb{R}\Gamma_{V^o}\mathbb{R}\theta_*\mathbb{R}\Gamma_{F+}(q_{M+}|_L))$. Therefore $f(x)g(x)$ is cohomologically well-defined as a section of $\Gamma(U, R^{n-1}\tau_{N*}\mathbb{R}\Gamma_{V^o}\mathbb{R}\theta_*\mathbb{R}\Gamma_{F+}(q_{M+}|_L))$. Thus $f(x)g(x) \in \Gamma(U, \mathcal{H}^0_{M+}(B_M)_N)$ and $\ell\text{-SS}(fg) \subset iS^*N - V^a$. On the other hand for any proper convex open subset V' of $iS^*N|_U$ satisfying that $V \subset V' \subset iS^*N - (\ell\text{-SS}(g))^a$, g is identi-

fied with a section of $\Gamma(U,\ R^{n-1}\tau_{N*}\mathbb{R}\Gamma_{(V')^{o}}\mathbb{R}\theta_{*}\mathbb{R}\Gamma_{F+}(q_{M+}|_{L}))$. Noting that $(V')^{o} \subset V^{o}$, this shows $f(x)g(x)$ is also defined as a section of $\Gamma(U,\ R^{n-1}\tau_{N*}\mathbb{R}\Gamma_{(V')^{o}}\mathbb{R}\theta_{*}\mathbb{R}\Gamma_{F+}(q_{M+}|_{L}))=\{h(x)\in\Gamma(U,\ \mathcal{H}^{0}_{M+}(B_{M})_{N});\ \ell\text{-}SS(h) \subset iS^{*}N-(V')^{a}\}$. Because these definitions of product are compatible with each other we obtain that

$$\ell\text{-}SS(fg) \subset \bigcap_{V'} (iS^{*}N|_{U} - (V')^{a}) ,$$

where V' moves over proper convex open subsets of $iS^{*}N|_{U}$ such that $V \subset V' \subset iS^{*}N-(\ell\text{-}SS(g))^{a}$. Let $(x';i\eta')$ be a point of $\bigcap_{V'}(iS^{*}N|_{U}-(V')^{a})$. Then, easily to see, $\gamma(\{(x';-i\eta')\}\cup V)\cap \ell\text{-}SS(g)^{a} \neq \phi$. In other words $(x';i\eta') \in V^{\vee}\ell\text{-}SS(g)$. Therefore we have the following estimate.

$$\ell\text{-}SS(fg) \subset V^{\vee}\,\ell\text{-}SS(g) \tag{2.21}$$

Before proceeding to the general case , it is noticed that the assumption of non-voidness of fibers of V is unnecessary. For the general case, express $f(x)$ in the form,

$$f(x) = \sum_{j}^{\text{finite}} f_{j}(x),$$

where $\{f_{j}\}_{j}$ are mild hyperfunctions defined on U such that for every j $\ell\text{-}SS(f_{j})$ is contained in a proper convex set which is separated rom $\ell\text{-}SS(g)^{a}$. In fact this is possible by the softness of $\hat{\mathcal{C}}_{N|M+}$ and the assumption : $\ell\text{-}SS(f)\cap\ell\text{-}SS(g)^{a}=\phi$. So we define fg by

$$\sum_{j} f_{j}(x)g(x).$$

From the edge of the wedge theorem for $\widetilde{A}_{M+}$ it follows that this definition does not depend on the expressions of $f(x)$. Further the estimate $\ell\text{-}SS(fg) \subset \ell\text{-}SS(f)^{\vee}\,\ell\text{-}SS(g)$ is easily obtained from (2.21).

Remark 2.2.4 i) These definitions of product satisfy Leibniz rule for differential calculations.

ii) If $g(x) \in \mathcal{H}^{0}_{M+}(B_M)_N$ is written as a boundary value of a section of $\widetilde{\widehat{B}}_{M+}$, the product of g(x) and a section of $\hat{B}_{N|M+}$ coincides with the product induced by the product : $\widetilde{\widehat{A}}_{M+} \underset{\mathbb{C}}{\otimes} \widetilde{\widehat{B}}_{M+} \longrightarrow \widetilde{\widehat{B}}_{M+}$. Especially we have the following equations for every $f \in \hat{B}_{N|M+}$.

$$\text{ext}(f(x)) = f(x)Y(x_1) \tag{2.22}$$

$$\text{Trace}(f(x)) = \int f(x)\delta(x_1)dx_1 \tag{2.23}$$

Using expressions by $\widetilde{\widehat{A}}_{M+}$, several operations on mild hyperfunctions other than products are defined in the following theorem (cf. S-K-K CH I [10] and § 2.3 [8]).

Theorem 2.2.5 Let M' M be real analytic manifolds, N' N be their submanifolds with codimension 1 respectively and φ; $M' \longrightarrow M$ be a real analytic map such that $\varphi(N') \subset N$ and $d\varphi$; $T_{N'}M'|_x \longrightarrow T_N M|_{\varphi(x)}$ is bijective for every $x \in N'$. So we may assume that $\varphi(M'_+) \subset M_+$. We denote by φ' the restriction of φ to N'.

i) Substitution is defined as the following sheaf homomorphism.

$$\varphi'^{-1}\hat{B}'_{N|M+} \ni f \longrightarrow f \circ \varphi \in \hat{B}_{N'|M'_+},$$

and

$$\iota\text{-SS}(f \circ \varphi) \subset \varphi'^{*}(\iota\text{-SS}(f)),$$

where $\hat{B}'_{N|M+} = \text{Kernel}(\hat{B}_{N|M+} \longrightarrow \pi_{N*}(\hat{C}_{N|M+}|_{iS^*_N,N}))$ and φ'^{*} ; $(iS^*N - iS^*_N,N) \underset{N}{\times} N' \longrightarrow iS^*N'$.

ii) Suppose that φ is a submersion. Then Integration along fibers is defined as follows.

$$\varphi_!\hat{B}_{N'|M'_+} \underset{\mathcal{A}_{M'}}{\otimes} \mathcal{V}_{M'}^{\dim M'} \ni g \longmapsto \int_{\varphi^{-1}} g \in \hat{B}_{N|M+} \underset{\mathcal{A}_M}{\otimes} \mathcal{V}_M^{\dim M}$$

and

$$\ell\text{-SS}\Big(\int_{\varphi^{-1}} g\Big) \subset \varpi\circ(\varphi'^{*})^{-1}(\ell\text{-SS}(g)),$$

where $\mathcal{V}_M^q$ is the sheaf of germs of real analytic q-forms on M and

$$iS^*N' \xleftarrow{\ \varphi'^{*}\ } iS^*N \underset{N}{\times} N' \xrightarrow{\ \varpi\ } iS^*N.$$

These operations commute with ext and Trace, that is,

(2.24) $$\operatorname{ext}(f)\circ\varphi = \operatorname{ext}(f\circ\varphi), \quad \operatorname{ext}\Big(\int_{\varphi^{-1}} g\Big) = \int_{\varphi^{-1}} \operatorname{ext}(g)$$

(2.25) $$\operatorname{Trace}(f)\circ\varphi' = \operatorname{Trace}(f\circ\varphi), \quad \operatorname{Trace}\Big(\int_{\varphi^{-1}} g\Big) = \int_{\varphi'^{-1}} \operatorname{Trace}(g)$$

hold.

<u>Proof</u> We prove only ii). We may assume that $M'=\mathbb{R}^{n+m}\ni(x,u) \xrightarrow{\varphi} x\in\mathbb{R}^n = M$ and $N'=\{(x,u)\in M'; x_1=0\}$, $N=\{x\in M; x_1=0\}$. Let $g(x,u)$ be a section of $\varphi_!\hat{B}_{N'|M'_+}$ defined in $\{x\in N; |x|<\delta\}$. Then by the softness of $\hat{C}_{N'|M'_+}$ there exist sections $F_1(z,w),\dots,F_{n+m}(z,w)$ of $\tilde{A}_{M'_+}$ defined on $\{(x',u;iy',iv)\in iSN'; |x'|<\frac{1}{2}\delta, u\in\mathbb{R}^m, (y',v)\in K_j\}\cup\{(x',u;iy',iv)\in iSN'; |x'|<\frac{1}{2}\delta, |u|\geq R, (y',v)\in S^{n+m-2}\}$ for $j=1,\dots,n+m$ respectively such that $\sum_{j=1}^{n+m} b_{K_j}(F_j(z,w))=g(x,u)$ as a section of $\hat{B}_{N'|M'_+}$, where $K_1,\dots,K_{n+m}$ are proper convex compact subsets of S^{n+m-2} satisfying $\bigcup_j \operatorname{int}(K_j^{o}) = S^{n+m-2}$ and $R>0$ is a large number satisfying that (the support of g in $\hat{B}_{N'|M'_+}$) $\cap\{(x',u)\in N'; |x'|<\frac{1}{2}\delta\}\subset\{(x',u)\in N'; |u|<R\}$. Then by the definition of integration of hyperfunctions (S-K-K CH I [10] and § 2.3 [8]) $\int g(x,u)du$ is written in the form

$$\sum_{j=1}^{n+m} b_{K'_j}\Big(\int_{|u|\leq R+1} F_j(z,u+i\alpha(|u|)v_j)\det\Big(\frac{\partial(u+i\alpha(|u|)v_j)}{\partial u}\Big)du\Big),$$

where $\alpha(t)$ is a smooth function on $[0,R+1]$ such that $1 \gg \alpha(t) \equiv \varepsilon > 0$ on $t \in [0,R]$, $0 \leq \alpha(t) \leq \varepsilon$ on $[R,R+1]$ and $\alpha(R+1)=0$, v_j is a unit vector of $\mathbb{R}^m$ such that $\{y' \in \mathbb{R}^{n-1}; (y',v_j) \in K_j\} \neq \emptyset$ for every j and $K'_j = \{y' \in \mathbb{R}^{n-1} - \{0\}/\mathbb{R}+;\ \exists v \in \mathbb{R}^m$ such that $(y',v) \in K_j\}$. In fact, since $F_j(z,w)$ is holomorphic on $\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m; z_1=0, \mathrm{Im} z'=0, \mathrm{Im} w=0, |\mathrm{Re} z'| < \frac{1}{2}\delta, |\mathrm{Re} w| \geq R\}$, the integral $G_j(z) = \int_{|u| \leq R+1} F_j(z, u+i\alpha(|u|)v_j) \det\left(\frac{\partial(u+i\alpha(|u|)v_j)}{\partial u}\right) du$ is well defined and does not depend on the choice of $\alpha(t)$ and v_j. So $G_j(z)$ is a section of $\widetilde{A}_{M+}$ on $\{(x';iy') \in iS\mathbb{R}^{n-1}; |x'| < \frac{1}{2}\delta, y' \in K'_j\}$ for every j. Thus the mildness of $\int g(x,u)du$ is proved. Furthermore from this concrete expression of the integral we obtain that $\mathrm{ext}(\int g(x,u)du) = \int \mathrm{ext}(g(x,u))du$, $\mathrm{Trace}(\int g(x,u)du) = \int \mathrm{Trace}(g(x,u))du$ and the estimate of the singular support.

Fix a coordinate system (x_1,x') of $\mathbb{R}^n$. Then for any germ $F(z)$ of $\widetilde{A}_{M+}$, $z_1^{-1}(F(z)-F(0,z'))$ and $F(z_1{}^q,z')$ belong to $\widetilde{A}_{M+}$ again, where q is a positive integer. Using these facts we obtain the following results.

<u>Proposition 2.2.6</u> Fix a coordinate (x_1,x') of M. Then we have the following sheaf homomorphisms.

i) $\hat{B}_{N|M+} \ni f(x) \longmapsto x_1^{-1}(f(x)-\mathrm{Trace}(f)(x')) \in \hat{B}_{N|M+}$

$$\ell\text{-SS}(x_1^{-1}(f(x)-\mathrm{Trace}(f)(x'))) \subset \ell\text{-SS}(f(x))$$

ii) $\hat{B}_{N|M+} \ni f(x) \longmapsto f(x_1{}^q,x') \in \hat{B}_{N|M+}$

$$\ell\text{-SS}(f(x_1{}^q,x')) \subset \ell\text{-SS}(f(x)),$$

where q is a positive integer. Furthermore, in connection with the trace operation, the following formulas hold.

$$\mathrm{Trace}(f(x)g(x)) = \mathrm{Trace}(f(x))\mathrm{Trace}(g(x))$$

$$\mathrm{Trace}(f(x_1{}^q, x')) = \mathrm{Trace}(f(x)),$$

where $f, g \in \hat{B}_{N|M+}$ such that $\tau\text{-}SS(f) \cap \tau\text{-}SS(g)^a = \phi$.

The following lemma is very useful to verify the mildness of boundary values of holomorphic functions.

Lemma 2.2.7 Let $F(z)$ be a holomorphic function defined on $\{z \in \mathbb{C}^n;\ |z| < \varepsilon, \mathrm{Re}z_1 > 0, \mathrm{Im}z_1 = \cdots = \mathrm{Im}z_{n-1} = 0, \mathrm{Im}z_n > 0\} \cup \{z \in \mathbb{C}^n;\ |z| < \varepsilon,\ \mathrm{Im}z_n > C(|z_1| + |\mathrm{Im}z_2| + \cdots + |\mathrm{Im}z_{n-1}|)\}$ for some positive numbers ε, C. Then the boundary value $F(x_1, \ldots, x_{n-1}, x_n + i0)$ is mild from $x_1 > 0$ at the origin.

Proof Consider $g(x_1, z') = Y(x_1)F(x_1, z')$. Then the boundary value of $g(x_1, z')$ is a hyperfunction with support in $\{x_1 \geqq 0\}$ and coincides with $F(x_1, \ldots, x_{n-1}, x_n + i0)$ in $\{x_1 > 0\}$. So it suffices to show that $g(x_1, z')$ defines a section of $C_{N|X}$ on $S_Y^*X \cap \pi^{-1}(0)$ (see Lemma 2.1.20). In particular $F(z)$ is extended analytically to $\{z \in \mathbb{C}^n;\ |z| < \delta, \mathrm{Im}z_n > \delta^{-1}(|\mathrm{Im}z_1| + \cdots + |\mathrm{Im}z_{n-1}| + |\mathrm{Re}z_1| \cdot Y(-\mathrm{Re}z_1))\}$ for sufficiently small $\delta > 0$.

The micro local versions of several operations are easily derived from these theorems. We omit the details.

We fix $M = \mathbb{R}^n$ $N = \mathbb{R}^{n-1} = \{x \in M; x_1 = 0\}$. Let $P(x,D) = D_1{}^m + P_1(x, D')D_1{}^{m-1} + \cdots + P_m(x, D')$ be a pseudo-differential operator of order m defined on a neighborhood of $\tau^{-1}((x_0'; i\eta_0'))$ and $f(x)$ be a germ of $\hat{C}_{N|M+}$ at $(x_0'; i\eta_0')$. Assume that $u(x)$ is a germ of $\hat{C}_{N|M+}$ at $(x_0'; i\eta_0')$ satisfying $Pu(x) = f(x)$. To calculate $P(x,D)\mathrm{ext}(u)$, we use (2.22) and (2.24). Indeed the operator $P_j(x, D')$ is considered as an integral transformation whose kernel is given by $P_j(x,D_{x'})\delta(x'-y')dy'$. So we have

$$P(x,D)\mathrm{ext}(u)(x) = \mathrm{ext}(f) + \sum_{j,k=0}^{m-1} \delta^{(j)}(x_1)Q_{jk}(x',D')(\frac{\partial^k u}{\partial x_1{}^k}(+0,x')),$$

where $\{Q_{jk}(x',D')\}_{j,k}$ are pseudo-differential operators of order less than m defined at $(x_0';i\eta_0')$ and they are induced by $P(x,D)$. In particular u is uniquely determined by f and $\partial^k u/\partial x_1{}^k(+0,x')$ for $k=0,1,\dots,m-1$.

<u>Theorem 2.2.8</u> We inherit the notations. Let $u(x)$ be a germ of $\hat{C}_{N|M+}$ at $(x_0';i\eta_0')$ such that $P(x,D)u=0$ and $v(x,y')$ be a germ of $\hat{C}_{N'|M_+'}$ at $(x_0',x_0';-i\eta_0',i\eta_0')$ such that ${}^tP(x,D_x)v=0$, where $M_+'=M_+\times\mathbb{R}^{n-1}$ $\supset N'=N\times\mathbb{R}^{n-1}\ni(x',y')=(x_2,\dots,x_n,y_2,\dots,y_n)$ and ${}^tP(x,D)$ is the formal adjoint of $P(x,D)$. Then we mean by "The micro local Green's formula" that

$$\begin{aligned} 0 &= \int P(x,D)u(x)\cdot\mathrm{ext}(v(x,y'))dx \\ &= \int u(x)\cdot {}^tP(x,D)\mathrm{ext}(v(x,y'))dx \\ &= \sum_{j=0}^{m-1} (-1)^j \int (D_1{}^j u)(+0,x')w_j(x',y')dx' \end{aligned}$$

holds at $(y';i\tau')=(x_0';i\eta_0')$ in the sense of microfunctions in y', where

$$(2.26) \qquad {}^tP(x,D_x)\mathrm{ext}(v(x,y')) = \sum_{j=0}^{m-1} w_j(x',y')\cdot\delta^{(j)}(x_1).$$

Two conditions are required for the micro local Green's formula. One is the well-definedness of the products $P_j(x,D')D_1^{m-j}u(x)\cdot\mathrm{ext}(v)$, ... and so on as microfunctions at $(x_0',x_0';\ 0,i\eta_0')$. The other is the integrability of the microfunctions $P_j(x,D')D_1^{m-j}u(x)\cdot\mathrm{ext}(v(x,y'))$, ... and so on. The following is a sufficient condition satisfying these requirements.

" $SS(w_j(x',y'))\subset\{(x',y';i\eta',i\tau')\in iS^*N';\ x'=y',\ \eta'+\tau'=0\}$ for every j and $SS(ext(u))\subset\{(x;i\eta)\in iS^*M;\ x_1=0\}$".

<u>Proof</u> By the assumption for $\{w_j(x',y')\}_j$ and (2.26), ℓ-$SS(ext(v))\subset\{x'=y',\ \eta'+\tau'=0\}$ holds (because ${}^tP(x,D_x): C_{M_+'|X'}\longrightarrow C_{M_+'|X'}$ is injective). So $(P_j(x,D_{x'})\delta(x'-x''))(D_1^{m-j}u)(x_1,x'')\times ext(v(x_1,x',y'))$ is well defined as a microfunction and is integrable with respect to the variables x',x''. Therefore,

$$\int P(x,D)u(x)\cdot ext(v(x,y'))dx'$$
$$=\sum_{j=0}^{m}\int P_j(x,D_{x'})\delta(x'-x'')(D_1^{m-j}u)(x_1,x'')\cdot ext(v(x_1,x',y'))dx'dx''$$
$$=\sum_{j=0}^{m}\int(D_1^{m-j}u)(x_1,x')\cdot{}^tP_j(x,D_{x'})ext(v(x_1,x',y'))dx'.$$

Since the support of $D_1{}^r u(x)\cdot D_1{}^s({}^tP_j(x,D'))ext(v(x,y'))$ in $C_{M'}$ is contained in $\{x_1=0\}$, by the Leibniz rule

$$\int P(x,D)u(x)\cdot ext(v(x,y'))dx$$
$$=\sum_{j=0}^{m}\int u(x)\cdot(-D_1)^{m-j}({}^tP_j(x,D'))ext(v(x,y'))dx$$
$$=\int u(x)\cdot{}^tP(x,D)ext(v(x,y'))dx$$
$$=\sum_{j=0}^{m-1}(-1)^j\int(D_1{}^j u)(+0,x')w_j(x',y')dx'.$$

Thus the micro local Green's formula holds at $\{(y';i\tau');\ y'=x_0',\ \tau'=\eta_0'\}$.

We will have several applications of this theorem in the next paper.

2.3 Topological properties of mild hyperfunctions.

In § 2.1, we defined boundary values of mild hyperfunctions purely algebraically. Our aim in this section is to show that, in some case, these boundary values coincide with corresponding topological boundary values.

Let $f(x)$ be a section of $\Gamma_c(\mathbb{R}^{n-1}, \hat{B}_{N|M+})$ $(N=\mathbb{R}^{n-1} \hookrightarrow M_+=\{x\in\mathbb{R}^n; x_1\geq 0\})$. That is, $f(x)$ is a hyperfunction on $U=\{x\in\mathbb{R}^n; 0<x_1<\delta\}$ with support in $\{x\in U; |x'|\leq 1/\delta\}$ which is mild at every point of N. Since ext(f) belongs to $C_{N|X} \subset \mathcal{H}^0_{N\times_M iS^*M}(C_M)$ at every point of $iS^*_N M$ by the definition of mildness, $f(x)$ depends real analytically on the variable x_1 in $\{x\in U; 0<x_1<\delta'\}$ for some small $\delta'\leq\delta$. So we can consider a one-parameter family of hyperfunctions in $\mathbb{R}^{n-1}$ with support in $\{x'\in\mathbb{R}^{n-1}; |x'|\leq 1/\delta\}$, $\{f(\varepsilon, x')\}_\varepsilon$.

<u>Lemma 2.3.1</u> $\displaystyle\lim_{\varepsilon\to+0} \int_{\mathbb{R}^n} \mathrm{ext}(f(x))Y(\varepsilon-x_1)dx = 0.$

<u>Proof</u> Recall Theorem 2.2.5 ii). By the softness of $\hat{C}_{N|M+}$ there exist sections $F_1(z),\dots,F_n(z)$ of $\tilde{A}_{M+}$ defined on $\{(x';iv')\in iSN; v'\in K_j \text{ or } |x'|\geq 2/\delta\}$ for $j=1,\dots,n$ respectively such that $\sum_{j=1}^n b_{K_j}(F_j(z))=f(x)$ as a section of $\hat{B}_{N|M+}$, where $K_1,\dots,K_n$ are proper convex compact subsets of S^{n-2} satisfying $\bigcup_j \mathrm{int}(K_j^o)=S^{n-2}$. Then the integral $\int_{\mathbb{R}^n} \mathrm{ext}(f(x))Y(\varepsilon-x_1)dx$ is written in the form,

$$\sum_{j=1}^n \int_{|x'|\leq\frac{2}{\delta}} dx' \int_0^\varepsilon F_j(x_1, x'+i\alpha(|x'|)v_j)\det\left(\frac{\partial(x'+i\alpha(|x'|)v_j)}{\partial x'}\right)dx_1 ,$$

where $\alpha(t)$ is a smooth function on $[0,2/\delta]$ such that $1\gg\alpha(t)>0$ on $[0,2/\delta)$ and $\alpha(2/\delta)=0$ and v_j is a unit vector of K_j for every j. The integrand in this representation is a continuous function on

$[0,\varepsilon]\times\{|x'|\leq 2/\delta\}$, and so the integral converges to zero as $\varepsilon\to+0$.

<u>Theorem 2.3.2</u> Let K be a compact subset of $\mathbb{R}^{n-1}$ and f(x) be a section of $\Gamma_K(\mathbb{R}^{n-1},\hat{B}_{N|M+})$. Then $f(\varepsilon,x')$ converges in the weak topology of $\mathcal{A}(K)'$ as $\varepsilon\to+0$ to Trace(f(x)).

<u>Proof</u> Let g(x') be an analytic function defined on K. Applying Lemma 2.3.1 to $D_1(g(x')f(x))$, we have

$$\lim_{\varepsilon\to+0}\int_{\mathbb{R}^n}\mathrm{ext}(D_1(g(x')f(x)))Y(\varepsilon-x_1)dx=0.$$

Use the formula $g(x')f(\varepsilon,x')\delta(\varepsilon-x_1) = g(x')\mathrm{Trace}(f(x))\delta(x_1) + \mathrm{ext}(D_1(g(x')f(x)))Y(\varepsilon-x_1) - D_1\{\mathrm{ext}(g(x')f(x))Y(\varepsilon-x_1)\}$. So we obtain,

$$\lim_{\varepsilon\to+0}\int_{\mathbb{R}^{n-1}} g(x')f(\varepsilon,x')dx' = \int_{\mathbb{R}^{n-1}} g(x')\mathrm{Trace}(f(x))dx'.$$

Considering the softness of the sheaf of mild hyperfunctions, we can generalize Theorem 2.3.2.

<u>Corollary 2.3.3</u> Let M be a real analytic manifold, Ω be an open subset of M with compact real analytic boundary $N=\partial\Omega$ and $h(t,x') : (-\varepsilon,\varepsilon)\times N\to M$ be an analytic diffeomorphism onto a neighborhood of N such that $h(0,x')=\mathrm{id}_N$ and $h(t,x')\in\Omega$ if $t>0$. We assume that $f(x)\in\Gamma(\Omega,B_M)$ is mild at every point of N. Then $f(h(t,x'))$ converges in the weak topology of $\mathcal{A}(N)'$ to Trace(f(x)) as $t\to+0$.

References

[1] Kaneko, A., Remarks on hyperfunctions with analytic parameters, J. Fac. Sci. Univ. Tokyo Sec. IA 22 (1975).

[2] Kaneko, A., Singular Spectrum of Boundary Values of Solutions of Partial Differential Equations with Real Analytic Coefficients, Sci. Pap. Coll. Gen. Educ. Univ. Tokyo 25, 59-68 (1975).

[3] Kaneko, A., Analyticity of Minimal Dimensional Singularity of Real Analytic Solutions, to appear.

[4] Kashiwara, M., Kawai, T., On the boundary value problem for elliptic system of linear differential equations, I, Proc. Japan Acad., 48 (1972), 712-715.

[5] _____, II, Proc. Japan Acad., 49 (1973), 164-168.

[6] Kashiwara, M., Oshima, T., Systems of differential equations with regular singularities and their boundary value problems, Ann. Math., 106 (1977), 145-200.

[7] Kataoka, K., A micro local approach to general boundary value problems, Publ. R. I. M. S. Kyoto Univ. Vol.12, Supplement (1977), 147-153.

[8] Kataoka, K., On the theory of Radon transformations of hyperfunctions, to appear.

[9] Komatsu, H., Kawai, T., Boundary values of hyperfunction solutions of linear differential equations, Publ. R.I.M.S., 7 (1971), 95-104.

[10] Sato, M., Kawai, T., Kashiwara, M., Microfunctions and pseudo-differential equations, Lecture Notes in Math. No.287, Springer (1973).

[11] Schapira, P., Hyperfonctions et problèmes aux limites elliptiques, Bull. Soc. Math. France, 99 (1971), 113-141.

[12] Schapira, P., Propagation au bord et reflexion des singularites analytiques des solutions des equations aux derivees partielles, Sem. Goulaouic-Schwartz 1975-1976.

[13] Schapira, P., ____________, II, Sem. Goulaouic-Schwartz 1976-1977.

[14] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations, Japanese Journal of Math., to appear.

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Bunkyo-ku, Tokyo 113.

Micro Local Theory of Boundary Value Problems II and a Theorem on Regularity of Diffractive Operators.

By Kiyômi KATAOKA

Abstract

It is proven that boundary value problems for partially micro-hyperbolic pseudo-differential operators in one side of the boundary are solvable in a micro local sense and that , for any micro local solution defined up to the boundary to anti partially micro-hyperbolic pseudo-differential operators in one side or diffractive pseudo-differential operators, micro-analyticicity propagates up to the boundary.

Introduction

In [17], we introduced the notion "mildness" of hyperfunctions on a real analytic boundary (say, $x_1=0$), which expressed hyperfunctions defined in one side of the boundary which have boundary values for any normal derivative of finite or infinite order, and developed several operations on mild hyperfunctions, which were simultaneously formulated micro-locally. In this paper, using these tools, we formulate boundary value problems in a micro local situation and study solvability or a kind of regularity for them.

Set $M=\mathbb{R}^n\ni(x_1,x')$, $M_+=\{x\in M; x_1\geq 0\}$ and $N=\{x\in M; x_1=0\}$. Let $P(x,D)$ $=D_1{}^m+P_1(x,D')D_1^{m-1}+\cdots+P_m(x,D')$ $(D_j=\partial/\partial x_j)$ be a differential operator of order m with real analytic coefficients defined on $\{x\in M; |x|<r\}$. Then, as seen in [17], every hyperfunction solution $f(x)$ to $Pf(x)=0$ on $\{x\in M; x_1>0, |x|<r\}$ is mild from the positive side of N on $\{x'\in N;$

$|x'|<r\}$. So the boundary values $(D_1{}^j f)(+0,x')$ $j=0,1,..$ are always well-defined. But, once you attempt to formulate these using only the theory of microfunctions, you will confront two essential difficulties. Actually you can not define boundary values of microfunction solutions (say, defined on $\{(x;i\eta)\in iS^*M; x_1>0, |x|<r\}$) in general, and even if so, these boundary values are not unique as microfunctions on N (consider the case $P=D_1{}^2+\cdots+D_n{}^2$). To avoid these difficulties we use the micro-localized notion of mildness, that is, the sheaf $\hat{C}_{N|M+}$ on iS^*N introduced in [17]. A germ $u(x)$ of $\hat{C}_{N|M+}$ at $p_0'=(x_0'; i\eta_0')\in iS^*N$ defines a section of microfunctions on $\{(x;i\eta)\in iS^*M; r>x_1>0, |x'-x_0'|<r, |\eta'-\eta_0'|<r\}$ for some small $r>0$, but the converse is not true. Further, sections of pseudo-differential operators defined on $\{(z;\zeta)\in T^*X; z_1=0, (z';\zeta')=p_0', \zeta_1\in\mathbb{C}\}$ operate on the stalk of $\hat{C}_{N|M+}$ at p_0'. As seen in [17], if $P(x,D)$ is a pseudo-differential operator of order m of the form $D_1{}^m+P_1(x,D')D_1^{m-1}+\cdots+P_m(x,D')$, the correspondence $\{u\in\hat{C}_{N|M+}; Pu=0\} \longrightarrow (u(+0,x'),..,(D_1^{m-1}u)(+0,x'))\in(C_N)^m$ is an injective sheaf homomorphism on iS^*N. Thus boundary value problems are micro-localized in a natural manner.

In the first section, we explicitly seek relations among boundary values of $\hat{C}_{N|M+}$ - solutions corresponding to elliptic factors, which have already been obtained in [15] and [26], and construct the solution $u(x)$ in ${}^t(\hat{C}_{N|M+})^k$ to the following system of pseudo-differential equations at $p_0'\in iS^*N$,

$$\begin{cases} (D_1 I - A(x,D'))u(x) = f(x) \\ u(+0,x') = u_0(x'), \end{cases}$$

where $A(x,D')$ is a (k,k)-matrix of first-order pseudo-differential operators defined at $(0,x_0';i\eta_0')\in R\times iS^*N$, $f(x)$ is a germ of ${}^t(\hat{C}_{N|M+})^k$ and $u_0(x')$ is a germ of ${}^t(C_N)^k$ at p_0' such that $D_1I-A(x,D')$ is partially micro-hyperbolic in $\{x_1>0\}$ at p_0', that is, $\det(\zeta_1 I-\sigma_1(A)(x,i\eta'))=0$ has no root with positive real part with recpect to ζ_1 when $0\leq x_1\leq \varepsilon$, $|x'-x_0'|\leq\varepsilon$, $|\eta'-\eta_0'|\leq\varepsilon$ for some $\varepsilon>0$. Theorems of this type have been obtained by many authers ([3], [12], [29], [8], [23], [14]), though they assumed that P was a differential operator or P was partially micro-hyperbolic in both sides or f=0. In the proof we employ the argument of analytic continuation of defining functions due to Bony-Schapira and Kawai-Kashiwara ([3], [12]). Furthermore, using micro local Green's formula in [17], we obtain the dual version of this theorem, that is, for any $\hat{C}_{N|M+}$-solution $u(x)$ at p_0' to $P(x,D)u(x)=0$, where $P(x,D_1,-D')$ is partially micro-hyperbolic in $\{x_1>0\}$ at p_0', micro-analyticity of $u(x)$ as a section of C_M in $\{x_1>0\}$ leads to micro-analyticity of all boundary values at p_0'. This is a generalization of the theorem by Kaneko [8], where P is a differential operator and $u(x)$ is a hyperfunction solution. For partially micro-hyperbolic operators in both sides ([26]) and non-micro-characteristic operators ([27]), theorems of the same type have been obtained by Schapira.

In the second section, we prove the N_+-regularity of diffractive pseudo-differential operators. That is, assume that $P(x,D)=D_1^2+P_1(x,D')D_1+P_2(x,D')$ is a second-order pseudo-differential operator with real principal symbol defined at $p_0=(0,x_0';i\eta_0)\in iS^*_M{}^{M\times N}$ satisfying : $\sigma(P)(p_0)=\{\sigma(P),x_1\}(p_0)=0$, $\{\{\sigma(P),x_1\},\sigma(P)\}(0,x_0',\eta_0)<0$ and $d\sigma(P)\wedge dx_1\wedge\omega(p_0)\neq 0$, where $\{\ ,\ \}$ is the Poisson bracket and ω is the fundamental 1-form. Then for every $\hat{C}_{N|M+}$ - solution

$u(x)$ to $Pu(x)=0$ at $p_0'=(x_0';i\eta_0')$, all boundary values $(D_1{}^j u)(+0,x')$ $(j=0,1)$ are micro-analytic at p_0' if and only if $u(x)$ is micro-analytic as a section of microfunctions on $\gamma_{p_0}-\{p_0\}$ $(\subset\{x_1>0\})$, where γ_{p_0} is the bicharacteristic strip through p_0 (cf. [21], [6], [28]). In the last step of the proof of this theorem we employ Bony and Schapira's results ([1],[27]) on non-micro-characteristic pseudo-differential operators.

§1 Micro local boundary value problems

Let $P(x,D)$ be a differential operator of order m defined in $M=\{x\in\mathbb{R}^n;\ |x|<r\}$. Suppose that $N=\{x\in M; x_1=0\}$ is non-characteristic with respect to P. Then, according to Komatsu-Kawai and Schapira's theory of boundary value problems (Komatsu-Kawai [19], Schapira [25]), any hyperfunction solution of $P(x,D)u=0$ in $\{x\in M;\ x_1>0\}$ has a unique extension $\tilde{u}(x)\in\Gamma_{M+}(M,B_M)$ and "boundary values" $(f_0,\dots,f_{m-1})\in\Gamma(N,B_N)^m$ such that $\tilde{u}$ coincides with u in $\{x\in M; x_1>0\}$ and $P\tilde{u}=\sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1)$. On the other hand this is directly explained by the theory of mild hyperfunctions. In fact, since u is mild on N from the positive side of N (see [17]), $\mathrm{ext}(u)=u(x)Y(x_1)\in\Gamma_{M+}(M,B_M)$ is well defined and satisfies

$$P(x,D)\mathrm{ext}(u) = \sum_{j,k=0}^{m-1} \delta^{(j)}(x_1)Q_{jk}(x',D')(D_1{}^k u)(+0,x'),$$

where $\{Q_{jk}(x',D')\}$ are differential operators of order less than m induced by $P(x,D)$ and N. So we know that $\tilde{u}=\mathrm{ext}(u)$ and $f_j(x')=\sum_{k=0}^{m-1} Q_{jk}(x',D')(D_1{}^k u)(+0,x')$. Furthermore, by the theory of the sheaf $\hat{C}_{N|M+}$ and the exact sequence

$$0 \longrightarrow \mathcal{A}_M|_N \longrightarrow \hat{B}_{N|M+} \longrightarrow \pi_{N*}\hat{C}_{N|M+} \longrightarrow 0,$$

we can treat the solution $u(x)$ or the boundary values $(f_0,\dots,f_{m-1})$ micro locally on iS^*N (recall that ext and Trace are defined for the sections of $\hat{C}_{N|M+}$). For example, the local hyperfunction solution $u(x)$ of the problem

$$(1.1)\qquad \begin{cases} P(x,D)u(x)=0 & x_1>0 \\ \partial^j u/\partial x_1{}^j(+0,x')=f_j(x') & j=0,\dots,m-1 \end{cases}$$

for given hyperfunctions $(f_0(x'),..,f_{m-1}(x'))$ exists if and only if the problem (1.1) has a $\hat{C}_{N|M+}$-solution at every point of iS^*N (such a solution is unique at every point of iS^*N because $P(x,D)ext(u)$ is uniquely determined by $(f_0,..,f_{m-1})$). The operator $P(x,D)$ is also micro localizable, too. Indeed $\hat{C}_{N|M+}$ is a $\iota_*\mathcal{P}_X$-module which contains $\pi_N^{-1}\mathcal{B}_X$ as a subsheaf. Thus micro local boundary value problems for pseudo-differential operators are formulated on iS^*N.

Definition 1.1 $P(x,D)\in \iota_*\mathcal{P}_X^f$ (ι; $S_N^*X\setminus S_Y^*X \to iS^*N$) is called to have N as a non-characteristic hypersurface if and only if the map ι; $(S_N^*X\setminus S_Y^*X)\cap\{\sigma(P)=0\} \longrightarrow iS^*N$ is proper. Easily to see, each fiber of ι ; $\{(0,x';\zeta_1,i\eta')\in S_N^*X\setminus S_Y^*X;\ \sigma(P)=0\} \longrightarrow (x';i\eta')\in iS^*N$ is finite and its number counting multiplicities is locally constant. Let m be this number. Then by Weierstrass' division theorem for pseudo-differential operators $P(x,D)$ is decomposed into the product $Q\cdot R$, where Q and R are sections of $\iota_*\mathcal{P}_X^f$, Q is invertible and R has the following form ;

$$R(x,D) = D_1^m + R_1(x,D')D_1^{m-1} + \cdots + R_m(x.D'),$$

order $R_j(x,D') \leqq j$. Therefore we will study essentially the pseudo differential operators as above.

Corollary 1.2 Let $P(x,D) = D_1^m + P_1(x,D')D_1^{m-1} + \cdots + P_m(x,D')$ (order $P_j \leq j$) be a section of $\iota_*\mathcal{P}_X^f$. Then the $\hat{C}_{N|M+}$ - solution of the problem : $P(x,D)u(x)=0$ and $\partial^j u/\partial x_1{}^j(+0,x')=f_j(x')$ $j=0,..,m-1$: for given microfunctions $(f_0,..,f_{m-1})\in C_N{}^m$ is unique at every point of iS^*N if it exists.

Considering ext(u) instead of u, the analysis of micro local boundary value problems is brought to the analysis on $S^*_{M+}X$. One of the advantages of the micro localization of boundary value problems is that we can use not only pseudo-differential operators, but also quantized contact transformations keeping $S^*_{M+}X$ (see Theorem 4.2.17 §4 [16]). We give easy applications of these tools.

Proposition 1.3 (Relations among boundary values, cf. [15] [26])

Let $P(x,D)=D_1{}^m+P_1(x,D')D_1^{m-1}+\cdots+P_m(x,D')$ (order $P_j \leqq j$) be a section of $\iota_*\mathcal{P}^f_X$ and s be the number of $\{\zeta_1\in\mathbb{C};\ \sigma(P)(0,x_0';\zeta_1,i\eta_0')=0,\ \mathrm{Re}\,\zeta_1>0\}$ counting multiplicities for a point $(x_0';i\eta_0')\in iS^*N$. Then there exist sections of $\mathcal{P}^f_Y$, $Q_{jk}(x',D')$ $(m-s\leqq j\leqq m-1,\ 0\leqq k\leqq m-s-1)$, defined in a neighborhood of $(x_0';i\eta_0')$ such that every $\hat{C}_{N|M+}$-solution $u(x)$ of $Pu(x)=0$ at $(x_0';i\eta_0')$ satisfies the following equations.

$$(1.2)\qquad (D_1{}^j u)(+0,x') = \sum_{k=0}^{m-s-1} Q_{jk}(x',D')(D_1{}^k u)(+0,x')$$

for $m-s\leqq j\leqq m-1$.

If $P(x,D)$ is elliptic, that is, $\{\zeta_1\in\mathbb{C};\ \sigma(P)(0,x_0';\zeta_1,i\eta_0')=0\}\cap i\mathbb{R}=\phi$, (1.2) is a necessary and sufficient condition for the solvability of the boundary value problem $P(x,D)u(x)=0$. We call the equations (1.2) " the relations among boundary values".

Proof Consider the canonical extension of u,

$$P(x,D)\mathrm{ext}(u) = \sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1).$$

By Weierstrass' division theorem for pseudo-differential operators $P(x,D)$ is decomposed into the product $P'P''$ of two pseudo-differential

operators $P', P'' \in \mathcal{C}_* \mathcal{D}_X^f$ which have the following properties :
P'' is invertible on $\{(0,x';\zeta_1, i\eta') \in S_N^* X;\ x'=x_0',\ \eta'=\eta_0',\ \mathrm{Re}\,\zeta_1 > 0\}$ and
$P'(x,D) = D_1{}^s + A_1(x,D')D_1^{s-1} + \cdots + A_s(x,D')$ (order $A_j \leq j$). Therefore
we have $\sum_{j=0}^{m-1} f_j(x')\delta^{(j)}(x_1) \in P'(x,D)\Gamma(\{x'=x_0', \eta'=\eta_0', \mathrm{Re}\,\zeta_1 > 0\}, \mathcal{C}_{N|X})$.
We may assume $(x_0'; i\eta_0') \in U_n^+$ and apply the quantized contact transform β_n^+. So it follows that

$$\sum_{j=0}^{m-1} (i\zeta_1)^j (-D_n)^{j+1} f_j(x') \in \tilde{P}'(\zeta_1, x', D_{\zeta_1}, D_{x'})\Gamma(\{x'=x_0', \eta'=\eta_0', \mathrm{Re}\,\zeta_1 > 0\}, \mathcal{C}\mathcal{O}),$$

where $\tilde{P}' = \beta_n^+ \cdot P'(\beta_n^+)^{-1}$ (see §1 in [17]). Now, using the formula (1.5) in Theorem 1.2.3 [17] we obtain the following equation,

$$\int_\gamma \sum_{r=0}^{s-1} (\zeta_1 - w)^r \Big(\sum_{l=-\infty}^{s} \sum_{j=0}^{\infty} \frac{(-1)^j}{(r+j+1)!} \frac{\partial^{2j+r+1} \tilde{P}_l'}{\partial w^{j+r+1} \partial \tau^j}(w, x'; 0, D_{x'}) \Big)$$

$$\times \Big\{ (\tilde{P}')^{-1} \cdot \sum_{q=0}^{m-1} (iw)^q (-D_n)^{q+1} f_q(x') \Big\}\, dw = 0,$$

where $\tilde{P}_l'(w, x'; \tau, \zeta')$ is the homogeneous part of $\tilde{P}'(w, x', D_w, D_{x'})$ of order l and γ is a real analytic closed curve in $\mathbb{C}$ enclosing all the zeros of $\sigma(P')(0, x_0'; \zeta_1, i\eta_0')$ $((\eta_0')_n = +1)$. Since the left-hand side of this equation is a polynomial in ζ_1 of degree less than s-1, it reduces to the following pseudo-differential equations for $(f_0, \ldots, f_{m-1})$.

$$(1.3) \qquad \sum_{q=0}^{m-1} B_{tq}(x', D') f_q(x') = 0 \quad \text{for } t = 0, \ldots, s-1,$$

where $\{B_{tq}(x', D')\}$ are pseudo-differential operators defined at $(x_0'; i\eta_0') \in iS^*N$ given as follows.

$$B_{tq}(x', D') f(x') = \frac{\partial^t}{\partial \zeta_1{}^t} \Big\{ \int_\gamma \sum_{r=0}^{s-1} (\zeta_1 - w)^r \Big(\sum_{l=-\infty}^{s} \sum_{j=0}^{\infty} \frac{(-1)^j}{(r+j+1)!} \times$$

$$\times \frac{\partial^{2j+r+1}\tilde{P}'_1}{\partial w^{j+r+1}\partial\tau^j}(w,x';0,D_{x'}))((\tilde{P}')^{-1}(iw)^q(-D_n)^{q+1}f(x'))dw\}_{\zeta_1=0}$$

Easily to see, the order of B_{tq} is less than $q+1$ and

$$\sigma_{q+1}(B_{tq})(x';\zeta') = \frac{\partial^t}{\partial\zeta_1{}^t}\{\int_\gamma \sum_{r=0}^{s-1}(\zeta_1-w)^r\frac{1}{(r+1)!}\frac{\partial^{r+1}\tilde{P}'_s}{\partial w^{r+1}}(w,x';0,\zeta')$$

$$\times(\tilde{P}'_s(w,x';0,\zeta'))^{-1}(iw)^q(-\zeta_n)^{q+1}dw\}_{\zeta_1=0}\ .$$

Let $a_1(x',\zeta'),\dots,a_s(x',\zeta')$ be zeros of $\tilde{P}'_s(w,x';0,\zeta')=\sigma(\tilde{P}')(w,x';0,\zeta')$ $=\sigma(P')(0,x';-i\zeta_n w,\zeta')$ with respect to w. Then, noting that $\tilde{P}'_s(w,x';0,\zeta')=(-i\zeta_n)^s(w-a_1(x',\zeta'))\text{-----}(w-a_s(x',\zeta'))$, we have

$$\sigma_{q+1}(B_{tq})(x',\zeta')= (-\zeta_n)^{q+1}\frac{\partial^t}{\partial\zeta_1{}^t}\{\int_\gamma(iw)^q\prod_{j=1}^{s}(w-a_j(x',\zeta'))^{-1}\times$$

$$\times\sum_{r=0}^{s-1}\frac{(\zeta_1-w)^r}{(r+1)!}\frac{\partial^{r+1}}{\partial w^{r+1}}(\prod_{j=1}^{s}(w-a_j(x',\zeta')))dw\}_{\zeta_1=0}$$

$$= (-\zeta_n)^{q+1}\frac{\partial^t}{\partial\zeta_1{}^t}\{\int_{|w|=R}(iw)^q\prod_{j=1}^{s}(w-a_j(x',\zeta'))^{-1}(\zeta_1-w)^{-1}(\prod_{j=1}^{s}(\zeta_1-a_j)-$$

$$\prod_{j=1}^{s}(w-a_j))dw\}_{\zeta_1=0}\ ,$$

where R is a sufficiently large number such that $|a_j|<R$ for every j. Therefore,

$$(1.4)\qquad \sigma_{q+1}(B_{tq})(x',\zeta')= 2\pi(-i\zeta_n)^{q+1}q!\ \delta_{tq}\quad \text{for } 0\leqq {}^{\forall}q\leqq s-1.$$

So the pseudo-differential equations in (1.3) are solvable with respect to $(f_0,\dots,f_{s-1})$, that is, there exist pseudo-differential operators $C_{jk}(x',D')\in\mathcal{P}^f_Y|_{(x'_0;i\eta'_0)}$ for $j=0,\dots,s-1,k=s,\dots,m-1$ such that (1.3) is equivalent to the following equations ;

$$(1.5)\qquad f_j(x') = \sum_{k=s}^{m-1} C_{jk}(x',D')f_k(x') \quad \text{for } j=0,..,s-1.$$

To obtain the relations among $u_j(x')=(D_1{}^j u)(+0,x')$, we write $f_j(x')$ as linear combinations of $u_0,..,u_{m-1}$;

$$(1.6)\qquad f_j(x')=u_{m-j-1}(x')+\sum_{k+l\leq m-j-2}(-1)^l\binom{j+l}{j}\frac{\partial^l P_{m-j-k-l-1}}{\partial x_1{}^l}(0,x',D')u_k(x').$$

(Use the formula : $D_1{}^k(Y(x_1)f(x))=Y(x_1)D_1{}^k f(x)+\sum_{j=1}^{k} D_1^{j-1}(\delta(x_1)D_1^{k-j}f(x))$.)

Remark that this equation is solvable with respect to $u_0,..,u_{m-1}$ in the following way.

$$(1.7)\qquad u_j(x')= f_{m-j-1}(x')+E_{j,m-j}(x',D')f_{m-j}(x')+ \cdots$$
$$+ E_{j,m-1}(x',D')f_{m-1}(x') \qquad \text{for every } j.$$

Combination of (1.5) $\sim$(1.7) yields the desired relations (1.2) . It is easy to see that, when $P(x,D)$ is elliptic, (1.5) is a necessary and sufficient condition for the solvability. Thus the proof is completed.

<u>Lemma 1.4</u> Let $(0,y_0';i\tau_0)$ and $(0,x_0';i\eta_0)$ be two points in $S_N^*X\cap iS^*M$ and Φ be a real quantized contact transformation from a neighborhood of $(0,y_0';i\tau_0)$ into a neighborhood of $(0,x_0';i\eta_0)$. Put $S^j(x,D_x)=\Phi\cdot y_j\cdot\Phi^{-1}$ and $R^j(x,D_x)=\Phi\cdot D_{y_j}\cdot\Phi^{-1}$ and assume $S^1(x,D_x)\in\mathcal{D}_X\cdot x_1$. Then the contact transformation φ induced by Φ keeps S_N^*X and so Φ defines a sheaf isomorphism $C_{N|X}\xrightarrow{\sim}C_{N|X}$ (or $C_{M+|X}\xrightarrow{\sim}C_{M+|X}$, see Lemma 4.2.13 and Theorem 4.2.17 §4 [16]). As for this sheaf isomorphism, the following formula holds.

$$(1.8)\qquad \Phi(\delta(y_1)f(y')) = \delta(x_1)\int k(x',y')f(y')dy' \quad \text{for } {}^\forall f(y')\in C_N,$$

where $k(x',y')$ is the kernel function of the real quantized contact transformation Φ' in N induced by Φ in the following way.

(1.9) $\qquad \Phi' y_j \Phi'^{-1} = S^j(*, D_{x'}, 0, x'), \ \Phi' D_{y_j} \Phi'^{-1} = R^j(*, D_{x'}, 0, x')$

for $j=2,\dots,n$, where $S^j(D_x, x)$ or $R^j(D_x, x)$ are the transposed normal expressions of S^j or R^j respectively, that is, all x-operators are disposed in the latter part of each term than D_x-operators. In fact we assert that $S^j(D_x, 0, x')$ and $R^j(D_x, 0, x')$ do not contain D_{x_1}-operator and that the commutation relations :

$[S^j(*, D_{x'}, 0, x'),\ S^k(*, D_{x'}, 0, x')] = 0 \quad j,k=2,..,n, \ \dots$ and so on :

hold.

Proof Let $K(x,y)$ be the kernel function of Φ . Then $\Phi(\delta(y_1) \times \delta(y'-\tilde{y}'))$ is given by $K(x,0,\tilde{y}')$. Use the theory on holonomic systems. We omit the details.

Remark i) An arbitrary real quantized contact transformation keeping $S^*_N X$ is written as the composite of an inner automorphism and a quantized contact transformation as above.

ii) Using this lemma we can calculate the transform of $F(x) = \sum_{j=0}^{m-1} \delta^{(j)}(x_1) f_j(x') = \sum_{j=0}^{m-1} D_1{}^j(\delta(x_1) f_j(x'))$ by Φ .

Example 1.5 The following boundary value problem $(k=0,1,..)$ is solvable microlocally at $(x_0'; i\eta_0') \in iS^*N$ $((\eta_0)_2 > 0)$ from the positive side of N

$$(\mathrm{I}) \begin{cases} Pu = (D_1{}^2 + x_1{}^k D_2{}^2) u(x) = 0 & x_1 > 0 \\ (D_1{}^j u)(+0, x') = u_j(x') & j=0,1 \end{cases}$$

if and only if the relation

$$(1.10)\qquad u_1(x') + \frac{\Gamma(1-1/(k+2))}{\Gamma(1+1/(k+2))}\left(\frac{1}{i(k+2)}D_2\right)^{\frac{2}{k+2}}u_0(x') = 0$$

holds at $(x_0';i\eta_0')$. Indeed the adjoint equation

$$\text{(II)}\quad \begin{cases} {}^tPu = (D_1{}^2 + x_1{}^kD_2{}^2)v(x,y') = 0 \qquad x_1>0 \\ v(+0,x',y') = \delta(x'-y') \end{cases}$$

has a hyperfunction solution $v(x,y') = C_k\cdot\delta(x_3-y_3)\cdot\cdots\cdot\delta(x_n-y_n)$ $\times\int_{-\infty}^{+\infty}|\eta|^{\frac{1}{k+2}}\sqrt{x_1}\; H^{(1)}_{1/k+2}\left(\frac{2i|\eta|}{k+2}x_1^{\frac{k+2}{2}}\right)e^{i\eta(x_2-y_2)}d\eta$, where $H^{(1)}_{\nu}(z)$ is the Hankel function of the first kind and C_k is a constant depending only on k. Since the singular support of $\partial v/\partial x_1(+0,x',y') = C_k'\cdot\delta(x''-y'')$ $\times\int_{-\infty}^{+\infty}|\eta|^{\frac{2}{k+2}}e^{i\eta(x_2-y_2)}d\eta$ is also contained in $\{(x',y';i\eta',i\tau');x'=y',$ $\eta'+\tau'=0\}$, we can apply the microlocal Green's formula (see [17]). Therefore $\int u_0(x')(D_1v)(+0,x',y')dx' - \int u_1(x')v(+0,x',y')dx' = 0$ holds at $(x_0';i\eta_0')$. This is just the relation (1.10). The sufficiency is also proved by using $v(x,y')$.

Next, we treat pseudo-differential operators which are partially micro-hyperbolic in the one side of the boundary (cf. [3],[12],[14], [8], [23]).

<u>Definition 1.6</u> Let $P(x,D) = D_1{}^m + P_1(x,D')D_1^{m-1} + \cdots + P_m(x,D')$ (order $P_j \leq j$) be a section of $\iota_*\mathcal{P}_X^f$. $P(x,D)$ is said to be partially micro-hyperbolic in the positive side of N at $(x_0';i\eta_0')\in iS^*N$ if the equation $\sigma(P)(x;\zeta_1,i\eta')=0$ with respect to ζ_1 has no root with positive real part when $\varepsilon\geq x_1\geq 0$ and $|x'-x_0'|\leq\varepsilon$, $|\eta'-\eta_0'|\leq\varepsilon$ for some positive constant ε. As typical examples, we have $P=D_1{}^2-x_1{}^kD_2{}^2$ at

$(0;\pm idx_2)$, $P=D_1-i(x_1+x_2^2)D_2$ at $(0;idx_2)$ and so on.

The following theorem is a generalization of the results obtained by many authers (see Introduction).

Lemma 1.7 Let U be an open subset of iS^*N with proper convex fibers. Then the following $\mathcal{O}_X$-sheaf isomorphism holds (cf. Proposition 2.1.21 in [17]).

$$(\pi_N|_U)_! R^1 \tau_! C_{N|X} \xleftarrow{\sim} (\tau_N|_{U^o})_*(\widetilde{\mathcal{A}}_{M+}|_{F_0})/\mathcal{A}_M|_N ,$$

where $F_0=F_+\cap F_- = \{(0,x';w_1,iv')\in S_NX; w_1=0\}\subset iSM$ and particularly $\widetilde{\mathcal{A}}_{M+}|_{F_0}=\widetilde{\mathcal{A}}_{M-}|_{F_0}$ holds (see Definition 2.1.15 in [17]).

Proof Calculate $\mathbb{R}(\pi_N|_U)_!\mathbb{R}\tau_! C_{N|X}$ in the same way as in Proposition 2.1.21 in [17]. We omit the details.

Theorem 1.8 Let $A(x,D')$ be a $k\times k$-matrix of pseudo-differential operators of order less than 1 defined in a neighborhood of $(0,x_0';i\eta_0')$ $\in\mathbb{R}\times iS^*N$. Suppose that $\det(\zeta_1 I_k-\sigma_1(A(x,D')))$ is partially micro-hyperbolic in the positive side of N at $(x_0';i\eta_0')$. Then for every germ $f(x)={}^t(f_1(x),\dots,f_k(x))\in{}^t(\hat{C}_{N|M+}^{\ k})$ and every data $v(x')={}^t(v_1(x'),\dots,v_k(x'))\in{}^t(C_N^{\ k})$ at $(x_0';i\eta_0')$, there exists a unique solution $u(x)={}^t(u_1(x),\dots,u_k(x))\in{}^t(\hat{C}_{N|M+}^{\ k})$ such that;

$$\left\{\begin{aligned} &(D_1I_k - A(x,D'))u(x) = f(x) \\ &\quad u(+0,x') = v(x') \end{aligned}\right. \qquad \text{at } (x_0';i\eta_0').$$

Remark Easily to see, inhomogeneous initial value problems for single pseudo-differential operators which are partially micro-

-hyperbolic in the positive side of N are reduced to this theorem.

Proof Without loss of generality we may assume $v(x')=0$, $(x_0';i\eta_0')=(0;idx_2)$ and that $A(x,D')$ is developed into the following power series

$$A(x,D') = \sum_{L=(l_2,\dots,l_n)} a_L(x)D_{x'}^{L},$$

where L moves over all multi-indices such that $l_2\in\mathbb{Z}$, $l_3\geq 0,\dots,l_n\geq 0$ and $|L|=l_2+\cdots+l_n\leq 1$, and $\{a_L(z)\}$ are holomorphic functions defined in $\Omega=\{z\in\mathbb{C}^n;\ |z_1|<R,\ |z'|<R\}$ satisfying that : ($|a_L(z)| = \max_{p,q}|a_L^{p,q}(z)|$)

$$\sup_{\Omega} |a_L(z)| \leq B(1-|L|)!\, b^{|L|} c^{l_2-|L|}$$

where B, b, c are positive constants (cf. §2 [4] and §2 [12]). In particular $A(z,D')$ is a section on $\{(z;\zeta)\in P^*X;\ |z_1|<R,\ |z'|<R,\ |\zeta_j|< c|\zeta_2|$ for $j=3,\dots,n\}$. Furthermore by the partially micro-hyperbolicity in the positive side of N we may assume that :

(1.11) $\det(\zeta_1 I_k-\sigma(A)(z,\zeta'))$ never vanishes on $\{(z;\zeta)\in S^*X;\ y_1=0,$ $0\leq x_1<R,\ |z'|<R,\ |\zeta_j|<c|\zeta_2|$ for $j=3,\dots,n$, $-\mathrm{Im}(\zeta_1/\zeta_2) > B(|y'| + \sum_{j=3}^{n}|\mathrm{Im}(\zeta_j/\zeta_2)|)\}$.

(Apply the ordinary hyperbolic inequality to $\det(\zeta_1 I_k-\sigma(A)(z_1{}^2,z_2,\dots,z_n,\zeta'))$; [3] , [12]). By the softness of $\hat{\mathcal{C}}_{N|M+}$, it may be assumed that $f(x)$ is a section of ${}^t(\hat{\mathcal{C}}_{N|M+}{}^k)$ on $\{|x'|<R\}$ with support in $U=\{(x';i\eta');\ |x'|<r,\ |\eta_j|<r\eta_2$ for $j=3,\dots,n\}$, where $r<\min(c/2,b/6,R/3)$ is taken small enough later (depending only on R,b,c,n).

Since $D_1I_k-A(x,D')$ is invertible on $\{(0,x';\zeta_1,i\eta')\in S_N^*X;\ |x'|<R,\ |\eta_j|< \frac{1}{2}c\eta_2$ for $j=3,\dots,n$ and $|\zeta_1|>T\eta_2\}$ for a sufficiently large number T, $u'(x)=(D_1I_k-A(x,D'))^{-1}\mathrm{ext}(f(x))$ are defined as a section of ${}^t(\mathcal{C}_{N|X}{}^k)$ on $\{(0,x';\zeta_1,i\eta')\in S_N^*X;\ |x'|<R,\ |\zeta_1|>T|\eta'|\}$ with support in

$\{(x';i\eta')\in U,\ |\zeta_1|>T|\eta'|\}$. Recalling the $\iota_*\mathcal{P}_X$-sheaf homomorphisms

$$C^{\infty}_{N|X}\big|_{iS^*N\times\infty} \longrightarrow C^{\infty}_{N|X}\big|_{iS^*N\times\infty}/\iota_*C_{N|X} \hookrightarrow R^1\iota_!C_{N|X},$$

${}^tu'(x)$ (modulo $(\iota_*C_{N|X})^k$) is identified with a section of $(R^1\iota_!C_{N|X})^k$ on $\{(x';i\eta');\ |x'|<R\}$ with support in U. By the preceding lemma and the cohomological triviality of the sheaf $\mathcal{A}_{M|N}$, we have a section ${}^tG(z)$ of $(\widetilde{\mathcal{A}}_{M+}|_{F_0})^k$ on $\{(x';iv')\in iSN;\ |x'|<R,\ (x';iv')\in U^o\}=\{(x';iv');\ r\leq|x'|<R\}\cup\{(x';iv');\ |x'|<R,\ v_2\geq r(|v_3|+\cdots+|v_n|)\}$ satisfying that $[-\frac{1}{2\pi i}{}^tG(z)\log z_1]$ coincides with ${}^tu'$ as a section of $(C^{\infty}_{N|X}|_{iS^*N\times\infty})^k$ modulo $(\iota_*C_{N|X})^k$. On the other hand by Prop.2.1.21 [17], ${}^tf(x)$ is identified with a section ${}^tF(z)$ of $(\widetilde{A}_{M+})^k$ on $\{(x';i\eta');\ |x'|<R,\ v_2\geq r(|v_3|+\cdots+|v_n|)\}$ modulo $(\mathcal{A}_{M|N})^k$. Especially for sufficiently small numbers R', R" G(z) is holomorphic on $\{z\in\mathbb{C}^n;\ |x'|<\frac{2}{3}R,\ R'>y_2>r(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''})\}\cup\{z\in\mathbb{C}^n;\ r<|x'|<\frac{2}{3}R,\ z_1=0,\ y'=0\}$ and F(z) is holomorphic on $\{z\in\mathbb{C}^n;\ |x'|<\frac{2}{3}R,\ |x_1|<R',\ |y|<R',\ R'>y_2>r(|y_3|+\cdots+|y_n|+\frac{|y_1|}{R''}+\frac{|x_1|}{R''}Y(-x_1))\}\cup\{z\in\mathbb{C}^n;\ r<|x'|<\frac{2}{3}R,\ z_1=0,\ y'=0\}$. From a technical reason we divide G(z) into a sum G'(z)+G"(z) of vectors of holomorphic functions such that G'(z) is holomorphic on $D'=\{z\in\mathbb{C}^n;\ |x'|<\frac{2}{3}R,\ R'>y_2>r(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''})\}\cup\{z\in\mathbb{C}^n;\ |x'|<\frac{2}{3}R,\ y_2>2r(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''})-R',\ y_2\geq R'\}$ and G" is holomorphic on $D''=\{z\in\mathbb{C}^n;\ |x'|<\frac{2}{3}R,\ R'>y_2>2r(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''})-R'\}$. In fact because $D'\cup D''=\{z\in\mathbb{C}^n;\ |x'|<\frac{2}{3}R,\ y_2>2r(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''})-R'\}$ is Stein, this is possible. Particularly G'(z) is holomorphic on $D=\{z\in\mathbb{C}^n;\ |x'|<\frac{2}{3}R,\ y_2>2r(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''})\}\cup\{r<|x'|<\frac{2}{3}R,\ z_1=0,\ y'=0\}$ and the boundary value of ${}^tG'(z)$ coincides with ${}^tu'(x)$ as a section of $(R^1\iota_!C_{N|X})^k$ on $\{(x';i\eta');\ |x'|<\frac{2}{3}R\}$. So, from now on, we use G'(z) instead of G(z).

Now we recall the operations of pseudo-differential operators on holomorphic functions ([4],[12]). Let ${}^tH(z)$ be a vector-valued holomorphic function defined in an open set W. Then for a positive number ε,

$$A^{\varepsilon}(z,D')H(z) = \sum_{l_2\geq 0} a_L(z)D_{z'}^{L}H(z) + \sum_{l_2\leq -1} \frac{a_L(z)}{(|l_2|-1)!}\int_{i\varepsilon}^{z_2} (z_2-s)^{|l_2|-1}$$
$$\times D_{z_3}^{l_3}\cdot\cdots\cdot D_{z_n}^{l_n}H(z_1,s,z_3,\ldots,z_n)ds$$

is well defined if $z\in W\cap\Omega$, $|z_2-i\varepsilon|<b$ and $\gamma(\{z\}\cup\{w\in\mathbb{C}^n; w_1=z_1,\ w_2=i\varepsilon,\ |w_j-z_j|\leq\frac{2}{c}|z_2-i\varepsilon|\quad j=3,\ldots,n\})\subset W$ (see [4], where γ denotes the convex hull). Set $\varepsilon=\min(b/2,cR/6n)$. Then $A^{\varepsilon}(z,D')G'(z)$ is holomorphic on $\{z\in\mathbb{C}^n;\ |z_1|<\delta,\ |x'|<2r,\ |y'|<r,\ y_2>2r(|y_3|+\cdots+|y_n|+\frac{|z_1|}{R''})\}\cup\{z\in\mathbb{C}^n;\ z_1=0,\ y'=0,\ r<|x'|<2r\}$ if $0<\delta<R$, $|\varepsilon+3r|<b$, $2r+\frac{2}{c}\sqrt{n}(3r+\varepsilon)<\frac{1}{2}R$ and $\frac{\delta}{R''}+\sqrt{n}(r+\frac{2}{c}\sqrt{n}(3r+\varepsilon))<\frac{\varepsilon}{16r}$. Easily to see, we can take r and δ satisfying these conditions. Note that the boundary value of $A^{\varepsilon}(z,D')G'(z)$ is equal to $A(x,D')u'(x)$ as a section of ${}^t(R^1\mathcal{L}_!C_{N|X}^{k})$ on $\{(x';i\eta');\ |x'|<2r\}$. So $(D_1I_k-A^{\varepsilon}(z,D'))G'(z)-F(z)$ is holomorphic on $\{z\in\mathbb{C}^n;\ z_1=0,\ y'=0,\ |x'|<2r\}$. Consequently for a sufficiently small positive number r_0 ($r_0<\frac{10nr}{c}(3r+\varepsilon)$, and $r_0<r$), $G'(z)$ is holomorphic on

$$V=\{z\in\mathbb{C}^n;\ |x'|<\tfrac{2}{3}R,\ y_1=0,\ y_2>3r(\sqrt{n}\,|y''|+\tfrac{|x_1|}{R''})\}\cup\{\tfrac{5}{4}r<|x'|<\tfrac{R}{2},\ y_1=0,\ y_2>3r(\sqrt{n}\,|y''|+\tfrac{|x_1|}{R''})-r_0\}$$

and $(D_1I_k-A^{\varepsilon}(z,D'))G'(z)$ is holomorphic on $\{z\in\mathbb{C}^n;\ |x'|<\frac{3}{2}r,\ y_1=0,\ 0\leq x_1<r_0,\ r_0>y_2>3r\sqrt{n}\,|y''|\}$. Now using the assumption (1.11) on partially micro-hyperbolicity of P, we will show that $G'(z)$ is extended analytically to $\{z\in\mathbb{C}^n;\ 0\leq x_1<r_1,\ |x'|<\frac{3}{2}r,\ y_1=y_3=\cdots=y_n=0,\ 0<y_2<r_1\}$ for a sufficiently small number $r_1>0$, by Lemma 2.2.7 [17] this implies that the boundary value $u(x)=G'(x_1,x_2+i0,x_3,\ldots,x_n)$ of $G'(z)$ defines a mild hyperfunction from the positive side of N and so u(x) is the solution in ${}^t(\hat{C}^k_{N|M+})$ to $(D_1 I_k - A(x,D'))u(x)=f(x)$ at $(0;idx_2)$. To do so, we employ Lemma 4.3 § 4 in [12]. Put a family of real analytic functions which are convex with respect to y" ;

$$\varphi_\lambda(x_1,y_3,\dots,y_n)=\{4r\sqrt{n}+(e^{Bx_1}-1)(4r\sqrt{n}+1)\}\sqrt{|y''|^2+\lambda^2}+\lambda\left(\frac{1}{\beta-x_1}-\frac{1}{\beta}\right)$$

for $1\geqq\lambda>0$, $\beta>0$. We claim that G'(z) is holomorphic on $V\cup\bigcup_{0<\lambda\leqq 1}$ $\{z\in\mathbb{C}^n; y_1=0,\ y_2=\varphi_\lambda,\ 0\leqq x_1<\beta,\ |x'|<R/2\}\supset\{z\in\mathbb{C}^n; y_1=0,\ y''=0,\ y_2>0,$ $0\leqq x_1<\beta,\ |x'|<R/2\}$ if the following conditions 1)～5) are all satisfied. Set $S_\lambda=\{z\in\mathbb{C}^n;\ y_1=0,\ y_2=\varphi_\lambda,\ 0\leqq x_1<\beta,\ |x'|<R/2\}$.

1) $S_1\subset V$: This is satisfied if $\beta^2<R''/3r$.

2) $S_\lambda\cap\{|x'|>5r/4$ or $|y''|>\mu\}\subset V$: This is satisfied if $\beta<r_0R''/3r$ and $3\beta/\sqrt{n}\,R''<\mu$.

3) $S_\lambda\setminus V\subset\{y_1=0,\ 0\leqq x_1<r_0,\ |x'|\leqq 5r/4,\ 3r\sqrt{n}\,|y''|<y_2<r_0\}$ for $\forall\lambda\in(0,1]$: This is satisfied if $\beta<r_0$ and $3r(\sqrt{n}\mu+\frac{\beta}{R''})<r_0$.

4) For every $z^0\in S_\lambda\setminus V$, $\gamma(\{z^0\}\cup\{w\in\mathbb{C}^n; w_1=z_1^0,\ w_2=i\varepsilon,\ |w_j-z_j^0|\leqq\frac{2}{c}|z_2^0-i\varepsilon|$ for $j=3,\dots,n\})\subset V\cup\bigcup_{\lambda\leqq\lambda'\leqq 1}S_{\lambda'}$: Taking account of the convexity of $\{y_2=\varphi_\lambda\}\cap\{z_1=z_1^0\}$ and the inequality $r_0+\frac{16rn}{c}(\varepsilon+2r+r_0)<\varepsilon$, this is satisfied if $(e^{B\beta}-1)(4r\sqrt{n}+1)<4r\sqrt{n}$.

5) At every point of $S_\lambda\setminus V$, the inequality $\partial\varphi_\lambda/\partial x_1>B(|y''|+\varphi_\lambda)$

holds (see Lemma 4.3 [12]). : This is satisfied if $\beta < 1/B$.

Clearly we can take β and μ as they satisfy all the conditions 1) $\sim$ 5). So by using Lemma 4.3 [12] and Holmgren's argument, our claim is justified. Since $\mathrm{ext}(u)-(D_1 I_k - A(x,D'))^{-1}\mathrm{ext}(f) \in {}^t(\tau_* C_{N|X}^{\,k})$, we have $(D_1 I_k - A(x,D'))\mathrm{ext}(u)-\mathrm{ext}(f) \in (D_1 I_k - A(x,D'))\cdot {}^t(\tau_* C_{N|X}^{\,k})$. On the other hand $(D_1 I_k - A(x,D'))\mathrm{ext}(u)-\mathrm{ext}(f) = \mathrm{Trace}(u) \times \delta(x_1)$ always holds. So from the division theorem for $C_{N|X}$ (matrix case) we obtain $\mathrm{Trace}(u)=0$. Thus the proof is completed.

<u>Corollary 1.9</u> (Half solvability, cf. [8], [23])

Let $P(x,D)$ be a partially micro-hyperbolic pseudo-differential operator of order m in the positive side of N defined on $\tau^{-1}((x_0';i\eta_0'))$. Then the sheaf homomorphism

$$P(x,D) \quad : \quad C_{M+|X} \ni u \longrightarrow Pu \in C_{M+|X}$$

is isomorphic on $(\tau^+)^{-1}((x_0';i\eta_0'))$. In particular when $P(x,D)$ is a hyperbolic differential operator in the positive side of N defined at $(0,x_0') \in N$, the sheaf homomorphism

$$P(x,D) \quad : \quad \mathcal{H}^0_{M+}(B_M) \ni u \longrightarrow Pu \in \mathcal{H}^0_{M+}(B_M)$$

is isomorphic at $(0,x_0')$.

<u>Proof</u> It suffices to show the solvability of $Pu=f$ in $C_{M+|X}$ at every point $p_0 \in (\tau^+)^{-1}((x_0';i\eta_0')) \cap iS^*M$ for every germ $f \in C_{M+|X}$. Considering the surjectivity $\mathcal{H}^0_{M+}(B_M) \longrightarrow C_{M+|X}/C_{N|X}$ at p_0 (Prop. 4.2.10 [16]), f is written as $f=[f']+g$ at p_0, where $f' \in \mathcal{H}^0_{M+}(B_M)$ at $(0,x_0')$ and $g \in C_{N|X}|_{p_0}$. Now we consider $f'(x)$ as a section of $C_{N|M+}$. Then according to Prop.2.1.10 [17], the quantized Legendre transform $\beta_k^\varepsilon(f)(\zeta_1,x')$ $((x_0';i\eta_0') \in V_k^\varepsilon)$ is represented by a section

$A(\zeta_1, x')$ of $\mathcal{BO}$ defined on $\{(\zeta_1, x') \in \mathbb{C}\times\mathbb{R}^{n-1};\ \mathrm{Re}\,\zeta_1>0,\ |x'-x_0'|<\delta\}$. Devide $A(\zeta_1, x')$ into a sum $A_1(\zeta_1, x')+A_2(\zeta_1, x')$ of sections of $\mathcal{BO}$, where A_1 is defined on $\{\zeta_1\in\mathbb{C};\ \mathrm{Re}\,\zeta_1>0 \text{ or } |\zeta_1|>2|t_0|\}\times\{x';\ |x'-x_0'|<\delta\}$ and A_2 is defined on $\{\zeta_1\in\mathbb{C};\ \mathrm{Re}\,\zeta_1>0 \text{ or } |\zeta_1|<2|t_0|\}\times\{x';\ |x'-x_0'|<\delta\}$ (where $p_0=(0, x_0';\ it_0, i\eta_0')$ $\eta_k=\pm 1=\varepsilon$). Note that $(\beta_k^{\varepsilon})^{-1}(A_2)\in \mathcal{C}_{N|X}\big|_{p_0}$ and $(\beta_k^{\varepsilon})^{-1}(A_1)\in (\iota_*^{+}\mathcal{C}_{M+|X}\cap \mathcal{C}_{N|X}^{\infty}\big|_{iS^*N\times\infty})(x_0';i\eta_0')$. Consequently f' is written as a sum $\mathrm{ext}(f'')+g'$ at p_0, where $f''\in\hat{\mathcal{C}}_{N|M+}\big|(x_0';i\eta_0')$ and $g'\in \mathcal{C}_{N|X}\big|_{p_0}$. So we have $f=\mathrm{ext}(f'')+(g+g')$ with $f''\in\hat{\mathcal{C}}_{N|M+}\big|(x_0';i\eta_0')$ and $g+g'\in\mathcal{C}_{N|X}\big|_{p_0}$. Furthermore by using the division theorem for $\mathcal{C}_{N|X}$, $g+g'$ is written as $Ph+\sum_{j=0}^{m-1} v_j(x')\delta^{(j)}(x_1)$ with $h\in\mathcal{C}_{N|X}\big|_{p_0}$ and $(v_j)_j\in\mathcal{C}_N{}^m\big|(x_0';i\eta_0')$. Thus the equation is reduced to $Pu=\mathrm{ext}(f'')+\sum_{j=0}^{m-1} v_j(x')\delta^{(j)}(x_1)$. This is solved by the preceding theorem.

At the end of this section we treat the problems of propagation of micro-analyticity of solutions up to the boundary. First of all we formulate these problems in a micro-local view point.

Definition 1.10 Let $P(x,D)=D_1{}^m+P_1(x,D')D_1^{m-1}+\cdots+P_m(x,D')$ be a section of $\iota_*\mathcal{D}_X^f$ of order m. Then $P(x,D)$ is said to be N_+-regular (N_--regular) at $p_0\in iS^*M\underset{M}{\times}N-S_Y^*X$ if the following condition is fulfilled : If a germ $u(x)$ of $\mathcal{C}_{M+|X}\cap\mathcal{H}^0_{iS^*M\underset{M}{\times}N}(\mathcal{C}_M)$ ($\mathcal{C}_{M-|X}\cap\{\ ''\ \}$, resp.) at p_0 satisfies $P(x,D)u\in\mathcal{C}_{N|X}$, u belongs to $\mathcal{C}_{N|X}\big|_{p_0}$. We remark that this concept is invariant under quantized contact transformations keeping S_{M+}^*X.

Corollary 1.11 The operator $P(x,D)$ is N-regular (see Schapira [26]) at p_0 if and only if P is N_+ and N_--regular at p_0.

Proof Recall the definition of N-regularity. This follows directly from the exact sequence,

$$0 \longrightarrow C_{N|X} \longrightarrow C_{M+|X} \oplus C_{M-|X} \longrightarrow C_M \longrightarrow 0$$

at p_0 (see Proposition 4.2.10 [16]).

The meaning of N_+-regularity is explained as follows (cf. Schapira [26]). We assume that $\sigma(P)$ has a zero of order s at $p_0 =(0,x_0';i\eta_{0,1},i\eta_0')$ with respect to ζ_1. Let $v(x)$ be a $\hat{C}_{N|M+}$-solution of $P(x,D)$ at $(x_0';i\eta_0')$. Suppose that $v(x)$ is micro-analytic near p_0 in the positive side of N, that is, ext(v) is zero as a microfunction on $\{(x;i\eta)\in iS^*M;\ \varepsilon > x_1 > 0,\ |x'-x_0'|<\varepsilon,\ |\eta-\eta_0|<\varepsilon\}$ for some $\varepsilon>0$. Remarking that $\mathrm{ext}(v)\in C_{M+|X}\cap \mathcal{H}^0_{iS^*M\times_M N}(C_M)$ at p_0, we obtain that $\mathrm{ext}(v)\in C_{N|X}$ at p_0 from the N_+-regularity of P at p_0. On the other hand by the same argument as in Proposition 1.3, we can show that this is equivalent to the s-relations among boundary values $v(+0,x'),\ldots, D_1^{m-1}v(+0,x')$. In other words s-boundary values corresponding to the zero $\zeta_1=i\eta_{0,1}$ of multiplicity s vanish at $(x_0';i\eta_0')$. Therefore this means propagation of micro-analyticity of solutions from the positive side of the boundary up to the boundary.

Theorem 1.12 Let $P(x,D)=D_1^{\,m}+ P_1(x,D')D_1^{m-1}+\cdots+ P_m(x,D')$ be a pseudo-differential operator of order m defined on $\tau^{-1}((x_0';i\eta_0'))$ with $(x_0';i\eta_0')\in iS^*N$. Suppose that ${}^tP(x,D)$ is partially micro-hyperbolic in the positive side of N at $(x_0';-i\eta_0')$, in other words, the equation $\sigma(P)(x;\zeta_1, i\eta')=0$ with respect to ζ_1 has no root with negative real part when $\varepsilon \geq x_1 \geq 0$, $|x'-x_0'| \leq \varepsilon$, $|\eta'-\eta_0'| \leq \varepsilon$ for some $\varepsilon > 0$. Then $P(x,D)$ is N_+-regular at every point of $\tau^{-1}((x_0';i\eta_0')) \cap iS^*M$. (Cf. Kaneko [8] and Schapira [26]).

<u>Proof</u> Fix a point $p_0=(0,x_0';i\eta_{0,1}, i\eta_0') \in iS^*M \underset{M}{\times} N$. To show the N_+-regularity of P at p_0, we may assume that $\{\zeta_1 \in \mathbb{C};\ \sigma(P)(0,x_0';\zeta_1, i\eta_0')=0\}=\{i\eta_{0,1}\}$. Set $M'=M\times R^{n-1} \ni (x,y')=(x_1,\ldots,x_n,y_2,\ldots,y_n)$ and $N'=N\times \mathbb{R}^{n-1}$. We remark that, as an operator on functions in (x,y'), ${}^tP(x,D)$ is partially micro-hyperbolic in the positive side of N' at $(x_0',x_0';-i\eta_0', i\eta_0') \in iS^*N'$. So the following boundary value problem has a $\hat{C}_{N'|M'_+}$-solution $u_k(x,y')$ for every $k=0,\ldots,m-1$;

$$\begin{cases} {}^tP(x,D)u_k(x,y')=0 \\ D_{x_1}^{\,j}u_k(+0,x',y')=\delta_{jk}\cdot\delta(x'-y') \qquad j=0,\ldots,m-1 \end{cases}$$

at $(x_0',x_0';-i\eta_0', i\eta_0')$. Let $v(x)$ be any germ of $C_{M+|X} \cap \mathcal{H}^0_{iS^*M\underset{M}{\times}N}(C_M)$ at p_0 with $Pv(x) \in C_{N|X}$. By the division theorem for $C_{N|X}$ we have a germ $v'(x) \in C_{N|X}$ at p_0 and germs $f_0(x'),\ldots,f_{m-1}(x')$ of C_N at $(x_0';i\eta_0')$ such that $P(x,D)(v(x)-v'(x))=\sum_j f_j(x')\delta^{(j)}(x_1)$. Recalling that P is invertible on $\tau^{-1}((x_0';i\eta_0'))-\{p_0\}$, $w(x)=v(x)-v'(x)$ is extended to a germ of $\tau_*^+ C_{M+|X} \cap C^\infty_{N|X}|_{iS^*N\times\infty}$ at $(x_0';i\eta_0')$. That is, $[w(x)]$ is a $\hat{C}_{N|M+}$-solution of P at $(x_0';i\eta_0')$. In order to prove this theorem it suffices to show that $f_1(x')=\cdots=f_{m-1}(x')=0$ at $(x_0';i\eta_0')$. We apply the micro-local Green's formula to this case (§2.2 [17]). Indeed since $SS(D_{x_1}^{\,j}u_k(+0,x',y')) \subset \{(x',y';i\eta',i\tau');x'=y',\ \eta'+\tau'=0\}$ for $j,k=0,1,\ldots,m-1$ and $SS(\mathrm{ext}[w(x)])=SS(w(x)) \subset \{x_1=0\}$, the conditions are all fulfilled. Therefore we have $f_j(y')=0$ at $(x_0';i\eta_0')$ for $j=0$,

$1,\ldots,m-1$. Thus the proof is completed.

Example 1.13 $P= D_1{}^2+x_1^{2k}D_2{}^2$ $(k=1,2,\ldots)$ is neither N_+- nor N_--regular at $(0;+idx_2)$ (see Example 1.5).

§2. An application to diffractive boundary value problems

We apply the results in §1 and [17] to prove the N_+-regularity of diffractive operators, for example $P=D_1{}^2-(x_1-x_2)D_3{}^2$ (cf. [6], [28]), which are neither operators treated in §1 nor operators studied by Schapira in [27].

Let $P(x,D)$ be a pseudo-differential operator of finite order with real principal symbol defined at $p_0=(0,x_0';i\eta_0)\in iS^*M\underset{M}{\times}N-iS_N^*M$. We consider the most generic case of diffraction, that is,

$$(2.1)\quad \begin{cases} \sigma(P)(p_0)=0, \quad \{\sigma(P),x_1\}(p_0)=0, \quad \{\{\sigma(P),x_1\},\sigma(P)\}(p_0)\neq 0 \\ \{\{\sigma(P),x_1\},x_1\}(p_0)\neq 0, \quad d\sigma(P)\wedge dx_1(p_0)\neq 0. \end{cases}$$

In fact, let $(x(t);i\eta(t))$ be the bicharacteristic strip for P passing through $p_0=(x(0);i\eta(0))$. Then we have $dx_1/dt(0)=c\{\sigma(P),x_1\}(p_0)$ $=0$ and $d^2x_1/dt^2(0)=c'\{\sigma(P),\{\sigma(P),x_1\}\}(p_0)\neq 0$ with some non zero constants c,c'. So the bicharacteristic strip is strictly tangent to $\{(x;i\eta);x_1=0\}$ at p_0.

By the condition $\{\{\sigma(P),x_1\},x_1\}=\partial^2\sigma(P)/\partial\zeta_1{}^2\neq 0$ we may assume that P is a pseudo-differential operator of second order written in the form, $\sigma(P)= \zeta_1{}^2+a_1(x,\zeta')\zeta_1+a_2(x,\zeta')$ (a_1,a_2 are real valued when x,ζ' are real), which has a double root $\zeta_1=i\eta_{0,1}$ for $(x;\zeta')$ $=(x_0;i\eta_0')$ by the condition $\{\sigma(P),x_1\}(p_0)=0$. Therefore by a suitable real contact transformation keeping $\{x_1=0\}$ invariant,

$\sigma(P)$ and p_0 are transformed into $\sigma(P)=\zeta_1^2+r(x,\zeta')$ and $p_0=(0;0,i\eta_0')$. Noting that $r(0,i\eta_0')=0$ and $\partial r/\partial x_1(0,i\eta_0')=\frac{1}{2}\{\{\sigma(P),x_1\},\sigma(P)\}\neq 0$, $r(x,\zeta')$ is written as $-(x_1-\varphi(x',\zeta'))a(x,\zeta')$ where $\varphi(x',\xi')$ and $a(x,\xi')$ are real valued analytic functions of homogeneous degree 0 and 2 with respect to ξ' respectively, and $\varphi(0,\eta_0')=0$, $a(0,\eta_0')\neq 0$. Since $d\varphi\neq 0$ follows from $d\sigma(P)\wedge dx_1\neq 0$, we can take $\varphi(x',\zeta')=x_2$. Thus $\sigma(P)$ is transformed into the following form ;

$$\sigma(P)=\zeta_1^2-(x_1-x_2)a(x,\zeta') \quad \text{at } p_0=(0;0,i\eta_0'), \tag{2.2}$$

where $a(x,\xi')$ is a positive valued real analytic function of homogeneous degree 2 with respect to ξ' defined on a neighborhood of $(x;\xi')=(0;\eta_0')$ (in the case $a>0$, the bicharacteristic strip passing through p_0 is contained in $\{x_1\geq 0\}$).

<u>Proposition 2.1</u> We inherit notations from above. Let $u(x)$ be a section of $\hat{C}_{N|M+}$ defined on a neighborhood U of $p_0'=\iota(p_0)$. Assume that $P(x,D)u(x)=0$ on U (particularly $P(x,D)$ is defined on $\iota^{-1}(U)$). Then there exists a section $v(x)$ of $\hat{C}_{N|M+}$ on U whose support is compactly contained in U such that $Pv(x)=0$ on U and $v(x)=u(x)$ at p_0'. (We may assume $P(x,D)=D_1^2-(x_1-x_2)a(x,D')$).

<u>Proof</u> We denote by $\gamma_p(t)=(x(t,p);i\eta(t,p))$ $(-\delta\leq t\leq\delta)$ the bicharacteristic strip passing through $p=\gamma_p(0)\in\{\sigma(P)=0\}$, that is, $(x(t,p);\eta(t,p))$ is the integral curve for $H_{\sigma(P)}$. Without loss of generality we may assume $\partial/\partial x_1((x_1-x_2)a(x,\eta'))>0$ on $\{|x_1|<\alpha,\ (x';i\eta')\in U\}$, which implies $d^2x_1/dt^2(t,p)=2d\eta_1/dt(t,p)>0$ there. Choose positive numbers δ, ε and a neighborhood $V\Subset U$ of p_0' such that the integral curve $(x(t,p);\eta(t.p))$ is defined as an analytic mapping from $[-\delta,\delta]\times\{(x;\eta);\ \sigma(P)(x,\eta)=0,\ |x_1|\leq\varepsilon,\ (x';i\eta')\in\overline{V}\}\ni(t,p)$

to $\{(x;\eta);\ |x_1|<\alpha,\ (x';i\eta')\in U\}$ and satisfies $x_1(\pm\delta,p)\geq 2\varepsilon$ for every $p\in\{\sigma(P)(x,\eta)=0,\ |x_1|\leq\varepsilon,\ (x';i\eta')\in\overline{V}\}$. By the softness of $\hat{C}_{N|M+}$, there exists a section $w(x)$ of $\hat{C}_{N|M+}$ on U with support in V such that $w(x)=u(x)$ on a neighborhood of p_0'. So the support of $P(x,D)w(x)$ is contained in $V\setminus\{p_0'\}$. Consider $f(x)=\mathrm{ext}(P(x,D)w(x))$ which is a section of $C_{M+|X}$ on $\{(x_1,x';\zeta_1,i\eta')\in S^*_{M+}X;\ x_1<\lambda,\ \eta'\neq 0\}$ with support in $\{(x_1,x';\zeta_1,i\eta')\in S^*_{M+}X;\ x_1=0,\ \eta'\neq 0,\ (x';i\eta')\in V\setminus\overline{W}\}\cup\{(x_1,x';\zeta_1,i\eta')\in S^*_{M+}X;\ 0\leq x_1<\lambda,\ \lambda|\eta_1|\leq|\eta'|,\ (x';i\eta')\in V\setminus\overline{W}\}$ for a small number $0<\lambda<\varepsilon$, and small neighborhood $W\Subset V$ of p_0' in iS^*N. By the flabbiness of C_M we can cut the support of $f(x)$ in $\{x_1>0\}$ such that (the support of $f(x))\cap\{x_1>0\}$ is contained in $\{(x_1,x';i\eta_1,i\eta')\in iS^*M;\ 0<x_1\leq\mu,\ \lambda|\eta_1|\leq|\eta'|,\ (x';i\eta')\in V\setminus\overline{W}\}$, where $\mu<\lambda$ is a positive number satisfying that the intersection of the bicharacteristic $\gamma_{p_0}(t)$ passing through p_0 with $\{0\leq x_1\leq\mu\}$ is contained in $\{(x;i\eta)\in iS^*M;\ (x';i\eta')\in W\}$. Since $P(x,D)$ is of real principal type and the support of $f(x)$ has a compact intersection with every bicharacteristic strip $\{\gamma_p(t)\}$, we can find a section $g(x)$ of C_M satisfying $P(x,D)g(x)=f(x)$ defined on $\{(x;i\eta)\in iS^*M;\ |x_1|<\lambda,\ \eta'\neq 0,\ (x';i\eta')\in U\}$ with support in $K=[(\text{support } f)\cap iS^*M]\cup[\{|x_1|<\lambda\}\cap\{\gamma_p(t);\ |t|\leq\delta,\ p\in(\text{support } f)\cap\{\sigma(P)(x,\eta)=0\}\}]$. By the flabbiness of C_M and B_M, there exist sections $G_+(x), G_-(x)$ of B_M on $\{|x_1|<\lambda,\ x'\in\pi_N(U)\}$ with support in $\{x_1\geq 0\}$, $\{x_1\leq 0\}$ respectively such that $g(x)=[G_+(x)]+[G_-(x)]$ holds as a microfunction on $\{|x_1|<\lambda,\ (x';i\eta')\in U\}$. Consider the difference $r(x)=\mathrm{ext}(w(x))-[G_+(x)]$ which is a section of $C_{M+|X}$ on $\{(x_1,x';\zeta_1,i\eta')\in S^*_{M+}X;\ 0\leq x_1<\lambda,\ \eta'\neq 0,\ (x';i\eta')\in U\}$ and satisfies $P(x,D)r(x)=w(+0,x')\delta'(x_1)+(D_1w)(+0,x')\delta(x_1)+f(x)-P(x,D)[G_+(x)]=w(+0,x')\delta'(x_1)+(D_1w)(+0,x')\delta(x_1)+P(x,D)[G_-(x)]\in\Gamma(U,(\iota^-)_*C_{M-|X})$. Noting that $(\iota^+)_*C_{M+|X}\cap(\iota^-)_*C_{M-|X}=\iota_*C_{N|X}$, we have $P(x,D)r(x)$

$\in\Gamma(U,\iota_* C_{N|X})$ and $r(x)\in\Gamma(U,(\iota^+)_* C_{M+|X}\cap C^{\infty}_{N|X}|_{iS^*N\times\infty})$. Since $[G_+]=-[G_-]\in C_{N|X}$ on $\{(x;i\eta)\in iS^*M;\ x_1=0,\ \eta'\neq 0,\ (x';i\eta')\in U\setminus\iota(K\cap\{x_1=0\})\}$, $[G_+(x)]$ represents a section of $\iota_* C_{N|X}$ on $U\setminus\iota(K\cap\{x_1=0\})$. Therefore $r(x)$ defines a section $v(x)$ of $\hat{C}_{N|M+}$ on U whose support is compactly contained in U satisfying that $Pv(x)=0$ on U and that $v(x)=u(x)$ at p_0', because $\iota(K\cap\{x_1=0\})$ is compactly contained in $U\setminus\{p_0'\}$. This completes the proof.

When $P(x,D)=D_1{}^2-(x_1-x_2)A(x,D')$ is a second-order differential operator defined in a neighborhood of the origin and $\sigma(A)(x,\xi')\geq 0$ for every x and every $\xi'\in\mathbb{R}^{n-1}$, making use of hyperbolicity of P in $\{x_1-x_2>0\}$, any hyperfunction solution to $Pu=0$ defined on $\{x_1>0,\ |x|<R\}$ can be continued to $\{|x|<r, x_1-x_2>0\}\cup\{|x|<R, x_1>0\}$ as a solution for a small $r>0$. Then this solution is identified with a solution defined on $\{(t,x)\in\mathbb{R}\times\mathbb{R}^n; 0<t<1,\ |x|<r,\ x_1-tx_2>0\}$ to the following system of differential equations ;

$$\begin{cases} (D_1{}^2-(x_1-x_2)A(x,D'))u(t,x)=0, \\ D_t u(t,x)=0. \end{cases}$$

To apply this argument to the case that $P(x,D)$ is a pseudo-differential operator in (2.2) and that $u(x)$ is a micro local solution (that is, $\hat{C}_{N|M+}$- solution) to $P(x,D)u=0$, we must employ the method used in Theorem 1.8.

<u>Lemma 2.2</u> Set $U=\{(0,x';\zeta_1,i\eta')\in S^*_N X;\ \eta_n=+1,\ |\zeta_1|^2+\eta_2{}^2+\cdots+\eta_{n-1}^2<\varepsilon^2\}$, and $K_+=\{(0,x';w_1,iv')\in S_{M+}X|_N;\ v_n\geq\varepsilon\sqrt{(-u_1)_+{}^2+v_1{}^2+\cdots+v_{n-1}^2}\}$ ($(t)_+=t$ if $t\geq 0$, $=0$ if $t<0$). Let $f(x)$ be a section of $\hat{B}_{N|M+}$ such

that $\mathrm{ext}(f(x))$ represents a section of $R^1(\pi_{N/X}|_U)_! C_{N|X}$, in other words, for a suitable closed set $A \subset U$ which is compact in every fiber of $\pi_{N/X}$, $\mathrm{ext}(f(x))$ can be continued to $S_N^*X - A$ as a section of $C_{N|X}$. Then $f(x)$ is written as a boundary value of a section of $(\tau|_{K+})_* \widetilde{\mathcal{a}}_{M+}$ (as for the definitions of $S_{M+}X$, $\widetilde{\mathcal{a}}_{M+}$, see § 2.1 in [17]).

<u>Proof</u> We denote $R^1\Gamma_{S_N X}\tau_{N/X}^{-1}\mathcal{O}_X[1]$ by $q_{N|X}$, where $\tau_{N/X}$ is the projection from the monoidal transform $\widetilde{^N X}$ of X with center N to X (see CH I [24]). Then, Proposition 1.2.2 in CH I [24] shows that $C_{N|X} = \mathbb{R}\tau'_* \pi'^{-1}_* q_{N|X} \otimes \omega_{N/X}[n-1]$, where τ' ; $D_N X = \frac{1}{2} S_N X \underset{N}{\times} S_N^* X \longrightarrow S_N^* X$, π' ; $D_N X \longrightarrow S_N X$ are canonical projections. Using this expression, after a direct calculation of derived functors (cf. Proposition 2.1.21 [17]), we obtain $\mathbb{R}(\pi_{N/X}|_U)_! C_{N|X} \simeq \mathbb{R}(\tau|_{K_0})_* q_{N|X}[-1]$ with $K_0 = \{(0,x'; w_1, iv') \in S_N X;\ v_n \geq \varepsilon\sqrt{|w_1|^2 + v_2^2 + \cdots + v_{n-1}^2}\}$ (the dual cone of U). Set $K_- = \{(0,x'; w_1, iv') \in S_{M-}X;\ v_n \gtrless \varepsilon\sqrt{(u_1)_+^2 + v_1^2 + \cdots + v_{n-1}^2}\}$. Note that $K_+ \cap K_- = K_0$, $q_{N|X} = q_{M+} = q_{M-} = q_{M+ \cup M-}$ on K_0 (as for the definitions of q_{M+}, q_{M-}, $q_{M+\cup M-}$, see Definition 2.1.15 and the proof of Proposition 2.1.21 in [17]) and that $K = K_+ \cup K_-$ is cohomologically trivial (that is, Stein) for the sheaf $q_{M+\cup M-}$. Thus we have the exact sequence ;

$$0 \longrightarrow (\tau|_K)_* q_{M+\cup M-} \longrightarrow (\tau|_{K_+})_* q_{M+} \oplus (\tau|_{K_-})_* q_{M-} \longrightarrow (\tau|_{K_0})_* q_{N|X} \longrightarrow 0.$$

This implies the proof. We omit the details.

From now on, we assume $P(x,D) = D_1^2 - (x_1 - x_2)a(x, D')$, where $a(x, \zeta')$ is the one defined in (2.2). In fact by inner automorphisms lower order terms are negligible because of $d\sigma(P) \wedge \omega \neq 0$, where $\omega = \zeta_1 dx_1 + \cdots + \zeta_n dx_n$ is the fundamental 1-form.

Proposition 2.3 Let $P(x,D)=D_1{}^2-(x_1-x_2)a(x,D')$ be the pseudo-differential operator as above, and $f(x)$ be a $\hat{C}_{N|M+}$-solution to $Pf(x)=0$ at $p_0'=(0;i\eta_0')$. Assume that $(\eta_{0,3}',\ldots,\eta_{0,n}')\neq 0$, in other words, $d\sigma(P)\wedge dx_1\wedge\omega\neq 0$ at p_0. Set $M'=\mathbb{R}\times M\ni(t,x_1,\ldots,x_n)$, $N'=\{(t,x)\in M';x_1-tx_2=0\}\ni(t,x')$ and $M_+'=\{(t,x)\in M';x_1-tx_2\geq 0\}$. Then there exists a hyperfunction $g(t,x)$ defined on $\Omega=\{(t,x)\in M';0<t<1,\ x_1-tx_2>0,\ |x|<r\}$ with small $r>0$ satisfying the following ;

i) $D_t g(t,x)=0$ on Ω and the canonical flabby extension $G(t,x)=g(t,x)Y(t)Y(1-t)$ is mild from the positive side of N' at every point of $\{(t,x')\in N';0\leq t\leq 1,\ |x'|<r\}$,

ii) $G(t,x)$ satisfies the pseudo-differential equation $P(x,D)\mathrm{ext}(G(t,x))=0$ as a section of $C_{M_+'|X'}/C_{N'|X'}$ in a neighborhood of $\{(t,x;i\tau dt+i\eta'dx'+\zeta_1 d(x_1-tx_2))\in S^*_{M_+'}X';\ x=0,\ 0\leq t\leq 1,\ \zeta_1=\tau=0,\ \eta'=\eta_0'\}$,

iii) $g(+0,x)$ and $g(1-0,x)$ are mild on $\{x\in N;\ |x|<r\}$ and $\{x\in M;\ |x|<r,\ x_1-x_2=0\}$ from the positive side of $x_1=0$ and $x_1-x_2=0$ respectively. $g(+0,x)$ coincides with $f(x)$ as a germ of $\hat{C}_{N|M+}$ at $p_0'=(0;i\eta_0')\in iS^*N$ and $g(1-0,x)$ coincides with $\mathrm{ext}(f(x))$ as a section of C_M on $\{(x;i\eta)\in iS^*M;\ x_1>0,\ x_1-x_2>0,\ |x|<r,\ |\eta-\eta_0|<r\}$.

Proof After a suitable change of coordinates, we can take $p_0=(0;idx_n)$ $(n\geq 3)$. In the coordinate system $u_1=x_1-x_2, u_2=x_2,\ldots,u_n=x_n$, $P(x,D)$ is written as $D_{u_1}{}^2-u_1a(u_1+u_2,u',D_{u_2}-D_{u_1},D_{u_3},\ldots,D_{u_n})$. So the Weierstrass' division theorem for pseudo-differential operators admits the following decomposition

$$P(x,D) = E(u,D_u)(D_{u_1}{}^2 - u_1B(u,D_{u'})D_{u_1} - u_1C(u,D_{u'}))$$

in $W=\{(w;\lambda)=(u+iv;\mu+i\nu)\in S^*X;\ |w_1|<2R,\ |w'|<R,\ |\lambda_j|<R|\lambda_n|$ for every $j\neq n\}$

for some $R>0$, where $E(u,D_u)$ is elliptic on W, $B(u,D_{u'})$ and $C(u,D_{u'})$ are pseudo-differential operators of order 1 and 2 respectively defined on $\overline{W}$ such that $\sigma_1(B)(w,\lambda)$, $\sigma_2(C)(w,\lambda)$ are real if w,λ are real and that $\sigma_2(C)(u,\mu)>0$ for every $(u;\mu)\in S^*M\cap\overline{W}$. Further, by taking R small enough, we may assume that :

(2.3) $\quad \lambda_1{}^2-u_1\sigma_1(B)(u_1,w',\lambda')\lambda_1-u_1\sigma_2(C)(u_1,w',\lambda')$ never vanishes on $\{(w;\lambda)\in\overline{W};\ \mathrm{Im}w_1=0,$ and $\mathrm{Im}(\lambda_1/\lambda_n)>I\sqrt{u_1}(|\mathrm{Im}w'|+|\mathrm{Im}(\lambda'/\lambda_n)|)$ if $u_1\geq 0$, $\mathrm{Im}(\lambda_1/\lambda_n)>I\sqrt{-u_1}$ if $u_1<0\}$.

Then, as in Theorem 1.8, there exists a constant $d>0$ such that, with respect to $\{w_n=i\varepsilon\}$, $B^{\varepsilon}(w,D_{w'})H(w)$ and $C^{\varepsilon}(w,D_{w'})H(w)$ are well-defined if $|w_n-i\varepsilon|<d$, $|w_1|<2R$, $|w'|<R$ and $H(\widetilde{w})$ is holomorphic on $\gamma(\{w\}\cup\{\widetilde{w};\ \widetilde{w}_1=w_1,\ \widetilde{w}_n=i\varepsilon,\ |\widetilde{w}_j-w_j|\leq\frac{1}{R}|\widetilde{w}_n-w_n|$ for every $j=2,..,n-1\})$.

Now return to the solution $f(x)$. By Proposition 2.1, we may assume that $f(x)$ is a section of $\hat{B}_{N|M+}$ on N whose support as a section of $\hat{C}_{N|M+}$ is contained in a sufficiently small neighborhood U of $p_0'=(0;idx_n)$ and that $P(x,D)f(x)=0$ holds as a section of $\hat{C}_{N|M+}$ everywhere on iS^*N. Since $P(x,D)\mathrm{ext}(f(x))\in\Gamma_c(U,\iota_*C_{N|X})$ and P is invertible on $\{(0;\zeta_1dz_1+idx_n)\in S_N^*X;\ \zeta_1\neq 0\}$, for every $k>0$ we can take U small enough such that $\mathrm{ext}(f(x))$ is extensible as a section of $C_{N|X}$ to $S_N^*X-\{(0,x';\zeta_1,i\eta')\in S_N^*X;\ |x'|<k,\ \eta_n>\frac{1}{k}\sqrt{|\zeta_1|^2+\eta_2{}^2+\cdots+\eta_{n-1}{}^2}\}$. Hereafter we fix k and $f(x)$, k $(<\frac{1}{2}R)$ will be chosen small enough depending only on n,R,d later. Then by Lemma 2.2 $f(x)$ is identified with a holomorphic function $F(z)$ defined on $\{z\in\mathbb{C}^n;\ \delta>y_n>k((-x_1)_++(x_1-\delta)_++|y_1|+\cdots+|y_{n-1}|),\ |x'|<R\}\cup\{z\in\mathbb{C}^n;y=0,\ x_1=0,\ k\leq|x'|<R\}$ for some $\delta>0$. Exchanging $F(z)$ modulo in $\mathcal{a}_M|_N$ due to the same argument as in Theorem 1.8, we may assume that $F(z)$ is holomorphic on $D=\{z\in\mathbb{C}^n;$

$y_n > 2k\{(-x_1)_+ + (x_1-\delta)_+ + |y_1| + \cdots + |y_{n-1}|\}$, $|x'| < R\} \cup \{y_n + \delta > 2k\{(-x_1)_+ + (x_1-\delta)_+ + |y_1| + \cdots + |y_{n-1}|\}$, $k < |x'| < 2R/3\}$ for some smaller $\delta > 0$. In the coordinates w, D is written as $\{w \in \mathbb{C}^n;\ v_n > 2k\{(-u_1-u_2)_+ + (u_1+u_2-\delta)_+ + |v_1+v_2| + |v_2| + \cdots + |v_{n-1}|\}$, $|u'| < R\} \cup \{v_n + \delta > 2k\{\cdots\}$, $k < |u'| < 2R/3\}$. Setting $\varepsilon = \min\{d/2,\ R^2/2\sqrt{n}\}$, $Q^{\varepsilon}(w, D_w)F(w) = (D_{w_1}^2 - w_1 B^{\varepsilon}(w, D_{w'})D_{w_1} - w_1 C^{\varepsilon}(w, D_{w'}))F(w)$ is holomorphic on $\{|u'| \leqq 2k,\ |w_1| < 2R,\ |w'| < R,\ k > v_n > 2k\{(-u_1-u_2)_+ + (u_1+u_2-\delta)_+ + |v_1+v_2| + |v_2| + \cdots + |v_{n-1}|\}\}$ if $2k + \frac{\sqrt{n}}{R}(3k+\varepsilon) < R$, $k + 2k\frac{(n+1)}{R}(3k+\varepsilon) < \varepsilon$, and $3k + \varepsilon < d$. These are fulfilled if k is taken small enough. Remark that the boundary value of $Q^{\varepsilon}(w, D_w)F(w)$ is equal to $Q(u, D_u)f = (D_{u_1}^2 - u_1 B(u, D_{u'})D_{u_1} - u_1 C(u, D_{u'}))f$ as a section of $R^1(\pi_{N/X}|_{W \cap S_N^* X})_! C_{N|X}$ on $\{|u'| \leq 2k\}$ (see Lemma 2.2). So, since $E(u, D_u)$ is elliptic on W and $E(u, D_u)Q(u, D_u)f = P(x, D_x)f = 0$ holds as a section of $R^1(\pi_{N/X}|_{W \cap S_N^* X})_! C_{N|X}$ (this is true when $k < R/2\sqrt{2n}$), $Q^{\varepsilon}(w, D_w)F(w)$ is holomorphic on $Z = \{|u_1+u_2| \leq \delta',\ |u'| \leq 2k,\ |v_n| \leq \delta',\ v_1^2 + \cdots + v_{n-1}^2 \leqq \delta'^2\}$ for some $\delta' > 0$. To prolong $F(u_1, w')$ analytically (cf. §4 [12]), we introduce a family of piecewise real analytic hypersurfaces $\{S_\lambda\}$ $(0 < \lambda \leqq 1)$; $S_\lambda = \{(u_1, w') \in \mathbb{R} \times \mathbb{C}^{n-1};\ v_n = \varphi_\lambda(u_1, u_2, v_2, \ldots, v_{n-1}) + 2k(u_1+u_2-\delta)_+ + 2k(-u_1-u_2-2\alpha)_+,\ |u'| < 2R/3,\ u_1 < \alpha\}$, where φ_λ is given by

$$\{(8k\sqrt{n}+1)(e^{I(\alpha-u_1)} - 1) + 8k\sqrt{n}\}\sqrt{\lambda^2 + v_2^2 + \cdots + v_{n-1}^2} + h_\lambda(u_1)$$
$$+\ h_\lambda(u_1 - 2\sqrt{u_2^2 + \tfrac{1}{8}\alpha^2} + \tfrac{3}{4}\alpha)\ ,$$

with $h_\lambda(t) = 8k(\sqrt{t^2+\lambda^2} - t)$ and α is a positive constant taken small later. Choose $\alpha < \min\{\delta/2, \delta/4k, \delta'/2, \delta'/12k\}$. Then we have $S_\lambda \setminus D \subset \{|u'| \leqq k,\ \alpha > u_1 \geqq 2\sqrt{u_2^2 + \frac{1}{8}\alpha^2} - \alpha,\ v_2^2 + \cdots + v_{n-1}^2 \leqq 4\alpha^2/n\}$ and $S_\lambda \setminus D \subset Z$ for every $\lambda \in (0,1]$. Therefore F is continued analytically to $D \cup \bigcup_{1 \geqq \lambda > 0} S_\lambda = D \cup \{|u'| < 2R/3,\ u_1 \leqq \alpha,\ v_n > ((8k\sqrt{n}+1)(e^{I(\alpha-u_1)} - 1) + 8k\sqrt{n})\sqrt{v_2^2 + \cdots + v_{n-1}^2}$

$+ 16k((-u_1)_+ + (-u_1 + 2\sqrt{u_2^2 + \frac{1}{8}\alpha^2} - \frac{3}{4}\alpha)_+)$ if the following conditions are satisfied.

1) $S_1 \subset D$: This is satisfied if $8k\sqrt{n} > 2k(2\alpha + 2\sqrt{n}\frac{2\alpha}{\sqrt{n}})$, that is, $\alpha < \frac{2}{3}\sqrt{n}$.

2) For every $(u_1^0, w^{0\prime}) \in S_\lambda \setminus D$, $\gamma(\{(u_1^0, w^{0\prime})\} \cup \{(u_1^0, w'); w_n = i\varepsilon, |w_j - w_j^0| \leq \frac{1}{R}|i\varepsilon - w_n^0|$ for every $j=2,..,n-1\})$ is contained in $\bigcup_{\lambda' \geq \lambda} S_{\lambda'}$ $(\subset \bigcup_{1 \geq \lambda' \geq \lambda} S_{\lambda'} \cup D)$: Considering the convexity of $S_\lambda \cap \{u_1 = u_1^0\}$, this is satisfied if $k + \frac{\sqrt{n}}{R}(\varepsilon + k + 12k\alpha) < 2R/3$ and $\varepsilon > 12k\alpha + [((8k\sqrt{n}+1)(e^{2\alpha I} - 1) + 8k\sqrt{n})\sqrt{n} + 32k]\frac{1}{R}(\varepsilon + k + 12k\alpha)$ (use the formula $h_\lambda(t_1 + t_2) \leq h_\lambda(t_1) + 16k|t_2|$).

3) At every $(u_1^0, w^{0\prime}) \in S_\lambda \setminus D$ the surface S_λ is real analytic and non-characteristic for $Q(w, D_w)$, that is, S_λ is written locally as $\{v_n = \varphi_\lambda\}$ and satisfies the following inequalities at this point (see Lemma 4.3 [12] and the assumption (2.3)),

$$\begin{cases} -\dfrac{\partial \varphi_\lambda}{\partial u_1} > I\sqrt{u_1}\ (|\varphi_\lambda| + \sqrt{v_2^2 + \cdots + v_{n-1}^2} + \left|\dfrac{\partial \varphi_\lambda}{\partial u_2}\right|) & \text{if} \quad u_1 \geq 0, \\ -\dfrac{\partial \varphi_\lambda}{\partial u_1} > I\sqrt{-u_1} & \text{if} \quad u_1 < 0. \end{cases}$$

Easily to see, these are fulfilled if $\alpha < 1$, $I\sqrt{\alpha} < \min\{1/6, 8k\}$. Surely we can take k and α small enough such that they satisfy all the conditions listed till now and that k depends only on n, R and d. Consequently it follows that F(z) is holomorphic on $\{z \in \mathbb{C}^n; |x'| < R, y_n > 2k((-x_1)_+ + (x_1 - \delta)_+ + |y_1| + \cdots + |y_{n-1}|)\} \cup \{|x'| < 2R/3, x_1 - x_2 \leq \alpha, y_1 = y_2, y_n > ((8k\sqrt{n}+1)(e^{I(\alpha - x_1 + x_2)} - 1) + 8k\sqrt{n})\sqrt{y_2^2 + \cdots + y_{n-1}^2} + 16k((x_2 - x_1 - \frac{3}{4}\alpha + 2\sqrt{x_2^2 + \frac{1}{8}\alpha^2})_+ + (x_2 - x_1)_+)\} \supset \{z \in \mathbb{C}^n; |x| < \beta, y_n > 16k \cdot \min\{(-x_1)_+, (x_2 - x_1)_+\}, y_1 = \cdots = y_{n-1} = 0\}$ for some small $\beta > 0$. So $G'(t,x) = F(x_1, \ldots, x_{n-1}, x_n + i0)Y(x_1 - tx_2)Y(t)Y(1-t)$ is a well-defined hyperfunction on $\{(t,x) \in \mathbb{R} \times \mathbb{R}^n; |x| < \beta\}$ with support in $\{x_1 - tx_2 \geq 0, 0 \leq t \leq 1\}$. Set $g(t,x) =$

$G'(t,x)\Big|_{\{0<t<1,\, x_1-tx_2>0\}}$. Then $g(t,x)$ is a hyperfunction defined in $\{0<t<1,\ x_1-tx_2>0,\ |x|<\beta\}$ and satisfies $D_t g(t,x)=0$ there. So $G'(t,x)\Big|_{\{x_1-tx_2>0\}}$ is equal to the canonical flabby extension $g(t,x)Y(t)Y(1-t)$. Further , though we omit the proof, we can show the mildness of $G'(t,x)$ on $\{x_1-tx_2=0\}$ by prolonging $F(w)$ analytically (cf. § 2 [12]). Since $G'(t,x)$ defines a section $[G'(t,x)]$ of $C_{M'_+|X'}/C_{N'|X'}$ on a neighborhood of $L=\{(t,x;i\tau dt+idx_n+\zeta_1 d(x_1-tx_2)\in S^*_{M'_+}X'$; $x=0,\ 0\leq t\leq 1,\ \zeta_1=\tau=0\}$ and $Q(u,D_u)[G'(t,x)]$ coincides on L with $[\{Q^{\varepsilon}(w,D_w)F(w)\}_{\mathrm{Im}w_1=\cdots=\mathrm{Im}w_{n-1}=0}\times Y(x_1-x_2t)Y(t)Y(1-t)]_{\mathrm{Im}w_n=+0}\in \mathcal{O}_{z=0}\otimes$ $Y(x_1-x_2t)Y(t)Y(1-t)$ (this is not trivial, but the proof is rather long and tedious), $Q(u,D_u)[G'(t,x)]=0$ holds on L , which is equivalent to $P(x,D_x)[G'(t,x)]=0$ on L. It is easy to verify the claim iii) in the statement of this proposition. Thus the proof is completed.

<u>Theorem 2.4</u> Let $P(x,D)$ be a pseudo-differential operator with real principal symbol defined at $p_0=(0,x_0';i\eta_0)\in iS^*M\underset{M}{\times}N-iS^*_N M$. Assume that ;

$$\begin{cases} \sigma(P)(p_0)=\{\sigma(P),\ x_1\}(p_0)=0,\quad \{\{\sigma(P),\ x_1\},\ x_1\}(p_0)\neq 0, \\ \{\{\sigma(P),\ x_1\},\ \sigma(P)\}(0,x_0',\eta_0)<0,\quad d\sigma(P)\wedge dx_1\wedge\omega\neq 0 \ \text{ at } p_0. \end{cases}$$

Then $P(x,D)$ is N_+-regular at p_0. In other words, this is equivalent to the following statement. Without loss of generality we may assume that $P(x,D)$ is a second-order pseudo-differential operator of the form $D_1{}^2+P_1(x,D')D_1+P_2(x,D')$. Then any boundary value of a

$\hat{C}_{N|M_+}$-solution $f(x)$ to $P(x,D)f(x)=0$ at $p_0'=\iota(p_0)$ is micro-analytic at p_0' if $SS(ext(f))\cap\gamma_{p_0}\cap\{x_1>0\}=\phi$, where γ_{p_0} is the bicharacteristic strip passing through p_0 defined in a small neighborhood of p_0.

Proof We may assume $P(x,D)=D_1{}^2-(x_1-x_2)a(x,D')$ and $p_0=(0;idx_n)$ as in Proposition 2.3. So by this proposition, for the solution $f(x)$ there exists a hyperfunction $g(t,x)$. Hereafter we use the coordinates $(s,u_1,\dots,u_n)=(t,x_1-x_2t,x_2,\dots,x_n)$. Therefore $P(x,D)$ and D_t are written in the form ,

$$\begin{cases} P = D_{u_1}^2 - (u_1-(1-s)u_2)a(u_1+su_2,u',D_{u_2}-sD_{u_1},D_{u_3},\dots,D_{u_n}), \\ D_t = D_s - u_2D_{u_1}. \end{cases}$$

By Weierstrass' division theorem for pseudo-differential operators P can be decomposed into the product $'R'R$ of pseudo-differential operators on $L=\{(s,u;idu_n)\in S^*_{M'_+}X';\ 0\leqq s\leqq 1,\ u=0\}$, where R' is elliptic on L and R is a second-order pseudo-differential operator of the form $R = D_{u_1}^2-(u_1-(1-s)u_2)B(s,u,D_{u'})D_{u_1}-(u_1-(1-s)u_2)C(s,u,D_{u'})$. Hence from ii) in Proposition 2.3 we obtain a pseudo-differential equation for sections of $C_{M'_+|X'}/C_{N'|X'}$ on L,

$$\{D_{u_1}^2 - (u_1-(1-s)u_2)B(s,u,D_{u'})D_{u_1} - (u_1-(1-s)u_2)C(s,u,D_{u'})\}[ext(G)] =0.$$

Since $R(s,u,D_u)$ is invertible on $\{(s,u;\lambda_1du_1+idu_n)\in S^*_{N'}X';\ u=0,\ 0\leqq s\leqq 1,\ \lambda_1\in\mathbb{C}\setminus\{0\}\}$, this implies $R(s,u,D_u)G=0$ as a section of $\hat{C}_{N'|M'_+}$ on $\{(s,u';idu_n)\in iS^*N';\ 0\leqq s\leqq 1,\ u'=0\}$. On the other hand from i) it follows that $(D_s-u_2D_{u_1})G=D_t(g(t,x)Y(t)Y(1-t))=g(+0,u)\delta(s)-g(1-0,u_1+u_2,u')\delta(s-1)$. Here $g(+0,u)=f(u)$ as a germ of $\hat{C}_{N|M_+}$ at

$(0;idu_n)\in iS^*N$ and $g(1-0,u_1+u_2,u')$ is a $\hat{C}_{N''|M''_+}$-solution to $R(1,u,D_u)\psi=0$ at $(0;idu_n)\in iS^*N''$ ($M''_+=\{(s,u)\in M';s=1,u_1\geq 0\}$, $N''=\{u\in M''_+;u_1=0\}$), because of $P(D_s-u_2D_{u_1})G=(D_s-u_2D_{u_1})PG=0$. Furthermore, recalling that $g(1-0,u_1+u_2,u')$ coincides with $f(u_1+u_2,u')$ as a section of $C_{M''}$ on $\{(u;i\nu du)\in iS^*M'';\ r>u_1>0,\ r>u_1+u_2>0,\ |u'|<r,\ |\nu-(0,..,0,1)|<r\}$, we can apply the N''_+-regularity of $R(1,u,D_u)$ to this case. In fact $R(1,u,D_u)$ is just the operator $D_{u_1}^2 - u_1B(u,D_{u'})D_{u_1}-u_1C(u,D_{u'})$ introduced in Proposition 2.3 which is hyperbolic to the codirection du_1 in $\{r>u_1>0\}$ for small $r>0$, so this is N''_+-regular by virtue of Theorem 1.12. Remarking that the deleted bicharacteristic strip $\gamma_{p_0}-\{p_0\}$ is contained not only in $\{x_1>0\}$, but also in $\{x_1-x_2>0\}$, the assumption $SS(ext(f))\cap\gamma_{p_0}\cap\{x_1>0\}=\phi$ leads to $SS(ext(g(1-0,u_1+u_2,u')))\subset\{u_1=0\}$ near $(0;idu_n)\in iS^*M''$. Therefore by the N''_+-regularity we have $g(1-0,u_1+u_2,u')=0$ at $(0;idu_n)\in iS^*N''$ as a germ of $\hat{C}_{N''|M''_+}$. Consequently it follows that $(D_s-u_2D_{u_1})G=f(u)\delta(s)$ holds on $\{(s,u';idu_n)\in iS^*N';\ 0\leq s\leq 1,\ u'=0\}$ as a section of $\hat{C}_{N'|M'_+}$. Hence, set $h_0(s,u')=G(s,+0,u')$ and $h_1(s,u')=D_{u_n}^{-1}(D_{u_1}G)(s,+0,u')$ and take boundary values on $u_1=+0$ in these equations, then we have the following system of pseudo-differential equations of first-order for sections of $C_{N'}$,

$$\{D_sI_2 - A(s,u',D_{u'})\}\begin{pmatrix} h_0 \\ h_1 \end{pmatrix}= \begin{pmatrix} f(+0,u')\delta(s) \\ D_{u_n}^{-1}(D_{u_1}f)(+0,u')\delta(s) \end{pmatrix}$$

on a neighborhood of $\{(s,u';idu_n)\in iS^*N';\ 0\leq s\leq 1,\ u'=0\}$, where

$$A = \begin{pmatrix} 0, & u_2 D_{u_n} \\ -(1-s)u_2{}^2 D_{u_n}^{-1} C(s,0,u',D_{u'}), & -(1-s)u_2{}^2 D_{u_n}^{-1} B(s,0,u',D_{u'})D_{u_n} \end{pmatrix}.$$

Recall $SS(h_0)$, $SS(h_1) \subset \{0 \leqq s \leqq 1\}$ by the definition. So, noting that the determinant of the principal symbol of $D_s I_2 - A(s,u',D_{u'})$ is given by :

$$\lambda_s{}^2 + (1-s)u_2{}^2 \sigma_1(B)(s,0,u',\lambda')\lambda_s + (1-s)u_2{}^3 \sigma_2(C)(s,0,u',\lambda') :$$

and that B, C have real principal symbols and $\sigma_2(C) > 0$ on $\{0 \leqq s \leqq 1,$ $u'=0,\ \lambda'=(0,..,0,1)\}$, we obtain $U \cap \{SS(h_0) \cup SS(h_1)\} \subset \{0 \leqq s \leqq 1,\ u_2=0,$ $i\nu_s=0\} \cup \{s=0\}$ for some neighborhood U of $\{(s,u';idu_n) \in iS^*N';\ u'=0,$ $0 \leqq s \leqq 1\}$, because $D_s I_2 - A(s,u',D_{u'})$ is invertible on $\{0<s<1,\ 0<u_2<\varepsilon\}$ and is a hyperbolic operator on $\{0<s<1,\ -\varepsilon<u_2<0\}$ with small velocity of order $|u_2|^{3/2}$. Furthermore in the regular involutory submanifold $V=\{(s,u';i\nu_s ds + i\nu' du') \in iS^*N';\ \nu_s=0,\ u_2=0\}$ the theorem of micro-Holmgren of Bony ([1]) is available. In fact since the determinant of the micro principal symbol of $D_s I_2 - A(s,u',D_{u'})$ along V is $\lambda_s{}^2$, the micro analyticity of (h_0,h_1) propagates along integral curves of $\partial/\partial s$ in $U \cap \{0<s \leqq 1,\ u_2=0,\ i\nu_s=0\}$. Thus we have $U \cap \{SS(h_0) \cup SS(h_1)\} \subset \{s=0\}$. At the last step of the proof, we use Schapira's theory in [27]. Since $N=\{x \in M; x_1=0\}=\{(s,u') \in N'; s=0\}$ is micro-non-characteristic for $D_s I_2 - A(s,u',D_{u'})$, according to his theory, $D_s I_2 - A(s,u',D_{u'})$ is a N-regular operater. So $(D_s I_2 - A) \times$ ${}^t(h_0,h_1) = {}^t(f(+0,u'),\ D_{u_n}^{-1}(D_{u_1} f)(+0,u')) \otimes \delta(s) \in (C_{N|Y'})^2$ (Y' is a complex neighborhood of N') leads to $(h_0,h_1) \in (C_{N|Y'})^2$ at $(0,0;$ $idu_n) \in iS^*N' \underset{N'}{\times} N$. Further by the division theorem for $C_{N|Y'}$ (see §1 [17]) we have $f(+0,u') = D_{u_n}^{-1}(D_{u_1} f)(+0.u')=0$ at $(0;idu_n) \in iS^*N$. This completes the proof.

References

[1] Bony, J.M., Extension du theorème de Holmgren. Sem. Goulaouic--Schwartz, 1975-76, exposé 17.

[2] Bony, J.M., Schapira, P., Existence et prolongement des solutions holomorphes des équations aux derivées partielles. Inv. Math., 17(1972), 95-105.

[3] Bony, J.M., Schapira, P., Solutions hyperfonctions du problème de Cauchy. Lecture Notes in Math. Springer, 287(1973), 82-98.

[4] Bony, J.M., Schapira, P., Propagation des singularités analytiques pour les solutions des équations aux derivées partielles. Ann. Inst. Fourier, 26(1976), 81-140.

[5] Friedlander, F.G., The wavefront set of a simple initial-boundary value problem with glancing rays. Math. Proc. Camb. Phil. Soc., 79(1976), 145-159.

[6] Friedlander, F.G., Melrose, R.B., The wavefront set of a simple initial-boundary value problem with glancing rays II. Math. Proc. Camb. Phil. Soc., 81(1977), 97-120.

[7] Hörmander, L., A remark on Holmgren's uniqueness Theorem. J. Diff. Geometry 6(1971), 129-134.

[8] Kaneko, A., Singular spectrum of boundary values of solutions of partial differential equations with real analytic coefficients. Sci. Pap. Coll. Gen. Educ. Univ. Tokyo 25(1975), 59--68.

[9] Kaneko, A., Estimation of singular spectrum of boundary values for some semihyperbolic operators (to appear).

[10] Kashiwara, M., Kawai, T., On the boundary value problem for elliptic system of linear differential equations, I. Proc. Japan Acad. 48(1972), 712-715.

[11] Kashiwara, M., Kawai, T., On the boundary value problem for elliptic system of linear differential equations, II. Proc. Japan Acad. 49(1973), 164-168.

[12] Kashiwara, M., Kawai, T., On microhyperbolic pseudo-differential operators I. J. Math. Soc. Japan 27, No 3 (1975), 359--404.

[13] Kashiwara, M., Oshima, T., Systems of differential equations with regular singularities and their boundary value problems. Ann. Math.,106(1977), 145-200.

[14] Kashiwara, M., Schapira, P., Micro-hyperbolic systems. Acta Math. 142(1979), 1-56.

[15] Kataoka, K., A micro local approach to general boundary value problems. Publ. R.I.M.S. Kyoto Univ. Vol.12, Supplement (1977), 147-153.

[16] Kataoka, K., On the theory of Radon transformations of hyperfunctions (to appear in J. Fac. Sci. Univ. Tokyo).

[17] Kataoka, K., Micro local theory of boundary value problems I (to appear in J. Fac. Sci. Univ. Tokyo).

[18] Kawai, T., Construction of elementary solutions of I-hyperbolic operators and solutions with small singularities. Proc. Japan Acad.,46(1970), 912-915.

[19] Komatsu, H., Kawai, T., Boundary values of hyperfunction solutions of linear partial differential equations. Publ. R.I.M.S., 7(1971), 95-104.

[20] Melrose, R.B., Microlocal parametrices for diffractive boundary value problems. Duke Math. J., 42(1975), 605-635.

[21] Melrose, R.B., Equivalence of glancing hypersurfaces. Inv. Math., 37(1976), 165-191.

[22] Melrose, R.B., Sjöstrand, J., Singularities of boundary value problems I. C.P.A.M., 31(1978), 593-617.

[23] Ôaku, T., Micro-local Cauchy problems and local boundary value problems. Proc. Japan Acad. 55(1979), 136-140.

[24] Sato, M., Kawai, T., Kashiwara, M., Microfunctions and pseudo--differential equations. Lecture Notes in Math. Vol.287, Springer (1973), 265-529.

[25] Schapira, P., Hyperfonctions et problèmes aux limites elliptiques. Bull. Soc. Math. France 99(1971), 113-141.

[26] Schapira, P., Propagation au bord et réflexion des singularités analytiques des solutions des équations aux dérivées partielles. Sem. Goulaouic-Schwartz 1975-76, exposé 6.

[27] Schapira, P., Propagation au bord et réflexion des singularités analytiques des solutions des équations aux dérivées partielles II. Sem. Goulaouic-Schwartz 1976-77, exposé 9.

[28] Sjöstrand, J., Sur la propagation des singularités analytiques pour certains problèmes de Dirichlet. C.R.Acad. Sc. Paris, t.288(1979), 673-674.

[29] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations (to appear in Jap. J. Math.).

[30] Tajima, S., Complex valued phase functions and their applications. Master's thesis in Univ. Tokyo (1978, in Japanese).

[31] Taylor, M.E., Grazing rays and reflection of singularities of solutions to wave equations. C.P.A.M. 29(1976), 1-38.