A Study of Standard Volume Table

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1) Introduction

It is well known that a standing tree volume is shown as

$$(\text{volume}) = m''(\text{d.b.h.})^{a''}(\text{height})^{b''} .$$

Based on the above expression, the standard volume table having two indexes of diameter at breast height and tree height is generally applied to estimate tree volume. Reasonably to apply this kind of volume table, however, both d.b.h. and height have to be measured accurately in practice.

The measurement of d.b.h. is comparatively easy, so that reliable results could be obtained practically. In general, however, there are various troubles in the measurement of height, and especially the error due to eye-measurement is remarkably larger (Ohtomo, 1956; Takata, 1956; Chō, 1959; Kajihara, 1965).

If there is a large error in the measurement of height, the estimate of volume obtained by using the standard volume table would be unreliable. In this paper, the writer presents how to reduce the error in volume resulting from mismeasurement of height, in other words, how to prepare the standard volume table when the measurement of height would have an error, from the angle of mean square error.

The writer wishes to express his sincere thanks to Professor Masaji SENDA and Dr. Gen-ichi TAGUCHI for their kind encouragement on this work. He is also much obliged to Mr. Shigeo YAMADA for his kind presentation of the materials.

2) The modificatory coefficient

Let us consider the equation for volume shown in the preceding section, where the logarithms of volume, d.b.h. and height in it are denoted as v, D and H, respectively. Further, instead of the next expansion of

$$v = m' + a''D + b''H , \tag{1}$$

let us consider the following orthogonal regression polynomial in which a simple regression is supposed for D and a partial regression is for H.

$$v = m + a(D-\bar{D}) + b\{(H-\bar{H}) - b'(D-\bar{D})\}, \tag{2}$$

where

$$m = \bar{v}, \quad a = \frac{(Dv)}{(DD)}, \quad b' = \frac{(DH)}{(DD)}, \quad b = \frac{(DD)(vH)-(DH)(Dv)}{(DD)(HH)-(DH)^2},$$

here (DD), (Dv), etc., show $\sum(D-\bar{D})^2$, $\sum(D-\bar{D})(v-\bar{v})$, etc., respectively.

In the above equation, let us suppose that the accurate values of m, a, b, b', $\bar{D}$, and $\bar{H}$ are known, and also that the accurate value of D is measured. Further, let the true value of v be V and let the mathematical expectation of square of $(v-V)$ be σ^2.

According to the theory of statistics (Taguchi, 1966), the error variance of the estimate is not minimal regarding the estimation of V, if actual measurement h is put into the equation (2) instead of the true value H, when the measurement of height would have an error. In order to make the mean square error minimal, it is necessary to introduce the following modificatory coefficient w to the equation.

Now consider that the relation between D and H is expressed as linear regression and let the error variance on the regression be σ_0^2, i.e.,

$$E\{H-\bar{H}-b'(D-\bar{D})\}^2=\sigma_0^2.$$

Moreover, let the variance of h be σ_h^2.

If h is given to the equation (2) instead of H, the difference between

$$\hat{v}=m+a(D-\bar{D})+bw\{(h-\bar{H})-b'(D-\bar{D})\} \tag{3}$$

and the equation (2) is shown as

$$\begin{aligned}\hat{v}-v&=bw\{(h-\bar{H})-b'(D-\bar{D})\}-b\{(H-\bar{H})-b'(D-\bar{D})\}\\&=b[(wh-H)+(1-w)\{\bar{H}+b'(D-\bar{D})\}].\end{aligned}$$

Then $E\{(\hat{v}-v)^2\}$ is shown as

$$\begin{aligned}E\{(\hat{v}-v)^2\}=b^2E[(wh-H)^2&+2(wh-H)(1-w)\{\bar{H}+b'(D-\bar{D})\}\\&+(1-w)^2\{\bar{H}+b'(D-\bar{D})\}^2].\end{aligned} \tag{4}$$

Regarding the above equation, the mathematical expectation of h is H and the variance of h is σ_h^2, then it is developed as follows:

$$E(wh-H)=(w-1)H$$

$$Var(wh-H)=w^2\sigma_h^2$$

$$E\{(wh-H)^2\}=(w-1)^2H^2+w^2\sigma_h^2.$$

Put those relations into the equation (4), then

$$\begin{aligned}E\{(\hat{v}-v)^2\}&=b^2E[w^2\sigma_h^2+(1-w)^2\{(H-\bar{H})-b'(D-\bar{D})\}^2]\\&=b^2\{w^2\sigma_h^2+(1-w)^2\sigma_0^2\}.\end{aligned} \tag{5}$$

Differentiate the above equation by w, then

$$\frac{d}{dw}b^2\{w^2\sigma_h^2+(1-w)^2\sigma_0^2\}=b^2\{2w\sigma_h^2-2(1-w)\sigma_0^2\}\ . \tag{6}$$

Let the equation (6) be zero, then

$$w=\frac{\sigma_0^2}{\sigma_0^2+\sigma_h^2}\ . \tag{7}$$

This is the coefficient taken up in this paper. In the equation mentioned above, σ_0^2 means the error variance of height which is estimated by using the relationship between d.b.h. and height, and σ_h^2 means the error variance of the measurement of height.

The mean square error in this case is shown as

$$\begin{aligned}E\{(\hat{v}-V)^2\}&=E\{(\hat{v}-v)+(v-V)\}^2=E\{(\hat{v}-v)^2\}+\sigma^2\\&=b^2\left\{\left(\frac{\sigma_0^2}{\sigma_0^2+\sigma_h^2}\right)^2\sigma_h^2+\left(\frac{\sigma_h^2}{\sigma_0^2+\sigma_h^2}\right)^2\sigma_0^2\right\}+\sigma^2=b^2\frac{\sigma_0^2\sigma_h^2}{\sigma_0^2+\sigma_h^2}+\sigma^2\ .\end{aligned} \tag{8}$$

3) Examples

The validity of the modificatory coefficient given in the preceding section was tested. The study was made by using the data collected from three stands for yield experiment at Oginoiri, Minamiyama and Yokoyama, within the province of Tokyo Regional Forest Office (Forestry Agency, 1961; Tokyo Regional Forest Office, 1964). Hereinafter those stands refer to as "Stand O", "Stand M" and "Stand Y", respectively.

The stands studied which were composed of *Cryptomeria japonica* had been artificially regenerated, and were even-aged. Because of the need for exact measurements, the height and the fixed diameters for sectional measurement to estimate volume were measured for each sample tree after cut over. Table 1 shows the stand age, the number of trees per ha, and the mean diameter at breast height, mean height, mean volume, and number of the sample trees.

Further, the number of sample trees classified by diameter for each stand is shown in Table 2.

In the preceding section, the variances, σ_0^2 and σ_h^2, were supposed theoretically for each surveyed tree and then the coefficient was expressed in general terms. Therefore,

Table 1. Characteristics of the stands studied.

Stand	Age	Number of trees per ha	Mean d. b. h. (cm)	Mean height (m)	Mean volume (m^3)	Number of sample trees
O	31	3500	12.7	11.6	0.0857	60
M	30	1400	16.5	16.0	0.2000	50
Y	30	2200	18.2	15.7	0.2311	33
O+M+Y			15.3	14.1	0.1592	143

Table 2. Number of trees classified by diameter.

D. b. h. class	Number of trees			
(cm)	Stand O	Stand M	Stand Y	Stand (O+M+Y)
8	6	1		7
10	13	5		18
12	13	7	5	25
14	12	6	5	23
16	13	8	4	25
18	3	7	4	14
20		7	5	12
22		4	5	9
24		5	4	9
26				
28			1	1
Total	60	50	33	143

if practical investigation is performed for individual trees contained within the objective stand, the coefficient ought to be also given to the individual ones. However, it would be meaningless in practice to estimate the variances for individual trees, and so the coefficient was applied to a stand in the experiments mentioned in this paper. Under the above consideration, therefore, the experiments were carried out on the assumption that the individual trees contained within a stand have the same variances.

Experiment (A)

In this experiment, the estimates based on eight processes were obtained for each objective stand. At first, the processes (I) and (II) are explained as follows:

Process (I): In the equation (1), the true value is taken for height. This process corresponds to the case where height is measured accurately and volume is estimated directly by using the regression equation. In this case, the resulting error would be due to a volume equation itself.

Process (II): Generally, the value of height is tabulated by the round number of every one meter. In this case, the rounded value is taken for height in the equation (1). This process corresponds in practice to the case where height is measured accurately and volume is estimated by using volume table.

The process (II) would be considered as the best way to estimate a tree volume by using volume table. However, it is quite difficult to measure a tree height accurately and the error resulting from mismeasurement is often run up into 10 to 20 percent of height (Mine, 1935; Kinashi and Uzaki, 1949). It may be difficult to assume an example of experimental mismeasurement. For the experiments mentioned in this paper, however, it is assumed that there would be an error of about one meter for every height measurement and it would happen evenly to plus and minus directions. More definitely, under the assumption that volume is estimated by using volume table, the calculation of volume based on the equation (3) was carried out with the following two cases. In one case, the value overestimated one meter as compared with the value H in the process (II) was supposed as the height. Similarly, the value underestimated one meter was supposed in another case. Then a mean in the above two cases was evaluated as the error.

Under the above consideration, the processes from (III) to (VIII) are introduced as follows:

Process (III): Zero is given as the value of w in the equation (3). This process means the case where volume is estimated by d.b.h. only. The value of w must be used for the case where σ_0^2 in objective stand is extremely small in comparison with σ_h^2.

Process (IV): 0.2 is given as the value of w.

Process (V): 0.4 is given as the value of w.

Process (VI): 0.6 is given as the value of w.

Process (VII): 0.8 is given as the value of w.

Process (VIII): 1.0 is given as the value of w. This process means the case where the mismeasured value is used as the estimate of height. Properly speaking, this value of the coefficient corresponds to the case where height is measured accurately, i.e., $\sigma_h^2=0$.

Further, the results in each process were logarithmically obtained under the assumption that d.b.h. was measured accurately.

The given values in orthogonal polynomials used in this experiment are summarized in Table 3.

Table 3. Given values of orthogonal polynomials used in the experiment (A).

Stand	m	a	$\bar{D}$	b	$\bar{H}$	b'
O	−1.1426	2.6833	1.0939	0.9665	1.0548	0.7403
M	−0.7903	2.4292	1.2013	1.1784	1.1956	0.6570
Y	−0.6919	2.2229	1.2479	1.0286	1.1915	0.4790

The mean square errors regarding the eight processes mentioned above are estimated for each objective stand as Table 4.

Table 4. Mean square errors estimated for each stand in the experiment (A).

Process	Stand O	Stand M	Stand Y
I	464	561	256
II	745	656	368
III	2330	2164	1016
IV	1734	1642	799
V	1358	1346	712
VI	1416	1229	755
VII	1694	1432	927
VIII	2233	1814	1227

All the figures are multiplied by 10^{-6}.

The above table shows that the best value of w in each stand would be contained within the following narrow limits, respectively.

Stand O: 0.4∼0.6

Stand M: 0.4∼0.6

Stand Y: 0.4∼0.6

On the other way, the estimates of σ_0^2 and σ_h^2 for individual stands are obtained practically as the columns (I) and (II) in Table 5.

Table 5. The values of σ_0^2, σ_h^2 and w in the experiment (A).

Stand	(I) σ_0^2	(II) σ_h^2	(III) w
O	0.002071	0.001800	0.54
M	0.001206	0.000925	0.57
Y	0.000755	0.000921	0.45

From the values of σ_0^2 and σ_h^2 shown in the above table, the coefficient w is given as the column (III). These values of w are well closed with the results in Table 4.

According to the results in Table 4, both the process (III) and the process (VIII) would have approximately three times the mean square error of the process (II) which is considered as most desirable way in practice. However, the values decrease down to two times when the best coefficient are applied.

Experiment (B)

Each given value in the volume equation was obtained for each objective stand separately in the experiment (A). Next, the variations regarding the mean square errors for each stand were studied under a consideration of the common volume equation in which the given values were obtained for the whole of three stands.

The given values in the common orthogonal polynomial used in this experiment are shown in Table 6.

In the same way as the former experiment, the mean square errors for each stand are estimated as Table 7.

On the other way, the estimates of σ_0^2, σ_h^2 and w are obtained practically as Table 8. The values of σ_0^2 are estimated by using the common regression equation of height for three stands. The values of σ_h^2 are the same as the former experiment.

To compare the results in Table 7 and Table 8, the same tendency as the experiment (A) is recognized as to the coefficient.

In Table 7, the mean square errors decrease when appropriate values of w are introduced. The variation of the values differs a little from that of the former experiment and the process (III) shows approximately four times the values in the process (II). The process (VIII) varies with the objective stands and comparatively small mean square

Table 6. Given values of orthogonal polynominal used in the experiment (B).

m	a	$\bar{D}$	b	$\bar{H}$	b'
−0.9154	2.6246	1.1670	1.1171	1.1356	0.7479

Table 7. Mean square errors estimated for each stand in the experiment (B).

Process	Stand O	Stand M	Stand Y	Stand (O+M+Y)
I	591	709	630	641
II	943	811	788	861
III	3610	3957	2765	3537
IV	2543	2872	2170	2572
V	1942	2136	1787	1974
VI	1809	1715	1617	1732
VII	2179	1627	1660	1866
VIII	2940	1853	1914	2323

All the figures are multiplied by 10^{-6}.

Table 8. The values of σ_0^2, σ_h^2 and w in the experiment (B).

Stand	(I) σ_0^2	(II) σ_h^2	(III) w
O	0.002673	0.001800	0.60
M	0.002457	0.000925	0.73
Y	0.001476	0.000921	0.62
O+M+Y	0.002371	0.001275	0.65

errors are given for some stands. If appropriate values of w are applied, however, better results are obtained even in these cases.

4) Volume table

The value of w shown in the column Stand O in Table 5, for example, is nearly equal to 0.5. Then the best volume equation in this case would be given as

$$\hat{v}=m+a(D-\bar{D})+0.5b\{(h-\bar{H})-b'(D-\bar{D})\},$$

under the assumption that the common value of coefficient is used for the stand.

Practically to apply this kind of method, the new index w should be added to a standard volume table having the indexes of d.b.h. and height. A standard volume table is made by the equation (3) in which the value of w is given as 1.0. Therefore, it is necessary to obtain the relationship of volume to d.b.h. and height for various values of w, to make a volume table under a consideration of the modificatory coefficient. For example, if eleven values of w, 0.0, 0.1, $\cdots$, and 1.0, are introduced to make the table, volume should be tabulated for each eleven case. To estimate a stand volume in practice, at first, the value of w is calculated from the variances of both σ_0^2 and σ_h^2 which are estimated for the objective stand, and then the table corresponding to the calculated value of w is used.

From the above reasons, it is necessary to consider the variance σ_0^2 or the variance σ_m^2 mentioned below in preparing a volume table. In other words, it is necessary to append, in the table, the regression equation of d.b.h. to height obtained by the same data from which the volume table is made. More practically, it may be possible to tabulate the anti-logarithmic estimate of mean height for each diameter class, which is obtained by the regression equation. Since σ_h^2 is an error variance due to observation, it is obtained for each objective stand.

The volume table in which the value of w is equal to zero corresponds to the local volume table having only the index of d.b.h.. This is the case where σ_h^2 is extremely larger than σ_0^2. Therefore, if there is a large error in the measurement of height, it would be meaningless in practice to consider the measurement of height. On the contrary, the volume table in which the value of w is equal to 1.0 corresponds to the case where the error in height would not be expected. It is not necessary to introduce the coefficient into volume estimation, if both d.b.h. and height are measured accurately. The above sensible results show that so-called standard volume table should be applied only to the case where errors would not be expected in the measurement of indexes.

5) Interpretation

Generally the variances, σ_0^2 and σ_h^2, are unknown in practice. To use the volume table mentioned in this paper, those variances have to be estimated for each objective stand. One of the practical methods of the estimation is to measure the heights of n sample trees selected from the objective stand by two ways; ordinary and accurate. Then the variances are denoted as follows:

$$\sigma_0^2=\frac{1}{n}\sum_1^n(H_i-\hat{H}_i)^2$$

$$\sigma_h^2=\frac{1}{n}\sum_1^n(h_i-H_i)^2,$$

where H_i means the values of height measured by accurate way, $\hat{H}_i$ means the mean values of height shown in the volume table concerned, and h_i means the values of height measured by ordinary way. To find out the values of $\hat{H}_i$, the regression equation of d.b.h. to height or the mean height for each diameter class has to be expressed clearly in the volume table having an additional index of w.

According to the theory mentioned in this paper, it is reasonable to calculate the variances σ_0^2 and σ_h^2 by logarithm. However, the logarithmic calculation has a lot of trouble in practice. If the values of w are taken at intervals of 0.1 or 0.2, the following approximation may be useful enough. That is the method directly using anti-logarithmic values.

From the following expansion,

$$E\{(h-\hat{H})^2\}=E\{(h-H)^2\}+E\{(H-\hat{H})^2\},$$

let $E\{(h-\hat{H})^2\}$ be σ_m^2, then

$$\sigma_m^2=\sigma_h^2+\sigma_0^2.$$

Therefore

$$w=\frac{\sigma_0^2}{\sigma_0^2+\sigma_h^2}=1-\frac{\sigma_h^2}{\sigma_m^2} \tag{9}$$

$$=1-\frac{1}{F}, \tag{10}$$

where

$$F=\frac{\sigma_m^2}{\sigma_h^2}.$$

On the other hand, let the anti-logarithmic values of h, H and $\hat{H}$ be Y_0, Y_t and Y_s, respectively,

$$E\{(\log Y_0-\log Y_t)^2\}=E\left\{\log\left(\frac{Y_0}{Y_t}\right)\right\}^2=E\left\{\log\frac{Y_t+\varepsilon}{Y_t}\right\}^2.$$

Since the above equation could be approximated roughly as $E(\varepsilon/Y_t)^2$ by Taylor's expansion,

$$E\left(\frac{\varepsilon}{Y_t}\right)^2=\left(\frac{1}{Y_t}\right)^2\sigma_Y^2.$$

Similarly the above approximate equation is made for σ_m^2, then the equation (9) is shown as

$$w=1-\frac{(1/Y_t)^2\sigma_Y^2}{(1/Y_s)^2\sigma_M^2}. \tag{11}$$

In the above equation, the values of Y_t, Y_s, σ_Y^2 and σ_M^2 are obtained anti-logarithmically.

Table 9 shows the values of w estimated by the above equation. In this example, the mean values for each stand are used as Y_t and Y_s. So that, in the experiment (A), the values obtained logarithmically are the same as those obtained anti-logarithmically because Y_t would be equal to Y_s. The following Table shows the results on the experiment (B).

Table 9. The values of w estimated anti-logarithmically.

Stand	σ_Y^2	σ_M^2	Y_t	Y_s	w
O	1.11	2.78	11.6	12.2	0.55
M	1.11	4.18	16.0	14.8	0.77
Y	1.11	3.29	15.7	16.0	0.65

In comparing the above values with the results in Table 8, the practicality of this method could be understood. If this is applied to a practical investigation, the mean heights appending in volume table could be shown in form of anti-logarithm. The value of σ_h^2 could be estimated roughly under the experiences of investigator.

Another method of estimation for the value of w is to use empirical knowledges. The desirable values of w, 0.4 to 0.6, are shown in the section (3), concerning the experiment (A). Therefore, roughly speaking, the substitute value 0.5 could be applied to these stands where the diameters at breast height are distributed between 10 and 30 cm and σ_h^2 is about 1 m.

Generally, the variance σ_0^2, obtained in the experiment (B) which is considered as more general case than the former experiment, would be larger in comparison with the former. This value is based on the difference of the relationship between d.b.h. and height in volume table and its relationship in objective stand.

From the matters mentioned above, it would be possible to estimate empirically the better value of w under the observation of characteristics for objective stand and σ_h^2.

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2 変数材積表についての一考察 (摘要)

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材積式 $V=mD^aH^b$ にもとづいた2変数材積表をもちいる場合，胸高直径の測定は比較的容易であるが，樹高については往々にして困難であり，ことに目測による場合には生ずる誤差も著しい。樹高の測定値に誤差があると考えられるとき，その誤測に基因する材積上の誤差を小さくするための方法として，つぎのような修正係数 w の利用が考えられる。

$$w=\frac{\sigma_0{}^2}{\sigma_0{}^2+\sigma_h{}^2}=1-\frac{\sigma_h{}^2}{\sigma_m{}^2}$$

ここに，$\sigma_0{}^2$ は胸高直径から樹高を推定したときの誤差分散，$E\{(H-\hat{H})^2\}$，であり，$\sigma_h{}^2$ は樹高を観測したときの誤差分散，$E\{(h-H)^2\}$，また，$\sigma_m{}^2=E\{(h-\hat{H})^2\}$ である。H，$\hat{H}$，h は，正しい樹高，使用する材積表における平均樹高，調査にもちいる方法によった測定値のそれぞれ対数である。

w をもとめるための一つの近似的な方法は，H，$\hat{H}$ の真数を Y_t，Y_s，また真数でもとめた $\sigma_h{}^2$，$\sigma_m{}^2$ の推定値を $\sigma_Y{}^2$，$\sigma_M{}^2$ として

$$w=1-\frac{(1/Y_t)^2\sigma_Y{}^2}{(1/Y_s)^2\sigma_M{}^2}$$

とすることである。

実際の調査においては，対象林分についての w をもとめ，その値に対応した表をもちいることになる。