

The nonlinear transformation of Gaussian measure on Banach
space and its absolute continuity の研究

(バナッハ空間上のガウス測度の非線型変換と
絶対連続性の研究)

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The nonlinear transformation of Gaussian measure
on Banach space and its absolute continuity (I)

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1. Introduction

To begin with, let us introduce some preliminary notions .
For real Banach spaces E and F , E^* denotes the dual space of E
with the strong topology , I_E denotes the identical map on E
and $L^\infty(E,F)$ denotes the Banach space consisting of all bounded
linear operators from E into F with the operator norm . For
real Hilbert spaces H and K , $L^2(H,K)$ denotes the Hilbert space
of all Hilbert-Schmidt operators from H into K with a Hilbert-
Schmidt norm .

We say that (μ, H, B) is an abstract Wiener space if μ , H
and B satisfy the following condition (W-1) and (W-2) .

(W-1) B is a real separable Banach space, and H is a real
separable Hilbert space densely and continuously included in H .

We identify H^* with H , then B^* is naturally regarded as
a dense subset of H and the inclusion map from B^* into H is
continuous . And moreover the relation ${}_{B^*}\langle u, v \rangle_B = (u, v)_H$
holds for any $u \in B^*$ and $v \in H$.

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(W-2) μ is a Gaussian probability measure on B such that

$$\int_B \exp(\sqrt{-1} {}_{B^*} \langle u, z \rangle_B) \mu(dz) = \exp(- \frac{1}{2} \|u\|_H^2) \quad \text{for each } u \in B^*$$

Throughout this paper we promise that (μ, H, B) denotes an abstract Wiener space .

In this paper we study the following problem . Let F be a measurable map from B into H . Our problem is to study when the image measure $(I_B - F)\mu$ on B induced by μ through $I_B - F : B \rightarrow B$ is absolutely continuous relative to μ , and to give the explicit form of its density function .

R.H.Cameron and W.T.Martin are the first that studied this problem . In their paper [2] , they dealt in the case that B is the space of all continuous functions on the interval $[0,1]$ and μ is an ordinary Wiener measure . L.Gross [5] and H.H.Kuo [8] extended the work of R.H.Cameron and W.T.Martin for general abstract Wiener spaces .

From a viewpoint of stochastic differential equation , this problem was studied by V.Girsanov [4] and M.Motoo [11] , and they showed that Ito integral appeared essentially in the density function .

In 1974 , R.Ramer [13] introduced an abstract version of Ito integral (we will call it Ito-Ramer integral) , and he solved our problem under some assumptions , one of which is that $F : B \rightarrow H$ is continuous . But his assumptions are sometimes so strong that we cannot apply his theorem for stochastic

differential equations and etc . He conjectured in his paper that we could solve our problem under some weaker assumptions than his . We give one of answer to his conjecture in this paper .

Now let us shortly summerize the content of our paper .

We give some tools for later use in Section 2, 3 and 4 . We also introduce an abstract version of Ito integral in Section 5 following the idea of R.Ramer [13] .

The main results are stated in Section 6, 7 and 8 . We say that a measurable map $F: B \rightarrow H$ is an $H-C^1$ map , if there exists a Hilbert Schmidt operator $DF(z): H \rightarrow H$ for each $z \in B$ such that (1) $\|F(z+h) - F(z) - DF(z)h\|_H = o(\|h\|_H)$ for each $z \in B$ as $\|h\|_H \rightarrow 0$, and (2) $DF(z+\cdot): H \rightarrow L^2(H,H)$ is continuous for each $z \in B$. We prove the following in Section 6 .

Theorem 6.2 If $F: B \rightarrow H$ is an $H-C^1$ map and $I_H - DF(z): H \rightarrow H$ is invertible for μ -a.e. z , then $(I_B - F)\mu$ is absolutely continuous relative to μ .

We give the explicit form of the density function $\frac{d(I_B - F)\mu}{d\mu}$

under some more assumptions in Theorem 6.3 . These results are extension of the work of R.Ramer [13] .

We give some extended theorems of Theorem 6.2 and 6.3 in Section 7 . There we don't need to assume that $F(z+\cdot): H \rightarrow H$ is continuous anymore .

In Section 8 , we prove some Sard type theorem and give some sufficient condition under which μ is absolutely continuous relative to $(I_B - F)\mu$.

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2. Preliminary material

In this section we note some known results about an abstract Wiener space (μ, H, B) . Let $P(E)$ denote the set of all projections of H with a finite dimensional range included in E for each vector subspace E of H . Then $P(E)$ is a directed set with usual order induced by the order of ranges. It is obvious that any projection P belonging to $P(B^*)$ is extensible to a bounded linear operator $\bar{P}: B \rightarrow B^*$. R. Carmona [3] proved

Proposition 2.1 Let $\{P_n\}_{n=1}^{\infty}$ be an increasing sequence of elements of $P(B^*)$ strongly converging to I_H , then

$$\|z - \bar{P}_n z\|_B \rightarrow 0, \quad n \rightarrow \infty \text{ for } \mu\text{-a.e. } z. \quad \text{Moreover, for any } 1 < q < \infty, \quad \int_B \|z - \bar{P}_n z\|_B^q \mu(dz) \rightarrow 0, \quad n \rightarrow \infty.$$

The following is well known. (See for an example R. Ramer [13].)

Proposition 2.2 For any Hilbert Schmidt operator K on H , there exists a measurable map $K: B \rightarrow H$ such that

$$\|Kz - K\bar{P}z\|_H \rightarrow 0, \quad P \in P(B^*), \text{ in probability with respect to } \mu, \text{ and } \int_B \|Kz\|_H^2 \mu(dz) = \text{trace } KK^*.$$

We call K the stochastic extension of K . For each $P \in P(B^*)$, it is easy to see that $Pz = \bar{P}z$ for μ -a.e. z , and so we identify \bar{P} with P .

The following are also well known. For the proof, see H. Kuo [9] Chap. 1 Section 4 and Chap. 2 Section 5.

Proposition 2.3 The inclusion map from H to B is a compact operator.

Proposition 2.4 For each $u \in B$, let N_u be an operator on B defined by $N_u z = u$ for any $z \in B$. Then the image measure

$(I_B - N_u)^{-1}\mu$ is absolutely continuous relative to μ , if and only if u belongs to H . Furthermore, for each $u \in B^*$,

$$\frac{d(I_B - N_u)^{-1}\mu}{d\mu}(z) = \exp({}_{B^*}\langle u, z \rangle_B - \frac{1}{2} \|u\|_H^2) .$$

Proposition 2.5 Let K be a bounded linear operator from B into B^* , and assume that $I_H - K$ is invertible as a map from H into itself. Then K is a nuclear operator on H , $I_B - K: B \rightarrow B$ is invertible and $(I_B - K)^{-1}\mu$ is absolutely continuous relative to μ . Furthermore,

$$\frac{d(I_B - K)^{-1}\mu}{d\mu}(z) = |\det K| \exp({}_{B^*}\langle Kz, z \rangle_B - \frac{1}{2} \|Kz\|_H^2) .$$

3. H^1 - class maps

In this section we introduce an infinite dimensional analogue of the classical Sobolev space . Let E denote a real Banach space .

Definition 3.1 (1) We say that a map $f: \mathbb{R} \rightarrow E$ is absolutely continuous if , for any $-\infty < a < b < \infty$ and $\varepsilon > 0$, there exists some $\delta(\varepsilon, a, b) > 0$ such that

$$\sum_{i=1}^n \|f(t_i) - f(s_i)\|_E < \varepsilon \text{ holds for any integer } n \text{ and}$$

$$a \leq t_1 < s_1 \leq t_2 < s_2 \dots t_n < s_n \leq b, \quad \sum_{i=1}^n |t_i - s_i| < \delta(\varepsilon, a, b) .$$

(2) We say that a map $f: \mathbb{R} \rightarrow E$ is strictly absolutely continuous, if $f: \mathbb{R} \rightarrow E$ is continuous , $f(t)$ is strongly differentiable for almost every t and it satisfies that

$$\int_a^b \left\| \frac{df}{dt}(t) \right\|_E dt < \infty \text{ and } f(b) - f(a) = \int_a^b \frac{df}{dt}(t) dt$$

for any $-\infty < a < b < \infty$, where $\frac{df}{dt}(t)$ denotes strong derivative of f at t .

Proposition 3.1 (1) Given a map $f: \mathbb{R} \rightarrow E$, f is absolutely continuous if f is strictly absolutely continuous .

(2) Assume that E is reflexive and $f: \mathbb{R} \rightarrow E$ is absolutely continuous , then f is strictly absolutely continuous .

See V.Barbu and Th.Precupanu [1] ,for the proof .

We say that $F: B \rightarrow E$ is strongly measurable, if there exists a Borel subset Ω of B and a separable closed subspace E_0 of E such that $F(\Omega) \subset E_0$, $F|_{\Omega}: \Omega \rightarrow E_0$ is measurable and $\mu(\Omega) = 1$.

Definition 3.2 We say that a strongly measurable map $F: B \rightarrow E$ is stochastic Gateaux H-differentiable (abbreviated by S.G.D.) if there exists a strongly measurable map $DF: B \rightarrow L^{\infty}(H, E)$ such that $\frac{1}{t} E^* \langle u, F(z+th) - F(z) \rangle_E \rightarrow E^* \langle u, DF(z)h \rangle_E$ in probability with respect to μ , $t \rightarrow 0$, for each $u \in E^*$ and $h \in H$. DF is called a stochastic Gateaux H-derivative of F .

Remark 3.1 By virtue of Proposition 2.4, $F(z+th)$ is determinate for μ -a.e. z without depending on a version of F , so is $DF(z)$.

Definition 3.3 We say that a strongly measurable map $F: B \rightarrow E$ is ray absolutely continuous with probability one (abbreviated by R.A.C.), if there exists a strongly measurable map $\tilde{F}_h: B \rightarrow E$ for each $h \in H$ such that $\tilde{F}_h(z) = F(z)$ for μ -a.e. $z \in B$ and $\tilde{F}_h(z+th)$ is strictly absolutely continuous in t for each $z \in B$.

Definition 3.4 We say that a map $F: B \rightarrow E$ belongs to $H^1(B \rightarrow E; d\mu)$ if $F: B \rightarrow E$ is strongly measurable, S.G.D. and R.A.C.

Proposition 3.2 Let $F: B \rightarrow E$ be an element of $H^1(B \rightarrow E; d\mu)$. Then for each $h \in H$ and $-\infty < a < b < \infty$, $\int_a^b \|DF(z+sh)h\|_E ds < \infty$ for μ -a.e. $z \in B$ and

$$\tilde{F}_h(z+bh) - \tilde{F}_h(z+ah) = \int_a^b DF(z+sh)h ds \quad \text{for } \mu\text{-a.e. } z.$$

Here \tilde{F}_h is a version of F as in Definition 3.3.

Before proving our proposition, we shall introduce some notion. For each finite dimensional subspace K of H , we define the probability measures μ_K and μ_K^\perp on B by

$\mu_K = \tilde{P}_K \mu$ and $\mu_K^\perp = (I_B - \tilde{P}_K) \mu$, where P_K is the orthogonal projection from H onto K and \tilde{P}_K is the stochastic extension of P_K . Let $\{k_1, \dots, k_n\}$ be an orthogonal base of K . Then it is easy to see that

$$(3.1) \quad \int_B f(z) \mu(dz) = \int_{B \times B} f(z + \tilde{z}) \mu_K^\perp(dz) \otimes \mu_K(d\tilde{z}) \\ = \int_{B \times \mathbb{R}^n} f(z + \sum_{j=1}^n x_j k_j) \mu_K^\perp(dz) \otimes \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{j=1}^n x_j^2\right) dx$$

for each bounded measurable function f on B .

Now let us prove our proposition. We may assume that

$\|h\|_H = 1$ without loss of generality. Let $K = \mathbb{R}h$.

Since $F_h(z+th)$ is strictly differentiable in t for a.e. t and

$\int_a^b \left\| \frac{d}{dt} F_h(z+th) \right\|_E dt < \infty$, it is easy to see that

$$(3.2) \quad \frac{1}{\tau} \{ \tilde{F}_h(z+(t+\tau)h) - \tilde{F}_h(z+th) \} \rightarrow \frac{d}{dt} \tilde{F}_h(z+th) \quad \text{for a.e. } (z, t)$$

with respect to $\mu_K^\perp(dz) \otimes \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}t^2} dt$. It follows from

Definition 3.2 and (3.1) that

$$\frac{1}{\tau} E^* \langle u, \tilde{F}_h(z+(t+\tau)h) - \tilde{F}_h(z+th) \rangle_E \rightarrow E^* \langle u, DF(z+th)h \rangle_E$$

in probability with respect to $\mu_K^\perp(dz) \otimes \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}t^2} dt$,

which implies that

(3.3) $\frac{d}{dt} \tilde{F}_h(z+th) = DF(z+th)h$ for a.e. (z,t) with respect to $\mu_K^1(dz) \otimes dt$. By the definition of μ_K^1 , it is easy to see that (3.3) holds for a.e. (z,t) with respect to $\mu(dz) \otimes dt$.

This completes the proof.

Remark 3.1 By (3.3) we get

$$\tilde{F}_h(z+bh) - \tilde{F}_h(z+ah) = \int_a^b DF(z+th)h dt \quad \text{for } \mu_K^1\text{-a.e. } z.$$

Definition 3.5 We say that a measurable function w defined on B is strictly positive, if there exists a measurable function \tilde{w}_h for each $h \in H$ such that

- (1) $\tilde{w}_h(z) = w(z)$ for μ -a.e. z and
- (2) $\inf \{ \tilde{w}_h(z+th); -T < t < T \} > 0$ for any $z \in B$ and $T > 0$.

The following theorem shows the stability of $H^1(B \rightarrow E; d\mu)$.

Theorem 3.1 Let F_n ($n=1,2,\dots$) be an element of $H^1(B \rightarrow E; d\mu)$, let F be a strongly measurable map from B to E , G be a strongly measurable map from B to $L^\infty(H, E)$, and let w be a strictly positive measurable function on B . Assume furthermore that

- (1) $\|F(z) - F_n(z)\|_E \rightarrow 0$ in probability with respect to μ as $n \rightarrow \infty$ and
- (2) $\int_B \|G(z)h\|_E w(z) \mu(dz) < \infty$ and

$$\int_B \|G(z)h - DF_n(z)h\|_E w(z) \mu(dz) \rightarrow 0, \quad n \rightarrow \infty, \quad \text{for each } h \in H.$$

Then F belongs to $H^1(B \rightarrow E; d\mu)$ and $DF(z) = G(z)$ for μ -a.e. z .

Proof. For simplicity, we assume $E = \mathbb{R}$. Take a measurable map $F_{n,h} : B \rightarrow \mathbb{R}$ as in Definition 3.3 for F_n . Let $K = \mathbb{R}h$ as in the proof of Proposition 3.2, then by Remark 3.1 we obtain

$$(3.4) \quad F_{n,h}(z+bh) - F_{n,h}(z+ah) = \int_a^b DF_n(z+sh)h ds \quad \text{for } \mu_K^\perp\text{-a.e. } z.$$

The right and left hands of (3.4) are continuous in a and b .

So there exists a σ -compact subset Ω_1 of B such that $\mu_K^\perp(\Omega_1) = 1$ and (3.4) holds for any $z \in \Omega_1$ and $a < b$.

Taking a subsequence if necessary, we may assume

$F_{n,h}(z) \rightarrow F(z)$ for μ -a.e. z . So there exists a σ -compact subset Ω_2 of B such that $\mu_K^\perp(\Omega_2) = 1$ and for each $z \in \Omega_2$

$$(3.5) \quad F_{n,h}(z+th) \rightarrow F(z+th), \quad n \rightarrow \infty, \quad \text{for a.e. } t.$$

By assumption (2), we get

$$\int_B \mu_K^\perp(dz) \int_{\mathbb{R}} |G(z+th)h - DF_n(z+th)h| \tilde{w}_h(z+th) e^{-\frac{1}{2}t^2} dt \rightarrow 0$$

where \tilde{w}_h is a function on B as in Definition 3.5. So we may

assume that there exists a σ -compact subset Ω_3 of B such that

$\mu_K^\perp(\Omega_3) = 1$ and $\int_{\mathbb{R}} |G(z+th)h - DF_n(z+th)h| \tilde{w}_h(z+th) e^{-\frac{1}{2}t^2} dt \rightarrow 0$, $n \rightarrow \infty$, for each $z \in \Omega_3$. By virtue of Definition 3.5 (2), we

obtain

$$(3.6) \quad \int_a^b |G(z+th) - DF_{n,h}(z+th)h| dt \rightarrow 0 \quad \text{for any } a < b \text{ and } z \in \Omega_3.$$

Let $\Omega_4 = \Omega_1 \cap \Omega_2 \cap \Omega_3$, then by (3.4), (3.5) and (3.6) it is easy to see that $\mu_K^\perp(\Omega_4) = 1$, $F_{n,h}(z+th)$ are convergent as $n \rightarrow \infty$ for any t and $z \in \Omega_4$ and $\lim F_{n,h}(z+th) = F(z+th)$ for a.e. (z, t) with respect to $\mu_K^\perp(dz) \otimes dt$.

Moreover we can see that

$\lim_{n \rightarrow \infty} \tilde{F}_{n,h}(z+bh) - \lim_{n \rightarrow \infty} \tilde{F}_{n,h}(z+ah) = \int_a^b G(z+th)h dt$ for any $a < b$
and $z \in \Omega_4$. Let $\Omega_0 = \Omega_4 + \mathbb{R}h$, then Ω_0 is σ -compact in B and
 $\mu(\Omega_0) = 1$. Define a map $F_h : B \rightarrow \mathbb{R}$ by

$$F_h(z) = \begin{cases} \lim_{n \rightarrow \infty} \tilde{F}_{n,h}(z) & \text{if } z \in \Omega_0 \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to see that $F_h(z) = F(z)$ for μ -a.e. z and

$$F_h(z+th) = \begin{cases} F_h(z) + \int_0^t G(z+sh)h ds & \text{if } z \in \Omega_0 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore F is R.A.C. . On the other hand, it is easy to see
that $\frac{1}{t} \{F_h(z+th) - F_h(z)\} \rightarrow G(z)h$, $t \rightarrow 0$, for μ -a.e. z .
This completes the proof .

4. The partition of unity

Definition 4.1 For any subset A of B , we define a function $\rho(\cdot; A) : B \rightarrow [0, \infty]$ by

$$\rho(z; A) = \begin{cases} \inf\{ \|h\|_H ; h \in (A-z) \cap H \} & \text{if } (A-z) \cap H \neq \emptyset, \\ \infty & \text{otherwise} \end{cases}$$

Then we get the following .

- Proposition 4.1 (1) If subsets A and A' of B satisfy $A \subset A'$, then $\rho(z; A) \geq \rho(z; A')$ for each $z \in B$.
- (2) For any subset A of B and $h \in H$, $\rho(z+h; A) \leq \|h\|_H + \rho(z; A)$ for each $z \in B$.
- (3) Let $\{A_n\}_{n=1}^{\infty}$ is increasing subsets of B and $A = \bigcup_{n=1}^{\infty} A_n$,

then $\rho(z; A_n) \uparrow \rho(z; A)$, $n \rightarrow \infty$, for each $z \in B$.

Proof. (1) and (2) are obvious, so it suffices us to show (4.1) $\lim_{n \rightarrow \infty} \rho(z; A_n) \leq \rho(z; A)$ for each $z \in B$. However, (4.1) is obvious when $(A-z) \cap H = \emptyset$. Therefore we will show (4.1) when $(A-z) \cap H \neq \emptyset$. For any $\varepsilon > 0$, there exists $h \in (A-z) \cap H$ such that $\|h\|_H \leq \rho(z; A) + \varepsilon$. Since $h+z \in A$ and $A_n \uparrow A$, there exists an integer n_0 such that $h+z \in A_n$ for each $n \geq n_0$. So $h \in (A_n - z) \cap H$, which implies $\rho(z; A_n) \leq \rho(z; A) + \varepsilon$ for any $n \geq n_0$. This completes the proof.

Theorem 4.1 (1) If K is a compact subset of B , then $\rho(\cdot; K) : B \rightarrow [0, \infty]$ is lower semi-continuous.

(2) If G is a σ -compact subset of B , then $\rho(\cdot; G) : B \rightarrow [0, \infty]$ is measurable.

Proof. (2) is an immediate consequence of (1) and Proposition 9.1 (3). So it is sufficient to show (1). Let $A_a = \{ z \in B; \rho(z;K) \leq a \}$ and $S_a = \{ h \in H; \|h\|_H \leq a \}$ for each $a \geq 0$. It is obvious that $A_a \supset K + S_a$. On the other hand let $z \in A_a$, then there exists a sequence $\{h_n\}_{n=1}^{\infty} \subset (K-z) \cap H$ such that $\|h_n\|_H \leq a + \frac{1}{n}$. By virtue of Proposition 2.3, taking a subsequence if necessary, there exists $h \in H$ such that h_n converges to h in B as $n \rightarrow \infty$ and $\|h\|_H \leq a$. In view of the closedness of K , we get $z+h \in K$, which shows that $z \in K + S_a$. So we obtain $A_a \subset K + S_a$. According to Proposition 2.3 S_a is compact in B , so is A_a . This completes the proof.

Remark 4.1 Let ϕ be a smooth function on \mathbb{R} with compact support and G be a σ -compact subset of B . Let $g(z) = \phi(\rho(z;G))$ for each $z \in B$ with the convention that $\phi(\infty) = 0$. Then g is a measurable function on B and $|g(z+h) - g(z)| \leq c \|h\|_H$ for each $z \in B$ and $h \in H$, where $c = \sup\{|\frac{d\phi}{dt}(t)|; t \in \mathbb{R}\}$.

Theorem 4.2 Let E be a separable reflexive Banach space and F be a measurable map from B to E and suppose furthermore that there exists a positive constant c such that

$\|F(z+h) - F(z)\|_E \leq c \|h\|_H$ for each $z \in B$ and $h \in H$. Then there exists a measurable subset D_0 of B and a map $DF: B \rightarrow L^{\infty}(H, E)$ such that (1) $\mu(D_0) = 1$

(2) $\lim_{t \rightarrow 0} \frac{1}{t} (F(z+th) - F(z)) = DF(z)h$ for any $z \in D_0$ and $h \in H$,

and

(3) $DF(\cdot)h: B \rightarrow E$ is measurable for each $h \in H$.

In particular if $DF: B \rightarrow L^\infty(B, E)$ is strongly measurable, F belongs to $H^1(B \rightarrow E; d\mu)$.

Proof. Let V be a countable subset of B^* such that V is Q -module and dense in H . Then $F(z+tv)$ is strictly absolutely continuous in t for any $z \in B$ and $v \in V$ by virtue of Proposition 3.1, and so $F(z+tv)$ is strongly differentiable in t for a.e. t . Let $D_V = \{z \in B; \frac{1}{r} \{F(z+rv) - F(z)\} \text{ is convergent as } r \rightarrow 0, r \in Q\}$, then D_V is measurable and $\mu(D_V) = 1$. Let $G(z; v) =$

$$\lim_{\substack{r \rightarrow 0 \\ r \in Q}} \frac{1}{r} \{F(z+rv) - F(z)\} \quad \text{for each } v \in V \text{ and } z \in D_V, \text{ then}$$

$G(\cdot; v): D_V \rightarrow E$ is measurable. Since $F(z+tv)$ is continuous in t , we obtain $G(z; v) = \lim_{t \rightarrow 0} \frac{1}{t} \{F(z+tv) - F(z)\}$.

We claim the following (4.2) and (4.3).

$$(4.2) \quad G(z; rv) = r G(z; v) \quad \text{for each } r \in Q \text{ and } z \in D_V = D_{rv}.$$

$$(4.3) \quad \text{Let } D_{v_1, v_2} = \{z \in D_{v_1} \cap D_{v_2} \cap D_{v_1+v_2}; G(z; v_1) + G(z; v_2) = G(z; v_1+v_2)\} \text{ for each } v_1, v_2 \in V, \text{ then } \mu(D_{v_1, v_2}) = 1.$$

(4.2) is obvious. (4.3) is also obvious whenever v_1 and v_2 are linearly dependent over \mathbb{R} . So we prove (4.3) when v_1 and v_2 are linearly independent over \mathbb{R} . Since E is separable and reflexive, E^* is also separable. Let $\{u_n\}_{n=1}^\infty$ be a countable dense subset of E^* and let $f_n(x, y; z) = {}_{E^*} \langle u_n, F(z+xv_1+yv_2) \rangle_E$ for each $x, y \in \mathbb{R}$ and $z \in B$. Then $f_n(x, y; z)$ is Lipschitz continuous in (x, y) .

According to H. Radmacher [12], $f_n(x, y; z)$ is totally differentiable in (x, y) for a.e. (x, y) . So let $A^n = \{ z \in D_{v_1} \cap D_{v_2} \cap D_{v_1+v_2} ;$

$$E^* \langle u_n, G(z; v_1+v_2) \rangle_E = E^* \langle u_n, G(z; v_1) \rangle_E + E^* \langle u_n, G(z; v_2) \rangle_E \}$$

and $A_z^n = \{ (x, y) ; z + xv_1 + yv_2 \in A^n \}$, then $\mathbb{R}^2 \setminus A_z^n$ is of Lebesgue measure zero for each $z \in B$, and accordingly

$$\mu(A^n) = \int_B \mu_{K^\perp}(dz) \mu_K(\{ xv_1 + yv_2 ; (x, y) \in A_z^n \}) = 1,$$

where $K = \mathbb{R}v_1 + \mathbb{R}v_2$. This shows (4.3).

Now we can prove Theorem 4.2. Let $D_0 = \cap \{ D_{v_1, v_2} ; v_1, v_2 \in V \}$,

then D_0 is measurable and $\mu(D_0)=1$. For each $z \in D_0$, $G(z, \cdot) : V \rightarrow E$ is a Q -linear map by virtue of (4.2) and (4.3), and furthermore

$\|G(z; v)\|_E = \lim_{t \rightarrow 0} \frac{1}{t} \|F(z+tv) - F(z)\|_E \leq c \|v\|_H$. So we may extend $G(z; \cdot)$ to a bounded linear operator $DF(z) : H \rightarrow E$. Let $DF(z) = 0$ for $z \in B \setminus D_0$, then $DF(\cdot)h$ is a measurable map from B to E for each $h \in H$. Moreover for any $h \in H$, $v \in V$ and $z \in D_0$,

$$\begin{aligned} & \overline{\lim}_{t \rightarrow 0} \left\| \frac{1}{t} (F(z+th) - F(z)) - DF(z)h \right\|_E \\ & \leq \overline{\lim}_{t \rightarrow 0} \left\| \frac{1}{t} (F(z+tv) - F(z)) - G(z; v) \right\|_E + \overline{\lim}_{t \rightarrow 0} \left\| \frac{1}{t} (F(z+th) - F(z+tv)) \right\|_E \\ & \quad + \|DF(z)(h - v)\|_E \\ & \leq 2c \|h - v\|_H. \end{aligned}$$

Since V is dense in H , we get

$$\overline{\lim}_{t \rightarrow 0} \left\| \frac{1}{t} (F(z+th) - F(z)) - DF(z)h \right\|_H = 0 \quad . \quad .$$

This completes the proof .

The following is immediate conclusion of Theorem 4.2 and

Remark 4.1 . .

Corollary to Theorem 4.2 Let G be a σ -compact subset of B and ϕ be a smooth function on \mathbb{R} with compact support . Then $g(\cdot) = \phi(\rho(\cdot; G)) : B \rightarrow \mathbb{R}$ belongs to $H^1(B \rightarrow \mathbb{R}; d\mu)$ and

$$\|Dg(z)\|_{L^\infty(H, \mathbb{R})} \leq \sup \left\{ \left| \frac{d\phi}{dt}(t) \right| ; t \in \mathbb{R} \right\} \quad \text{for } \mu\text{-a.e. } z \quad .$$

5. Ito-Ramer integral and H - C^1 maps

In this section we introduce Ito-Ramer integral which is an extension of Ito's stochastic integral in some sense . This was first introduced by R.Ramer [13] and the results in this section owes much to Ramer's results .

Let F be an element of $H^1(B \rightarrow H; d\mu)$. We define a measurable function $L_P F$ on B by

$$L_P F(z) = (F(z), \tilde{P}z)_H - \text{trace PDF}(z) \quad \text{for each } P \in \mathcal{P}(H) ,$$

where \tilde{P} is a stochastic extension of P . Then $L_P F(z)$ is defined for μ -a.e. z .

Definition 5.1 We say that a map $F : B \rightarrow H$ belongs to $\mathcal{D}(L)$, the domain of L , if

- (1) F belongs to $H^1(B \rightarrow H; d\mu)$,
- (2) $DF(z)$ belongs to $L^2(H, H)$ for μ -a.e. z and
- (3) there exists a measurable function LF on B such that

$$L_P F(z) \rightarrow LF(z) , \quad P \in \mathcal{P}(H) , \quad \text{in probability with respect to } \mu .$$

We call LF the Ito-Ramer integral of F .

Remark 5.1 If F belongs to $\mathcal{D}(L)$, then DF is a strongly measurable map from B to $L^2(H, H)$.

The following is due to R.Ramer .

Theorem 5.1 Let F be an element of $H^1(B \rightarrow H; d\mu)$ and assume that $DF(z)$ belongs to $L^2(H, H)$ for μ -a.e. z and

$$\int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H, H)}^2 \} \mu(dz) < \infty .$$

Then F belongs to $\mathcal{D}(L)$ and

$$\int_B |LF(z)|^2 \mu(dz) \leq \int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H,H)}^2 \} \mu(dz) .$$

For the proof , see R.Ramer[13] .

Definition 5.2 We say that a measurable function w on B is a positive weight function , if

- (1) $w(z) > 0$ for each $z \in B$ and
- (2) $w(z+\cdot) : H \rightarrow \mathbb{R}$ is continuous for each $z \in B$.

Remark 5.2 Any positive weight function is strictly positive .

Theorem 5.2 Let F be an element of $H^1(B \rightarrow H; d\mu)$ and w be a positive weight function . Assume that $DF(z)$ belongs to $L^2(H,H)$ for μ -a.e. z and

$$\int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H,H)}^2 \} w(z) \mu(dz) < \infty .$$

Then F belongs to $\mathcal{D}(L)$. Furthermore there exists a positive measurable function k defined on B dependent only on w such that

$$\int_B |LF(z)|^2 k(z) \mu(dz) \leq \int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H,H)}^2 \} w(z) \mu(dz) .$$

Proof. Let $A_n = \{z \in B ; w(z+h) \geq \frac{1}{n} \text{ for any } h \in H \text{ such that } \|h\|_H \leq \frac{1}{n}\}$, $n = 1, 2, \dots$. The continuity of $w(z+\cdot) : H \rightarrow (0, \infty)$ assures the measurability of A_n . Since μ is a Radon measure on B , there exists a σ -compact subset G_n of B such that $G_n \subset A_n$ and $\mu(A_n \setminus G_n) = 0$.

Now let ϕ be a smooth function on \mathbb{R} such that $|\phi(t)| \leq 1$ and $|\phi'(t)| \leq 4$ for any $t \in \mathbb{R}$, $\phi(t) = 1$ for $|t| \leq \frac{1}{3}$, and $\phi(t) = 0$ for $|t| \geq \frac{2}{3}$. Let $\psi_n(z) = \phi(n\rho(z; G_n))$, then ψ_n belongs to $H^1(B \rightarrow \mathbb{R}; d\mu)$ and $\|D\psi_n(z)\|_{L^\infty(H, \mathbb{R})} \leq 4n$ for μ -a.e. z by virtue of Corollary to Theorem 3.2. Let $F_n(z) = \psi_n(z)F(z)$ furthermore. Obviously F_n belongs to $H^1(B \rightarrow H; d\mu)$ and it is easy to see that $DF_n(z)h = (D\psi_n(z)h)F(z) + \psi_n(z)DF(z)h$ for each $h \in H$ and μ -a.e. z . Since $D\psi_n(z) = 0$ and $\psi_n(z) = 1$ for μ -a.e. $z \in G_n$, we get

(5.1) $F_n(z) = F(z)$ and $DF_n(z) = DF(z)$ for μ -a.e. $z \in G_n$.

It is easily seen that $\|F_n(z)\|_H \leq \|F(z)\|_H$ and

$$\begin{aligned} \|DF_n(z)\|_{L^2(H, H)}^2 &\leq \|F(z)\|_H^2 \|D\psi_n(z)\|_{L^\infty(H, \mathbb{R})}^2 + \|DF(z)\|_{L^2(H, H)}^2 \\ &\leq 4n^2 \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H, H)}^2. \end{aligned}$$

On the other hand, $\psi_n(z) = 0$ and $D\psi_n(z) = 0$ for μ -a.e. $z \in B$ satisfying $\rho(z; G_n) > \frac{2}{3n}$. Therefore we get $F_n(z) = 0$ and $DF_n(z) = 0$ for μ -a.e. $z \in B$ satisfying $w(z) < \frac{1}{n}$.

Hence we obtain

$$\begin{aligned} &\int_B \{ \|F_n(z)\|_H^2 + \|DF_n(z)\|_{L^2(H, H)}^2 \} \mu(dz) \\ &\leq n \int_B \{ \|F_n(z)\|_H^2 + \|DF_n(z)\|_{L^2(H, H)}^2 \} w(z) \mu(dz) \\ &\leq 33n^2 \int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H, H)}^2 \} w(z) \mu(dz) < \infty, \end{aligned}$$

which implies $F_n \in \mathcal{D}(L)$ in view of Theorem 5.1. Moreover let

$$k_n(z) = \begin{cases} \frac{1}{2^n} \frac{1}{33n^2} & \text{if } z \in G_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} & \int_B |LF_n(z)|^2 k_n(z) \mu(dz) \\ & \leq \frac{1}{2^n} \int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H,H)}^2 \} w(z) \mu(dz) . \end{aligned}$$

However, (5.1) shows that

$$(5.2) \quad \chi_{G_n}(z) L_P F(z) = \chi_{G_n}(z) L_P F_n(z) \quad \text{for } \mu\text{-a.e. } z \in B \text{ and any}$$

$P \in \mathcal{P}(H)$. We define a measurable function LF on B by

$$LF(z) = \begin{cases} LF_1(z) & \text{if } z \in G_1 , \\ LF_{n+1}(z) & \text{if } z \in G_{n+1} \setminus G_n , \quad n=1,2,\dots , \\ 0 & \text{otherwise} . \end{cases}$$

Noticing that $\mu(\bigcup_n G_n) = 1$, we see that $L_P F(z)$ converges to

$LF(z)$, $P \in \mathcal{P}(H)$, in probability with respect to μ . So F belongs to $\mathcal{D}(L)$. Furthermore let $k(z) = \sum_{n=1}^{\infty} k_n(z)$, then $k(z) > 0$ for

$\mu\text{-a.e. } z$, and it is easy to see that

$$\begin{aligned} & \int_B |LF(z)|^2 k(z) \mu(dz) \\ & \leq \int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H,H)}^2 \} w(z) \mu(dz) . \end{aligned}$$

This completes the proof.

Definition 5.3 We say that a measurable map $F : B \rightarrow H$ is an H - C^1 map, if

(1) for each $z \in B$, there exists a Hilbert Schmidt operator $DF(z) : H \rightarrow H$ such that

$$\|F(z+h) - F(z) - DF(z)h\|_H = o(\|h\|_H) \text{ as } \|h\|_H \rightarrow 0, \text{ and}$$

(2) for any $z \in B$, $DF(z+\cdot) : H \rightarrow L^2(H, H)$ is continuous.

Corollary to Theorem 4.2 Any H - C^1 map belongs to $\mathcal{D}(L)$.

Proof. For any H - C^1 map $F : B \rightarrow H$, let

$$w(z) = \{1 + \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H, H)}^2\}^{-1}. \text{ Then it is}$$

easy to see that w is a positive weight function and

$\int_B \{\|F(z)\|_H^2 + \|DF(z)\|_{L^2(H, H)}^2\} w(z) \mu(dz) < \infty$. So our assertion has been proved.

6. Nonlinear transformation of μ and its absolute continuity

Definition 6.1 For each element F of $\mathcal{D}(L)$, we define

$$d(z; F) = \delta(I_H - DF(z)) \exp\left\{ LF(z) - \frac{1}{2} \|F(z)\|_H^2 \right\}.$$

Here $\delta(A)$ denotes the Carleman-Fredholm determinant of an operator $A: H \rightarrow H$. It is well known that $\delta(I_H - \cdot): L^2(H, H) \rightarrow \mathbb{R}$ is continuous and $\delta(I_H - K) = \det(I_H - K) \exp(\text{trace } K)$ for any nuclear operator $K: H \rightarrow H$. It is also well known that

$\delta((I_H - K_1)(I_H - K_2)) = \delta(I_H - K_1) \delta(I_H - K_2) \exp(-\text{trace } K_1 K_2)$ holds for any $K_1, K_2 \in L^2(H, H)$.

Theorem 6.1 Let $F: B \rightarrow H$ be a measurable map belonging to $\mathcal{D}(L)$. Assume that there exists a constant c , $0 < c < 1$, such that $\|F(z+h) - F(z)\|_H \leq c \|h\|_H$ for each $z \in B$ and $h \in H$. Then (1) $I_B - F: B \rightarrow B$ is bijective, (2) the image measure $(I_B - F)\mu$ induced by μ through $I_B - F$ is absolutely continuous relative to μ , and (3) $\int_B f(z) \mu(dz) \geq \int_B f((I_B - F)z) |d(z; F)| \mu(dz)$ for any bounded positive measurable function f on B .

Proof. Step 1. Let $F = \{G: B \rightarrow H; G \text{ is measurable and it satisfies that } \|G(z+h) - G(z)\|_H \leq c \|h\|_H \text{ for any } z \in B \text{ and } h \in H \text{ and for any } G \in F\}$, we put inductively

$$\begin{cases} u_0(z; G) = 0, \\ u_{n+1}(z; G) = G(z + u_n(z; G)) \text{ for each } z \in B \text{ and } n = 1, 2, \dots \end{cases}$$

It is easy to see that

$$\begin{aligned}\|u_{n+1}(z;G) - u_n(z;G)\|_H &\leq c \|u_n(z;G) - u_{n-1}(z;G)\|_H \\ &\leq c^n \|G(z)\|_H.\end{aligned}$$

So there exists $u_\infty(z;G) = \lim_{n \rightarrow \infty} u_n(z;G)$ in H . We also get

$$(6.1) \quad \|u_\infty(z;G)\|_H \leq \frac{1}{1-c} \|G(z)\|_H \quad \text{and}$$

$$(6.2) \quad \|u_\infty(z;G) - u_n(z;G)\|_H \leq \frac{c^n}{1-c} \|G(z)\|_H.$$

Obviously $u_\infty(z;G) = G(z + u_\infty(z;G))$, which implies

$$(6.3) \quad (I_B - G)(z + u_\infty(z;G)) = z.$$

This shows that $I_B - G : B \rightarrow B$ is surjective.

On the other hand, suppose that $(I_B - G)z_1 = (I_B - G)z_2$ for some $z_1, z_2 \in B$. Then $z_1 - z_2 = G(z_1) - G(z_2) \in H$ and

$$\|z_1 - z_2\|_H = \|G(z_2 + z_1 - z_2) - G(z_2)\|_H \leq c \|z_1 - z_2\|_H.$$

This proves $z_1 = z_2$, and so $I_B - G : B \rightarrow B$ is injective.

Hence we have proved that

$$(6.4) \quad I_B - G : B \rightarrow B \text{ is bijective for each } G \in F, \text{ and}$$

$$(6.5) \quad (I_B - G)^{-1}z = z + u_\infty(z;G).$$

This proves (1) in our assertion.

Step 2. Take $P_n \in \mathcal{P}(B^*)$, $n=1,2,\dots$, such that $P_n \uparrow I_H$ strongly and $L_{P_n} F(z) \rightarrow LF(z)$ for μ -a.e. z , and let $F_n(z) = P_n F(z)$.

It is obvious that $F_n \in F$.

Since $u_0(z; F) = u_0(z; F_n) = 0$ and

$$u_{m+1}(z; F) - u_{m+1}(z; F_n) = F(z + u_m(z; F)) - F_n(z + u_m(z; F_n)) ,$$

we see inductively that $u_m(z; F_n) \rightarrow u_m(z; F)$ in H , $n \rightarrow \infty$, for each m . By virtue of (6.2), we get

$$\|u_\infty(z; F) - u_\infty(z; F_n)\|_H \leq \frac{2c^m}{1-c} \|F(z)\|_H + \|u_m(z; F) - u_m(z; F_n)\|_H .$$

This proves that $u_\infty(z; F_n) \rightarrow u_\infty(z; F)$ in H , $n \rightarrow \infty$. Therefore by (6.5) we obtain

$$(6.6) \quad (I_B - F_n)^{-1}z \rightarrow (I_B - F)^{-1}z \text{ in } B, n \rightarrow \infty, \text{ for each } z \in B .$$

Step 3. We will prove the following .

Proposition 6.1 For any continuous bounded function g on B ,

$$\begin{aligned} & \int_B g((I_B - F_n)^{-1}z) \mu(dz) \\ &= \int_B g(z) |\delta(I_H - P_n D F(z))| \exp\{L_{P_n} F(z) - \frac{1}{2} \|P_n F(z)\|_H^2\} \mu(dz) . \end{aligned}$$

Let $\{e_1, \dots, e_n\}$ be an orthogonal base of $E = P_n H$, and let

$F_n(z_1, z_2) = F_n(z_1 + z_2)$ for each $z_1 \in E$ and $z_2 \in B$. Then

$F_n(\cdot, \cdot) : E \times B \rightarrow E$ is measurable. It is easy to see that

(6.7) $I_E - F_n(\cdot, z_2) : E \rightarrow E$ is bijective and Lipschitz continuous for each $z_2 \in B$, and

$$(6.8) \quad (I_E - F_n(\cdot, z_2))^{-1}z_1 = (I_B - F_n)^{-1}(z_1 + z_2) - z_2 \text{ for each } z_1 \in E \text{ and } z_2 \in B .$$

Therefore we obtain

$$\begin{aligned}
 & \int_B g((I_B - F_n)^{-1} z) \mu(dz) \\
 &= \int_{B \times \mathbb{R}^m} g((I_B - F_n)^{-1}(\sum_{j=1}^m x_j e_j + z)) \mu_{E^\perp}(dz) \otimes \\
 & \quad \left(\frac{1}{2\pi}\right)^{m/2} \exp\left\{-\frac{1}{2} \sum_{j=1}^m x_j^2\right\} dx_1 \dots dx_m \\
 &= \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int_B \mu_{E^\perp}(dz) \int_{\mathbb{R}^m} g((I_E - \tilde{F}_n(\cdot, z))^{-1}(\sum_{j=1}^m x_j e_j) + z) \\
 & \quad \exp\left\{-\frac{1}{2} \sum_{j=1}^m x_j^2\right\} dx_1 \dots dx_m .
 \end{aligned}$$

Since $I_E - \tilde{F}_n(\cdot, z_2) : E \rightarrow E$ is homeomorphic and Lipschitz continuous, by virtue of R. Radmacher [12]

$$\begin{aligned}
 & \int_{\mathbb{R}^m} g((I_E - F_n(\cdot, z))^{-1}(\sum_{j=1}^m x_j e_j)) \exp\left\{-\frac{1}{2} \sum_{j=1}^m x_j^2\right\} dx_1 \dots dx_m \\
 &= \int_{\mathbb{R}^m} g(\sum_{j=1}^m x_j e_j + z) \left| \det\left(\delta_{i,j} - \frac{\partial f_i}{\partial x_j}(x; z)\right)_{i,j=1,\dots,m} \right| \\
 & \quad \exp\left\{-\frac{1}{2} \sum_{j=1}^m (x_j - f_j(x; z))^2\right\} dx_1 \dots dx_m
 \end{aligned}$$

where $\delta_{i,j}$ is Kronecker's delta and $f_i(x; z) = (e_i, F_n(\sum_{j=1}^m x_j e_j + z))_H$

According to Proposition 3.2, we get

$$\frac{\partial f_i}{\partial x_j}(x; z) = (e_i, DF_n(\sum_{k=1}^m x_k e_k + z) e_j)_H \quad \text{for a.e. } (x, z) \text{ with}$$

respect to $dx_1 \dots dx_m \otimes \mu_E(dz)$.

So we obtain

$$\begin{aligned}
& \left| \det \left(\delta_{i,j} - \frac{\partial f_i}{\partial x_j} \right) \right| \exp \left\{ -\frac{1}{2} \sum_{j=1}^m (x_j - f_j(x; z))^2 \right\} \\
&= \left| \det \left(I_H - P_n DF \left(\sum_{j=1}^m x_j e_j + z \right) \right) \right| \exp \left(-\frac{1}{2} \sum_{j=1}^m x_j^2 \right) \\
&\quad \times \exp \left\{ \left(P_n F \left(\sum_{j=1}^m x_j e_j + z \right), P_n \left(\sum_{j=1}^m x_j e_j \right) \right)_H \right. \\
&\quad \left. - \frac{1}{2} \left\| P_n F \left(\sum_{j=1}^m x_j e_j + z \right) \right\|_H^2 \right\} \\
&= \left| \delta \left(I_H - P_n DF \left(\sum_{j=1}^m x_j e_j + z \right) \right) \right| \exp \left(-\frac{1}{2} \sum_{j=1}^m x_j^2 \right) \\
&\quad \times \exp \left\{ L_{P_n} F \left(\sum_{j=1}^m x_j e_j + z \right) - \frac{1}{2} \left\| P_n F \left(\sum_{j=1}^m x_j e_j + z \right) \right\|_H^2 \right\}
\end{aligned}$$

for a.e. (x, z) with respect to $dx_1 \dots dx_m \times \mu_{E^1}(dz)$.

This proves our proposition .

Step 4. Let g be a positive bounded continuous function on B . Then we get

$$\begin{aligned}
& \int_B g \left((I_B - F)^{-1} z \right) \mu(dz) \\
&= \lim_{n \rightarrow \infty} \int_B g \left((I_B - F_n)^{-1} z \right) \mu(dz) \\
&= \lim_{n \rightarrow \infty} \int_B g(z) \left| \delta \left(I_H - P_n DF(z) \right) \right| \exp \left\{ L_{P_n} F(z) - \frac{1}{2} \left\| P_n F(z) \right\|_H^2 \right\} \mu(dz) \\
&\geq \int_B g(z) |d(z; F)| \mu(dz)
\end{aligned}$$

by (6.6) and Proposition 6.1 .

So for any positive bounded measurable function f on B we get

$$\int_B f((I_B - F)^{-1}z) \mu(dz) \geq \int_B f(z) |d(z; F)| \mu(dz) .$$

Replacing $f(z)$ by $f((I_B - F)z)$, we obtain

$$(6.9) \quad \int_B f(z) \mu(dz) \geq \int_B f((I_B - F)z) |d(z; F)| \mu(dz) \quad \text{for any positive measurable function } f \text{ on } B .$$

This proves (3) in our assertion .

In view of Theorem 4.2 , we get $\|DF(z)\|_{L^\infty(H, H)} \leq c$ for μ -a.e. z , and so $\delta(I_H - DF(z)) > 0$ for μ -a.e. z . Hence $|d(z; F)| > 0$ for μ -a.e. z . Now (2) in our assertion is an easy conclusion of (6.9). This completes the proof .

Lemma 6.1 Let F_1 and F_2 be elements of $\mathcal{D}(L)$ and F_3 be an element of $H^1(B \rightarrow H; d\mu)$. Assume furthermore that

- (1) $(I_B - F_2)\mu$ is absolutely continuous relative to μ ,
- (2) $(I_B - F_3)z = (I_B - F_1)(I_B - F_2)z$ for μ -a.e. z , and
- (3) $I_H - DF_3(z) = (I_H - DF_1((I_B - F_2)z))(I_H - DF_2(z))$ for μ -a.e. z .

Then F_3 belongs to $\mathcal{D}(L)$ and it satisfies that

$$d(z; F_3) = d((I_B - F_2)z; F_1) d(z; F_2) \quad \text{for } \mu\text{-a.e. } z .$$

Proof. Notice that $DF_1((I_B - F_2)z)$ and $d((I_B - F_2)z; F_1)$ are well-defined for μ -a.e. z , because $(I_B - F_2)\mu$ is absolutely continuous relative to μ .

According to our assumptions (2) and (3) ,

$$F_3(z) = F_1((I_B - F_2)z) + F_2(z) \quad \text{and}$$

$$DF_3(z) = DF_1((I_B - F_2)z) + DF_2(z) \in L^2(H, H) \quad \text{for } \mu\text{-a.e. } z.$$

$$\text{So } L_P F_3(z) = (F_3(z), \tilde{P}z)_H - \text{trace PDF}_3(z)$$

$$\begin{aligned} &= L_P F_1((I_B - F_2)z) + L_P F_2(z) + (F_1((I_B - F_2)z), PF_2(z))_H \\ &\quad + \text{trace PDF}_1((I_B - F_2)z)DF_2(z) \end{aligned}$$

Since $L_P F_1(z) \rightarrow LF_1(z)$, $P \in \mathcal{P}(H)$, in probability with respect to μ and $(I_B - F_2)\mu$ is absolutely continuous relative to μ , $L_P F_1((I_B - F_2)z) \rightarrow LF_1((I_B - F_2)z)$, $P \in \mathcal{P}(H)$, in probability with respect to μ . Hence we obtain

$$\begin{aligned} L_P F_3(z) &\rightarrow LF_1((I_B - F_2)z) + LF_2(z) + (F_1((I_B - F_2)z), F_2(z))_H \\ &\quad + \text{trace DF}_1((I_B - F_2)z)DF_2(z), \end{aligned}$$

$P \in \mathcal{P}(H)$, in probability with respect to μ . This shows that F_3 belongs to $\mathcal{D}(L)$ and

$$\begin{aligned} LF_3(z) &= LF_1((I_B - F_2)z) + LF_2(z) + (F_1((I_B - F_2)z), F_2(z))_H \\ &\quad + \text{trace DF}_1((I_B - F_2)z)DF_2(z). \end{aligned}$$

Using this fact, we can obtain

$$d(z; F_3) = d((I_B - F_2)z; F_1) d(z; F_2) \quad \text{by easy calculation.}$$

Theorem 6.2 Let $F : B \rightarrow H$ be an H - C^1 map and D be a measurable subset of B , and assume that $I_H - DF(z) : H \rightarrow H$ is invertible for μ -a.e. $z \in D$. Then $(I_B - F)(\mu|_D)$ is absolutely continuous relative to μ , where $\mu|_D$ is the restricted measure of μ to D . In particular, if $I_B - F|_D : D \rightarrow B$ is injective, then it satisfies that

$$\int_{(I_B - F)D} f(z) \mu(dz) \geq \int_D F((I_B - F)z) |d(z; F)| \mu(dz)$$

for any positive bounded function f on B .

Proof. Step 1. let K_0 be a countable dense subset of $L^2(H, H)$, and let $\{P_n\}_{n=1}^\infty$ is an increasing subsequence of $P(B^*)$ strongly converging to I_H , and put

$$K = \{P_n K P_n; K \in K_0, n = 1, 2, \dots\} \cap \{K \in L^2(H, H); I_H - K \in GL(H)\},$$

where $GL(H)$ is a set of all invertible bounded operator from H onto H . Then it is obvious that K is dense in $L^2(H, H) \cap (I_H - GL(H))$ and every element K of K is extensible to a bounded linear operator \tilde{K} from B to B^* . Moreover let V be a countable subset of B^* such that V is dense in H .

For any $K \in K$ and $n = 1, 2, \dots$, we define a subset $A_K^{1,n}$ of B by

$$(6.10) \quad A_K^{1,n} = \left\{ z \in B; \quad \|DF(z+h) - DF(z)\|_{L^2(H, H)} \leq \frac{1}{24} \|(I_H - K)^{-1}\|_{L^\infty(H, H)}^{-1} \right. \\ \left. \text{for any } h \in H \text{ such that } \|h\|_H \leq \frac{1}{n} \right\}.$$

Since $DF(z+\cdot) : H \rightarrow L^2(H, H)$ is continuous for each $z \in B$, $A_K^{1,n}$ is a measurable subset of B .

We also define a measurable subset $A_K^{2,n}$ of B by

$$(6.11) \quad A_K^{2,n} = \{ z \in B; \quad \|F(z+h) - F(z) - DF(z)h\|_H \leq \frac{1}{24n} \| (I_H - K)^{-1} \|_{L^\infty(H,H)}^{-1} \\ \text{for any } h \in H \text{ with } \|h\|_H \leq \frac{1}{n} \} .$$

It is obvious that $A_K^{1,n} \uparrow B$ and $A_K^{2,n} \uparrow B$ as $n \rightarrow \infty$ for each $K \in K$.

Furthermore, for each $K \in K$, $v \in V$ and $n = 1, 2, \dots$, we define a measurable subset $\tilde{A}(n, K, v)$ of B by

$$(6.12) \quad \tilde{A}(n, K, v) = A_K^{1,n} \cap A_K^{2,n} \cap \{ z \in B; \\ \|DF(z) - K\|_{L^2(H,H)} \leq \frac{1}{24} \| (I_H - K)^{-1} \|_{L^\infty(H,H)}^{-1} \text{ and} \\ \|F(z) - Kz - v\|_H \leq \frac{1}{24n} \| (I_H - K)^{-1} \|_{L^\infty(H,H)}^{-1} \} .$$

It is easy to see that $D \subset \cup \{ \tilde{A}(n, K, v); K \in K, v \in V, n = 1, 2, \dots \}$.

By (6.10), (6.11) and (6.12), for any $z \in \tilde{A}(n, K, v)$ and $h \in H$ with

$$\|h\|_H \leq \frac{1}{n}, \text{ we get}$$

$$(6.13) \quad \|DF(z+h) - K\|_{L^2(H,H)} \\ \leq \|DF(z+h) - DF(z)\|_{L^2(H,H)} + \|DF(z) - K\|_{L^2(H,H)} \\ \leq \frac{1}{12} \| (I_H - K)^{-1} \|_{L^\infty(H,H)}^{-1} \text{ and}$$

$$\begin{aligned}
(6.14) \quad & \| F(z+h) - K(z+h) - v \|_H \\
& \leq \| F(z+h) - F(z) - DF(z)h \|_H + \| F(z) - Kz - v \|_H \\
& \quad + \| DF(z) - K \|_{L^\infty(H,H)} \| h \|_H \\
& \leq \frac{1}{8n} \| (I_H - K)^{-1} \|_{L^\infty(H,H)}^{-1}
\end{aligned}$$

Step 2. By virtue of Proposition 2.5, $I_B - \tilde{K} : B \rightarrow B$ is homeomorphism for each $K \in K$ and

$$(6.15) \quad (I_B - \tilde{K})^{-1} \mu(dz) = |d(z; K)| \mu(dz) .$$

For any $K \in K$ and $v \in V$, let

$$\begin{aligned}
\tilde{F}_{K,v}(z) &= F((I_B - \tilde{K})^{-1}z) - \tilde{K}(I_B - \tilde{K})^{-1}z - v \\
&= (F - \tilde{K} - N_v)(I_B - \tilde{K})^{-1}z, \quad \text{where } N_v z = v \text{ for each } z \in B.
\end{aligned}$$

It is obvious that $\tilde{F}_{K,v} : B \rightarrow H$ is a H - c^1 map and that

$$(6.16) \quad (I_B - F)(z) = (I_B - N_v)(I_B - \tilde{F}_{K,v})(I_B - \tilde{K})z$$

for each $z \in B$. It is also easy to see that

$$(6.17) \quad D\tilde{F}_{K,v}(z) = (DF((I_B - \tilde{K})^{-1}z) - K)(I_H - K)^{-1} \quad \text{for each } z \in B.$$

Assume that $z \in (I_B - \tilde{K})\tilde{A}(n, K, v)$ and $\|h\|_H \leq \frac{1}{n} \| (I_H - K)^{-1} \|_{L^\infty(H,H)}^{-1}$,

then in view of (6.13) and (6.14) we get

$$(6.18) \quad \| \tilde{F}_{K,v}(z+h) \|_H \leq \frac{1}{8n} \| (I_H - K)^{-1} \|_{L^\infty(H,H)}^{-1} \quad \text{and}$$

$$(6.19) \quad \| D\tilde{F}_{K,v}(z+h) \|_{L^2(H,H)} \leq \frac{1}{12}$$

Since $\tilde{A}(n, K, v)$ is measurable, there exists a σ -compact subset $A(n, K, v)$ of B such that $A(n, K, v) \subset \tilde{A}(n, K, v)$ and $\mu(\tilde{A}(n, K, v) \setminus A(n, K, v)) = 0$.

Then we obtain

$$(6.20) \quad \mu(D \setminus \cup \{A(n, K, v) ; K \in \mathcal{K}, v \in \mathcal{V}, n=1, 2, \dots\}) = 0.$$

Let ϕ be a smooth function on \mathbb{R} such that $\phi(t) = 1$ for $|t| \leq \frac{1}{3}$, $\phi(t) = 0$ for $|t| \geq \frac{2}{3}$ and $|\phi'(t)| \leq 4$ for any $t \in \mathbb{R}$, and let

$$\psi(z) = \psi_{n, K, v}(z) = \phi(n \| (I_H - K)^{-1} \|_{L^\infty(H, H)} \rho(z; (I_B - \tilde{K})A(n, K, v))).$$

Noticing that $\psi(z+th)\tilde{F}_{K, v}(z+th)$ is strictly absolutely continuous in t for any $z \in B$ and $h \in H$, by virtue of (6.16), (6.17) and (6.18) we get

$$\begin{aligned} (6.21) \quad & \| \psi(z+h)\tilde{F}_{K, v}(z+h) - \psi(z)\tilde{F}_{K, v}(z) \|_H \\ & \leq \int_0^1 \left\| \frac{d}{dt} \psi(z+th)\tilde{F}_{K, v}(z+th) + \psi(z+th)D\tilde{F}_{K, v}(z+th)h \right\|_H dt \\ & \leq \int_0^1 |\phi'(n \| (I_H - K)^{-1} \|_{L^\infty(H, H)} \rho(z; (I_B - \tilde{K})A(n, K, v)))| \\ & \quad \times \| \tilde{F}_{K, v}(z+th) \|_H dt + n \| (I_H - K)^{-1} \|_{L^\infty(H, H)} \| h \|_H \\ & \quad + \int_0^1 \psi(z+th) \| D\tilde{F}_{K, v}(z+th) \|_{L^2(H, H)} \| h \|_H dt \\ & \leq \frac{7}{12} \| h \|_H. \end{aligned}$$

According to Theorem 4.2, $\psi\tilde{F}_{K, v} : B \rightarrow H$ belongs to $H^1(B \rightarrow H; d\mu)$ and

$D(\psi\tilde{F}_{K, v})(z)h = (D\psi(z)h)\tilde{F}_{K, v}(z) + \psi(z)D\tilde{F}_{K, v}(z)h$ for any $h \in H$ and μ -a.e. z . So by virtue of (6.17) and (6.18) we obtain

$$\begin{aligned} \| D(\psi\tilde{F}_{K, v})(z) \|_{L^2(H, H)} & \leq \| D\psi(z) \|_{L^\infty(H, \mathbb{R})} \| \tilde{F}_{K, v}(z) \|_H \\ & \quad + \| \psi(z)D\tilde{F}_{K, v}(z) \|_{L^2(H, H)} \\ & \leq \frac{7}{12}, \quad \text{for } \mu\text{-a.e. } z. \end{aligned}$$

Hence in view of Theorem 5.1 $\psi \tilde{F}_{K,v}$ belongs to $\mathcal{D}(L)$. Thus $\psi \tilde{F}_{K,v}$ satisfies the assumption of Theorem 6.1, and accordingly

$$(6.22) \quad \int_B f(z) \mu(dz) \geq \int_B f((I_B - \psi \tilde{F}_{K,v})z) |d(z; \psi \tilde{F}_{K,v})| \mu(dz)$$

for any positive measurable function f on B , and $(I_B - \psi \tilde{F}_{K,v})\mu$ is absolutely continuous relative to μ . So it follows from Proposition 2.4 and 2.5 that $(I_B - N_v)(I_B - \psi \tilde{F}_{K,v})(I_B - \tilde{K})\mu$ is absolutely continuous relative to μ .

For each Borel measurable set A of B , by virtue of (6.20) we get $\mu((I_B - F)^{-1}A \cap D) \leq \sum_{n,K,v} \mu((I_B - F)^{-1}A \cap A(n,K,v))$.

Since $(I_B - F)z = (I_B - N_v)(I_B - \psi \tilde{F}_{K,v})(I_B - \tilde{K})z$ for each $z \in A(n,K,v)$ by the definition of $\psi_{n,K,v}$ and $\tilde{F}_{K,v}$, $\mu((I_B - F)A \cap D) = 0$ provided that $\mu(A) = 0$. This shows that $(I_B - F)(\mu|_D)$ is absolutely continuous relative to μ .

Step 3. Let us define a measurable map $G_{n,K,v} : B \rightarrow H$ by $G_{n,K,v}z = z - (I_B - N_v)(I_B - \psi_{n,K,v} \tilde{F}_{K,v})(I_B - \tilde{K})z$ for each $z \in B$. Then Lemma 6.1 proves that $G_{n,K,v}$ belongs to $\mathcal{D}(L)$ and

$$(6.23) \quad d(z; G_{n,K,v}) = d((I_B - \psi \tilde{F}_{K,v})(I_B - \tilde{K})z; N_v) d((I_B - \tilde{K})z; \psi \tilde{F}_{K,v}) d(z; \tilde{K})$$

By the definition of $\psi_{n,K,v}$ and $\tilde{F}_{K,v}$, we also obtain

$G_{n,K,v}(z) = F(z)$ and $DG_{n,K,v}(z) = DF(z)$ for μ -a.e. $z \in A(n,K,v)$, and consequently we get

$$(6.24) \quad d(z; G_{n,K,v}) = d(z; F) \quad \text{for } \mu\text{-a.e. } z \in A(n,K,v).$$

Proposition 2.4 and 2.5 shows that

$$\int_B g(z) \mu(dz) = \int_B g((I_B - N_V)z) |d(z; N_V)| \mu(dz) \quad \text{and}$$

$\int_B g(z) \mu(dz) = \int_B g((I_B - \tilde{K})z) |d(z; \tilde{K})| \mu(dz)$ for any positive function g on B . Using (6.22) and (6.23), we get

$$\begin{aligned} \int_B g(z) \mu(dz) &= \int_B g((I_B - N_V)z) |d(z; N_V)| \mu(dz) \\ &\geq \int_B g((I_B - N_V)(I_B - \tilde{F}_{K,V})z) |d((I_B - \tilde{F}_{K,V})z; N_V)| \\ &\quad \times |d(z; \tilde{F}_{K,V})| \mu(dz) \\ &\geq \int_B g((I_B - G_{n,K,V})z) |d(z; G_{n,K,V})| \mu(dz) \end{aligned}$$

for any positive measurable function g on B .

Assume that $I_B - F|_D : D \rightarrow B$ is injective. Then $(I_B - F)(A \cap D)$ is measurable for any measurable set A . By virtue of (6.20), there exists mutually disjoint sets $C(n, K, v)$, $K \in \tilde{K}$, $v \in V$, $n=1, 2, \dots$ such that $C(n, K, v) \subset A(n, K, v) \cap D$, $C(n, K, v)$'s are measurable and $\mu(D \setminus \bigcup_{n,K,v} C(n, K, v)) = 0$. Then according to (6.24), we get

$$\begin{aligned} &\int_{(I_B - F)D} f(z) \mu(dz) \\ &\geq \sum_{n,K,v} \int_B \chi_{(I_B - F)C(n,K,v)}(z) f(z) \mu(dz) \\ &\geq \sum_{n,K,v} \int_B \chi_{(I_B - F)C(n,K,v)}((I_B - G_{n,K,V})z) f((I_B - G_{n,K,V})z) \\ &\quad \times |d(z; G_{n,K,V})| \mu(dz) \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{n,K,v} \int_B \chi_{C(n,K,v)}(z) f((I_B - F)z) |d(z;F)| \mu(dz) \\
&\geq \int_D f((I_B - F)z) |d(z;F)| \mu(dz)
\end{aligned}$$

for any positive measurable function f on B .

This completes the proof.

Theorem 6.3 Let F and G be H - C^1 maps from B to H , and assume that $(I_B - F)(I_B - G)z = z$ and $(I_B - G)(I_B - F)z = z$ for μ -a.e. z . Assume furthermore that $I_H - DF(z) : H \rightarrow H$ and $I_H - DG(z) : H \rightarrow H$ are invertible for μ -a.e. z . Then $(I_B - F)\mu = |d(z;G)|\mu(dz)$ and $(I_B - G)\mu = |d(z;F)|\mu(dz)$.

Proof. Let $D = \{ z \in B ; (I_B - F)(I_B - G)z = (I_B - G)(I_B - F)z = z \}$. Then $\mu(D) = 1$, and it is easy to see that $I_B - F|_D$ and $I_B - G|_D$ are injective on D . By our assumption, we get

$$(6.25) \quad G((I_B - F)z) + F(z) = z - (I_B - G)(I_B - F)z = 0 \text{ for } \mu\text{-a.e. } z \in B.$$

Since the left hand of (6.25) is Frechet differentiable along H -direction and the right hand is stochastic H -Gateux differentiable we obtain $DG((I_B - F)z)(I_H - DF(z)) + DF(z) = 0$, which implies

$I_H = (I_H - DG((I_B - F)z))(I_H - DF(z))$ for μ -a.e. $z \in B$. By virtue of Lemma 6.1, we get $1 = d(z;0) = d((I_B - F)z;G)d(z;F)$ for μ -a.e. z . Similarly we get $d((I_B - G)z;F)d(z;G) = 1$ for μ -a.e. z .

In view of Theorem 6.2, for any positive measurable function f on B , we obtain

$$\begin{aligned}
& \int_B f(z) \mu(dz) \\
&= \int_{(I_B - F)D} f(z) \mu(dz) \\
&\geq \int_D f((I_B - F)z) |d(z; F)| \mu(dz) \\
&= \int_B f((I_B - F)z) |d(z; F)| \mu(dz) \\
&\geq \int_B f((I_B - F)(I_B - G)z) |d((I_B - G)z; F)| |d(z; G)| \mu(dz) \\
&= \int_B f(z) \mu(dz)
\end{aligned}$$

So $\int_B f(z) \mu(dz) = \int_B f((I_B - F)z) |d(z; F)| \mu(dz)$. Replacing $f(z)$ by $f((I_B - G)z)$, we get

$\int_B f((I_B - G)z) \mu(dz) = \int_B f(z) |d(z; F)| \mu(dz)$. This proves our assertion .

Theorem 6.4 Let F be an $H-C^1$ map from B to H , and assume that $I_B - F : B \rightarrow B$ is bijective and $I_H - DF(z) : H \rightarrow H$ is invertible for each $z \in B$. Then $(I_B - F)^{-1} \mu(dz) = |d(z; F)| \mu(dz)$.

Proof. Let $G(z) = z - (I_B - F)^{-1}z = -F((I_B - F)^{-1}z)$ for each $z \in B$. Then the implicit functional theorem (see J.Schwartz [14] for an example) assures us that $G : B \rightarrow H$ is an $H-C^1$ map and $I_H - DG(z) = (I_H - DF((I_B - F)^{-1}z))^{-1}$ for each $z \in B$. So our assertion is an easy consequence of Theorem 6.3 .

7. Some more results about nonlinear transformation of μ and its absolute continuity

Definition 7.1 We say that a measurable map $F: B \rightarrow H$ is regular, if

- (1) F belongs to $H^1(B \rightarrow H; d\mu)$,
- (2) $I_H - DF(z) : H \rightarrow H$ is invertible and $DF(z) : H \rightarrow H$ is a Hilbert-Schmidt operator for μ -a.e. z , and
- (3) There exist a weight function w , a measurable set D_0 of μ -measure one and a sequence $\{F_n\}_{n=1}^\infty$ of H - C^1 maps, such that
 - (i)
$$\int_B \{ \|F(z)\|_H^2 + \|DF(z)\|_{L^2(H,H)}^2 \} w(z) \mu(dz) < \infty,$$
 - (ii)
$$\int_B \{ \|F(z) - F_n(z)\|_H^2 + \|DF(z) - DF_n(z)\|_{L^2(H,H)}^2 \} w(z) \mu(dz) \rightarrow 0 \text{ as } n \rightarrow \infty,$$
 - (iii) $I_H - DF_n(z) : H \rightarrow H$ is invertible for μ -a.e. z , and
 - (iv) $I_B - F_n|_{D_0} : D_0 \rightarrow B$ is injective.

Remark 7.1 If $F : B \rightarrow H$ is regular, then F belongs to $\mathcal{D}(L)$ by virtue of Theorem 5.2.

Theorem 7.1 Suppose that $F : B \rightarrow H$ is regular. Then $(I_B - F)\mu$ is absolutely continuous relative to μ , and

$$\int_B f(z) \mu(dz) \geq \int_B f((I_B - F)z) |d(z; F)| \mu(dz) \quad \text{for any positive bounded measurable function } f \text{ on } B.$$

Proof. Let F_n 's be $H-C^1$ maps and D_0 be a measurable set as in Definition 7.1. Then in view of Theorem 5.2, we get $LF_n(z) \rightarrow LF(z)$ in probability with respect to $\mu(dz)$.

According to Theorem 6.2, we obtain

$$\begin{aligned} \int_B f(z) \mu(dz) &\geq \int_{(I_B - F_n)D_0} f(z) \mu(dz) \\ &\geq \int_{D_0} f((I_B - F_n)z) |d(z; F_n)| \mu(dz) \\ &\geq \int_B f((I_B - F_n)z) |d(z; F_n)| \mu(dz). \end{aligned}$$

So using Fatou's lemma, we have

$\int_B f(z) \mu(dz) \geq \int_B f((I_B - F)z) |d(z; F)| \mu(dz)$ for any positive bounded measurable function f on B . By the similar argument to the proof of Theorem 6.1 we can easily see that $(I_B - F)\mu$ is absolutely continuous relative to μ . This completes the proof.

Theorem 7.2 Suppose that $F: B \rightarrow H$ and $G: B \rightarrow H$ are regular and that $(I_B - F)(I_B - G)z = z$ and $(I_B - G)(I_B - F)z = z$ for μ -a.e. z . Suppose furthermore that $d(\cdot; F): B \rightarrow \mathbb{R}$ and $d(\cdot; G): B \rightarrow \mathbb{R}$ are strictly positive functions. Then $(I_B - F)\mu(dz) = |d(z; G)| \mu(dz)$ and $(I_B - G)\mu(dz) = |d(z; F)| \mu(dz)$.

Proof. If we have proved that

$$(7.1) \quad I_H = (I_H - DF((I_B - G)z))(I_H - DG(z)) \quad \text{for } \mu\text{-a.e. } z \quad \text{and}$$

$$(7.2) \quad I_H = (I_H - DG((I_B - F)z))(I_H - DF(z)) \quad \text{for } \mu\text{-a.e. } z,$$

then our theorem will be proved by the similar argument to the proof of Theorem 6.3. (7.1) and (7.2) are similar, so we will prove (7.1).

According to Definition 7.1 , there exist a sequence $\{F_n\}_{n=1}$ of H - C^1 maps and a weight function $w: B \rightarrow \mathbb{R}$ such that

$$\int_B \{ \|F(z) - F_n(z)\|_H^2 + \|DF(z) - DF_n(z)\|_{L^2(H,H)}^2 \} w(z) \mu(dz) \rightarrow 0$$

as $n \rightarrow \infty$. On the other hand there exists a weight function $v: B \rightarrow \mathbb{R}$ such that

$$\int_B \{ \|G(z)\|_H^2 + \|DG(z)\|_{L^2(H,H)}^2 \} v(z) \mu(dz) < \infty . \quad \text{Replacing}$$

$v(z)$ by $\min\{v(z), 1\}$ if necessary, we may assume that $v(z) \leq 1$ for each $z \in B$.

Since $(I_B - F)(I_B - G)z = z$ for μ -a.e. z , we get

(7.3) $G(z) = -F((I_B - G)z)$ for μ -a.e. z . By virtue of Theorem 7.1 $(I_B - G)\mu$ is absolutely continuous relative to μ , which shows that

(7.4) $F_n((I_B - G)z) \rightarrow F((I_B - G)z)$ in probability with respect to μ .

Take an arbitrary element h of H and fix it for a moment .

Since $G: B \rightarrow H$ is R.A.C. , there exists a measurable map $\tilde{G}_h: B \rightarrow H$ such that $\tilde{G}_h(z) = G(z)$ for μ -a.e. z and $\tilde{G}_h(z+th)$ is strictly absolutely continuous in t for each $z \in B$. Then we get

$$F_n((I_B - \tilde{G}_h)z) = F_n((I_B - G)z) \quad \text{for } \mu\text{-a.e. } z \quad \text{and}$$

$$F_n((I_B - \tilde{G}_h)(z+th)) = F_n(z+th - \tilde{G}_h(z+th)) \quad \text{for each } z \in B \text{ and } t \in \mathbb{R} .$$

According to the assumption of $F_n, F_n(z+\cdot): H \rightarrow H$ is continuously Fréchet differentiable , which shows that $F_n((I_B - \tilde{G}_h)(z+th))$ is strictly absolutely continuous in t for each $z \in B$.

So $F_n((I_B - G)(\cdot)) : B \rightarrow H$ is R.A.C. . By the similar argument we can easily see that $w((I_B - G)(\cdot)) : B \rightarrow \mathbb{R}$ is strictly positive . On the other hand, we get

$$\begin{aligned} \frac{d}{dt} F_n((I_B - \tilde{G}_h)(z + th)) &= DF_n((I_B - \tilde{G}_h)(z + th)) \frac{d}{dt} \{z + th - \tilde{G}_h(z + th)\} \\ &= DF_n((I_B - G)(z + th)) (I_H - DG(z + th)) h \end{aligned}$$

for a.e. $(z, t) \in B \times \mathbb{R}$ with respect to $\mu(dz) \times dt$. This shows that $F_n((I_B - G)(\cdot)) : B \rightarrow H$ is S.G.D. and

$$(7.5) \quad D(F_n((I_B - G)(\cdot)))(z) = DF_n((I_B - G)z)(I_H - DG(z)) \in L^2(H, H)$$

for μ -a.e. z .

Let $\rho(z) = \{|d(z; G)| w((I_B - G)z) v(z)\}^{1/2}$ for each $z \in B$.

It is easy to see that $\rho(z)$ is strictly positive . Using Theorem 7.1 , we get

$$\begin{aligned} (7.6) \quad & \left\{ \int_B \|D(F_n((I_B - G)(\cdot)))(z) - DF((I_B - G)z)(I_H - DG(z))\|_{L^2(H, H)}^2 \rho(z) \mu(dz) \right\}^2 \\ & \leq \left\{ \int_B \|DF_n((I_B - G)z) - DF((I_B - G)z)\|_{L^2(H, H)}^2 \right. \\ & \quad \times (1 + \|DG(z)\|_{L^2(H, H)})^2 \rho(z) \mu(dz) \left. \right\}^2 \\ & \leq \int_B \|DF_n((I_B - G)z) - DF((I_B - G)z)\|_{L^2(H, H)}^2 w((I_B - G)z) |d(z; G)| \mu(dz) \\ & \quad \times \int_B (1 + \|DG(z)\|_{L^2(H, H)})^2 v(z) \mu(dz) \\ & \leq \int_B \|DF_n(z) - DF(z)\|_{L^2(H, H)}^2 w(z) \mu(dz) \\ & \quad \times \left\{ 2 + 2 \int_B \|DG(z)\|_{L^2(H, H)}^2 v(z) \mu(dz) \right\} \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty . \end{aligned}$$

By virtue of Theorem 3.1, (7.4) and (7.6) , we can see that

$F((I_B - G)(\cdot)) : B \rightarrow H$ belongs to $H^1(B \rightarrow H; d\mu)$ and

$$(7.7) \quad D(F((I_B - G)(\cdot)))(z) = DF((I_B - G)z)(I_H - DG(z)) \quad \text{for } \mu\text{-a.e. } z .$$

This and (7.3) prove that $DG(z) = -DF((I_B - G)z)(I_H - DG(z))$

for $\mu\text{-a.e. } z$, which implies (7.1) . This completes the proof .

8. The Sard type theorem and its application

We prove the following Sard type theorem .

Theorem 8.1 Let $F : B \rightarrow H$ be an H - C^1 map , and let A be a measurable subset of B defined by

$$A = \{ z \in B ; I_H - DF(z) : H \rightarrow H \text{ is invertible} \}$$

$$= \{ z \in B ; \delta(I_H - DF(z)) \neq 0 \} \quad . \quad \text{Assume that } \nu \text{ is}$$

a probability measure on B such that the restricted measures $\nu|_A$ and $\mu|_A$ on A are mutually singular . Then $(I_B - F)\nu$ and μ are mutually singular .

Proof. Step 1. We prove the following.

Proposition 8.1 There exists a σ -compact subset Ω_0 of B such that $\mu(\Omega_0) = \nu(\Omega_0) = 1$ and $(I_B - F)(\Omega_0 \cap G)$ is σ -compact for any σ -compact subset G of B .

Proof of Proposition 8.1 By virtue of Lusin's theorem , there exists a compact subset K_n for each $n = 1, 2, \dots$ such that $F|_{K_n} : K_n \rightarrow H$ is continuous , $\mu(K_n) > 1 - \frac{1}{n}$ and $\nu(K_n) > 1 - \frac{1}{n}$. Then $(I_B - F)(K_n \cap K)$ is compact for any compact subset K of B . Let $\Omega_0 = \bigcup_n K_n$, then $\mu(\Omega_0) = \nu(\Omega_0) = 1$ and $(I_B - F)(\Omega_0 \cap K)$ is σ -compact for each compact subset K of B . This proves our proposition .

Step 2. Let ε be an arbitrary positive number and fix it throughout the proof . Then $\mu(\Omega_m) > 1 - \frac{\varepsilon}{2}$ and $\nu(\Omega_m) > 1 - \frac{\varepsilon}{2}$ for some integer m , where

$\Omega_m = \{ z \in B ; \quad \| DF(z+h) - DF(z) \|_{L^2(H,H)} \leq \frac{1}{120} \text{ for any } h \in H$
such that $\| h \|_H \leq \frac{8}{m} \}$. Notice that Ω_m is measurable because
 F is an H - C^1 map.

Take a subsequence $\{P_n\}_{n=1}^\infty$ of $\mathcal{P}(B^*)$ such that $P_n \uparrow I_H$ strongly,
 $n \rightarrow \infty$. Then $\mu(\Omega_{n,m}) > 1-\varepsilon$ and $\nu(\Omega_{n,m}) > 1-\varepsilon$ for some integer n ,
where $\Omega_{n,m} = \Omega_m \cap \{ z \in B ; \quad \| (I_H - P_n)F(z) \|_H \leq \frac{1}{120m} \}$ and

$\{ \| (I_H - P_n)DF(z) \|_{L^2(H,H)} \leq \frac{1}{120} \}$. Let Ω be a σ -compact subset
of B such that

$$(8.1) \quad \Omega \subset \Omega_{n,m} \cap \Omega_0, \quad \mu(\Omega) > 1-\varepsilon \text{ and } \nu(\Omega) > 1-\varepsilon.$$

It is easy to see that

$$(8.2) \quad \| (I_H - P_n)DF(z+h) \|_{L^2(H,H)} \leq \frac{1}{60} \quad \text{for any } h \in H, \quad \| h \|_H \leq \frac{8}{m}$$

and $z \in \Omega$. We also obtain

$$(8.3) \quad \| (I_H - P_n)F(z+h) \|_H \leq \frac{17}{120m} \quad \text{for any } h \in H, \quad \| h \|_H \leq \frac{8}{m} \text{ and } z \in \Omega,$$

because $\| (I_H - P_n)F(z+h) \|_H$

$$\leq \| (I_H - P_n)F(z) \|_H + \int_0^1 \| DF(z+th) \|_{L^2(H,H)} \| h \|_H dt.$$

Take a smooth function ϕ on \mathbb{R} such that $0 \leq \phi(t) \leq 1$ and
 $|\phi'(t)| \leq 2$ for $t \in \mathbb{R}$, $\phi(t) = 1$ for $|t| \leq 6$ and $\phi(t) = 0$ for
 $|t| \geq 7$, and let $g(z) = \phi(m\rho(z;\Omega))$ and $G(z) = g(z) \cdot (I_H - P_n)F(z)$.

It is easy to see that G belongs to $H^1(B \rightarrow H; d\mu)$ and

$$(8.4) \quad DG(z)h = (Dg(z)h) \cdot (I_H - P_n)F(z) + g(z) \cdot (I_H - P_n)DF(z)h$$

for μ -a.e. z and each $h \in H$.

Hence we get by (8.1) and (8.2)

$$\begin{aligned}
 (8.5) \quad \|DG(z)\|_{L^2(H,H)} &\leq \|DG(z)\|_{L^\infty(H,\mathbb{R})} \cdot \|(I_H - P_n)F(z)\|_H \\
 &\quad + \|(I_H - P_n)DF(z)\|_{L^2(H,H)} \\
 &\leq \frac{3}{10} \quad \text{for } \mu\text{-a.e. } z, \text{ and}
 \end{aligned}$$

$$(8.6) \quad \|G(z)\|_H \leq \frac{17}{120m} < \frac{1}{6m} \quad \text{for each } z \in B.$$

In view of Theorem 5.1, (8.5) and (8.6) imply that G belongs to $\mathcal{D}(L)$. We can also see that $I_B - G: B \rightarrow B$ is bijective by virtue of Theorem 6.1.

Let $\tilde{E} = (I_B - G)\Omega$, then \tilde{E} is measurable because $I_B - G: B \rightarrow B$ is injective. Suppose that $\rho(x; \tilde{E}) \leq \frac{3}{m}$ for some $x \in B$. Then there exists some $z \in \Omega$ such that $x - (I_B - G)z \in H$ and $\|x - (I_B - G)z\|_H < \frac{17}{5m}$. According to the proof of Theorem 6.1 and (8.5), we obtain

$$\|(I_B - G)^{-1}x - z\|_H \leq \frac{1}{1 - \frac{3}{10}} \|x - (I_B - G)z\|_H < \frac{5}{m},$$

which shows that $\rho((I_B - G)^{-1}x, \Omega) < \frac{5}{m}$. Thus we get

$$(8.7) \quad (I_B - (I_H - P_n)F)(I_B - G)^{-1}x = x \quad \text{for any } x \in B \text{ such that}$$

$$\rho(x; \tilde{E}) \leq \frac{3}{m}.$$

Let E be a σ -compact subset of B such that $E \subset \tilde{E}$ and $\mu(\tilde{E} \setminus E) = (I_B - G)\nu(\tilde{E} \setminus E) = 0$ and take a smooth function ψ on \mathbb{R}

such that $0 \leq \psi(t) \leq 1$ and $|\psi'(t)| \leq 2$ for $t \in \mathbb{R}$, $\psi(t) = 1$

for $|t| \leq 1$, and $\psi(t) = 0$ for $|t| \geq 2$. Let $k(x) = \psi(m\rho(x;E))$

and $K(x) = k(x)\{x - (I_B - G)^{-1}x\} = -k(x)G((I_B - G)^{-1}x)$ for each $x \in B$.

Since $F: B \rightarrow H$ is an H - C^1 map, it is easy to see that K belongs to $H^1(B; H; d\mu)$ by virtue of (8.7), and we obtain

$$\begin{aligned} DK(x)h &= (Dk(x)h)\{x - (I_B - G)^{-1}x\} \\ &\quad + k(x)[I_H - \{I_H - (I_H - P_n)DF((I_B - G)^{-1}x)\}^{-1}]h \\ &= -(Dk(x)h)G((I_B - G)^{-1}x) \\ &\quad + k(x)(I_H - P_n)DF((I_B - G)^{-1}x)\{I_H - (I_H - P_n)DF((I_B - G)^{-1}x)\}^{-1}h \end{aligned}$$

for any $h \in H$ and μ -a.e. x . So we get

$$(8.8) \quad \|K(x)\|_H \leq \frac{1}{6m} \text{ for any } x \in B \text{ and}$$

$$(8.9) \quad \|DK(x)\|_{L^2(H,H)} \leq 2m \cdot \frac{1}{6m} + \frac{\frac{1}{60}}{1 - \frac{1}{60}} < \frac{1}{2} \text{ for } \mu\text{-a.e. } x.$$

This shows that K belongs to $\mathcal{D}(L)$ and

$$(8.10) \quad \|K(x+h) - K(x)\|_H \leq \frac{1}{2} \|h\|_H \text{ for any } x \in B \text{ and } h \in H.$$

Step 3. Let $S = -(I_B - F)(I_B - K) + I_B$, then it is easy to see that S belongs to $\mathcal{D}(L)$ and $d(z; S) = d((I_B - K)z; F)d(z; K)$

for μ -a.e. z by virtue of Lemma 6.1. Let M be a σ -compact subset such that $M \subset (I_B - G)^{-1}E \subset \Omega$, $\mu(M \cap A) = 0$ and $\nu(M) = \nu(\Omega)$.

The existence of such M is guaranteed by the singularity between $\mu|_A$ and $\nu|_A$. Let $N = (I_B - G)M = (I_B - K)^{-1}M$. According to

Theorem 6.1 and (8.10), we get

$$(8.11) \quad \mu(N \cap (I_B - K)^{-1}A) = (I_B - K)\mu(M \cap A) = 0.$$

Since $M \subset \Omega$, $(I_B - S)N = (I_B - F)M$ is σ -compact. Notice that $Sx = - (I_B - F)(I_B - K)x + x = P_n F(I_B - K)x \in P_n H$ for any $x \in B$ such that $\rho(x; E) < \frac{1}{m}$. By Fubini's theorem, we get

$$\begin{aligned} & \mu((I_B - S)N) \\ &= \int_{B \times B} \chi_{(I_B - S)N}(z_1 + z_2) \mu_{P_n H^\perp}(dz_1) \mu_{P_n H}(dz_2) \\ &= \int_B \mu_{P_n H^\perp}(dz_1) \int_{P_n H} \chi_{(I_B - S)N - z_1}(z_2) \mu_{P_n H}(dz_2) \\ &= \int_B \mu_{P_n H^\perp}(dz_1) \mu_{P_n H}((I_{P_n H} - S(\cdot + z_1))((N - z_1) \cap P_n H)) . \end{aligned}$$

Thus using usual Sard's lemma (see J. Schwartz [14] for an example),

$$\begin{aligned} & \mu((I_B - S)N) \\ &\leq \int_B \mu_{P_n H^\perp}(dz_1) \int_{(N - z_1) \cap P_n H} |\delta(I_{P_n H} - DS(z_1 + z_2)|_{P_n H})| \\ &\quad \times \exp[(S(z_1 + z_2), z_2)_H - \text{trace } DS(z_1 + z_2)|_{P_n H} \\ &\quad \quad - \frac{1}{2} \|S(z_1 + z_2)\|_H^2] \mu_{P_n H}(dz_2) \\ &\leq \int_B \mu_{P_n H^\perp}(dz_1) \int_{(N - z_1) \cap P_n H} |d(z_1 + z_2; S)| \mu_{P_n H}(dz_2) \\ &\leq \int_N |d(z; S)| \mu(dz) \\ &\leq \int_N |d((I_B - K)z; F)| |d(z; K)| \mu(dz) . \end{aligned}$$

In view of (8.11) , $d((I_B - K)z; F) = 0$ for μ -a.e. $z \in N$, and accordingly $\mu((I_B - F)M) = \mu((I_B - S)N) = 0$. On the other hand , $(I_B - F)\nu((I_B - F)M) \geq \nu(M) = \nu(\Omega) > 1 - \varepsilon$. Since ε is arbitrary , this shows that μ and $(I_B - F)\nu$ are mutually singular . This completes the proof .

Using Theorem 8.1 , we can prove the following .

Theorem 8.2 Suppose that $F: B \rightarrow H$ is an H - C^1 map such that

(1) $F(z + h_n) \rightarrow F(z)$ in H , $n \rightarrow \infty$, for each $z \in B$, whenever $h_n \rightarrow 0$ weakly in H , and

(2) $\limsup_{r \rightarrow \infty} \left\{ \frac{\|F(z+h)\|_H}{\|h\|_H} ; h \in H \text{ and } \|h\|_H \geq r \right\} < 1$ for each

$z \in B$. Then μ is absolutely continuous relative to $(I_B - F)\mu$.

Proof. Step 1. Let $U_r = \{h \in H; \|h\|_H \leq r\}$. Since U_r is weakly compact in H , the image of U_r through $F(z+\cdot)$ is compact in H . So $\sup\{\|F(z+h)\|_H; h \in U_r\} < \infty$ for each $z \in B$.

According to our assumption (2) , there exists some $r > 0$ for each $z \in B$ such that $\|F(z+h)\|_H \leq r$ for any $h \in U_r$. Since $F(z+\cdot)|_{U_r}: U_r \rightarrow U_r$ is continuous and the image is compact , there exists some $h \in H$ such that $F(z+h) = h$ by virtue of Schauder's fixed point theorem .

Step 2 . Let $K_z = \{h \in H; F(z+h) = h\}$ for each $z \in B$.

Then $K_z \neq \emptyset$ for any $z \in B$ by Step 1 . In view of our assumption (2) , there exists some $r > 0$ such that $K_z \subset U_r$.

Suppose that $\{h_n\}_{n=1}^\infty \subset K_z$, then $\{h_n\}_{n=1}^\infty \subset U_r$, and so there exists a subsequence $\{h_{n_j}\}$ and $h_0 \in H$ such that

$h_{n_j} \rightarrow h_0$, $j \rightarrow \infty$, weakly in H . This implies that $F(z+h_{n_j})$

$\rightarrow F(z+h_0)$, $j \rightarrow \infty$, strongly in H . Since $h_n = F(z+h_n)$, we get $h_{n_j} \rightarrow h_0$, $j \rightarrow \infty$, strongly in H and $h_0 \in K_z$. So K_z

is compact in the strong topology of H . Let K be a map from B to $\text{comp}(H)$ such that z corresponds to K_z through K , where $\text{comp}(H)$ is a space of all compact subsets of H .

(See D.W.Stroock and S.R.S.Varadhan [16] Chapter 12 about the topology of $\text{comp}(H)$ and its property .)

Step 3. We prove the following in this step .

Proposition 8.2 Let G be an open set in H . Then

$$\{z \in B; K_z \subset G\} = \{z \in B; \inf\{\|F(z+h) - h\|_H; h \in H \setminus G\} > 0\}.$$

Proof of Proposition 8.2 . Suppose that $\inf\{\|F(z+h) - h\|_H; h \in H \setminus G\} > 0$, then $F(z+h) \neq h$ for any $h \in H \setminus G$, and so $K_z \subset G$. Conversely suppose that $\inf\{\|F(z+h) - h\|_H; h \in H \setminus G\} = 0$.

Then there exists a sequence $\{h_n\}_{n=1}^\infty \subset H \setminus G$ such that

$$\|F(z+h_n) - h_n\|_H \rightarrow 0, n \rightarrow \infty. \quad \text{According to our assumption (2),}$$

$\{\|h_n\|_H\}$ must be bounded. So taking a subsequence if necessary, we may assume that $h_n \rightarrow h_0$, $n \rightarrow \infty$, weakly in H for some $h_0 \in H$. Since $F(z+h_n) - h_n \rightarrow 0$ strongly, $n \rightarrow \infty$, we obtain that $h_n \rightarrow h_0$, $n \rightarrow \infty$, strongly and $F(z+h_0) = h_0$. By the closedness of $H \setminus G$, we get $h_0 \in H \setminus G$, which shows that $K_z \not\subset G$. This completes the proof.

Let $\{h_n\}_{n=1}^{\infty}$ be a dense countable subset of $H \setminus G$, then
 $\inf\{\|F(z+h) - h\|_H; h \in H \setminus G\} = \inf\{\|F(z+h_n) - h_n\|_H; n=1,2,\dots\}$
 This shows that $\{z \in B; K_z \subset G\}$ is measurable. According
 to D.W.Stroock and S.R.S.Varadhan [16] Chapter 12, we have got
 the following.

Corollary to Proposition 8.2 $K: B \rightarrow \text{comp}(H)$ is measurable.

Step 4. By the measurable selection theorem, there exists
 a measurable map $G: B \rightarrow H$ such that $G(z) \in K_z$ for each $z \in B$.
 Then $(I_B - F)(I_B + G)z = z + Gz - F(z + Gz) = z$ for each $z \in B$.
 Let $\nu = (I_B + G)\mu$ and $\nu = \nu_1 + \nu_2$ be the Lebesgue decomposition
 of ν relative to μ , i.e. ν_1 is absolutely continuous and ν_2
 is singular relative to μ . Then $\mu = (I_B - F)\nu = (I_B - F)\nu_1$
 $+ (I_B - F)\nu_2$. But $(I_B - F)\nu_2$ and μ must be mutually singular
 in view of Theorem 8.1. So we have got $\nu_2 = 0$, and accordingly
 ν is absolutely continuous relative to μ . Thus $\mu = (I_B - F)\nu$
 is absolutely continuous relative to $(I_B - F)\mu$. This completes
 the proof.

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The nonlinear transformation of Gaussian measure
on Banach space and its absolute continuity (II)

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1. Introduction.

Let (μ, H, B) be an abstract Wiener space in the sense of the previous paper [4]. (See Kuo [3] for its property.) Let $F : B \rightarrow B$ be a Borel map such that $I_B - F : B \rightarrow B$ is bijective, and let $\nu = (I_B - F)^{-1} \mu$ be the image measure on B of μ under $(I_B - F)^{-1} : B \rightarrow B$. In the previous paper [4], the author gave some sufficient condition on F for the image measure ν to be absolutely continuous relative to μ . However, in the case where B is a Banach space included in $S'(\mathbb{R}^d)$, the space of tempered distributions over \mathbb{R}^d , and μ and ν can be regarded as stationary ergodic probability measures on $S'(\mathbb{R}^d)$, μ and ν are identical or mutually singular. Therefore we can not expect that ν is absolutely continuous to μ in the important case.

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But even in the case where μ and ν are mutually singular, there sometimes exists a sub- σ -field F of $\mathcal{B}(B)$, the Borel field over B , such that F is not so small and the restricted measures $\mu|_F$ and $\nu|_F$ of μ and ν to the σ -field F are mutually absolutely continuous. The purpose of the present paper is to find such a σ -field F .

Now let us show the results in our paper. Let $H_1 + H_2$ be an orthogonal decomposition of H , and let F_1 and F_2 be the sub- σ -fields of $\mathcal{B}(B)$ generated by Borel functions

$$\{ {}_{B^*} \langle u, \cdot \rangle_B ; u \in H_1 \cap B^* \} \text{ and } \{ {}_{B^*} \langle u, \cdot \rangle_B ; u \in H_2 \cap B^* \}$$

respectively. We will show in Theorem 1 that on some condition for F , H_1 and H_2 , there exist Borel maps $\pi_1 : B \rightarrow B$ and $\pi_2 : B \rightarrow B$, and an $F_1 \times F_2$ -measurable function $\tilde{H} : B \times B \rightarrow \mathbb{R}$ such that for any bounded Borel function f defined on B , the conditional expectation $E_\nu[f|F_2]$ of f relative to the σ -field F_2 under the image measure $\nu = (I_B - F)^{-1} \mu$ is represented by

$$E_\nu[f|F_2](z) = \int_B f(\pi_1 \tilde{z} + \pi_2 z) \frac{\exp \tilde{H}(\tilde{z}, z)}{\int_B \exp \tilde{H}(\tilde{z}, z) \mu(d\tilde{z})} \mu(d\tilde{z})$$

for ν -a.e. z . We will also give the explicit form of $\tilde{H}(\tilde{z}, z)$.

In Section 5, we will consider the stochastic non-linear pseudo-differential equation introduced in the author [5]:

$$p(D_x)X - b(q_1(D_x)X, \dots, q_n(D_x)X) = W,$$

where W is a Gaussian white noise with d -dimensional parameter. We will set some assumption for p , q_j , $j = 1, \dots, n$, and b .

It has been shown in [5] that there exists a unique solution X for the equation on such assumption. Let $Y = p(D_X)^{-1}W$. Then X and Y are $S'(\mathbb{R}^d)$ -valued random variables. Let $\tilde{\mu}$ and $\tilde{\nu}$ denote the probability laws of Y and X respectively.

For any domain D in \mathbb{R}^d , let F_D denote the sub- σ -field of the Borel field $B(S'(\mathbb{R}^d))$ over $S'(\mathbb{R}^d)$ generated by Borel functions $\{S\langle u, \cdot \rangle_S; u \in S(\mathbb{R}^d) \text{ and the support of } u \text{ is contained in } D\}$, and I_D denote the sub- σ -field of $B(S'(\mathbb{R}^d))$ generated by Borel functions $\{S\langle u, \cdot \rangle_S; u \in S(\mathbb{R}^d) \text{ and the support of } (p(D_X)p(-D_X))^{-1}u \text{ is contained in } D\}$, where $S(\mathbb{R}^d)$ denotes the space of rapidly decreasing smooth functions. In the case where $p(D_X)p(-D_X)$ is a differential operator, $\{I_D; D \text{ is a domain in } \mathbb{R}^d\}$ is an innovating system for $\{F_D; D \text{ is a domain in } \mathbb{R}^d\}$ under the probability measure $\tilde{\mu}$ in the sense of Dobrushin and Surgailis [2].

Now let D be a bounded domain in \mathbb{R}^d with smooth boundary, and let D^e denote the exterior of D . Moreover let $\tilde{\nu}(\cdot | I_{D^e})$ denote the conditional probability measure of relative to the σ -field I_{D^e} . Then we will show in Theorem 2 that

(1) the restricted measures $\tilde{\mu}|_{F_D}$ and $\tilde{\nu}|_{F_D}$ are mutually

absolutely continuous, and

(2) there exists an $F_D \times I_{D^e}$ -measurable function

$\tilde{H} : S'(\mathbb{R}^d) \times S'(\mathbb{R}^d) \rightarrow \mathbb{R}$ such that for any $E \in F_D$,

$$\tilde{\nu}(E|I_D e)(w) = \frac{\int_E \exp \tilde{H}(\tilde{w}, w) \tilde{\mu}(d\tilde{w})}{\int_{S'} \exp \tilde{H}(\tilde{w}, w) \tilde{\mu}(d\tilde{w})} \quad \text{for } \tilde{\nu}\text{-a.e. } w.$$

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Notation.

For any Banach space E , E^* denotes the dual Banach space of E and I_E denotes the identity map in E .

For any Hilbert spaces H and K , $L^\infty(H, K)$ denotes the Banach space consisting of all bounded linear maps from H into K with the operator norm, and $L^2(H, K)$ denotes the Hilbert space consisting of all Hilbert-Schmidt operators with Hilbert-Schmidt norm.

For any σ -fields F and F' , $F \vee F'$ denotes the σ -field generated by $F \cup F'$.

$$\langle x \rangle = \sqrt{1 + \sum_{j=1}^d x_j^2} \quad \text{for any } x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

2. The Borel maps π_1 and π_2 .

Let (μ, H, B) denote an abstract Wiener space throughout this paper. Let H_1 and H_2 be mutually orthogonal closed linear subspaces of H satisfying $H = H_1 + H_2$. Let P_1 (resp. P_2) denote the orthogonal projection defined in H onto H_1 (resp. H_2), and let B_1 and B_2 be the closure of H_1 and H_2 in B respectively. Now let B_0 be a real separable reflexive Banach space such that H is densely, continuously included in B_0 and B_0 is also densely, continuously included in B . Then it is clear that B_0^* is densely included in H , and that $B^* \subset B_0^* \subset H \subset B_0 \subset B$.

We assume the following two assumptions through this section, Section 3 and 4:

(A-1) $B_0 \cap B_1 \cap B_2 = \{0\}$, and

(A-2) the orthogonal projection $P_1 : H \rightarrow H_1$ is extensible to a bounded linear map $\bar{P}_1 : B_0 \rightarrow H_1$.

Then we get the following.

Proposition 2.1. $B^* \cap H_1 + B^* \cap H_2$ is dense in B_0^* . Therefore $B^* \cap H_1 + B^* \cap H_2$ is dense in H .

Proof. It is obvious that

$$B^* \cap H_1 = \{ u \in B^* ; B^* \langle u, z \rangle_B = 0 \text{ for any } z \in B_2 \}, \text{ and}$$

$$B^* \cap H_2 = \{ u \in B^* ; B^* \langle u, z \rangle_B = 0 \text{ for any } z \in B_1 \}.$$

Then it is easy to see that

$$(2.1) \quad B_1 = \{ z \in B ; B^* \langle u, z \rangle_B = 0 \text{ for any } u \in B^* \cap H_2 \}, \text{ and}$$

$$(2.2) \quad B_2 = \{ z \in B ; B^* \langle u, z \rangle_B = 0 \text{ for any } u \in B^* \cap H_1 \}.$$

(See Yoshida [7, Appendix to Chapter 5] for instance.)

Now suppose that $B^* \cap H_1 + B^* \cap H_2$ is not dense in B_0^* . Then the Hahn-Banach theorem and the reflexivity of B_0 imply that there exists some $z \in B_0$ such that $z \neq 0$ and $B_0^* \langle u, z \rangle_{B_0} = B^* \langle u, z \rangle_B = 0$ for any $u \in B^* \cap H_1 + B^* \cap H_2$. Thus it follows from (2.1) and (2.2) that $z \in B_0 \cap B_1 \cap B_2$. But this contradicts the assumption (A-1). This completes the proof.

For any subspace E of H , let $\mathcal{P}(E)$ denote the set of all orthogonal projections on H with a finite dimensional range contained in E . It is easy to see that any projection P , $P \in \mathcal{P}(B^*)$, is extensible to a bounded linear map from B into B^* , which will be denoted by \tilde{P} .

Take such sequences $\{P_1^{(n)}\}_{n=1}^{\infty}$ and $\{P_2^{(n)}\}_{n=1}^{\infty}$ of increasing orthogonal projections on H that $\{P_1^{(n)}\}_{n=1}^{\infty} \subset \mathcal{P}(B^* \cap H_1)$ and $\{P_2^{(n)}\}_{n=1}^{\infty} \subset \mathcal{P}(B^* \cap H_2)$, and that $P_1^{(n)} \uparrow P_1$ and $P_2^{(n)} \uparrow P_2$ strongly as $n \rightarrow \infty$, and fix them through this paper. The existence of such sequences are guaranteed by Proposition 2.1.

Definition 2.1. We define a Borel subset $\mathcal{D}(\pi_1)$ of B by $\mathcal{D}(\pi_1) = \{ z \in B ; \{ \tilde{P}_1^{(n)} z \}_{n=1}^{\infty} \text{ is convergent in } B \}$,

and a Borel map $\pi_1 : \mathcal{D}(\pi_1) \rightarrow B_1$ by

$$\pi_1 z = \lim_{n \rightarrow \infty} \tilde{P}_1^{(n)} z \quad \text{for each } z \in \mathcal{D}(\pi_1).$$

We define a Borel subset $\mathcal{D}(\pi_2)$ of B by

$\mathcal{D}(\pi_2) = \{ z \in \mathcal{D}(\pi_1) ; z - \pi_1 z \in B_2 \}$, and a Borel map

$\pi_2 : \mathcal{D}(\pi_2) \rightarrow B_2$ by $\pi_2 z = z - \pi_1 z$ for each $z \in \mathcal{D}(\pi_2)$.

Proposition 2.2. (1) $\mathcal{D}(\pi_1)$ and $\mathcal{D}(\pi_2)$ are linear subspaces of B , and $\pi_1 : \mathcal{D}(\pi_1) \rightarrow B_1$ and $\pi_2 : \mathcal{D}(\pi_2) \rightarrow B_2$ are linear.

(2) $B_0 \subset \mathcal{D}(\pi_2) \subset \mathcal{D}(\pi_1)$ and $\pi_1 z = \bar{P}_1 z$ for each $z \in B_0$.

(3) $B_2 \subset \mathcal{D}(\pi_2)$, and $\pi_1 z = 0$ and $\pi_2 z = z$ for each $z \in B_2$.

(4) If $z \in \mathcal{D}(\pi_1)$, then $\pi_1 z \in \mathcal{D}(\pi_2)$, $\pi_1 \pi_1 z = \pi_1 z$ and $\pi_2 \pi_1 z = 0$.

Proof. Our assertion (1) is obvious. It is clear that $P_1^{(n)} h = P_1^{(n)} \bar{P}_1 h$ and $B_0^* \langle u, h - \bar{P}_1 h \rangle_{B_0} = 0$ for any $h \in H$ and $u \in B^* \cap H_1$. Thus we get for any $z \in B_0$,

$$\lim_{n \rightarrow \infty} \tilde{P}_1^{(n)} z = \lim_{n \rightarrow \infty} \tilde{P}_1^{(n)} \bar{P}_1 z = \bar{P}_1 z \quad \text{and} \quad B_0^* \langle u, z - \bar{P}_1 z \rangle_{B_0} = 0$$

for any $u \in B^* \cap H_1$. Therefore by (2.2) we see that $z \in \mathcal{D}(\pi_1)$, $\pi_1 z = \bar{P}_1 z$ and $z - \bar{P}_1 z \in B_2$ for any $z \in B_0$. This proves our assertion (2).

It is obvious that $\tilde{P}_1^{(n)} z = 0$, $n=1,2,\dots$, for any $z \in B_2$. This proves our assertion (3). Let $z \in \mathcal{D}(\pi_1)$. Then we see that $\tilde{P}_1^{(n)} \pi_1 z = \lim_{m \rightarrow \infty} \tilde{P}_1^{(n)} \tilde{P}_1^{(m)} z = \tilde{P}_1^{(n)} z$, $n=1,2,\dots$, which

shows our assertion (4). This completes the proof.

Proposition 2.3. (1) $\mu(\mathcal{D}(\pi_1)) = \mu(\mathcal{D}(\pi_2)) = 1$ and

$\tilde{P}_2^{(n)} z \rightarrow \pi_2 z$ in B , $n \rightarrow \infty$, for μ -a.e. $z \in \mathcal{D}(\pi_2)$.

(2) The probability law on B of $\pi_1 z_1 + \pi_2 z_2$ under $\mu(dz_1) \otimes \mu(dz_2)$ is equal to μ . That is,

$$\int_{B \times B} f(\pi_1 z_1 + \pi_2 z_2) \mu(dz_1) \otimes \mu(dz_2) = \int_B f(z) \mu(dz)$$

for any bounded Borel function f on B .

Proof. By virtue of Carmona[1], we see that $\{\tilde{P}_1^{(n)} z\}_{n=1}^{\infty}$ and $\{\tilde{P}_2^{(n)} z\}_{n=1}^{\infty}$ are convergent in B for μ -a.e. z , and that $\tilde{P}_1^{(n)} z + \tilde{P}_2^{(n)} z \rightarrow z$ in B , $n \rightarrow \infty$, for μ -a.e. z .

Thus we have $\mu(\mathcal{D}(\pi_1)) = \mu(\mathcal{D}(\pi_2)) = 1$ and $\tilde{p}_2^{(n)} z \rightarrow \pi_2 z$ in B , $n \rightarrow \infty$, for μ -a.e. $z \in \mathcal{D}(\pi_2)$. This proves our assertion (2).

Let f be a bounded continuous function defined on B . Since $\tilde{p}_1^{(n)} z$ and $\tilde{p}_2^{(n)} z$ are independent under $\mu(dz)$, we obtain

$$\begin{aligned} \int_{B \times B} f(\tilde{p}_1^{(n)} z_1 + \tilde{p}_2^{(n)} z_2) \mu(dz_1) \times \mu(dz_2) \\ = \int_B f(\tilde{p}_1^{(n)} z + \tilde{p}_2^{(n)} z) \mu(dz). \end{aligned}$$

Letting $n \rightarrow \infty$, we have got

$$\int_{B \times B} f(\pi_1 z_1 + \pi_2 z_2) \mu(dz_1) \times \mu(dz_2) = \int_B f(z) \mu(dz).$$

This completes the proof.

The probability measure on B_1 (resp. B_2) induced by μ through $\pi_1 : \mathcal{D}(\pi_1) \rightarrow B_1$ (resp. $\pi_2 : \mathcal{D}(\pi_2) \rightarrow B_2$) will be denoted by μ_1 (resp. μ_2).

3. The σ -fields F_1 and F_2 .

Let F_1 (resp. F_2) denote the sub- σ -field of $B(B)$, the Borel field over B , generated by Borel functions

$\{ {}_{B*}\langle u, \cdot \rangle_B : B \rightarrow \mathbb{R}; u \in B^* \cap H_1 \}$ (resp. $\{ {}_{B*}\langle u, \cdot \rangle_B : B \rightarrow \mathbb{R}; u \in B^* \cap H_2 \}$). For each probability measure ν on B , N_ν will denote the σ -field generated by ν -null sets, i.e.

$N_\nu = \{ A; A \text{ is a subset of } B \text{ and there exists a Borel subset } C \text{ of } B \text{ of } \nu\text{-measure zero such that } A \subset C \text{ or } B \setminus A \subset C \}$.

Proposition 3.1. (1) If $g : B \rightarrow \mathbb{R}$ is F_1 -measurable, then

$g(z+z') = g(z)$ for any $z \in B$ and $z' \in B_2$.

(2) If $g : B \rightarrow \mathbb{R}$ is F_2 -measurable, $g(z+z') = g(z)$ for any $z \in B$ and $z' \in B_1$.

Proof. It is clear that if $u \in B^* \cap H_1$, then

${}_{B*}\langle u, z+z' \rangle_B = {}_{B*}\langle u, z \rangle_B$ for any $z \in B$ and $z' \in B_2$. Thus we get our assertion (1) by the definition of the σ -field F_1 .

The proof of our assertion (2) goes similarly. This completes the proof.

Let $F : B \rightarrow B_0$ be a Borel map such that $I_B - F : B \rightarrow B$ is bijective, and let $\nu = (I_B - F)^{-1} \mu$ be the image probability measure of μ under $(I_B - F)^{-1} : B \rightarrow B$. Then we have the following.

Proposition 3.2. $\nu(\mathcal{D}(\pi_1)) = \nu(\mathcal{D}(\pi_2)) = 1$.

Proof. It is clear that

(3.1) $(I_B - F)^{-1} z = z + F(I_B - F)^{-1} z$ for any $z \in B$. Thus it

follows from Proposition 2.2. (1) and (2) that

$(I_B - F)^{-1} \mathcal{D}(\pi_2) = \mathcal{D}(\pi_2)$. This and Proposition 2.3 lead to our assertion.

Proposition 3.3. (1) If $f : B_1 \rightarrow \mathbb{R}$ is a Borel function, then $f(\pi_1 \cdot) : \mathcal{D}(\pi_1) \rightarrow \mathbb{R}$ is F_1 -measurable.

(2) If $f : B_2 \rightarrow \mathbb{R}$ is a Borel function, then $f(\pi_2 \cdot) : \mathcal{D}(\pi_2) \rightarrow \mathbb{R}$ is $F_2 \vee N_v$ -measurable.

(3) $F_1 \vee F_2 \vee N_v = \mathcal{B}(B) \vee N_v$.

Proof. It is obvious that $\mathcal{D}(\pi_1) \in F_1$ and $f(\pi_1 z) = \lim_{n \rightarrow \infty} f(\tilde{P}_1^{(n)} z)$ for any $z \in \mathcal{D}(\pi_1)$ and any bounded continuous function f defined on B . This shows our assertion (1).

Next let us prove our assertion (2). By virtue of Proposition 3.2, we see that $\mathcal{D}(\pi_2) \in N_v$. Let \tilde{u} be an arbitrary element of B_2^* and $g : B_2 \rightarrow \mathbb{C}$ be a continuous function given by $g(w) = \exp(\sqrt{-1} {}_{B_2^*} \langle \tilde{u}, w \rangle_{B_2})$ for each $w \in B_2$. The Hahn-Banach

theorem implies that there exists some $u \in B^*$ such that $g(w) = \exp(\sqrt{-1} {}_{B^*} \langle u, w \rangle_B)$ for any $w \in B_2$. Observing $u \in B^* \subset B_0^*$, we see by Proposition 2.1 that there exist sequences

$\{v_n\}_{n=1}^\infty \subset B^* \cap H_1$ and $\{u_n\}_{n=1}^\infty \subset B^* \cap H_2$ such that

$$(3.2) \quad v_n + u_n \rightarrow u \text{ in } B_0^*, \quad n \rightarrow \infty.$$

It is easy to see by (2.1) and (2.2) that

$$(3.3) \quad {}_{B^*} \langle v_n + u_n, \pi_2 z \rangle_B = {}_{B^*} \langle u_n, \pi_2 z \rangle_B \\ = {}_{B^*} \langle u_n, z \rangle_B$$

for any $z \in \mathcal{D}(\pi_2)$.

Let $g_n : B \rightarrow \mathbb{C}$ be a function given by $g_n(z) = \exp(\sqrt{-1} {}_{B^*} \langle u_n, z \rangle_B)$ for any $z \in B$. Then it is obvious that g_n is F_2 -measurable. It follows from Proposition 2.2, Proposition 3.2, (3.1) and (3.3) that

$$\begin{aligned}
& \int_B |g(\pi_2 z) - g_n(z)| \, \nu(dz) \\
& \leq \int_{\mathcal{D}(\pi_2)} |\exp(\sqrt{-1} B^* \langle u - (u_n + v_n), \pi_2 z \rangle_B) - 1| \, \nu(dz) \\
& = \int_{\mathcal{D}(\pi_2)} |\exp(\sqrt{-1} B^* \langle u - (u_n + v_n), \pi_2 z + (I_{B_0} - \bar{P}_1) F(I_B - F)^{-1} z \rangle_B) \\
& \quad - 1| \, \mu(dz) \\
& \leq \int_{\mathcal{D}(\pi_2)} |\exp(\sqrt{-1} B^* \langle u - (u_n + v_n), \pi_2 z \rangle_B) - 1| \, \mu(dz) \\
& \quad + \int_B |\exp(\sqrt{-1} B_0^* \langle u - (u_n + v_n), (I_{B_0} - \bar{P}_1) F(I_B - F)^{-1} z \rangle_{B_0}) \\
& \quad - 1| \, \mu(dz) .
\end{aligned}$$

Proposition 2.3 implies that

$$\begin{aligned}
& \int_{\mathcal{D}(\pi_2)} |\exp(\sqrt{-1} B^* \langle u - (u_n + v_n), \pi_2 z \rangle_B) - 1| \, \mu(dz) \\
& \leq \left\{ \int_{\mathcal{D}(\pi_2)} |B^* \langle u - (u_n + v_n), \pi_2 z \rangle_B|^2 \, \mu(dz) \right\}^{1/2} \\
& = \|P_2(u - (u_n + v_n))\|_H .
\end{aligned}$$

Therefore by (3.2) we see that

$$\int_B |g(\pi_2 z) - g_n(z)| \, \nu(dz) \rightarrow 0 \text{ as } n \rightarrow \infty .$$

This shows that $g(\pi_2 \cdot) : \mathcal{D}(\pi_2) \rightarrow \mathbb{C}$ is $F_2 \vee N_\nu$ -measurable.

Let V be the set of linear combinations of

$\{ \cos(B_2^* \langle u, \cdot \rangle_{B_2}), \sin(B_2^* \langle u, \cdot \rangle_{B_2}) ; u \in B_2^* \}$. Then

$g(\pi_2 \cdot) : \mathcal{D}(\pi_2) \rightarrow \mathbb{R}$ is $F_2 \vee N_\nu$ -measurable for any $g \in V$. Let

$f : B_2 \rightarrow \mathbb{R}$ be a bounded continuous function and let

$C = \sup \{ |f(w)| ; w \in B_2 \}$. Since the image measure π_2^ν on B_2

of ν under $\pi_2 : \mathcal{D}(\pi_2) \rightarrow B$ is a Radon measure, there exists

a sequence $\{K_m\}_{m=1}^\infty$ of increasing compact subsets of B_2

such that $\pi_2^\nu(B_2 \setminus K_m) \rightarrow 0$ as $m \rightarrow \infty$.

By virtue of the Stone-Weierstrass theorem, we see that there exists a sequence $\{\tilde{f}_n\}_{n=1}^{\infty} \subset V$ such that $\tilde{f}_n(w) \rightarrow f(w)$, $n \rightarrow \infty$, uniformly for $w \in K_m$, $m = 1, 2, \dots$. Let $f_n : B_2 \rightarrow \mathbb{R}$, $n = 1, 2, \dots$, be functions given by

$f_n(w) = \min\{C, \max\{-C, \tilde{f}_n(w)\}\}$ for each $w \in B_2$. Then it is obvious that $f(\pi_2 \cdot) : \mathcal{D}(\pi_2) \rightarrow \mathbb{R}$ is $F_2 \vee N_v$ -measurable, and

we get

$$\begin{aligned} & \int_{\mathcal{D}(\pi_2)} |f(\pi_2 z) - f_n(\pi_2 z)| \nu(dz) \\ &= \int_{B_2} |f(w) - f_n(w)| \pi_2 \nu(dw) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad \text{Therefore}$$

$f(\pi_2 \cdot) : \mathcal{D}(\pi_2) \rightarrow \mathbb{R}$ is $F_2 \vee N_v$ -measurable. This proves our assertion (2). Our assertion (3) follows immediately from our assertions (1), (2) and the fact that $z = \pi_1 z + \pi_2 z$ for any $z \in \mathcal{D}(\pi_2)$. This completes the proof.

4. Gibbs representation of $(I_B - F)^{-1}\mu$.

In this section we assume that a Borel map $F : B \rightarrow B_0$ satisfies the following five assumptions.

(F-1) $F(z+h) - F(z) \in H$ for any $z \in B$ and $h \in H$, and there exists a map $DF : B \rightarrow L^\infty(H, H)$ (not necessarily Borel) such that $\|F(z+h) - F(z) - DF(z)h\|_H = o(\|h\|_H)$, $\|h\|_H \rightarrow 0$, and $DF(z+\cdot) : H \rightarrow L^\infty(H, H)$ is continuous for any $z \in B$.

(F-2) $I_B - F : B \rightarrow B$ is bijective and $I_H - DF(z) : H \rightarrow H$ is invertible for any $z \in B$.

(F-3) $P_1 DF(z) : H \rightarrow H$ and $DF(z)P_1 : H \rightarrow H$ are Hilbert-Schmidt operators for any $z \in B$, and $P_1 DF(z+\cdot) : H \rightarrow L^2(H, H)$ and $DF(z+\cdot)P_1 : H \rightarrow L^2(H, H)$ are continuous for any $z \in B$.

(F-4) $I_B - F_2 : B \rightarrow B$ is bijective, where F_2 denotes a Borel map $(I_{B_0} - \bar{P}_1)F : B \rightarrow B_0$, and $I_H - P_2 DF(z) : H \rightarrow H$ is invertible for any $z \in B$.

(F-5) For any $z \in B$ and $x \in B_1$, $F(x+z) - F(z) \in H$ and $DF(x+z) - DF(z) : H \rightarrow H$ is a Hilbert-Schmidt operator, and moreover $DF(x+z+\cdot) - DF(z+P_2\cdot) : H \rightarrow L^2(H, H)$ is continuous.

Remark 4.1. Since $F : B \rightarrow B_0$ is a Borel map and $L^2(H, H)$ is a separable Hilbert space, $P_1 DF(\cdot) : B \rightarrow L^2(H, H)$, $DF(\cdot)P_1 : B \rightarrow L^2(H, H)$ and $DF(x+\cdot) - DF(\cdot) : B \rightarrow L^2(H, H)$, $x \in B_1$, are Borel maps.

Let $H_0^{(n)} : B_1 \times B_2 \rightarrow \mathbb{R}$, $n=1,2,\dots$, be Borel functions given by

$$H_0^{(n)}(x, y) = B_* \langle P^{(n)}(F(x+y) - F_2(y)), x+y - F_2(y) \rangle_B \\ - \text{trace}_H P^{(n)}(DF(x+y) - P_2 DF(y)P_2)(I_H - P_2 DF(y)P_2)^{-1}$$

for any $x \in B_1$ and $y \in B_2$, where $P^{(n)} = P_1^{(n)} + P_2^{(n)}$. And let $\nu = (I_B - F)^{-1}\mu$.

The following is our main result.

Theorem 1. (1) There exists a Borel function $H_0 : B_1 \times B_2 \rightarrow \mathbb{R}$ such that $H_0^{(n)}(x, y) \rightarrow H_0(x, y)$, $n \rightarrow \infty$, in probability with respect to $\pi_1 \mu(dx) \otimes \pi_2 \nu(dy)$.

(2) For any bounded function $f : B \rightarrow \mathbb{R}$, the conditional expectation $E_\nu[f|F_2]$ of f relative to the σ -field F_2 under the probability measure ν is given by

$$E_\nu[f|F_2](z) = \int_B f(\pi_1 \tilde{z} + \pi_2 z) \frac{\exp(H(\pi_1 \tilde{z}, \pi_2 z))}{\int_B \exp(H(\pi_1 \tilde{z}, \pi_2 z)) \mu(d\tilde{z})} \mu(d\tilde{z})$$

for ν -a.e. z , where

$$H(x, y) = H_0(x, y) - \frac{1}{2} \|F(x+y) - F_2(y)\|_H^2 + \log |\delta_H((I_H - DF(x+y))(I_H - P_2 DF(y)P_2)^{-1})|$$

for each $x \in B_1$ and $y \in B_2$.

Here $\delta_H(A)$ denotes the Carleman-Fredholm determinant of an operator $A : H \rightarrow H$ (see [4, Definition 6.1] for the detail).

In particular, the restricted measures $\mu|_{F_1}$ and $\nu|_{F_1}$ of μ and ν to the σ -field F_1 are mutually absolutely continuous.

Remark 4.2. Suppose that

$$\|(DF(x+y) - P_2 DF(y)P_2)(I_H - P_2 DF(y)P_2)^{-1}\|_{L^\infty(H, H)} < 1$$

for any $x \in B_1$ and $y \in B_2$. Noting that

$$\delta_H(I_H - K) = \exp\left(-\sum_{n=2}^{\infty} \frac{1}{n} \text{trace}_H K^n\right), \quad K \in L^2(H, H) \text{ such that}$$

$$\|K\|_{L^\infty(H, H)} < 1, \text{ we get}$$

$$H(x, y) = H_0(x, y) - \frac{1}{2} \|F(x+y) - F_2(y)\|_H^2 - \sum_{n=2}^{\infty} \frac{1}{n} \text{trace}_H [(DF(x+y) - P_2 DF(y)P_2)(I_H - P_2 DF(y)P_2)^{-1}]^n.$$

We will prove Theorem 1 in several steps.

Step 1. First we prove the following.

Proposition 4.1. (1) The image of $F_2 : B \rightarrow B_0$ is contained in B_2 .

(2) $I_{B_2} - F_2 : B_2 \rightarrow B_2$ is bijective.

Proof. (1) is obvious. For any $u \in B_2$, there exists $v \in B$ such that $(I_B - F)v = u$ by the assumption (F-4). Since $v = F_2 v + u \in B_2$, we see that $I_{B_2} - F_2 : B_2 \rightarrow B_2$ is surjective.

On the other hand, the injectivity of $I_B - F_2 : B \rightarrow B$ leads to that of $I_{B_2} - F_2 : B_2 \rightarrow B_2$. This completes the proof.

Let $G_1 : B_1 \oplus B_2 \rightarrow B_1 \oplus B_2$ and $G_2 : B_1 \oplus B_2 \rightarrow B_1 \oplus B_2$ be Borel maps given by

$$(4.1) \quad G_1(x, y) = (x, y - F_2(y)) \text{ and}$$

$$(4.2) \quad G_2(x, y) = (G_2^{(1)}(x, y), G_2^{(2)}(x, y)) \\ = (x - \bar{P}_1 F(x+y), y - F_2(x+y))$$

for each $(x, y) \in B_1 \oplus B_2$.

Then we have the following

Proposition 4.2. (1) $G_1 : B_1 \oplus B_2 \rightarrow B_1 \oplus B_2$ is bijective.

(2) $G_2 : B_1 \oplus B_2 \rightarrow B_1 \oplus B_2$ is bijective and

$$G_2^{-1}(x, y) = (x + \bar{P}_1 F(I_B - F)^{-1}(x+y), y + F_2(I_B - F)^{-1}(x+y))$$

for any $(x, y) \in B_1 \oplus B_2$.

$$(3) \quad G_2^{(1)}(x, y) + G_2^{(2)}(x, y) = (I_B - F)(x+y) \text{ for any}$$

$(x, y) \in B_1 \oplus B_2$.

Proof. The assertions (1) and (3) are obvious. Let us prove our assertion (2). Let $J : B_1 \oplus B_2 \rightarrow B_1 \oplus B_2$ be a Borel map given by

$$\begin{aligned}
J(x,y) &= (J^{(1)}(x,y), J^{(2)}(x,y)) \\
&= (x + \bar{P}_1 F(I_B - F)^{-1}(x+y), y + F_2(I_B - F)^{-1}(x+y))
\end{aligned}$$

for any $(x,y) \in B_1 \oplus B_2$. Then it is obvious that

$$J^{(1)}(x,y) + J^{(2)}(x,y) = (I_B - F)^{-1}(x+y). \quad \text{Therefore we get}$$

$$\begin{aligned}
J \circ G_2(x,y) &= (G_2^{(1)}(x,y) + \bar{P}_1 F(x+y), G_2^{(2)}(x,y) + F_2(x+y)) \\
&= (x,y), \text{ and}
\end{aligned}$$

$$\begin{aligned}
G_2 \circ J(x,y) &= (J^{(1)}(x,y) - \bar{P}_1 F(I_B - F)^{-1}(x+y), \\
&\quad J^{(2)}(x,y) - F_2(I_B - F)^{-1}(x+y)) \\
&= (x,y).
\end{aligned}$$

This completes the proof.

Step 2. It is clear that $(\mu_1 \otimes \mu_2, H_1 \oplus H_2, B_1 \oplus B_2)$ is an abstract Wiener space. Let $K : B_1 \oplus B_2 \rightarrow B_1 \oplus B_2$ be a Borel map given by

$$K(x,y) = (x,y) - G_2 \circ G_1^{-1}(x,y) \quad \text{for each } (x,y) \in B_1 \oplus B_2.$$

Then it is obvious that

$$\begin{aligned}
(4.3) \quad K(x,y) &= (\bar{P}_1 F(x + (I_{B_2} - F_2)^{-1}y), F_2(x + (I_{B_2} - F_2)^{-1}y) - F_2((I_{B_2} - F_2)^{-1}y))
\end{aligned}$$

for each $(x,y) \in B_1 \oplus B_2$. Thus by the assumption (F-5), we

see that K is a Borel map defined on $B_1 \oplus B_2$ into $H_1 \oplus H_2$.

For each $(x,y) \in B_1 \oplus B_2$, let $DK(x,y) : H_1 \oplus H_2 \rightarrow H_1 \oplus H_2$ be a bounded linear operator given by

$$(4.4) \quad DK(x,y)(h_1, h_2) = (DK^{(1)}(x,y)(h_1, h_2), DK^{(2)}(x,y)(h_1, h_2))$$

for each $(h_1, h_2) \in H_1 \oplus H_2$, where

$$\begin{aligned}
&DK^{(1)}(x,y)(h_1, h_2) \\
&= P_1 DF(x + (I_{B_2} - F_2)^{-1}y) (I_H - P_2 DF((I_{B_2} - F_2)^{-1}y) P_2)^{-1} (h_1 + h_2),
\end{aligned}$$

and

$$\begin{aligned}
& DK^{(2)}(x,y)(h_1,h_2) \\
&= P_2 DF(x+(I_{B_2}-F_2)^{-1}y) (I_H - P_2 DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2) \\
&\quad - P_2 DF((I_{B_2}-F_2)^{-1}y)P_2 (I_H - P_2 DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2).
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
& \|K(x+h_1,y+h_2) - K(x,y) - DK(x,y)(h_1,h_2)\|_{H_1 \oplus H_2} \\
&= o(\|h_1\|_{H_1} + \|h_2\|_{H_2}), \quad \|h_1\|_{H_1} + \|h_2\|_{H_2} \rightarrow 0,
\end{aligned}$$

for each $(x,y) \in B_1 \oplus B_2$. By the assumptions on F , we also see that $DK(x,y) : H_1 \oplus H_2 \rightarrow H_1 \oplus H_2$ is a Hilbert-Schmidt operator and $DK(x+\cdot, y+\cdot) : H_1 \oplus H_2 \rightarrow L^2(H_1 \oplus H_2, H_1 \oplus H_2)$ is continuous for each $(x,y) \in B_1 \oplus B_2$.

Note that

$$\begin{aligned}
(4.5) \quad & (DK^{(1)}(x,y) + DK^{(2)}(x,y))(h_1,h_2) \\
&= (DF(x+(I_{B_2}-F_2)^{-1}y) - P_2 DF((I_{B_2}-F_2)^{-1}y)P_2) \\
&\quad \cdot (I_H - P_2 DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2), \text{ and that}
\end{aligned}$$

$$\begin{aligned}
(4.6) \quad & (h_1+h_2) - (DK^{(1)}(x,y) + DK^{(2)}(x,y))(h_1,h_2) \\
&= (I_H - DF(x+(I_{B_2}-F_2)^{-1}y)) (I_H - P_2 DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2)
\end{aligned}$$

for each $(x,y) \in B_1 \oplus B_2$ and $(h_1,h_2) \in H_1 \oplus H_2$. Thus

$I_{H_1 \oplus H_2} - DK(x,y) : H_1 \oplus H_2 \rightarrow H_1 \oplus H_2$ is invertible for any $(x,y) \in B_1 \oplus B_2$.

$$\begin{aligned}
& \text{Let } \bar{H}_0^{(n)} : B_1 \oplus B_2 \rightarrow \mathbb{R}, \quad n=1,2,\dots, \text{ be Borel functions given by} \\
& \bar{H}_0^{(n)}(x,y) = B_1 * \oplus B_2 * \langle P_1^{(n)}, P_2^{(n)} \rangle K(x,y), (x,y) \rangle_{B_1 \oplus B_2} \\
& \quad - \text{trace}_{H_1 \oplus H_2} (P_1^{(n)}, P_2^{(n)}) DK(x,y)
\end{aligned}$$

for each $(x,y) \in B_1 \oplus B_2$, where $(P_1^{(n)}, P_2^{(n)})$ denotes the orthogonal projection on $H_1 \oplus H_2$ such that

$(P_1^{(n)}, P_2^{(n)})(h_1, h_2) = (P_1^{(n)}h_1, P_2^{(n)}h_2)$ for each

$(h_1, h_2) \in H_1 \oplus H_2$. Then we have

$$\begin{aligned} & \bar{H}_0^{(n)}(x, y) \\ = & B^* \langle P^{(n)}(F(x + (I_{B_2} - F_2)^{-1}y) - F_2((I_{B_2} - F_2)^{-1}y)), x + y \rangle_B \\ & - \text{trace}_H P^{(n)}(DF(x + (I_{B_2} - F_2)^{-1}y) - P_2 DF((I_{B_2} - F_2)^{-1}y)P_2) \\ & \cdot (I_H - P_2 DF((I_{B_2} - F_2)^{-1}y)P_2)^{-1}. \end{aligned}$$

Therefore we have got

$$(4.7) \quad \bar{H}_0^{(n)}(x, y) = H_0^{(n)}(x, (I_{B_2} - F_2)^{-1}y)$$

for each $(x, y) \in B_1 \oplus B_2$ and $n = 1, 2, \dots$. According to [4, Corollary to Theorem 4.2], we see that there exists a Borel function $\bar{H}_0 : B_1 \oplus B_2 \rightarrow \mathbb{R}$ such that

$$(4.8) \quad \bar{H}_0^{(n)}(x, y) \rightarrow \bar{H}_0(x, y), \quad n \rightarrow \infty, \text{ in probability with respect to } \mu_1(dx) \otimes \mu_2(dy).$$

Furthermore by virtue of [4, Theorem 6.3], we see that $(I_{B_1 \oplus B_2} - K)^{-1} \mu_1 \otimes \mu_2$ and $\mu_1 \otimes \mu_2$

are mutually absolutely continuous, and that

$$\begin{aligned} & (I_{B_1 \oplus B_2} - K)^{-1} \mu_1 \otimes \mu_2(dx \times dy) \\ = & |\delta_{H_1 \oplus H_2}(I_{H_1 \oplus H_2} - DK(x, y))| \exp(\bar{H}_0(x, y) - \frac{1}{2} \|K(x, y)\|_H^2) \\ & \mu_1(dx) \otimes \mu_2(dy). \end{aligned}$$

Thus by (4.3) and (4.6), we obtain

$$\begin{aligned} (4.9) \quad & G_1 \circ G_2^{-1} \mu_1 \otimes \mu_2(dx \times dy) \\ = & |\delta_H((I_H - DF(x + (I_{B_2} - F_2)^{-1}y))(I_H - P_2 DF((I_{B_2} - F_2)^{-1}y)P_2)^{-1})| \\ & \exp(\bar{H}_0(x, y) - \frac{1}{2} \|F(x + (I_{B_2} - F_2)^{-1}y) - F_2((I_{B_2} - F_2)^{-1}y)\|_H^2) \\ & \mu_1(dx) \otimes \mu_2(dy). \end{aligned}$$

So it is easy to see that

$$(4.10) \quad G_2^{-1} \mu_1 \otimes \mu_2 (dx \times dy) = \rho(x, y) \mu_1(dx) \otimes (I_{B_2} - F_2)^{-1} \mu_2(dy),$$

where

$$\rho(x, y) = \left| \delta_H \left((I_H - DF(x+y)) (I_H - P_2 DF(y) P_2)^{-1} \right) \right| \\ \cdot \exp \left(\bar{H}_0(x, (I_{B_2} - F_2)y) - \frac{1}{2} \|F(x+y) - F_2(y)\|_H^2 \right)$$

Note that (4.7) and (4.8) implies that

$$(4.11) \quad H_0^{(n)}(x, y) \rightarrow \bar{H}_0(x, (I_{B_2} - F_2)y) \quad , \quad n \rightarrow \infty, \text{ in probability}$$

with respect to $\mu_1(dx) \otimes (I_{B_2} - F_2)^{-1} \mu_2(dy)$.

Step 3. Let us prove the following.

Proposition 4.3.

$$(1) \quad \int_{B_1 \oplus B_2} f(x+y) G_2^{-1} \mu_1 \otimes \mu_2 (dx \times dy) = \int_B f(z) \nu(dz)$$

for any bounded Borel function $f : B \rightarrow \mathbb{R}$.

$$(2) \quad \int_{B_1 \oplus B_2} g(x, y) G_2^{-1} \mu_1 \otimes \mu_2 (dx \times dy) = \int_B g(\pi_1 z, \pi_2 z) \nu(dz)$$

for any bounded Borel function $g : B_1 \oplus B_2 \rightarrow \mathbb{R}$.

Proof. Let $f : B \rightarrow \mathbb{R}$ be a bounded Borel function. By Proposition 2.3.(2) and Proposition 4.2.(1), we see that

$$\begin{aligned} & \int_{B_1 \oplus B_2} f(x+y) G_2^{-1} \mu_1 \otimes \mu_2 (dx \otimes dy) \\ &= \int_{B_1 \oplus B_2} f(x + y + F(I_B - F)^{-1}(x+y)) \mu_1(dx) \otimes \mu_2(dy) \\ &= \int_B f((I_B - F)^{-1}z) \mu(dz) \\ &= \int_B f(z) \nu(dz). \end{aligned}$$

This proves our assertion (1).

Now let $g_1 : B_1 \rightarrow \mathbb{R}$ and $g_2 : B_2 \rightarrow \mathbb{R}$ be bounded Borel functions. Then it follows from Proposition 2.2, 3.2, (4.10) and our assertion (1) that $\pi_1(x+y) = x$ and $\pi_2(x+y) = y$ for $G_2^{-1} \mu_1 \otimes \mu_2$ -a.e. (x, y) . Thus we have got by Proposition 2.2 and our assertion (1),

$$\begin{aligned} & \int_{B_1 \oplus B_2} g_1(x) g_2(y) G_2^{-1} \mu_1 \otimes \mu_2(dx \times dy) \\ = & \int_{B_1 \oplus B_2} g_1(\pi_1(x+y)) g_2(\pi_2(x+y)) G_2^{-1} \mu_1 \otimes \mu_2(dx \times dy) \\ = & \int_B g_1(\pi_1 z) g_2(\pi_2 z) \nu(dz). \end{aligned}$$

This proves our assertion (2). This completes the proof.

Now we will complete the proof of Theorem 1. It follows from Proposition 4.3.(2) and (4.10) that $\pi_2 \nu$ and $(I_{B_2} - F_2)^{-1} \mu_2$ are mutually absolutely continuous. Therefore (4.11) implies that $H_0^{(n)}(x, y) \rightarrow \bar{H}_0(x, (I_{B_2} - F_2)y)$, $n \rightarrow \infty$, in probability with respect to $\pi_1 \mu(dx) \otimes \pi_2 \nu(dy)$. This shows Theorem 1.(1) and $H_0(x, y) = \bar{H}_0(x, (I_{B_2} - F_2)y)$.

Let $f : B \rightarrow \mathbb{R}$ be a bounded Borel function and $g : B \rightarrow \mathbb{R}$ be an F_2 -measurable bounded function. Then it follows from Proposition 3.1, 4.3 and (4.10) that

$$\begin{aligned} & \int_B f(z) g(z) \nu(dz) \\ = & \int_{B_1 \oplus B_2} f(x+y) g(x+y) G_2^{-1} \mu_1 \otimes \mu_2(dx \times dy) \\ = & \int_{B_1 \oplus B_2} f(x+y) g(y) \rho(x, y) \mu_1(dx) \otimes (I_{B_2} - F_2)^{-1} \mu_2(dy) \end{aligned}$$

$$\begin{aligned}
&= \int_{B_1 \oplus B_2} g(y) \frac{\int_{B_1} f(\tilde{x} + y) \rho(\tilde{x}, y) \mu_1(d\tilde{x})}{\int_{B_1} \rho(\tilde{x}, y) \mu_1(d\tilde{x})} \rho(x, y) \\
&\quad \mu_1(dx) \otimes (I_{B_2} - F_2)^{-1} \mu_2(dy) \\
&= \int_B g(\pi_2 z) \tilde{f}(z) \nu(dz), \text{ where} \\
\tilde{f}(z) &= \frac{\int_B f(\pi_1 \tilde{z} + \pi_2 z) \rho(\pi_1 \tilde{z}, \pi_2 z) \mu(d\tilde{z})}{\int_B \rho(\pi_1 \tilde{z}, \pi_2 z) \mu(d\tilde{z})}.
\end{aligned}$$

Since $g(\pi_2 z) = g(z)$ for ν -a.e. z and \tilde{f} is $F_2 \vee N_\nu$ -measurable by Proposition 3.2 and 3.3, we have got $E[f|F_2](z) = \tilde{f}(z)$ for ν -a.e. z . This completes the proof.

Proposition 4.4. Suppose that there exists a constant C , $0 < C < 1$, such that $\|DF(z)\|_{L^\infty(H, H)} \leq C$ for any $z \in B$.

Then (F-1) and (F-2) lead to (F-4).

Proof. Since $\|P_2 DF(z)\|_{L^\infty(H, H)} \leq C$ for any $z \in B$,

$I_H - P_2 DF(z) : H \rightarrow H$ is invertible for any $z \in B$. Therefore it suffices to prove that $I_B - F_2 : B \rightarrow B$ is bijective under the assumptions (F-1) and (F-2). It is easy to see that

$$\|F(z+h) - F(z)\|_H = \left\| \int_0^1 DF(z+th)h dt \right\|_H \leq C \|h\|_H, \text{ and}$$

$$\|F_2(z+h) - F_2(z)\|_H = \|P_2(F(z+h) - F(z))\|_H \leq C \|h\|_H$$

for any $z \in B$ and $h \in H$. Therefore $I_H - (F(z+\cdot) - F(z)) : H \rightarrow H$ and $I_H - (F_2(z+\cdot) - F_2(z)) : H \rightarrow H$ are bijective for any $z \in B$ by virtue of the fixed point theorem for contraction map.

Now let us prove the injectivity of $I_B - F_2 : B \rightarrow B$.

Suppose that $(I_B - F_2)z_1 = (I_B - F_2)z_2$ for some $z_1, z_2 \in B$.

Then we get $(I_B - F)z_2 = (I_B - F)z_1 + k$, where

$k = \bar{P}_1 F(z_1) - \bar{P}_1 F(z_2) \in H$. Since $I_H - (F(z_1 + \cdot) - F(z_1)) : H \rightarrow H$

is bijective, there exists some $h \in H$ such that

$h - (F(z_1 + h) - F(z_1)) = k$. Thus $(z_1 + h) - F(z_1 + h) = z_1 - F(z_1) + k$.

Since $I_B - F : B \rightarrow B$ is bijective by (F-1), we get $z_2 = z_1 + h$.

Hence $h - (F_2(z_1 + h) - F_2(z_1)) = (I_B - F_2)z_2 - z_1 + F_2(z_1) = 0$.

The injectivity of $I_H - (F_2(z_1 + \cdot) - F_2(z_1)) : H \rightarrow H$ implies

$h = 0$, and accordingly we have got $z_1 = z_2$. This shows the injectivity of $I_B - F_2 : B \rightarrow B$.

Let w be an arbitrary element of B . Let $z = (I_B - F)^{-1}w$.

Since $\bar{P}_1 F(z) \in H$, there exists some $h \in H$ such that

$h - (F_2(z + h) - F_2(z)) = -\bar{P}_1 F(z)$. Then we obtain

$$\begin{aligned} (I_B - F_2)(z + h) &= z - F_2 z - \bar{P}_1 F(z) \\ &= (I_B - F)z = w. \end{aligned}$$

This shows the surjectivity of $I_B - F_2 : B \rightarrow B$. This completes the proof.

By using Schwartz [6, 1.22. Theorem], we can also prove the following similarly to Proposition 4.4.

Proposition 4.5. Suppose that $I_H - DF(z) : H \rightarrow H$ and

$I_H - P_2 DF(z) : H \rightarrow H$ are invertible for any $z \in B$ and that

there exists a constant $K > 0$ such that

$$\|(I_H - DF(z))^{-1}\|_{L^\infty(H, H)} \leq K \quad \text{and} \quad \|(I_H - P_2 DF(z))^{-1}\|_{L^\infty(H, H)} \leq K$$

for any $z \in B$. Then (F-1) and (F-2) lead to (F-4).

5. Application.

In this section we will consider the solution of the stochastic pseudo-differential equation treated in [5, Section 5]. We will use the notation introduced in [5] sometimes without explanation.

Let $p(\xi) \in \tilde{S}^m$, $m \in \mathbb{R}$, such that $p(\xi) \neq 0$ for any $\xi \in \mathbb{R}^d$ and $p(\xi)^{-1} \in \tilde{S}^{-m}$, and let $q_j(\xi) \in \tilde{S}^r$, $j=1, \dots, n$ and $r \in \mathbb{R}$. Moreover let $b: \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded smooth function such that

$$\|\partial_j b\|_\infty = \sup\left\{\left|\frac{\partial b}{\partial y_j}(y)\right|; y \in \mathbb{R}^n\right\} < \infty, \text{ and}$$

$$\|\partial_{ij} b\|_\infty = \sup\left\{\left|\frac{\partial^2 b}{\partial y_i \partial y_j}(y)\right|; y \in \mathbb{R}^n\right\} < \infty$$

for any $i, j = 1, \dots, n$. Now let us consider the following stochastic pseudo-differential equation

$$(5.1) \quad p(D_x)X - b(q_1(D_x)X, \dots, q_n(D_x)X) = W,$$

where W is a Gaussian white noise with d -dimensional parameter. Let $Y = p(D_x)^{-1}W$. Then we get

$$(5.2) \quad X - p(D_x)^{-1}b(q_1(D_x)X, \dots, q_n(D_x)X) = Y.$$

Assume that $m > r + \frac{d}{2}$ and $\sum_{j=1}^n \|\partial_j b\|_\infty \cdot \|q_j p^{-1}\|_{L^\infty} < 1$.

Then according to [5, Theorem 3], there exists the unique solution X of the equation (5.1). Let D be a bounded domain in \mathbb{R}^d with smooth boundary. Let us make some preparation to study about the σ -fields F_D and I_D as in Introduction.

Let $\sigma^t(x) = \langle x \rangle^t$ and $\rho^s(x) = \langle x \rangle^s$, $x \in \mathbb{R}^d$, for each

for each $t, s \in \mathbb{R}$. Let $W_2^{\sigma^t, \rho^s}$ be a Banach space with a

norm $\| \cdot \|_{W_2^{\sigma^t, \rho^s}}$, the same as in [5], given by

$$W_2^{\sigma^t, \rho^s} = \{ u \in S'(\mathbb{R}^d) ; \rho^s(X) \sigma^t(D_x) u \in L^2(\mathbb{R}^d) \}, \text{ and}$$

$$\|u\|_{W_2^{\sigma^t, \rho^s}} = \| \rho^s(X) \sigma^t(D_x) u \|_{L^2} \text{ for each } u \in W_2^{\sigma^t, \rho^s}.$$

The following has been shown in [5, Theorem 2].

Proposition 5.1. For any $s, t \in \mathbb{R}$ and any pseudo-differential operator P belonging to \tilde{S}^0 , there exists a constant $C > 0$ such that

$$\|Pu\|_{W_2^{\sigma^t, \rho^s}} \leq C \|u\|_{W_2^{\sigma^t, \rho^s}} \text{ for any } u \in W_2^{\sigma^t, \rho^s}. \text{ Therefore}$$

P can be considered a bounded linear operator in $W_2^{\sigma^t, \rho^s}$.

Let $\sigma_\eta^t(x) = \langle \eta \cdot x \rangle^t$ and $\rho_\lambda^s(x) = \langle \lambda \cdot x \rangle^s$, $x \in \mathbb{R}^d$, for each $t, s \in \mathbb{R}$ and $\eta, \lambda \in (0, 1]$, and let $W_2^{\sigma_\eta^t, \rho_\lambda^s}$ be a Banach space with a norm $\| \cdot \|_{W_2^{\sigma_\eta^t, \rho_\lambda^s}}$ given by

$$W_2^{\sigma_\eta^t, \rho_\lambda^s} = \{ u \in S'(\mathbb{R}^d) ; \rho_\lambda^s(X) \sigma_\eta^t(D_x) u \in L^2(\mathbb{R}^d) \}, \text{ and}$$

$$\|u\|_{W_2^{\sigma_\eta^t, \rho_\lambda^s}} = \| \rho_\lambda^s(X) \sigma_\eta^t(D_x) u \|_{L^2} \text{ for each } u \in W_2^{\sigma_\eta^t, \rho_\lambda^s}.$$

Then we have got the following.

Proposition 5.2. $W_2^{\sigma_\eta^t, \rho_\lambda^s} = W_2^{\sigma^t, \rho^s}$ as a set and the norms

$$\| \cdot \|_{W_2^{\sigma_\eta^t, \rho_\lambda^s}} \text{ and } \| \cdot \|_{W_2^{\sigma^t, \rho^s}} \text{ are equivalent for any } s, t \in \mathbb{R}$$

and $\eta, \lambda \in (0, 1]$.

Proof. It is obvious that $\|u\|_{W_2^{\sigma_\eta^t, \rho_\lambda^s}} =$

$$\| \rho_\lambda^s(X) \rho^s(X)^{-1} (\rho^s(X) \sigma_\eta^t(D_x) \sigma^t(D_x)^{-1} \rho^s(X)) \rho^s(X) \sigma^t(D_x) u \|_{L^2}$$

for any $u \in S(\mathbb{R}^d)$. Since $\rho_\lambda^s(x) \rho^s(x)^{-1}$ and $\rho^s(x) \sigma_\eta^t(D_x) \sigma^t(D_x)^{-1} \rho^s(x)^{-1}$ are pseudo-differential operators belonging to \tilde{S}^0 by virtue of [5, corollary to Lemma 4.1], it follows from Proposition 5.1 that there exists a constant $C > 0$ such that

$$\begin{aligned} \|u\|_{W_2^{\sigma_\eta^t, \rho_\lambda^s}} &\leq C \|\rho^s(x) \sigma^t(D_x) u\|_{L^2} \\ &= C \|u\|_{W_2^{\sigma_\eta^t, \rho^s}} \quad \text{for any } u \in S(\mathbb{R}^d). \end{aligned}$$

Similarly we see that there exists a constant $C' > 0$ such that

$$\|u\|_{W_2^{\sigma_\eta^t, \rho^s}} \leq C' \|u\|_{W_2^{\sigma_\eta^t, \rho_\lambda^s}} \quad \text{for any } u \in S(\mathbb{R}^d). \quad \text{This}$$

proves our assertion.

Let $t_0 = -\frac{1}{2}(m-r+\frac{d}{2})$ and $s_0 = -\frac{d}{2}-1$. Then it is obvious that $\sigma^{t_0, \rho^{s_0}} \in L^2(\mathbb{R}^d)$. Let U_1 and U_2 be bounded domains in \mathbb{R}^d such that $\bar{D} \subset U_1 \subset \bar{U}_1 \subset U_2$, and let $g: \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function such that $g(x) > 0$, $x \in \mathbb{R}^d$, $g(x) = 1$ for any $x \in U_1$ and $g(x) = \rho^{s_0}(x) = \langle x \rangle^{s_0}$ for any $x \in U_2^c$, where \bar{D} and \bar{U}_1 denote the closure of D and U_1 and U_2^c denotes the complement of U_2 in \mathbb{R}^d . Note that $g \in L^2(\mathbb{R}^d)$.

Let $r(\xi) = p(\xi)p(-\xi)$, $\xi \in \mathbb{R}^d$, and let

$$\begin{aligned} A_1 &= g(X)^{-1} r(D_x)^{-1} g(X) r(D_x), \quad A_2 = p(D_x) g(X) p(D_x)^{-1} g(X)^{-1} \\ \text{and } A_3 &= g(X)^{-1} p(D_x) g(X) p(D_x)^{-1}. \end{aligned}$$

Then we get the following.

Proposition 5.3. (1) A_1, A_2 and A_3 are pseudo-differential operators belonging to \tilde{S}^0 .

(2) $g(X)$ can be considered a continuous linear map from $W_2^{\sigma^t, \rho^{s_0}}$ into $W_2^{\sigma^t, \rho^0}$ for any $t \in \mathbb{R}$.

Proof. Our assertion (1) is an immediate consequence of [5, Corollary to Lemma 4.1]. It is obvious that

$$\begin{aligned} \|g(X)u\|_{W_2^{\sigma^t, \rho^0}} &= \|\sigma^t(D_x)g(X)u\|_{L^2} \\ &= \|\sigma^t(D_x)g(X)\sigma^t(D_x)^{-1}g(X)^{-1}(g(X)\rho^{s_0}(X)^{-1})\rho^{s_0}(X)\sigma^t(D_x)u\|_{L^2} \end{aligned}$$

for any $u \in S(\mathbb{R}^d)$. Therefore the proof of our assertion (2) goes similarly to that of Proposition 5.2.

Now let $G: W_2^{\sigma^{t_0}, \rho^{s_0}} \rightarrow W_2^{\sigma^0, \rho^{s_0}}$ be a continuous map given by

$$Gu(x) = b(q_1(D_x)p(D_x)^{-1}u(x), \dots, q_n(D_x)p(D_x)^{-1}u(x)),$$

$x \in \mathbb{R}^d$, for each $u \in W_2^{\sigma^{t_0}, \rho^{s_0}}$. Then it follows from the

proof of [5, Theorem 3] and Proposition 5.2 that

$$I_{W_2^{\sigma^{t_0}, \rho^{s_0}}} - G : W_2^{\sigma^{t_0}, \rho^{s_0}} \rightarrow W_2^{\sigma^{t_0}, \rho^{s_0}} \text{ is bijective.}$$

Let $t_1 = t_0 + m = \frac{1}{2}(m + r - \frac{d}{2})$, and let B denote $W_2^{\sigma^{t_1}, \rho^{s_0}}$ and B_0 denote $W_2^{\sigma^m, \rho^{s_0}}$. By virtue of [5, Theorem 2], $p(D_x)$

can be considered a bijective bicontinuous linear map from B onto $W_2^{\sigma^{t_0}, \rho^{s_0}}$ and also considered a bijective bicontinuous

linear map from B_0 onto $W_2^{\sigma^0, \rho^0}$. Therefore we can define a continuous linear map $F : B \rightarrow B_0$ by

$$Fu = p(D_x)^{-1} G p(D_x) u \quad \text{for each } u \in B, \text{ and we see that}$$

$I_B - F : B \rightarrow B$ is bijective.

Let μ be a probability measure on $S'(\mathbb{R}^d)$ such that

$$\begin{aligned} \int_{S'(\mathbb{R}^d)} \exp(\sqrt{-1} \langle f, w \rangle_{S'}) \mu(dw) \\ = \exp\left(-\frac{1}{2} \|p(-D_x)^{-1} f\|_{L^2}^2\right) \end{aligned}$$

for any $f \in S(\mathbb{R}^d)$. Then μ is the probability law of Y .

It follows from [5, Theorem 1] that $\mu(B) = 1$. Thus by (5.2), we see that $\nu = (I_B - F)^{-1} \mu$ is the probability law of X . Let H be a Hilbert space with an inner product

$(\cdot, \cdot)_H$ given by

$$H = \{ u \in S'(\mathbb{R}^d) ; p(D_x)u \in L^2(\mathbb{R}^d) \}, \text{ and}$$

$$(u, v)_H = (p(D_x)u, p(D_x)v)_{L^2} \quad \text{for each } u, v \in H. \quad \text{Then it is}$$

easy to see that $H = W_2^{\sigma^m, \rho^0}$ as a set.

Let us identify the dual space H^* with H . Then it is easy to see that $S(\mathbb{R}^d) \subset B^* \subset H \subset B_0 \subset B$ and

$$(5.3) \quad (u, v)_H = {}_B \langle u, v \rangle_{B^*} = (u, r(D_x)v)_{L^2}$$

for any $u, v \in S(\mathbb{R}^d)$. Therefore for any $u \in S(\mathbb{R}^d)$, we obtain

$$\begin{aligned} \int_B \exp(\sqrt{-1} {}_B \langle u, w \rangle_{B^*}) \mu(dw) \\ = \int_{S'(\mathbb{R}^d)} \exp(\sqrt{-1} \langle r(D_x)u, w \rangle_{S'}) \mu(dw) \end{aligned}$$

$$= \exp\left(-\frac{1}{2} \|p(-D_x)^{-1} r(D_x)u\|_L^2\right)$$

$$= \exp\left(-\frac{1}{2} \|u\|_H^2\right).$$

Therefore (u, H, B) is an abstract Wiener space.

Recall that D is a bounded domain in \mathbb{R}^d with smooth boundary, and let H_1 and H_2 be closed linear subspaces of H given by

(5.4) $H_1 = \{ u \in H \subset S'(\mathbb{R}^d) ; \text{ the support of } r(D_x)u \text{ is contained in the closure } \bar{D} \text{ of } D \}, \text{ and}$

(5.5) $H_2 = \{ u \in H \subset S'(\mathbb{R}^d) ; \text{ the support of } u \text{ is contained in the complement } D^c \text{ of } D \}.$

Then it is obvious that H_1 and H_2 are orthogonal and $H = H_1 + H_2$. Let B_1 and B_2 be the closure of H_1 and H_2 in B respectively. Then it is easy to see that

(5.6) $B_1 = \{ u \in B \subset S'(\mathbb{R}^d) ; \text{ the support of } r(D_x)u \text{ is contained in } \bar{D} \}, \text{ and}$

(5.7) $B_2 = \{ u \in B \subset S'(\mathbb{R}^d) ; \text{ the support of } u \text{ is contained in } D^c \}.$

Now we get the following.

Proposition 5.4. The assumptions (A-1) and (A-2) hold.

That is,

(1) $B_0 \cap B_1 \cap B_2 = \{0\}$, and

(2) the orthogonal projection $P_1 : H \rightarrow H_1$ is extensible to a bounded linear map $\bar{P}_1 : B_0 \rightarrow H_1$.

Proof. Since $g(x)^{-1} = 1$ around D , we get

$$\begin{aligned} (5.8) \quad g(X)^{-1}u &= g(X)^{-1}r(D_x)^{-1}g(X)g(X)^{-1}r(D_x)u \\ &= A_1 u \end{aligned}$$

for any $u \in B_1$.

Suppose that $u \in B_0 \cap B_1 \cap B_2$. Then Proposition 5.1 and 5.3

(1) show that $g(X)^{-1}u = A_1 u \in B_0$. Thus by Proposition 5.3

(2), we see that $u = g(X)g(X)^{-1}u \in H$. However, it is obvious that $H \cap B_1 = H_1$ and $H \cap B_2 = H_2$. Therefore $u \in H_1 \cap H_2 = \{0\}$.

This proves (A-1).

Now let us prove (A-2). By (5.3), we see that for any $u \in S(\mathbb{R}^d)$ and $v \in H$,

$$\begin{aligned} (P_1 u, v)_H &= S \langle u, r(D_x) P_1 v \rangle_S, \\ &= S \langle g(X) u, r(D_x) P_1 v \rangle_S, \\ &= (P_1 g(X) u, v)_H. \end{aligned}$$

Therefore we get

$$(5.9) \quad P_1 u = P_1 g(X) u \quad \text{for any } u \in S(\mathbb{R}^d).$$

Hence due to Proposition 5.3 (2), we obtain (A-2). This completes the proof.

Since B_0 is reflexive, we see that B_0, H_1 and H_2 satisfy all the assumptions in Section 2. Now let us study about the property of the Borel map $F: B \rightarrow B_0$. For each $w \in B$, let $f(x; w) = b(q_1(D_x)w(x), \dots, q_n(D_x)w(x))$, $x \in \mathbb{R}^d$, and

$$f_j(x; w) = \frac{\partial b}{\partial y_j}(q_1(D_x)w(x), \dots, q_n(D_x)w(x)), \quad j = 1, \dots, n$$

and $x \in \mathbb{R}^d$, and let $T_j(w) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, $j = 1, \dots, n$, be bounded linear operators given by

$$T_j(w)u(x) = f_j(x; w)u(x), \quad x \in \mathbb{R}^d, \quad \text{for each } u \in L^2(\mathbb{R}^d).$$

Note that $p(D_x)$ can be considered a bijective isometry from H into $L^2(\mathbb{R}^d)$. Now let $DF(w) : H \rightarrow H$ be a bounded linear operator given by

$$(5.10) \quad DF(w)h = \sum_{j=1}^n p(D_x)^{-1} T_j(w) q_j(D_x) p(D_x)^{-1} p(D_x) h,$$

$h \in H$, for each $w \in B$. It is obvious that DF is well-defined.

It is easy to see that for any $w \in B$,

$$\begin{aligned} (5.11) \quad & \| DF(w) \|_{L^\infty(H,H)} \\ &= \| p(D_x) DF(w) p(D_x)^{-1} \|_{L^\infty(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} \\ &\leq \sum_{j=1}^n \| T_j(w) \|_{L^\infty(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} \\ &\quad \times \| q_j(D_x) p(D_x)^{-1} \|_{L^\infty(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))} \\ &\leq \sum_{j=1}^n \| \partial_j b \|_\infty \| q_j \cdot p^{-1} \|_{L^\infty} < 1. \end{aligned}$$

Since $m-r > \frac{d}{2}$, by virtue of Sobolev's lemma there exists a constant $C > 0$ such that

$$\| q_j(D_x) p(D_x)^{-1} u \|_{L^\infty} \leq C \| u \|_{L^2}, \quad j=1, \dots, n, \text{ for any } u \in L^2(\mathbb{R}^d).$$

Thus we get for any $w \in B$ and $h \in H$,

$$\begin{aligned} (5.12) \quad & \| DF(w+h) - DF(w) \|_{L^\infty(H,H)} \\ &\leq \sum_{j=1}^n \| f_j(\cdot; w+h) - f_j(\cdot; w) \|_{L^\infty} \| q_j \cdot p^{-1} \|_{L^\infty} \\ &\leq \sum_{i,j=1}^n \| \partial_{ij} b \|_\infty \| q_i(D_x) p(D_x)^{-1} p(D_x) h \|_{L^\infty} \| q_j \cdot p^{-1} \|_{L^\infty} \\ &\leq C \left(\sum_{i,j=1}^n \| \partial_{ij} b \|_\infty \| q_j \cdot p^{-1} \|_{L^\infty} \right) \| h \|_H. \end{aligned}$$

Therefore $DF(w+\cdot) : H \rightarrow L^\infty(H,H)$ is continuous for any $w \in B$.

It is obvious that for any $w \in B$ and $h \in H$,

$$F(w+h) - F(w) = \int_0^1 DF(w+th) h \, dt,$$

which implies that $F(w+h) - F(w) \in H$ and

$$\|F(w+h) - F(w) - DF(w)h\|_H = o(\|h\|_H), \quad \|h\|_H \rightarrow 0. \quad \text{Thus}$$

by (5.11) and Proposition 4.4, we get the following.

Proposition 5.5. The Borel map F satisfies the assumptions (F-1), (F-2) and (F-4).

Now let us prove the following.

Proposition 5.6. The Borel map F satisfies the assumption (F-3).

Proof. It follows from (5.8) and (5.9) that

$$\begin{aligned} (5.13) \quad P_1 DF(w) &= \sum_{j=1}^n P_1 g(X) p(D_x)^{-1} T_j(w) q_j(D_x) p(D_x)^{-1} p(D_x) \\ &= \sum_{j=1}^n P_1 p(D_x)^{-1} A_2 T_j(w) (g(X) q_j(D_x) p(D_x)^{-1}) p(D_x) \end{aligned}$$

and

$$\begin{aligned} (5.14) \quad DF(w) P_1 &= \sum_{j=1}^n p(D_x)^{-1} T_j(w) q_j(D_x) p(D_x)^{-1} p(D_x) g(X) g(X)^{-1} P_1 \\ &= \sum_{j=1}^n p(D_x)^{-1} T_j(w) (q_j(D_x) p(D_x)^{-1} g(X)) A_3 p(D_x) A_1 P_1. \end{aligned}$$

Note that A_1 can be considered a bounded linear operator in H and that A_2 and A_3 can be considered bounded linear

operators in $L^2(\mathbb{R}^d)$, due to Proposition 5.1 and 5.3.

Since g and $q_j \cdot p^{-1}$, $j=1, \dots, n$, belong to $L^2(\mathbb{R}^d)$, we see that $g(X) q_j(D_x) p(D_x)^{-1}$, $q_j(D_x) p(D_x)^{-1} g(X)$, $j=1, \dots, n$, can be considered Hilbert-Schmidt operators in $L^2(\mathbb{R}^d)$.

Therefore $P_1 DF(w) : H \rightarrow H$ and $DF(w)P_1 : H \rightarrow H$ are Hilbert-Schmidt operators for each $w \in B$. Similarly to (5.12), we can see that $T_j(w+\cdot) : H \rightarrow L^\infty(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$, $j = 1, \dots, n$, are continuous for any $w \in B$. Thus $P_1 DF(w+\cdot) : H \rightarrow L^2(H, H)$ and $DF(w+\cdot)P_1 : H \rightarrow L^2(H, H)$ are continuous for any $w \in B$.

This completes the proof.

Proposition 5.7. $DF(w+u) - DF(w) : H \rightarrow H$ is a Hilbert-Schmidt operator for any $w \in B$ and $u \in B_1$. Furthermore,

$DF(\cdot+u) - DF(\cdot) : B \rightarrow L^2(H, H)$ is continuous for any $u \in B_1$, and there exists a constant $C > 0$ such that

$\| DF(w+u) - DF(w) \|_{L^2(H, H)} \leq C \|u\|_B$ for any $w \in B$ and $u \in B_1$. Therefore the map from $B \times B_1$ into $L^2(H, H)$ under which (w, u) corresponds to $DF(w+u) - DF(w)$ is continuous. In particular, the Borel map F satisfies (F-5).

Proof. It is obvious that

$$(5.15) \quad DF(w+u) - DF(w) = p(D_x)^{-1} \sum_{j=1}^n (T_j(w+u) - T_j(w)) q_j(D_x) p(D_x)^{-1} p(D_x)$$

for any $w \in B$ and $u \in B_1$. It is easy to see that

$$(5.16) \quad |f_j(x; w+u) - f_j(x; w)| \leq \sum_{j=1}^n \|\partial_{ij} b\|_\infty |q_i(D_x) u(x)|,$$

$x \in \mathbb{R}^d$ and $j = 1, \dots, n$, for each $w \in B$ and $u \in B_1$.

It follows from (5.8) that

$$(5.17) \quad \begin{aligned} q_i(D_x) u &= q_i(D_x) g(X) A_1 u \\ &= g(X) g(X)^{-1} q_i(D_x) g(X) A_1 u \end{aligned}$$

for any $u \in B_1$ and $i = 1, \dots, n$.

By virtue of [5, Corollary to Lemma 4.1 and Theorem 2], $g(X)^{-1}q_i(D_x)g(X)$, $i = 1, \dots, n$, can be considered a continuous

linear map from B into $W_2^{\sigma^0, \rho^s 0}$. Proposition 5.3 (2) shows

that $g(X)$ can be considered a continuous linear map from

$W_2^{\sigma^0, \rho^s 0}$ into $L^2(\mathbb{R}^d)$, and Proposition 5.1 and 5.3 (1)

show that A_1 can be considered a bounded linear operator

in B . Therefore by (5.17) we see that there exists a

constant $C'' > 0$ such that

$$(5.18) \quad \|q_i(D_x)u\|_{L^2} \leq C'' \|u\|_B, \quad i = 1, \dots, n,$$

for any $u \in B_1$. Thus by virtue of Lebesgue's convergence theorem, (5.16) and (5.18), we get

$$(5.19) \quad \int_{\mathbb{R}^d} | (f_j(x; w' + u) - f_j(x; w')) - (f_j(x; w + u) - f_j(x; w)) |^2 dx \\ \rightarrow 0, \quad w' \rightarrow w \text{ in } B,$$

for any $u \in B_1$. Moreover (5.16) and (5.18) imply that

there exists a constant $C' > 0$ such that

$$(5.20) \quad \left\{ \int_{\mathbb{R}^d} |f_j(x; w + u) - f_j(x; w)|^2 dx \right\}^{1/2} \leq C' \|u\|_B$$

for any $u \in B_1$. Since $q_j \cdot p^{-1} \in L^2(\mathbb{R}^d)$, $j = 1, \dots, n$, we

see by (5.15), (5.19) and (5.20) that $DF(w+u) - DF(w) : H \rightarrow H$

is a Hilbert-Schmidt operator for any $w \in B$ and $u \in B_1$,

$$\| (DF(w'+u) - DF(w')) - (DF(w+u) - DF(w)) \|_{L^2(H,H)} \rightarrow 0$$

as $w' \rightarrow w$ in B for any $u \in B_1$, and that there exists a constant $C > 0$ such that

$$\| DF(w+u) - DF(w) \|_{L^2(H,H)} \leq C \|u\|_B \quad \text{for any } u \in B_1.$$

This proves the first part of our assertion. The latter part is obvious. This completes the proof.

Let F_D and I_D^e be σ -fields as in Introduction. By ignoring $S'(\mathbb{R}^d) \setminus B$, we obtain $F_1^v N_\mu = F_D^v N_\mu$, $F_1^v N_\nu = F_D^v N_\nu$, $F_2^v N_\mu = I_D^e v N_\mu$ and $F_2^v N_\nu = I_D^e v N_\nu$. Thus according to Theorem 1, Proposition 3.3, 5.5, 5.6 and 5.7, we get the following by letting $\tilde{H}(\tilde{w}, w) = H(\pi_1 \tilde{w}, \pi_2 w)$ as in Theorem 1.

Theorem 2. Let μ and ν be the probability laws of X and Y respectively, and let D be a bounded domain with smooth boundary. Moreover let $\nu(\cdot | I_D^e)$ denote the conditional probability measure relative to the σ -field I_D^e under ν . Then

- (1) the restricted measures $\mu|_{F_D}$ and $\nu|_{F_D}$ relative to the σ -field F_D are mutually absolutely continuous, and
- (2) there exists an $F_D \times I_D^e$ -measurable function

$\tilde{H} : S'(\mathbb{R}^d) \times S'(\mathbb{R}^d) \rightarrow \mathbb{R}$ such that for any $E \in F_D$,

$$\nu(E | I_D^e)(w) = \frac{\int_E \exp \tilde{H}(\tilde{w}, w) \mu(dw)}{\int_{S'} \exp \tilde{H}(\tilde{w}, w) \mu(dw)} \quad \text{for } \nu\text{-a.e. } w.$$

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