A Note on the KATO-HORI's Tention Formulas for the Calculation of the Skyline Cables

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I. The Formulas

How to calculate the accurate values of the tension of the skyline-cables used for the logging cableways and cablecranes might be one of the important problems relating to construction and maintenance of the cable-installations. This problem had been considerably discussed by many researchers in the past. Resulting from such discussions, now most Forest Engineers of Japan are used to apply the theoretical formulas given by the writer and his colaborator, T. HORI, Assistant Professor of the Kyoiku University Tokyo, who contributed great parts of the mathematial computations. Use of these formulas, which are adequately applicable for the skyline-cable carrying any number of carriages, has been also recommeded by the Forestry Agency of Japan from the view point of pracical convenience.

The KATO-HORI's formulas are based on the theory of the "Parabolic Cable" and the method of calculation by means of these formulas is called the "Method of equivalent uniform load and equivalent sag-span ratio". Now, we shall consider a cable which is hung and tensioned between two points, then a certain amount of the sag or the dip will be given to the cable line, and the cable line will show a certain curve called "Catenary", providing that the cable is perfectly flexible and the linear density of the cable is quite uniform along the longitudinal axis of it. If the curve of the cable line is very flat, e.i. the amount of the central sag is relatively small compared with the span-distance, the "Catenary" is approximately coincident with a "Parabola" which has the same amount of the central sag. And this fact is recognized in the most cases of the skylines of the logging cable-installations. The statical meaning of a parabola is a curve of the flexible cable loaded with any uniformly distributed load along a straight line which represents the vertical projection of the cable line. The formulas were derived by the analysis of this approximate parabolic cable.

The fundamental structure of the KATO-HORI's tension formula could be given by the following expression:

$$(\text{Maximum tension}) = (\text{Total load}) \times (\text{Coefficient})$$

In this expression, the term "maximum tenssion" means the greatest tension of the skyline-cable at the upper end of the span under consideration; the term "total load" means the total sum of the vertical loads including the weight of the cable and the weight of the carriage-loads within the coresponding span; and the term "coefficient" is such a variable coefficient or a parameter as to be expressed by a function of the sag-span ratio of the original empty cable, weight of the carriage-loads, number and interval of the carriages, etc.

The general expression for the maximum tension $T_{i\max}$ of the cabel, which is hung over a certain span and loaded with several carriages, is given by the formula:

$$T_{i\,\max}=(W+i\cdot P)\Phi \tag{A}$$

where, W=Weight of the cable within the span under consideration

P=Weight of the single carriage-load, assumed to be equal to each other carriage.

i=Numbers of the carriages hung on the cable within the same span

Φ=Coefficient

The coefficient Φ is given by the expression:

$$\Phi=\frac{\sqrt{1+(4s_i+\tan\alpha)^2}}{8s_i} \tag{A·a}$$

in which, $\tan\alpha$=Inclination of the span

$s\cdot z=s_i$=Equivalent sag-span ratio

$\dfrac{f}{l_0}=s$=Central sag-span ratio of the empty cable

f=Central sag of the empty cable

l_0=Horizontal distance of the span

z=Coefficient of equivalency

and this coefficient z is given by the following expression;

$$z=\frac{1+in}{\sqrt{1+n[3i-i(i+1)(i-1)q^2]+n^2[3i^2-2i(i+1)(i-1)q]}} \tag{A·b}$$

where, $n=\dfrac{P}{W}$=Moving load ratio

$q=\dfrac{\text{Distance between two adjacent carriages}}{\text{Span length}}$

=Ratio of the interval of the carriages to the total length of the span

For the case of the cable loaded with a single carriage, put $i=1$ in the formula (A), then the maximum tension;

$$\left.\begin{aligned} &T_1=(W+P)\Phi\\ &\Phi=\frac{\sqrt{1+(4s_1\tan\alpha)^2}}{8s_1}\\ &s_1=z_1\cdot s\\ &z_1=\frac{1+n}{\sqrt{1+3n+3n^2}} \end{aligned}\right\} \tag{B}$$

For the case of the empty cable, put $i=0$ in the formula (A), then the maximum tension or the tension of the cable at the upper end;

$$\begin{aligned} &T_0=W\cdot\Phi_0\\ &\Phi_0=\frac{\sqrt{1+(4s+\tan\alpha)^2}}{8s} \end{aligned} \tag{C}$$

These formulas (A),(B), (C) are called the KATO-HORI's tension formula.

II. Derivation of the general formula

(1) Behavior of the cable and the carriage-loads

As shown in Fig. 1, a cable is hung and tensioned between two points A and B. The horizontal distance from A to B is l_0. When the cable is empty, e.i. there is no concentrated load on the cable, the amount of the sag of the cable at the span center is f, and consequently the sag-span ratio $s=f/l_0$. On this cable any number of the vertical concentrated loads P_1 P_2, $\cdots\cdots$ P_i, are applied at the corresponding points 1, 2, $\cdots\cdots i$.

Denote $k\cdot l_0$ the horizontal distance from A to each loaded point, then, for example, $l_1=k_1\cdot l_0$, $l_2=k_2\cdot l_0$, $\cdots\cdots$, $l_j=k_j\cdot l_0$, $\cdots\cdots$, in general $l_r=k_r\cdot l_0$ and in the extreme cases $k=[k_r]_{(A)}=0$ and $k_{i+1}=[k_r]_{(B)}=1$.

When the cable length from A to B is L, and the weight of the cable per unit length of the cable is p, the total weight W of the cable is given by $W=p\cdot L$. If we assume that the weight of the cable is distributed along the horizontal straight line, the density w of the assumed uniform load is given by $w=W/l_0=pL/l_0$.

The cable is tensioned and fixed at A and B. And the carriages are hung on the cable. The tensions of the cable at A and B and their vertical components are denoted by T_a, T_b and V_a, V_b correspondingly, while the horizontal component, which is constant at every point on the cable, is denoted by H_i.

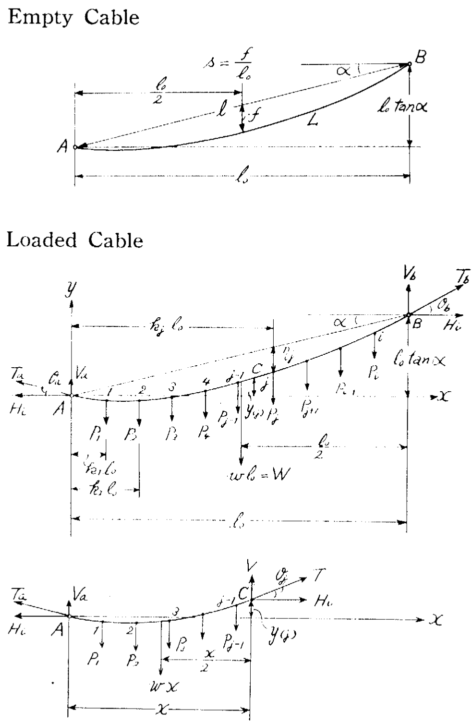

Fig. 1.

(2) Equation of the cable curve

Now we can consider the statical equilibrium condition ($\sum H=0$, $\sum V=0$ and $\sum M=0$), and taking the moments of all external forces in respect to the point B;

$$M_{(B)}=H_i\cdot l_0\tan\alpha+V_a\cdot l_0-\frac{wl_0^2}{2}-\sum_{r=1}^{i}P_r(1-k_r)\,l_0=0$$

from which we obtain

$$\frac{V_a}{H_i}=-\tan\alpha+\frac{wl_0}{2H_i}+\sum_{r=1}^{i}\frac{P_r(1-k_r)}{H_i} \tag{1}$$

The original curve of the empty cable is assumed to be a parabolic curve, but when this cable is loaded by the concentrated carriage-loads, the curve is angled at each points where each carriage-load is applied. Take the origine of $x-y$ axis at A, and consider that the cable is cut at any point C on the cable between $(j-1)$ and (j), and all forces are still remain acting on the portion A~C of the cable. Then we obtain the following equation from the statical equilibrium condition $\sum M_{(C)}=0$;

$$H_i \cdot y_{(j)} + V_a \cdot x - \frac{wx^2}{2} - \sum_{r=1}^{j-1} [P_r(x-k_r l_0] = 0$$

Substituting eq. (1), the equation of the cable curve is derived;

$$y_{(j)} = \frac{wx^2}{2H_i} + x\left\{\tan\alpha - \frac{wl_0}{2H_i} - \sum_{r=1}^{i} \frac{P_r(1-k_r)}{H_i} + \sum_{r=1}^{j-1} \frac{P_r}{H_i}\right\} - \sum_{r=1}^{j-1} P_r k_r l_0 \qquad (2\cdot a)$$

If we put the parameter,

$$m_i = \frac{H_i}{w} \qquad (2\cdot b)$$

and the moving load ratio,

$$n_r = \frac{P_r}{W} \qquad (2\cdot c)$$

eq. (2·a) can be written as follows;

$$y_{(j)} = \frac{x^2}{2m_i} + x\left\{\tan\alpha - \frac{l_0}{2m_i} - \frac{l_0}{m_i}\sum_{r=1}^{i} n_r(1-k_r) + \frac{l_0}{m_i}\sum_{r=1}^{j-1} n_r\right\} - \frac{l_0}{m_i}\sum_{r=1}^{j-1} n_r k_r \qquad (2)$$

This is the general equation of the cable curve of any portion of the loaded cable. And taking the derivative of eq. (2), we obtain the inclination of the curve:

$$\tan\theta_{(j)} = \frac{dy_{(j)}}{dx} = \tan\alpha - \frac{l_0}{m_i}\left\{\frac{1}{2} - \frac{x}{l_0} + \sum_{r=1}^{i} n_r(1-k_r) - \sum_{r=1}^{j-1} n_r\right\} \qquad (3)$$

For the empty cable, put $n_r=0$ and $m_i=m_0$ in eq. (2), then we have the equation of the original curve.

$$y = \frac{x^2}{2m_0} + x\left(\tan\alpha - \frac{l_0}{2m_0}\right) \qquad (2)'$$

Denote the horizontal component of the tension for the case of the empty cable by H_0, then the parameter m_0 is H_0/w. The amount of the sag of the original cable can be obtained by use of eq. (2)′;

$$f = [x\tan\alpha - y]_{x=l_0/2} = \frac{l_0^2}{8m_0}$$

hence we have the relation

$$m_0 = \frac{l_0^2}{8f} = \frac{l_0}{8s} \qquad (4)$$

(3) Curve length

The curve length of the cable between the two supporting points A and B is given by

$$L = \int_0^l \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sum_{j=1}^{i+1} \int_{k_{j-1}l_0}^{k_j l_0} \sqrt{1+\left(\frac{dy_{(j)}}{dx}\right)^2}\,dx \qquad (5\cdot a)$$

The cable curve is generally very flat, e.i., the value of sag-span ratio is usually small, say less than 0.08. Then, the second term in eq. (3) is always very small when it is compared with 1. In general, following relation has been mathematically recognized, if Δ is relatively small amount when it is compared with 1.

$$\sqrt{1+(\tan\alpha-\Delta)^2}=\sec\alpha-\Delta\sin\alpha+\frac{1}{2}\Delta^2\cos^3\alpha \tag{5·b}$$

Applying this approximate calculation, we obtain from eqs (3) and (5·a) the equation for the curve length;

$$L=\sum_{j=1}^{i+1}\int_{k_{j-1}l_0}^{k_j l_0}\left[\sec\alpha-\frac{l_0}{m_i}\left\{\frac{1}{2}-\frac{x}{l_0}+\sum_{r=1}^{i}n_r(1-k_r)-\sum_{r=1}^{j-1}n_r\right\}\sin\alpha\right.$$
$$\left.+\frac{1}{2}\frac{l_0^2}{m_i^2}\left\{\frac{1}{2}-\frac{x}{l_0^2}+\sum_{r=1}^{i}n_r(1-k_r)-\sum_{r=1}^{j-1}n_r\right\}^2\cos^3\alpha\right]dx \tag{5·c}$$

The procedure of integration in this equation is rather complicated, so that only the result will be written in this note as follows.

$$L=D_1+D_2+D_3 \tag{i}$$

$$D_1=\sum_{j=1}^{i+1}\int_{k_{j-1}l_0}^{k_j l_0}\sec\alpha\,dx=l_0\sec\alpha \tag{ii}$$

$$D_2=\sum_{j=1}^{i+1}\frac{l_0}{m_i}\left\{\frac{1}{2}-\frac{x}{l_0}+\sum_{r=1}^{i}n_r(1-k_r)-\sum_{r=1}^{j-1}n_r\right\}\sin\alpha\,dx$$
$$=\frac{{}_il_0}{m_i}\sin\alpha\left\{\frac{1}{2}-\frac{1}{2}+l_0\sum_{j=1}^{i}n_r(1-k_r)-l_0\sum_{j=1}^{i+1}(k_j-k_{j-1})\sum_{r=1}^{j-1}n_r\right\}=0 \tag{iii}$$

$$D_3=\frac{1}{2}\frac{l_0^2}{m_i^2}(D_3'+D_3''+D_3''')\cos^3\alpha$$

$$D_3'=\sum_{j=1}^{i+1}\int_{k_{j-1}l_0}^{k_j l_0}\left(\frac{1}{2}-\frac{x}{l_0}\right)^2dx=\int_0^{l_0}\left(\frac{1}{2}-\frac{x}{l_0}\right)^2dx=\frac{l_0}{12} \tag{iv·a}$$

$$D_3''=2\sum_{j=1}^{i+1}\int_{k_{j-1}l_0}^{k_j l_0}\left(\frac{1}{2}-\frac{x}{l_0}\right)\left\{\sum_{r=1}^{i}n_r(1-k_r)-\sum_{r=1}^{j-1}n_r\right\}dx=l_0\sum_{r=1}^{i}n_rk_r(1-k_r) \tag{iv·b}$$

$$D_3'''=\sum_{j=1}^{i+1}\int_{k_{j-1}l_0}^{k_j l_0}\left\{\sum_{r=1}^{i}n_r(1-k_r)-\sum_{r=1}^{j-1}n_r\right\}^2dx$$
$$=l_0\left[\sum_{j=1}^{i+1}(k_j-k_{j-1})\left(\sum_{r=1}^{j-1}n_r\right)^2-\left\{\sum_{r=1}^{i}n_r(1-k_r)\right\}^2\right]$$
$$=l_0\sum_{r=1}^{i}n_r^2k_r(1-k_r)+2l_0\sum_{r=1}^{i-1}\sum_{g=r+1}^{i}n_rn_gk_r(1-k_g) \tag{iv·c}$$

$$D_3=\frac{1}{2}\frac{l_0^2}{m_i^2}(D_3'+D_3''+D_3''')\cos^3\alpha=\frac{l_0^3K_i}{24m_i^2}\cos^3\alpha \tag{iv}$$

where,

$$K_i=1+12\sum_{r=1}^{i}(n_r+n_r^2)k_r(1-k_r)+24\sum_{r=1}^{i-1}\sum_{g=r+1}^{i}n_rn_gk_r(1-k_g) \tag{v}$$

Substituting (ii), (iii), (iv), (v) in eq. (i), we obtain the total curve length

$$L=l_0\sec\alpha\left(1+\frac{l_0^2K_i}{24m_i^2}\cos^4\alpha\right)=l\left(1+\frac{l_0^2K_i}{24m_i^2}\cos^4\alpha\right) \tag{5}$$

where, $\quad l=l_0 \sec\alpha=$direct distance from A to B

$\quad\quad K_i=$given by eq. (v)

(4) Relation between sag-span ratio s of the original empty cable and the parameter m_i.

For the empty cable, $P_r=0$ and $n_r=0$, so that $[K_i]_{i=0}=1$ from eq. (v). Hence, put $K_i=1$ in eq. (5), then we have the curve length from A to B.

$$L_0=l\left(1+\frac{l_0^2}{24m_0^2}\cos^4\alpha\right) \tag{6}$$

Substituting the relation (4) into this, we get the formula for the curve length of the empty cable;

$$L_0=l\left(1+\frac{8}{3}s^2\cos^4\alpha\right) \tag{7}$$

If it could be assumed that, there is no change in the length of the cable and no displacement of the suporting points A and B, the curve length of the loaded cable given by eq. (5) must be equal to the curve length of the empty cable given qy eq. (7). Hence we have

$$m_i=\frac{\sqrt{K_i}}{8s}\cdot l_0 \tag{8}$$

(5) Inclination and sag of cable

The inclination of the cable at the lower end A and uper end B is derived from eq. (3).

$$\left.\begin{aligned}\tan\theta_a&=\left[\frac{dy_{(1)}}{dx}\right]_{x=0}=\tan\alpha-\frac{4s}{\sqrt{K_i}}\left\{1+2\sum_{r=1}^{i}n_r(1-k_r)\right\}\\ \tan\theta_b&=\left[\frac{dy_{(i+1)}}{dx}\right]_{x=l_0}=\tan\alpha+\frac{4s}{\sqrt{K_i}}\left\{1+2\sum_{r=1}^{i}n_rk_r\right\}\end{aligned}\right\} \tag{9}$$

The amount of sag η_j at the point j of the loaded cable is derived from eqs. (2) and (8);

$$\begin{aligned}\eta_j&=[x\tan\alpha-y_{(j)}]_{x=k_jl_0}\\ &=\frac{k_jl_0^2}{2m_i}\left\{1+2\sum_{r=1}^{i}n_r(1-k_r)-2\sum_{r=1}^{j-1}n_r-k_j\right\}+\frac{l_0^2}{m_i}\sum_{r=1}^{j-1}n_rk_r\\ &=\frac{4s\,l_0}{\sqrt{K_i}}\left\{k_j(1-k_j)+2(1-k_j)\sum_{r=1}^{j-1}n_rk_r+2k_j\sum_{r=j}^{i}n_r(1-k_r)\right\}\end{aligned} \tag{10}$$

(6) Maximum tension

The horizontal component of the tension of the loaded cable can be expressed in the term of the sag-span ratio of the empty cable from eq. (8);

$$H_i=m_iw=\frac{wl_0}{8s}\sqrt{K_i} \tag{11}$$

Then, the tension of the cable at A (lower end) and B (upper end);

$$\left.\begin{aligned}T_a&=H_i\sqrt{1+\tan^2\theta_a}\\ T_b&=H_i\sqrt{1+\tan^2\theta_b}\end{aligned}\right\} \tag{12}$$

The position of a system of the carriage loads moving over the span, in which K_i

will be a maximum, can be found by setting $dK_i/dk_1=0$, where $k_r=k_1+(k_r-k_1)$, and (k_r-k_1) is constant.

$$\frac{dK_i}{dk_1}=12\left(\sum_{r=1}^{i}n_r-2\sum_{r=1}^{i}n_r k_r\right)\left(1+\sum^{i}n_r\right)=0$$

therefore,

$$\frac{\sum_{r=1}^{i}n_r k_r}{\sum_{r=1}^{i}n_r}=\frac{1}{2}\quad\text{or}\quad k_1=\frac{1}{2}-\frac{\sum_{r=1}^{i}n_r(k_r-k_1)}{\sum_{r=1}^{i}n_r} \tag{13}$$

This means that the value of K_i is maximum when the center of gravity of the whole moving loads comes to the middle point of the span, and the maximum value of K_i in this case is

$$K_{i0}=1+3\sum_{r=1}^{i}n_r+12\frac{\left\{\sum_{r=1}^{i}n_r(k_r-k_1)\right\}^2}{\sum_{r=1}^{i}n_r}-12\sum_{r=1}^{i}n_r(k_r-k_1)^2+3\left(\sum_{r=1}^{i}n_r\right)^2+12\sum_{r=1}^{i-1}\sum_{g=r+1}^{i}n_r n_g(k_r-k_g) \tag{14}$$

In this position of the load-system, the horizontal component of the tension is also a maximum and the cable tension at the upper end closely approximates to the maximum value of the tension. Therefore, the approximate maximum tension of the loaded cable is given by the formula;

$$T_{i\max}=[T_b]_{K_{i0}}=\frac{wl_0\sqrt{K_{i0}}}{4s}\sqrt{1+\left\{\tan\alpha+\frac{4s}{\sqrt{K_i}}\left(1+\sum_{r=1}^{i}n_r\right)\right\}^2} \tag{15}$$

(7) Method of equivalent uniform load and equivalent sag-span ratio.

In eq. (15), put

$$z=\frac{1+\sum_{r=1}^{i}n_r}{\sqrt{K_{i0}}}=\text{Coefficient of equivalency} \tag{16}$$

and

$$s_i=z\cdot s=\text{Equivalent sag-span ratio} \tag{17}$$

then,

$$T_{i\max}=\left(W+\sum_{r=1}^{i}P_r\right)\frac{\sqrt{1+(\tan\alpha+4s_i)^2}}{8s_i} \tag{18}$$

Put

$$\Phi=\frac{\sqrt{1+(\tan\alpha+4s_i)^2}}{8s_i}=\text{Coefficient of maximum tension} \tag{19}$$

then,

$$T_{i\max}=\left(W+\sum_{r=1}^{i}P_r\right)\Phi=(\text{Total load})\times\text{Coefficient.} \tag{20}$$

If the weight of the carriage-load is equal to each other, as it is practically so in the case of the cableway, and the number of the carriages is i,

$$P_1=P_2=\cdots\cdots\cdots\cdots=P_i=P$$

$$n_1=n_2=\cdots\cdots\cdots\cdots=n_i=n=P/W$$

and the total load$=(W+i\cdot P)=W(1+in)$.

For the case of the equal interval distance of the carriages, as it is quite common in the cableway practice, denote that,

$$q=\frac{\text{Average horizontal distance between two adjacent carriages}}{\text{Horizontal distance of span}} \tag{21}$$

then,

$$k_r=k_1+(r-1)q, \quad k_1=\frac{1-(i-1)q}{2}$$

Substituting these in eqs. (14) and (16), we obtain the practical formula for the coefficient of equivalency,

$$z=\frac{1+in}{\sqrt{1+n\{3i-i(i+1)(i-1)q^2\}+n^2\{3i^2-2i(i+1)(i-1)q\}}} \tag{22}$$

Summerizing the results of the theoretical analysis mentioned above, the writer could generally state the KATO-HORI's practical tension formula:

$$\left.\begin{aligned}
&T_{i\max}=(W+i\cdot P)\Phi \\
&\quad \Phi=\frac{\sqrt{1+(4s_i+\tan\alpha)^2}}{8s_i} \\
&\quad s_i=z\cdot s \\
&\quad z=\frac{1+in}{\sqrt{1+n\{3i-i(i+1)(i-1)q^2\}+n^2\{3i^2-2i(i+1)(1-1)q\}}}
\end{aligned}\right\} \tag{A}$$

III. Correction-factors

Becuse of the relatively small amount of the sag-span ratio, for instance $s<0.08$, which is very common in the case of the skyline-cable, the change in the cable length due to the elastic elongation caused by the application of the carriage-loads or due to the temperature change, and also the displacements of the supporting poits of the cable caused by the loading should be taken into consideration in most cases.

A certain correction is inevitably needed for the values of the cable tension calculated by the formula (A) or (B). For this purpose the use of the correction-factors had been recommended. In general, for the calculation of the tension-formula the corrected values of the equivalent sag-span ratio $s_i'=\varepsilon\cdot s_i$ should be used instead of s_i. Here, ε is the compound correction-factor;

$$\varepsilon=\varepsilon_e\times\varepsilon_d\times\varepsilon_t \tag{a}$$

in which, ε_e: for the elastic elongation of the cable.
ε_d: for the displacements of the supporting points.
ε_t: for the temperature change.

The practical formulas for the calculation of these correction-factors are given as follows;

$$\left.\begin{aligned}
&\varepsilon_e=\frac{1}{2}\left\{1+\sqrt{1+\left(1+\frac{3}{8s^2\cos^4\alpha}\right)\Delta_e}\right\} \\
&\quad \Delta_e=T_d/AE \\
&\quad T_d=T_{i\max}-T_0
\end{aligned}\right\} \tag{b}$$

$$\varepsilon_d=\sqrt{\frac{1+\dfrac{3\Delta_d}{8s^2\cos^4\alpha}}{1-\Delta_d}} \tag{c}$$

$$\varepsilon_t=1\pm\frac{3}{16s^2\cos^4\alpha}\,\omega t^0\,. \tag{d}$$

The same correction-factors can be applicable for the corrective calculation of the sag η of the loaded points of the cable. (See reference 2.)

References :

1) Kato, Seihei and Hori, Takao: How to estimate the tension of the skyline cable with multiple concentrated loads on it. Journal of the Forestry Society, Vol. 36, No. 12, Vol. 37, No. 1, 2, 1954/55.

2) Kato, Seihei: Studies on the Skyline Logging Cables for their Planning and Inspection, Bulletin of the Tokyo University Forests, No. 40, 1951.