

SOLUTIONS FOR SEMILINEAR PARABOLIC EQUATIONS IN  $L^p$   
AND  
REGULARITY OF WEAK SOLUTIONS OF THE NAVIER-STOKES SYSTEM

半線型放物型方程式の $L^p$ 解およびナヴィエ・ストークス系の弱解の正則性

by

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## Semilinear Parabolic Equations in $L^p$

### Abstract

We construct a unique local regular solution in  $L^q(0,T; L^p)$  for a class of semilinear parabolic equations which includes the semilinear heat equation  $u_t - \Delta u = |u|^\alpha u$  ( $\alpha > 0$ ) and the Navier-Stokes system. Here  $p$  and  $q$  are so chosen that the norm of  $L^q(0,T; L^p)$  is dimensionless or scaling invariant. The main relation between  $p$  and  $q$  for the semilinear heat equation is  $1/q = (1/r - 1/p)n/2$ ,  $p > r$  provided that initial data is in  $L^r$  with  $r = n\alpha/2 > 1$ , where  $n$  is the space dimension. Applying our regular solutions to the Navier-Stokes system, we show that the Hausdorff dimension of possible time singularities of a turbulent solution is less than  $k/2$  if the turbulent solution is in  $L^q(0,T; L^p)$  where  $k = 2 - q + nq/p$ ,  $p \geq n$ ,  $1 \leq q < \infty$ . We show, moreover, that a turbulent solution is regular if it is in  $C((0,T); L^n)$ .

## 1. INTRODUCTION

We consider semilinear parabolic equations of type

$$u_t + Au = Fu, \quad u(0) = a \quad (1.1)$$

where  $Fu$  represents the nonlinear part of the equation and  $A$  is an elliptic operator. We study this initial value problem in  $L^p$  spaces. A standard theory (e.g. [8,29]) shows for a large class of  $Fu$  that there is a local solution  $u(t)$  which is continuous from  $[0, T)$  to  $L^r$  for  $a \in L^r$ ; here  $r > 1$  is the exponent determined by the structure of the nonlinear term  $Fu$ . The solution  $u(t)$  can be extended globally, namely,  $T$  can be taken infinity provided that  $\|a\|_r$ ,  $L^r$  norm of  $a$ , is sufficiently small. In this paper we show that the above constructed solution  $u(t)$  belongs to  $L^q(0, T; L^p)$  with suitably chosen  $q$ , where  $p > r$ . Since  $u \in L^q(0, T; L^p)$  is equivalent to  $\|u\|_p(t) \in L^q(0, T)$ , this result gives the asymptotic behavior of  $\|u\|_p(t)$  as  $t \rightarrow 0$  and  $t \rightarrow \infty$  if  $T = \infty$ . We prove, moreover, that the uniqueness holds in the class  $L^q(0, T; L^p)$ . As is seen later, these results are not only interesting by itself but also applicable to the regularity theory for weak solutions of the nonstationary Navier-Stokes system.

A simple example of (1.1) is a semilinear heat equation

$$u_t - \Delta u = |u|^\alpha u, \quad u(x, 0) = a(x), \quad x \in \mathbb{R}^n \quad (1.2)$$

where  $\alpha > 0$ . Existence results due to Weissler [31] say that there is a local solution  $u$  in  $C([0, T); L^r)$  for  $a \in L^r$  if  $r = p_0 := n\alpha/2 > 1$  and that  $T$  can be taken infinity provided

that  $\|a\|_r$  is sufficiently small. Applying our theory to (1.2) yields  $u \in L^q(0, T; L^p)$  with  $1/q = (1/r - 1/p)n/2$ ,  $q, p > r$ ,  $q > \alpha + 1$ . Moreover, the above class  $L^q(0, T; L^p)$  guarantees uniqueness of solutions of (1.2); here, we have to replace  $q > r$  by  $p > \alpha + 1$ . Our results seem new even for (1.2); see section 4.

To explain the meaning of these results conceptually, it is convenient to recall dimensional analysis of (1.2); see [2, 11]. If  $u(x, t)$  solves (1.2), then for each  $\lambda > 0$ ,

$$u_\lambda(x, t) = \lambda^{2/\alpha} u(\lambda x, \lambda^2 t)$$

also solves (1.2) unless we consider the initial condition. We describe this scaling property by assigning a scaling dimension to each quantity:

$$t: 2, \quad x_i: 1, \quad u: -2/\alpha$$

so that each term in (1.2) has dimension  $-2 - 2/\alpha$ . Clearly, the norm  $\|a\|_r$  with  $r = p_0$  has zero-dimension, so conceptually the existence results read: if a zero-dimensional integral of initial data is finite, then a solution  $u$  exists at least locally; if  $\|a\|_{p_0}$  is small,  $u$  can be extended globally. Our results read: many zero-dimensional integrals of solution are finite and that the class  $L^q(0, T; L^p)$ ,  $p > r$  having zero-dimension guarantees the uniqueness if  $p, q > \alpha + 1$ . However, I suspect that in general the class  $C([0, T]; L^r)$  with  $r = p_0$  is not sufficient to guarantee the uniqueness although the norm  $\sup_t \|u\|_p(t)$  is dimensionless. There is a counterexample due to Ni and Sacks [18] for the initial-boundary value problem of (1.2) on a ball. Although it

is convenient to use scaling dimension to explain the meaning of  $p_0$ ,  $p$ ,  $q$  our methods are not based on the scaling property of the equations. Our theory is also applicable to more general equations which do not have the scaling property.

Another typical example of (1.1) covered by our theory is the nonstationary Navier-Stokes system which has also the scaling property. The dimension of  $u$  should be replaced by  $-1$  and  $p_0$  should equal  $n$ , the space dimension; see [2, 8]. The solution  $u$  we discussed above is called a regular solution since  $u$  is smooth both in space and time variables for  $t > 0$ . Because of a special property of the nonlinear term there is a global weak solution constructed by Leray [17] and Hopf [12]; we call this solution a turbulent solution following Leray's definition. If  $n \geq 3$  we do not know whether turbulent solutions are regular. As is pointed out, however, by Leray [17] properties of regular solutions are very useful to study partial regularity of turbulent solutions.

In this paper we give proofs of regularity criteria for turbulent solutions which are announced in [10]. Let us roughly and briefly review our results. To fix the idea we consider the system on a smoothly bounded domain in  $\mathbb{R}^n$ . Let  $k \geq 0$  be the scaling dimension of  $\int (\int |v|^p dx)^{q/p} dt$ , i.e.  $k = 2 - q + nq/p$ . One of our results read: if a turbulent solution belongs to  $L^q(0, T; L^p)$ ,  $p > n$ , then the set of possible time singularity of it has  $k/2$ -dimensional Hausdorff measure zero. If  $k = 0$ , the turbulent solution should be smooth. The case  $k = 0$  is recently proved also by Sohr [21] by a different method. To show our re-

sult we estimate life span of regular solutions from below if  $a \in L^r$ ,  $r > n$ . If we consider the marginal case  $p = n$ , the set of possible time singularity has Lebesgue measure zero. This result is recently proved also by Sohr and von Wahl [23]. Their method is based on an improvement of Sather and Serrin's uniqueness result [21] whereas our method depends on the existence of regular solution  $u$  in  $L^q(0, T; L^p)$   $p > n$  whose scaling dimension is zero provided that  $u(0) \in L^n$ . As a by-product we show that turbulent solutions belonging to  $C((0, T); L^n)$  are regular; this is proved by von Wahl [27]. Relations to other regularity results will be discussed in section 5.

Another interesting application of regular solution  $u$  in  $L^q(0, T; L^p)$  with  $p > n$ ,  $k = 0$  is discussed in Kato [15]. Using the fact  $\|u\|_p \in L^q(0, \infty)$ , he in particular proves that the energy of the turbulent solution tends to zero as  $t \rightarrow \infty$  if the domain is  $\mathbb{R}^2$ .

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In section 2 we study (1.1) in an abstract setting. Actually, we consider an integral form of (1.1) instead of (1.1). We do not use any fractional powers but various  $L^p$  spaces so that our analysis works even for unbounded domains; this is important if we discuss the asymptotic behavior of solution as  $t \rightarrow \infty$ . Also the assumptions are so chosen that results are directly applicable both to the semilinear heat equation (1.2) and the Navier-Stokes system. We state the existence and the uniqueness of solutions and estimate for the life span of solutions in Theorem 1. The results in  $L^q(0, T; L^p)$  framework are stated in Theorem 2.

Both theorems are important in sections 4 and 5.

Our analysis is based on the regularization property of linear part  $e^{-tA}$  and successive approximations. The basic idea is similar to [8,14,17,23,29,30,31] but we use no fractional powers nor derivatives. To show Theorem 2, however, new idea is required. Proofs of Theorems 1 and 2 are given in section 3. The proof of Theorem 1 is more or less known; however, arguments are scattered in many papers cited above, so we give the proof for completeness.

In section 4 we show that Theorems 1 and 2 are applicable to the semilinear heat equation (1.2) and the Navier-Stokes system. For (1.2) we also compare our results with other works so that most of restrictions on exponents are really necessary.

In section 5 we study the regularity of turbulent solution by using results in section 4. This section gives a proof of the results announced in [10].

## 2. ABSTRACT EXISTENCE THEOREMS

This section states existence theorems for semilinear parabolic evolution equations of type

$$u_t + Au = Fu, \quad u(0) = a \quad (2.1)$$

in various  $L^p$  type function spaces; here  $Fu$  represents the non-linear part of the equation. As is standard practice, we study (2.1) via the corresponding integral equation

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-\tau)A}Fu(\tau)d\tau. \quad (2.2)$$

The solutions of this equation are often called mild solutions. We shall construct mild solutions only because in many examples one can prove that mild solutions are differentiable in  $t$  and are strong solutions of (2.1) as far as (2.1) is a parabolic equation [8, 14, 29, 30]. For later use we study (2.2) in an abstract setting.

Let  $X$  be a locally compact Hausdorff space and let  $\mu$  be a Radon measure. Let  $L^p$  denote the set of  $\mu$ -measurable functions on  $X$  with  $\mu$ -integrable  $p$ -th power, where  $1 \leq p < \infty$ . Let  $C_c$  be the space of continuous functions on  $X$  with compact support. For applications it is convenient to consider a direct sum decomposition of  $(L^p)^\ell$ , the set of  $\ell$ -vector valued  $L^p$  functions. Let  $P$  be a continuous projector from  $(L^p)^\ell$  to a closed subspace  $E^p$  of  $(L^p)^\ell$  for  $1 < p < \infty$  such that the restriction of  $P$  on  $(C_c)^\ell$  is independent of  $p$ . For technical reason we assume  $(C_c)^\ell \cap E^p$  is dense in  $E^p$ . Let  $e^{-tA}$  ( $t \geq 0$ ) be a strongly continuous semigroup simultaneously on all  $E^p$ ,  $1 < p < \infty$ . Since

problems we consider are parabolic, it is natural to assume the following estimates for  $e^{-tA}$ . There are constants  $n, m \geq 1$  such that for a fixed  $T, 0 < T < \infty$  the estimate

$$(A) \quad \|e^{-tA}f\|_p \leq M\|f\|_s/t^\sigma, \quad f \in E^s, \quad 0 < t < T$$

holds with  $\sigma = (1/s - 1/p)n/m, p \geq s > 1$  and constant  $M$  depending only on  $p, s, T$ , where  $\|f\|_s$  denotes the norm of  $f$  in  $L^s$ . A typical example of such semigroups is the solution operator of the heat equation in  $\mathbb{R}^d$ ; it is easy to check that (A) holds with  $n = d, m = 2, E^p = L^p$ .

Having the Navier-Stokes system in mind, we give assumptions on the nonlinear term  $Fu$ . Let  $\Gamma$  be a closed linear operator densely defined in  $(L^p)^k$  to  $E^q$  with some  $q > 1$  such that for some  $\gamma, 0 \leq \gamma < m$  the estimate

$$(N1) \quad \|e^{-tA}\Gamma f\|_p \leq N_1\|f\|_p/t^{\gamma/m}, \quad f \in E^p, \quad 0 < t < T$$

holds with  $N_1$  depending only on  $T$  and  $p, 1 < p < \infty$ . We assume  $Fu$  can be written as

$$Fu = \Gamma Gu \tag{2.3}$$

and  $G$  is a nonlinear mapping from  $E^p$  to  $(L^h)^k$  such that for some  $\alpha > 0$  the estimate

$$(N2) \quad \|Gv - Gw\|_h \leq N_2\|v - w\|_p(\|v\|_p^\alpha + \|w\|_p^\alpha), \quad G0 = 0$$

holds with  $1 \leq h = p/(1+\alpha)$  and  $N_2$  depending only on  $p, 1 < p < \infty$ . For example, let  $g(y)$  be a mapping from  $\mathbb{R}^l$  to  $\mathbb{R}^k$  satisfying

$$|g(y)-g(z)| \leq N_2 |y-z| (|y|^\alpha + |z|^\alpha), \quad g(0) = 0 \quad (2.4)$$

and put  $(Gu)(x) = g(u(x))$ . This  $G$  satisfies (N2) which follows (2.4) by applying the Hölder inequality. Heuristically,  $\Gamma$  has a role of differential operator of order  $\gamma$  and  $Gu$  behaves like  $|u|^\alpha u$ .

We now state the existence of mild solutions of (2.2), at least locally, assuming (A) and (N1,2). In what follows  $BC$  denotes the class of bounded and continuous functions and  $C$  denotes positive constant whose value may change from one line to the next.

THEOREM 1. (i) (Existence). Let  $p_0$  denote

$$p_0 = \frac{n\alpha}{m-\gamma} \quad (2.5)$$

and assume  $p_0 > 1$ . Suppose  $a \in E^r$  for a fixed  $r \geq p_0$ . Then,  
there is  $T_0$ ,  $0 < T_0 \leq T$  and a solution  $u$  of (2.2) on  $[0, T_0)$  such that

$$t^\sigma u(t) \in BC([0, T_0); E^p) \quad \text{for } r \leq p < \infty \quad (2.6)$$

$$t^\sigma \|u(t)\|_p \rightarrow 0 \quad \text{as } t \rightarrow 0 \quad \text{for } r < p \quad (2.7)$$

with  $\sigma = (1/r - 1/p)n/m$ ,  $0 \leq \sigma < 1/(\alpha+1)$ .

(ii) (Estimate for  $T_0)$ . If  $r > p_0 > 1$ ,

$$T_0 \geq C \|a\|_r^{-\alpha/(1-\beta(r))} \quad (2.8)$$

$$\beta(r) = (\gamma + n\alpha/r)/m \quad (2.9)$$

with  $C$  independent of  $a$ .

(iii) (Global existence for small initial data). There is a positive constant  $\epsilon$  such that if  $\|a\|_{p_0} < \epsilon$ , then  $T_0$  equals  $T$ . In the case  $T = \infty$  we have

$$\|u\|_p(t) \leq C/t^\sigma, \quad 0 < t < \infty \quad (2.10)$$

with  $C$  independent of  $t$ , provided that  $p \geq p_0$ .

(iv) (Uniqueness). Solutions of (2.2) satisfying (2.6) and (2.7) for some  $0 < \sigma < 1/(\alpha+1)$ ,  $p > \alpha+1$ ,  $\sigma = (1/r - 1/p)n/m$  are unique. If  $r > p_0$ ,  $\sigma$  may equal zero and (2.7) is not necessary to guarantee the uniqueness. In particular, if  $r > p_0$  solutions are unique in  $BC([0, T_0]; E^r)$  provided that  $r > \alpha+1$ .

(v) Let  $(0, T_*)$  be the maximal interval such that  $u$  solves (2.2) in  $C((0, T_*); E^r \cap E^{r'})$ ,  $r > p_0 > 1$ ,  $r' > \max(\alpha+1, p_0)$ . Then

$$\|u(s)\|_r \geq C/(T_* - s)^{(1-\beta(r))/\alpha} \quad (2.11)$$

with constant  $C$  independent of  $T_*$  and  $s$ .

The results in this generality are new although some parts are more or less known. If  $\Gamma$  is a bounded operator from  $(L^p)^k$  to  $E^p$ , (i) is proved by Weissler [29]. Theorem 1 gives the global existence for small initial data in  $E^{p_0}$ , where  $p_0$  is the marginal number defined by (2.5). Also it gives estimates for  $u$  from below near the blow up point; see (v). In section 4 we shall discuss examples and compare them with previously known results.

We next state that  $u$  in Theorem 1 is also in  $L^{1/\sigma}(0, T_0; E^p)$  provided that  $p$  is close to  $r$ ; this result does not directly follow from (2.6) and (2.7) so a new idea is required to prove it.

THEOREM 2. (i). Let  $u$  be the solution of (2.2) constructed in Theorem 1(i). Then,  $u$  belongs to  $L^q(0, T_0; E^p)$  with  $q, p > r$ ,  $q > \alpha+1$ ,  $1/q = (1/r - 1/p)n/m$ .

(ii). Assume  $T = \infty$ ,  $r = p_0$ . If  $\|a\|_{p_0}$  is sufficiently small,  $u$  is in  $L^q(0, \infty; E^p)$ .

(iii). (Uniqueness). Solutions of (2.2) belonging to  $L^q(0, T_0; E^p)$  for some  $p > r$ ,  $1/q = (1/r - 1/p)n/m$ ,  $q, p > \alpha+1$  are unique.

## 3. PROOFS OF THEOREMS 1 AND 2

To solve the integral equation (2.2), i.e.,

$$u = u_0 + Su \quad (3.1)$$

$$Su(t) = \int_0^t e^{-(t-\tau)A} Fu(\tau) d\tau \quad (3.2)$$

$$u_0(t) = e^{-tA} a \quad (3.3)$$

we use successive approximation

$$u_{j+1} = u_0 + Su_j, \quad j \geq 0 \quad (3.4)$$

and estimate them in various norms. Since Theorem 1 improves some known results we give the proof for completeness although the basic idea is nowadays standard (cf. [8,14,17,23,28]). The proof given below is technically different from those of [8,9,14,23,30] because we do not use any estimates for spatial derivatives of  $u$ .

~~Proof of Theorem 1.~~ We begin with estimates for  $e^{-tA} Fv$ . The assumptions (A), (N1,2) give

$$\|e^{-tA}(Fv-Fw)\|_s \leq \frac{M'}{t^{\beta(p)-\delta}} \|v-w\|_p (\|v\|_p^\alpha + \|w\|_p^\alpha), \quad 0 < t < T \quad (3.5)$$

with  $\delta = (1/s - 1/p)n/m$ ,  $M' = 2^{\beta-\delta} M N_1 N_2$ ,  $v, w \in E^p$  provided that  $p > 1+\alpha$ ,  $s \geq p/(1+\alpha)$ ; here  $\beta = \beta(p)$  is defined by (2.9). In fact using (A), we have

$$\begin{aligned} \|e^{-tA}(Fv-Fw)\|_s &= \|e^{-tA/2} e^{-tA/2} (Fv-Fw)\|_s \\ &\leq \frac{2^{\theta} M}{t^{\theta}} \|e^{-tA/2} (Fv-Fw)\|_h \quad \text{with } \theta = \frac{n\alpha}{mp} - \delta, \quad h = \frac{p}{\alpha+1}. \end{aligned}$$

This is dominated by

$$\frac{2^{\beta-\delta} M_N}{t^{\beta-\delta}} \|Gv - Gw\|_h$$

since  $F$  has the form (2.3) and (N1) holds for  $\Gamma$ . The estimate (3.5) now follows from (N2).

We next derive an a priori estimate for

$$K_j = K_j(T_0) = \sup_{0 < t \leq T_0} t^\sigma \|u_j\|_p(t), \quad j \geq 0$$

for  $\sigma, p$  such that

$$\sigma = \left(\frac{1}{r} - \frac{1}{p}\right)\frac{n}{m}, \quad 0 \leq \sigma < \frac{1}{\alpha+1}, \quad p > \alpha+1, \quad p \geq r, \quad p \neq p_0 \quad (3.6)$$

We wrote that the numbers  $\sigma, p$  satisfying (3.6) do exist because we have assumed  $p_0 > 1$ . In fact, the definition of  $p_0$  in (2.5) shows

$$\frac{\alpha+1}{r} < \frac{\alpha+1}{p_0} = \frac{1}{p_0} + \frac{m}{n} - \frac{\gamma}{n} \leq \frac{1}{p_0} + \frac{m}{n}$$

which gives

$$\frac{1}{\alpha+1} > \left(\frac{1}{r} - \frac{1}{\alpha+1}\right)\frac{n}{m},$$

provided that  $p_0 > 1$ . This shows there are  $\sigma$  and  $p$  satisfying (3.6). To estimate  $K_j$  let us recall the scheme (3.4):

$$u_{j+1} = u_0 + Su_j.$$

We apply (3.5) with  $v = u_j$ ,  $w = 0$  to the second term  $Su_j$  and get

$$t^\sigma \|Su_j\|_p(t) \leq t^\sigma \int_0^t \frac{M'}{(t-\tau)^{\beta(p)}} \|u_j\|_p^{1+\alpha}(\tau) d\tau; \quad (3.7)$$

here  $p > 1+\alpha$  is used. This gives an iterative estimate

$$K_{j+1} \leq K_0 + M' B K_j^{1+\alpha} T_0^{1-\beta(r)} \quad (3.8)$$

with

$$B = \int_0^1 \frac{1}{(1-\tau)^{\beta(p)}} \frac{1}{\tau^{\sigma(1+\alpha)}} d\tau$$

since  $\beta(r) = \beta(p) + \sigma\alpha$ ; the assumptions  $\sigma < 1/(\alpha+1)$  and  $p > p_0$  ensures the convergence of  $B$  since  $\beta(p_0) = 1$ . For a technical reason we use a less sharp estimate but essentially same as (3.8):

$$K_{j+1} \leq K_j + 2M' B T_0^{1-\beta(r)} K_j^{1+\alpha}.$$

An elementary calculation shows that there is a constant  $K(T_0)$  such that

$$K_j < K \text{ satisfying } 2M' B T_0^{1-\beta(r)} K^\alpha < \frac{1}{1+\alpha} \quad (3.9)$$

and

$$K \rightarrow 0 \text{ as } K_0 \rightarrow 0, \quad (3.10)$$

provided that

$$T_0^{1-\beta(r)} K_0^\alpha \leq c = \left( \frac{1}{\alpha+1} \right)^\alpha \frac{1}{2(1+\alpha)M'B}. \quad (3.11)$$

We thus have an a priori estimate for  $K_j$  under the condition (3.11).

We next study what conditions for  $T_0$  and  $\alpha$  guarantees (3.11). First we prove that for  $\sigma > 0$

$$t^\sigma \|e^{-tA}\|_p \rightarrow 0 \text{ as } t \rightarrow 0. \quad (3.12)$$

Since we have assumed  $(C_c)^{\ell} \cap E^r$  is dense in  $E^r$  there is a sequence  $\{a_i\}$  in  $(C_c)^{\ell}$  such that  $a_i \rightarrow a$  in  $E^r$ . Applying (A) gives

$$\begin{aligned} t^{\sigma} \|e^{-tA} a\|_p &\leq t^{\sigma} \|e^{-tA} (a - a_i)\|_p + t^{\sigma} \|e^{-tA} a_i\|_p \\ &\leq C \|a - a_i\|_r + t^{\sigma} \|e^{-tA} a_i\|_p \end{aligned}$$

with constant  $C$  independent of  $i$  and  $t$ . Since  $a_i \in E^p$ , (A) implies that the second term tends to zero as  $t \rightarrow 0$ . We thus have (3.12) which particularly implies that

$$K_0 \rightarrow 0 \text{ as } T_0 \rightarrow 0 \quad (3.13)$$

for  $\sigma > 0$ . If  $r > p_0$  (consequently  $\beta(r) < 1$ ), the condition (2.8) ensures (3.11) since (A) implies that  $K_0/\|a\|_r$  is bounded independent of  $a$  and  $T_0$ . In the case  $r = p_0$ , (3.13) shows that for small  $T_0$  we have (3.11) for every  $T_0 > 0$ . Moreover, since  $\beta(p_0) = 1$ , (3.9) includes no  $T_0$  explicitly, so  $K$  is bounded on  $(0, T)$  even if  $T = \infty$ . We thus see (3.11) holds under the assumptions on  $a$ ,  $T_0$  in (i), (ii) or (iii) of Theorem 1.

So far we proved a priori estimates (3.9) and (3.10) in the situation of (i), (ii) or (iii). To see the existence it remains to prove the convergence of  $\{u_j\}$  as  $j \rightarrow \infty$ . Actually, we shall first prove  $t^{\sigma} u_j$  converges in  $BC([0, T_0]; E^p)$  provided that  $p, \sigma$  satisfy (3.6). Note that (3.7) implies each  $t^{\sigma} u_j \in BC((0, T_0); E^p)$  because  $e^{-tA}$  is strongly continuous in any  $E^q$ ,  $1 < q < \infty$ ; for example see [28, Lemma 2.1]. Moreover, from (3.10) and (3.13) it follows that  $t^{\sigma} u_j \in BC([0, T_0]; E^p)$  and if

$\sigma > 0$ ,  $t^\sigma u_j$  is zero at  $t = 0$ . To show the convergence we consider the successive difference of  $u_j$  constructed by (3.4):

$$u_{j+1} - u_j = Su_j - Su_{j-1}.$$

Just like deriving (3.8), (3.5) with  $p = s$  gives

$$t^\sigma \|u_{j+1} - u_j\|_p(t) \leq 2M'T_0^{1-\beta(r)} K^\alpha \sup_{0 \leq \tau \leq T_0} \tau^\sigma \|u_j - u_{j-1}\|_p(\tau); \quad (3.14)$$

here  $p > p_0$  is used. Since  $2M'T_0^{1-\beta(r)} K^\alpha < 1/(\alpha+1) < 1$ , this shows that there is a function  $u$  such that  $\lim_{j \rightarrow \infty} t^\sigma u_j = t^\sigma u$  in  $BC([0, T_0]; E^p)$  which solves

$$u = u_0 + Su.$$

Also if  $\sigma > 0$ ,  $t^\sigma \|u\|_p$  takes zero at  $t = 0$ , since each  $t^\sigma u_j$  has the same property. To complete the proof of (2.6) and (2.7) we have to relax the condition on  $p$ . Let  $p'$  be  $r \leq p' \leq p$  and  $\sigma' = (1/r - 1/p')n/m$ . We shall prove that  $t^{\sigma'} u_j$  converges in  $BC([0, T_0]; E^p)$  and that  $t^{\sigma'} \|u\|_{p'}$  takes zero at  $t = 0$  if  $\sigma' > 0$ . Applying (3.5) to  $Su$  with  $s = p'$  yields

$$t^{\sigma'} \|u_{j+1} - u_j\|_{p'}(t) \leq 2M''T_0^{1-\beta(r)} K^\alpha \sup_{0 \leq \tau \leq T_0} \tau^{\sigma'} \|u_j - u_{j-1}\|_{p'}(\tau); \quad (3.15)$$

with a different constant  $M''$ . Since  $t^\sigma u_j$  converges in  $BC([0, T_0]; E^p)$  this implies  $t^{\sigma'} u_j \rightarrow t^{\sigma'} u$  in  $BC([0, T_0]; E^p)$ . Thus we have proved (2.6). Since  $t^\sigma \|u_j\|_p$  and  $t^{\sigma'} \|u_0\|_{p'}$  take zero at  $t = 0$  by (3.12), (3.15) implies (2.7). The asymptotic behavior (2.10) comes from (3.9) if  $p, \sigma$  satisfy (3.6). For general  $p$  in Theorem 1, it is not difficult to

see (2.10) also holds by using (3.12) and (3.15). Thus we have proved (i), (ii) and (iii) of Theorem 1.

Since (v) easily follows from (ii) <sup>and (iv)</sup>, it remains to prove the uniqueness (iv). Let  $u, v$  be two solutions of (2.2) satisfying assumptions of (iv). We may assume that  $u$  and  $v$  satisfy (2.6-7) for same  $\sigma$ ,  $0 < \sigma < 1/(\alpha+1)$ ,  $p > \alpha+1$ ; if  $r > p_0$ ,  $\sigma$  may equal zero and we assume (2.6) only. In fact, if  $u$  satisfies (2.6-7) for some  $\sigma, p$  such that  $0 < \sigma < 1/(\alpha+1)$ ,  $p > \alpha+1$ ,  $\sigma = (1/r - 1/p)n/m$ , we see  $u$  also satisfies (2.6-7) for every  $\sigma', p'$  such that  $0 \leq \sigma' < \sigma$ ,  $\sigma' = (1/r - 1/p')n/m$ . To see this we just use the estimate

$$t^{\sigma'} \|u\|_{p'}(t) \leq t^{\sigma'} \|u_0\|_{p'} + M'' T_0^{1-\beta(r)} \left( \sup_{0 \leq \tau \leq T_0} \tau^{\sigma} \|u\|_p(\tau) \right)^{1+\alpha},$$

which is proved similarly to (3.15).

We first consider the case  $r > p_0$ . Let  $K$  be a constant such that  $t^{\sigma} \|u\|_p, t^{\sigma} \|v\|_p \leq K$ ,  $0 \leq t \leq T_0$ , where  $\sigma, p$  satisfy  $0 \leq \sigma < 1/(\alpha+1)$ ,  $p > \alpha+1$ ,  $\sigma = (1/r - 1/p)n/m$ . Since  $u - v = Su - Sv$  by definition, we have the estimate

$$t^{\sigma} \|u-v\|_p(t) \leq 2M' t_0^{1-\beta(r)} K^{\alpha} \sup_{0 \leq \tau \leq t_0} \tau^{\sigma} \|u-v\|_p(\tau), \quad 0 \leq \tau \leq t_0 \quad (3.16)$$

which follows from (3.5) similarly to (3.15). Since  $r > p_0$ , we can take  $t_0$  small so that  $2M' t_0^{1-\beta(r)} K^{\alpha} < 1$ . This implies that  $u = v$  on  $[0, t_0)$ . Since  $u, v \in BC([\varepsilon, T_0], E^p)$  for every  $\varepsilon > 0$ , the above argument with initial data  $u(\varepsilon) = v(\varepsilon)$  shows that if  $u = v$  on  $[0, \tau)$  for some  $0 < \tau < T_0$ , then  $u$  agrees with  $v$  on  $[\tau, \tau+t_0)$  for some  $t_0 > 0$ . This shows  $u = v$  on  $[0, T_0)$ .

It remains to discuss the case  $r = p_0$ . Let  $K(t_0)$  be a constant such that  $t^\sigma \|u\|_p, t^\sigma \|v\|_p \leq K(t_0)$ ,  $0 \leq t \leq t_0$  where  $0 < \sigma < 1/(\alpha+1)$ ,  $p > \alpha+1$ ,  $\sigma = (1/r - 1/p)n/m$ . Here by (2.7)  $K(t_0)$  tends to zero as  $t_0 \rightarrow 0$ . Instead of (3.16) we have

$$t^\sigma \|u-v\|_p(t) \leq 2M'K(t_0)^\alpha \sup_{0 \leq \tau \leq t_0} \tau^\sigma \|u-v\|(\tau), \quad 0 \leq t \leq t_0$$

since  $\beta(p_0) = 1$ . Take  $t_0 > 0$  small so that  $2M'K(t_0)^\alpha < 1$ . As is seen in the preceeding paragraph we have  $u = v$  on  $[0, t_0)$ . Since  $u, v \in BC([t_0/2, T_0), E^p)$ ,  $p > \alpha+1$ ,  $p > r$  with  $u(t_0/2) = v(t_0/2)$ , the uniqueness of the case  $r > p_0$  implies  $u = v$  on  $[t_0/2, T_0)$ . Thus we have proved the uniqueness.  $\square$

To prove Theorem 2 a new idea is necessary. We begin with estimates for the linear part  $u_0$  which are simple but important.

LEMMA. Under the condition (A) we have

$$\int_0^t \|e^{-\lambda A} a\|_p^q dt \leq C \|a\|_r^q, \quad 0 \leq t \leq T$$

$$\frac{1}{q} = \left(\frac{1}{r} - \frac{1}{p}\right) \frac{n}{m}, \quad q > r > 1$$

with  $C = C(p, q, M)$ .

Proof. We apply the Marcinkiewicz interpolation theorem [25, Appendix]. The idea of the proof is essentially the same as Weissler's for the case  $A = -\Delta$ ,  $p = \infty$  [31, p. 39(6)]. However, we give it for completeness. Consider the map  $U$  defined by  $Ua = \|e^{-tA} a\|_p$  from  $(L^p)^k$  to functions on  $[0, T)$ . The assumption (A) shows that  $U$  is of weak type  $(r, q)$ , where  $1/q = (1/r - 1/p)n/m$ ,  $1 < r, q$ . Clearly  $U$  is subadditive and of

weak type  $(p, \infty)$ . If  $r \leq q$ , the interpolation theorem is applicable. So  $U$  is of strong type  $(r_1, q_1)$ ,  $r_1 < q_1$  with  $1/q_1 = (1/r_1 - 1/p)n/m$ , which is the desired result.

**Proof of Theorem 2.** Let  $\{u_j\}$  be the sequence defined by (3.4). We shall prove

$$\|u_{j+1}\|_{p,q,T'} \leq \|u_0\|_{p,q,T'} + CT'^{1-\beta(r)} \|u_j\|_{p,q,T'}^{1+\alpha}, \quad (3.17)$$

$$T' \geq 0$$

with  $C = C(M', p, q)$ ,  $p > \alpha+1$ ,  $q > \alpha+1$ ,  $1/q = (1/r - 1/p)n/m$ , where  $\|v\|_{p,q,T}$  denotes the norm  $\left( \int_0^T \|v\|_p^q ds \right)^{1/q}$ . Recall the inequality (3.7):

$$\|Su_j\|_p(t) \leq M' \int_0^t \frac{1}{(t-\tau)^{\beta(p)}} \|u_j\|_p^{1+\alpha}(\tau) d\tau.$$

Applying the Hardy-Littlewood inequality [19, p. 31] to this yields

$$\|Su_j\|_{p,q,T'} \leq CM'^{1-\beta(r)} \|u_j\|_{p,q,T'}^{1+\alpha}.$$

Using this estimate to  $Su_j$  in (3.4), we get (3.17).

If  $q > r$ , Lemma shows that  $\|u_0\|_{p,q,T'}$  is finite and  $\|u_0\|_{p,q,T'} \leq C\|a\|_r$ . Just like the proof of Theorem 1 (i) (ii) (iii), this with (3.17) implies that  $\|u_j\|_{p,q,T'}$  is bounded in  $j$  provided that  $T'$  is sufficiently small or  $\|a\|_{p_0}$  is sufficiently small for  $T' = T$ ,  $r = p_0$ . Since  $u \in BC((0, T_0); E^p)$  and  $u$  is the limit of  $u_j$ , this implies

$$u \in L^q(0, T_0; E^p) \text{ with } q, p > \max\{r, \alpha+1\} \quad (3.18)$$

If  $\|a\|_{p_0}$  is sufficiently small and  $T = \infty$ , we have

$$u \in L^q(0, T_0; E^p) \text{ with } q, p > \max(p_0, \alpha+1) \quad (3.19)$$

To complete the proof of (i) (ii) it remains to prove (3.18), (3.19) without assuming  $p > \alpha+1$ . Let  $p'$  be  $r < p' \leq p$  and  $1/q' = (1/r - 1/p')n/m$ . Similarly to deriving (3.15) and (3.17), we have

$$\begin{aligned} \|u_{j+1}\|_{p', q', T'} &\leq \|u_0\|_{p', q', T'} \\ &+ CT^{1-\beta(r)} \|u_j\|_{p, q, T'}^\alpha \|u_j\|_{p', q', T'} \end{aligned} \quad (3.20)$$

As we have seen before, if  $T'$  is sufficiently small,  $CT^{1-\beta(r)} \times \|u_j\|_{p, q, T'}$  is small, say, less than  $1/2$ . In the case  $r = p_0$ ,  $T = \infty$ , if  $\|a\|_{p_0}$  is sufficiently small then  $\|u_j\|_{p, q, \infty}$  is small, less than  $1/2C$ . This shows that  $\|u_j\|_{p', q', T'}$  is bounded under the assumptions of (i), (ii). We thus have proved (3.18-19) without assuming  $p > \alpha+1$ , which completes the proof of (i), (ii).

It remains to prove the uniqueness. Let  $u$  and  $v$  be solutions of (2.2) satisfying the assumption of (iii). We may assume that  $u$  and  $v$  are in  $L^q(0, T_0; E^p)$  for same  $q, p$  such that  $q, p > \max(r, \alpha+1)$ ,  $1/q = (1/r - 1/p)n/m$ . In fact if  $u$  is in  $L^q(0, T_0; E^p)$  for some  $p, q$  such that  $p > r$ ,  $q, p > \alpha+1$ ,  $1/q = (1/r - 1/p)n/m$ ,  $u$  is in  $L^{q'}(0, T_0; E^{p'})$  for all  $p', q' > \max(r, \alpha+1)$ ,  $p' \leq p$ ,  $1/q' = (1/r - 1/p')n/m$ . This easily follows from the estimate

$$\|u\|_{p', q', T_0} \leq \|u_0\|_{p', q', T_0} + CT_0^{1-\beta(r)} \|u\|_{p, q, T_0}^\alpha \|u\|_{p', q', T_0}$$

which is proved similarly to (3.20).

By (3.4) we see  $u - v = Su - Sv$ . Apply (3.5) and the Hardy-Littlewood inequality to get

$$\|u-v\|_{p,q,T'} \leq CT'^{1-\beta(r)} (\|u\|_{p,q,T'}^\alpha + \|v\|_{p,q,T'}^\alpha) \|u-v\|_{p,q,T'} \quad (3.21)$$

where  $0 < T' \leq T_0$ . If  $T'$  is sufficiently small so that  $\|u\|_{p,q,T'}^\alpha, \|v\|_{p,q,T'}^\alpha \leq 1/4CT_0^{1-\beta(r)}$ , then (3.21) implies that  $u = v$  on  $[0, T')$ .

Set

$$T'' = \sup \{t; u(s) = v(s), \text{ a.e.s. } 0 \leq s \leq t\}.$$

We have proved  $T'' \geq T' > 0$ . To show the uniqueness on  $(0, T_0)$ , it is enough to prove  $T'' = T_0$ . Suppose not, i.e.  $T'' < T_0$ . Since  $u = v$  on  $(0, T_1)$ , estimating  $Su - Sv$  just like (3.21) yields

$$\begin{aligned} \|u-v\|_{p,q,T'',t} &\leq CT_0^{1-\beta(r)} (\|u\|_{p,q,T'',t}^\alpha + \|v\|_{p,q,T'',t}^\alpha) \\ &\quad \times \|u-v\|_{p,q,T'',t} \end{aligned}$$

where

$$\|w\|_{p,q,T,t} = \left( \int_T^t \|w\|_p^q ds \right)^{1/q}.$$

It is easy to see that there is a constant  $\varepsilon > 0$  such that for  $t - T'' \leq \varepsilon$  the coefficient of  $\|u-v\|_{p,q,T'',t}$  of the right hand side is small, say, less than  $1/2$ . The inequality now shows  $u = v$  on  $[T'', T'' + \varepsilon)$  which contradicts the maximality of  $T''$ . We thus have proved  $T'' = T_0$  and the uniqueness of solutions. This completes the proof of Theorem 2.

## 4. THE SEMILINEAR HEAT EQUATION AND THE NAVIER-STOKES SYSTEM

In this section we give examples of (2.1) satisfying all assumptions for (2.1) of section 2. A simple but important example is the initial value problem for the semilinear heat equation in  $\Omega \subset \mathbb{R}^n$ :

$$u_t - \Delta u = |u|^\alpha u \quad (\alpha > 0), \quad u(0, x) = a(x), \quad x \in \Omega \quad (4.1)$$

with boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega, \quad (4.2)$$

where  $\Omega$  is a smoothly bounded domain or  $\mathbb{R}^n$  itself; if  $\Omega = \mathbb{R}^n$  we just consider (4.1). We now check assumptions for (2.1) in section 2. Since  $C_c(\Omega)$  is dense in  $L^p(\Omega)$ , assumptions on  $E^p$  are verified if we put  $E^p = L^p(\Omega)$  and  $P = \text{identity}$ . If we put  $e^{-tA} = e^{t\Delta}$ ,  $e^{-tA}$  is strongly continuous in  $L^p(\Omega)$ . It is easy to check (A) with  $m = 2$  if  $\Omega = \mathbb{R}^n$  because  $e^{t\Delta}$  can be written explicitly by using the Gaussian kernel. Also for a bounded domain  $\Omega$  it is known that (A) holds; see [28] for example. The assumptions for the nonlinear term are easily verified. The operator  $\Gamma$  in (N1) should be the identity operator and  $\gamma = 0$ . We thus have checked all assumptions for (2.1). Let us pick up some results of Theorems 1 and 2 for (4.1-2).

**THEOREM 3. (Existence).** Suppose  $a \in L^{p_0} = L^{p_0}(\Omega)$  and  $p_0 = n\alpha/2 > 1$ . Then there is  $T_0 > 0$  and a mild solution of (4.1-2) on  $[0, T_0)$  such that

$$u \in BC([0, T_0); L^{p_0}) \cap L^q(0, T_0; L^p)$$

with  $1/q = (1/p_0 - 1/p)n/2$ ,  $q, p > p_0$ ,  $q > \alpha+1$ . There is a positive constant  $\epsilon$  such that if  $\|a\|_{p_0} < \epsilon$  then  $T_0$  can be taken infinity.

(Uniqueness). Mild solutions of (4.1-2) are unique in  $L^q(0, T_0; L^p)$  with  $1/q = (1/p_0 - 1/p)n/2$ ,  $q, p > \alpha+1$ ,  $p > p_0$ .

(Estimate near the blow up). Let  $(0, T_*)$  be the maximal interval such that  $u$  solves (4.1-2) in  $C((0, T_*); L^r \wedge L^{r'})$ ,  $r > p_0 > 1$ ,  $r' > \max(\alpha+1, p_0)$ .  
Then

$$\|u(s)\|_r \geq C/(T_* - s)^{(2r - n\alpha)/2r\alpha}$$

with constant  $C$  independent of  $T_*$  and  $s$ .

Remark. The mild solution  $u$  constructed above is a classical solution of (4.1-2); see also [29]. In fact  $u \in L^q(0, T_0; L^p)$  with  $p = q$  implies that  $u \in L^p((0, T_0) \times \Omega)$  with  $p = p_0 + \alpha > \alpha+1$ . So we see  $|u|^\alpha u \in L^{p/(\alpha+1)}((0, T_0) \times \Omega)$ . An  $L^p$ -estimate [16] for the heat equation gives  $u \in W_{p/(\alpha+1)}^{2,1}((\delta, T_0) \times \Omega)$  for  $\delta > 0$ . A standard bootstrap argument yields  $u \in W_r^{2,1}$  for every  $r > 1$ . Applying the Schauder estimate [16], we have  $\nabla^2 u, u_t \in C((0, T_0) \times \bar{\Omega})$ .

Remark. Except the fact that  $u \in L^q(0, T_0; L^p)$  the existence part of Theorem 3 is known by Weissler [29, 31]; in [31] he assumed  $a \geq 0$ , however his proof holds for general  $a$ .

Remark. The assumption  $p_0 > 1$  is necessary for the global existence for small initial data. In the case  $p_0 \leq 1$  and  $\Omega = \mathbb{R}^n$  solutions blow up even if  $a \geq 0$  is small; see Fujita [6] and Weissler [31].

Remark. The smallness assumption is necessary for the global existence. If  $\Omega$  is a bounded domain, for initial data  $\phi = k\psi$ ,  $\psi \geq 0$ ,  $\psi \not\equiv 0$ ,  $k \in \mathbb{R}$ , solutions blow up in a finite time provided that  $k > 0$  is sufficiently large; see, for example [7]. If  $\Omega$  is  $\mathbb{R}^n$ , choose  $a \in C_c^\infty(\mathbb{R}^n)$  such that  $a > \phi$  in  $\bar{\Omega}$  and  $a \geq 0$ . A standard comparison argument shows that the solution  $u$  of (4.1) with such initial data  $a$  blows up.

Remark. As is seen in Theorem 1 the local existence holds even if  $p_0$  is replaced by  $r > p_0$ . However, for the global existence the assumption  $\|a\|_{p_0} < \varepsilon$  cannot be replaced by  $\|a\|_r < \varepsilon$  if  $\Omega = \mathbb{R}^n$ . In fact since  $\|a\|_r$  is not invariant under scaling  $a_\lambda(x) = \lambda^{2/\alpha} a(\lambda x)$ , the assumption  $\|a\|_r < \varepsilon$  is equivalent to  $a \in L^r(\mathbb{R}^n)$ . However, smallness of  $a$  is necessary for the global existence since there is  $a \in C_c^\infty(\mathbb{R}^n)$  such that the solution blows up; see the preceeding remark.

Remark. As is pointed out in the Introduction, note that the norm of  $L^q(0, T_0; L^p)$  in our uniqueness result has zero scaling dimension. This improves Baras' result [1]: the uniqueness holds in  $C([0, T_0]; L^p)$ ,  $p > p_0$ , the scaling dimension of which is less than zero. The class  $C([0, T_0]; L^{p_0})$  does not guarantee the uniqueness although the norm is dimensionless. In fact Ni and Sacks [18] proved the nonuniqueness in  $C([0, T_0]; L^{p_0})$  if  $\Omega$  is a ball and  $\alpha = 2/(n-2)$  (consequently,  $p_0 = \alpha + 1$ ).

Remark. The estimates from below near the blow-up is also proved in Baras [1] by using essentially same methods. Special cases are previously proved in [31].

We next consider the initial value problem for the Navier-Stokes system in  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ):

$$\begin{aligned} u_t - \Delta u + (u, \nabla)u + \nabla p &= 0, & \nabla \cdot u &= 0 \\ u(x, 0) &= a(x), & x &\in \Omega \end{aligned} \quad (4.3)$$

with boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \quad (4.4)$$

where  $(u, \nabla) = \sum_{j=1}^n u^j (\partial/\partial x_j)$  and  $\Omega$  is a smoothly bounded domain or  $\mathbb{R}^n$  itself; if  $\Omega = \mathbb{R}^n$ , we only consider (4.3). This system describes the motion of viscous incompressible fluid filling a rigid vessel  $\Omega$ . The function  $u = (u^1(x, t), \dots, u^n(x, t))$  represents the velocity of the fluid and  $p(x, t)$  is the pressure. The function  $a = (a^1(x), \dots, a^n(x))$  is given initial velocity. For simplicity external force is assumed zero.

For a suitable choice of function spaces the system (4.3-4) can be written as a form (2.1). This is nowadays very standard; see [9] and papers cited there. However, we briefly review it for completeness. Let  $E^p$  be the closure in  $(L^p(\Omega))^n$  of all divergence-free vector fields with compact support in  $\Omega$ . It is known that there is a continuous projector  $P$  from  $(L^p(\Omega))^n$  to  $E^p$  and that  $P$  is independent of  $1 < p < \infty$  on  $(C_c)^\infty$ . Clearly,  $(C_c)^\infty \cap E^p$  is dense in  $E^p$ . The Stokes operator  $A$  in  $E^p$  is defined by  $A = -PA$  with dense domain

$$D(A) = E^p \cap \{u \in (W^{2,p}(\Omega))^n; u = 0 \text{ on } \partial\Omega\}.$$

Applying  $P$  to both sides of the Navier-Stokes system gives

$$u_t + Au = -P(u, \nabla)u, \quad u(0) = a \in E^p \quad (4.5)$$

for some  $p > 1$ ; obviously, this is a form (2.1) with  $Fu = -P(u, \nabla)u$ .

We now verify assumptions on  $e^{-tA}$  in section 2. For a bounded domain  $\Omega$ , we have

$$\|A^\sigma e^{-tA} f\|_s \leq C \|f\|_s t^\sigma$$

since  $e^{-tA}$  is a bounded holomorphic semigroup in  $E^p$ . Since  $D(A^\sigma)$  is continuously embedded in  $(H^{2\sigma, 2}(\Omega))^n$  this together with the Sobolev embedding theorem yields (A) with  $m = 2$ ; for more detail see [8, 9]. For  $\Omega = \mathbb{R}^n$ ,  $e^{-tA}$  is the solution operator of the heat equation so (A) can be directly verified [15].

We next verify the assumptions for the nonlinear term  $Fu = -P(u, \nabla)u$ . Since  $\nabla \cdot u = 0$ , we have  $(u, \nabla)u^j = \sum_{i=1}^n \nabla_j u^i u^j$ . We put  $\Gamma = \sum_{j=1}^n P \nabla_j$  which is a linear operator from  $(L^p)^{n^2}$  to  $E^q$ .

The nonlinear term  $Fu$  is expressed by  $\Gamma Gu$  if we define  $g(u): \mathbb{R}^n \rightarrow \mathbb{R}^{n^2}$  by  $g(u)^{ij} = -u^i u^j$ . It is easy to see  $g$  satisfies (2.4) with  $\alpha = 1$  which implies that  $(Gu)(x) = g(u(x))$  satisfies (N2). For a bounded domain  $\Omega$ , since

$$\begin{aligned} \|e^{-tA} \Gamma f\|_p &= \|A^{1/2} e^{-tA} A^{-1/2} \Gamma f\|_p \\ &\leq \frac{C}{t^{1/2}} \|A^{-1/2} \Gamma f\|_p \end{aligned}$$

and since  $A^{-1/2} \Gamma$  is bounded in  $L^p$  [8], the assumption (N1) with  $\gamma = 1$  is verified. If  $\Omega = \mathbb{R}^n$ ,  $\nabla_j (= \partial/\partial x_j)$  and  $P$  commute with  $e^{-tA}$  so (N1) is directly verified. We thus have checked all assumptions in section 2. Let us pick up some results

from Theorems 1 and 2 for (4.5) which are important in sequel.

THEOREM 4. (Existence and Uniqueness) Suppose  $a \in E^r$ ,  $r > n$ . Then there is  $T_0 > 0$  and a unique mild solution of (4.5) on  $[0, T_0)$  such that

$$u \in BC([0, T_0); E^n) \cap L^q(0, T_0; E^p) \quad (4.6)$$

$$t^{1/q} u \in BC([0, T_0); E^p) \text{ and } t^{1/q} u \text{ takes zero at } t = 0 \quad (4.7)$$

with  $2/q + n/p = n/r$ ,  $q, p > r$ . There is a positive constant  $\epsilon$  such that if  $\|a\|_r < \epsilon$  then  $T_0$  can be taken infinity for  $r = n$ . (Estimate near the blow up). Let  $(0, T_*)$  be the maximal interval such that  $u$  solves (4.5) in  $C((0, T_*); E^r)$ ,  $r > n > 1$ . Then

$$\|u(s)\|_r \geq C/(T_* - s)^{(r-n)/2r} \quad (4.8)$$

with constant  $C$  independent of  $T_*$  and  $s$ .

Remark. The mild solution  $u$  constructed above is a classical smooth solution; see [8]. More precisely,  $u$  belongs to  $C^\infty(\bar{\Omega} \times (0, T_0))$ . We call  $u$  a regular solution.

Remark. The advantage of theorem 4 is that  $u$  belongs to  $L^q(0, t_0; E^p)$  having zero scaling dimension for  $a \in L^n$ . In [4] Fabes, Lewis and Rivere constructed  $L^q(0, T_0; E^p)$  solution. However they are forced to assume  $a \in E^r$ ,  $r > n$  because they do not use Lemma in section 3; see also [3] for  $\Omega = \mathbb{R}^n$ .

Remark. As is seen in our proofs of Theorems 1, 2 and 4, we do not use a priori estimates for the nonstationary Stokes equation due to Solonnikov [24]:

$$\int_0^t \|Ae^{-tA}a\|_q^q dt \leq C \|a\|_{W^{2-2/q,q}}^q,$$

where  $\Omega$  is a bounded domain. However, we note that for  $e^{-tA}$  Lemma in section 3 follows from this estimate and the characterization of  $D(A^\alpha)$ ; this method also shows that the inequality in Lemma holds for the Stokes operator if  $q = r$ .

Remark. In the case  $\Omega = \mathbb{R}^3$ , (4.8) was given by Leray [17]. He also gave the estimate from below for  $\|\nabla u\|_2$  and  $\|u\|_\infty$  near the blow-up. In [5] Foias and Temam gave the corresponding estimate for  $\|u\|_{\dot{H}^1}$  when  $\Omega$  is a bounded domain in  $\mathbb{R}^3$ .

Remark. In [15] Kato studied the asymptotic behavior of  $\|u\|_n$  as  $t \rightarrow \infty$  when  $\Omega = \mathbb{R}^n$ . He proved

$$\frac{1}{T} \int_0^T \|u\|_n(t) dt \rightarrow 0 \quad \text{as } T \rightarrow \infty \quad (4.9)$$

if  $u(0) = a$  is small in  $E^n$ . When  $n = 2$ , this implies  $\|u\|_2(t) \rightarrow 0$  as  $t \rightarrow \infty$  since the energy cannot increase. To show (4.9) he used the fact that  $u \in L^q(0, \infty; E^p)$  in (4.6). Note that his method works in a more general situation. In fact, under the assumptions of Theorem 1 we have

$$\frac{1}{T} \int_0^T \|u\|_{p_0}(t) dt \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

## 5. REGULARITY OF WEAK SOLUTIONS OF THE NAVIER-STOKES SYSTEM

We shall prove some sufficient conditions for regularity announced in [10] by using regular solutions in Theorem 4; the case  $\Omega = \mathbb{R}^n$  is included here. The results can be extended to the case having non-zero external force  $f$ ;

$$u_t + Au = Fu + f$$

under an appropriate restriction on  $f$ ; however, we omit  $f$  for simplicity.

We begin by showing that regular solutions satisfy energy equality provided that the initial data are in  $E^2$ . For  $\Omega = \mathbb{R}^n$  this is important because  $u(t) \in E^n$  does not imply  $u(t) \in E^2$ .

PROPOSITION 1. (i) Let  $u$  be the regular solution of (4.5) on  $(0, T_0)$  ( $T_0 < \infty$ ) satisfying (4.6). Suppose  $u(0) = a \in E^2$ . Then

$$u \in L^\infty(0, T_0; E^2) \cap L^2(0, T_0; H^1), \quad (5.1)$$

where  $H^1 = H^1(\Omega)$  is the Sobolev space of order one.

(ii) The above regular solution  $u$  satisfies the energy equality

$$\|u\|_2^2(t) + 2 \int_0^t \|\nabla u\|_2^2(s) ds = \|a\|_2^2 \quad (5.2)$$

Proof. We begin with estimates for  $Su$  in (3.4), where  $A$  is the Stokes operator and  $Fu = -P(u, \nabla)u$ . We shall prove

$$\|Su\|_2(t) \leq C \sup_{0 < \tau \leq t} s^{1/q} \|u\|_p \|u\|_2 \quad (5.3)$$

$$\|\nabla Su\|_{2,2,t} \leq C \|u\|_{p,q,t} \|\nabla u\|_{2,2,t} \quad (5.4)$$

with  $2/q + n/p = 1$ ,  $p > n$ , where  $\|v\|_{p,q,t} = \left( \int_0^t \|v\|_p^q(s) ds \right)^{1/q}$

and  $C$  is independent of  $u$  and  $t$ . As in section 3, we have

$$\|Su\|_2(t) \leq C \int_0^t (t-s)^{-1/2 - n/2p} \|u\|_p(s) \|u\|_2(s) ds,$$

which yields (5.3). Since  $\|Ve^{-tA}a\|_2 \leq C\|a\|_2/t^{1/2}$  [9,15], it is not difficult to see

$$\|\nabla Su\|_2(t) \leq C \int_0^t (t-s)^{-1/2 - n/2p} \|u\|_p \|\nabla u\|_2(s) ds.$$

Applying the Hardy-Littlewood inequality gives (5.4).

Since  $u$  satisfies (3.4), (5.3-4) yields

$$\|u\|_{2,\infty,t} \leq C\|a\|_2 + C\|s^{1/q}u\|_{p,\infty,t} \|u\|_{2,\infty,t},$$

$$\|\nabla u\|_{2,2,t} \leq C\|Ve^{-tA}a\|_{2,2,t} + C\|u\|_{p,q,t} \|\nabla u\|_{2,2,t}.$$

In the second inequality, since  $v = e^{-tA}a$  solves  $v_t + Av = 0$ , taking the inner product of  $v$  and the equation eventually gives  $2\|Ve^{-tA}a\|_{2,2,t}^2 \leq \|a\|_2^2$ . By (4.6-7) both  $\|s^{1/q}u\|_{p,\infty,t}$  and  $\|u\|_{p,q,t}$  are small, say, less than  $1/2C$  if  $t$  is sufficiently small.

Hence the above two inequalities imply that

$$u \in L^\infty(0, t_0; E^2) \cap L^2(0, t_0; H^1)$$

for small  $t > 0$ . This argument can be repeated for initial data  $u(t)$  since  $u \in BC([0, T_0]; E^n)$ , so we eventually have (5.1).

The proof of (ii) is given by Prodi; see [21]. The crucial point is that  $(Fu, u)_{L^2}$  makes sense and vanishes.  $\square$

Remark. The proof given here is similar to that of Kato [15].

If  $\|a\|_n$  is sufficiently small, so do  $\|u\|_{p,q,\infty}$ ,  $\|s^{1/q}u\|_{p,\infty,\infty}$ .

If so, we have (5.1) for  $T_0 = \infty$ .

Let us recall properties of weak solutions of (4.5) constructed by Leray [17] and Hopf [12]. A weak solution  $v$  is supposed to satisfy the following properties [26]:

$$v \text{ is weakly continuous from } [0, \infty) \text{ to } E^2 \quad (5.5)$$

$$v \in L^\infty(0, \infty; E^2) \cap L^2(0, \infty; H^1) \quad (5.6)$$

$$\int_0^\infty \{(v, \partial\phi/\partial t) + (v, \Delta\phi) + (v, (v, \nabla)\phi)\} dt + (a, \phi(x, 0)) = 0 \quad (5.7)$$

$$\int_0^\infty (v, \nabla\psi) dt = 0, \quad v = 0 \text{ on } \partial\Omega \times (0, \infty) \text{ if } \partial\Omega \neq \emptyset,$$

for all  $\phi^i, \psi \in C_c^\infty(\Omega \times [0, \infty))$ ,  $\nabla \cdot \phi = 0$ , where  $(,)$  denotes the standard  $L^2$  inner product. Moreover, the energy inequality holds for  $v$ ;

$$\|v\|_2^2(t) + 2 \int_{t_0}^t \|\nabla v\|_2^2(s) ds \leq \|v\|_2^2(t_0) \quad \text{for } t \geq t_0 \quad (5.8)$$

$$\text{a.e. } t_0 \geq 0$$

$$\|v\|_2^2(t) + 2 \int_0^t \|\nabla v\|_2^2(s) ds \leq \|a\|_2^2 \quad \text{for } t \geq 0 \quad (5.8')$$

Up to now we do not know the uniqueness nor the regularity of solutions satisfying (5.5-8) if  $n \geq 3$ . However, as far as a regular solution exists all weak solutions should agree with the regular solution. More precisely, we have

PROPOSITION 2. ([21], see also [26, Theorem 3.9]). Let  $v$  be a weak solution satisfying (5.5-7, 8'). Suppose that  $u$  is a solution with  $u(0) = v(0) \in E^2$  satisfying (5.6-7) and

$$u \in L^q(0, T; E^p) (\equiv L_T^{p, q}) \quad (5.9)$$

with some  $p, q$ ,  $2/q + n/p = 1$ ,  $p > n$ . Then  $v$  agrees with  $u$  in  $\bar{\Omega} \times [0, T]$ .

Proof. The crucial step is to show the identity

$$- \int_0^t (2(\nabla u, \nabla v) + b(w, w, u)) ds = (u(t), v(t)) - \|a\|_2^2 \quad (5.10)$$

with  $w = v - u$ , where  $(\cdot, \cdot)$  denotes the  $L^2$ -inner product and

$$b(u_1, u_2, u_3) = ((u_1, \nabla) u_2, u_3). \quad (5.11)$$

Admitting (5.10), we have the estimate

$$\|w\|_2^2(t) + 2 \int_0^t \|\nabla w\|_2^2 ds \leq 2 \int_0^t b(w, w, u) ds, \quad t \geq 0 \quad (5.12)$$

which follows from  $2 \times (5.10) + (5.2) + (5.8)$ . Applying the standard argument to (5.12) eventually gives

$$\|w\|_2^2(t) \leq \|w\|_2^2(0) \left( \exp C \int_0^t \|u\|_p^q ds \right)$$

$$2/q + n/p = 1, \quad p > n$$

which yields  $w \equiv 0$ , since  $w(0) = v(0) - u(0) = 0$ ; see [21].

It remains to prove (5.10). Applying Theorem 4 in [21], (5.7) gives

$$\int_0^t ((v, \phi_{1t}) - (\nabla v, \nabla \phi_1) + b(v, \phi_1, v)) ds$$

$$= (v, \phi_1(\cdot, t)) - (a, \phi_1(\cdot, 0)) \quad (5.13)$$

$$\int_0^t ((u, \phi_2) - (\nabla u, \nabla \phi_2) + b(u, \phi_2, u)) ds \quad (5.14)$$

$$= (u, \phi_2(\cdot, t)) - (a, \phi_2(\cdot, 0))$$

where  $\phi_i \in C_c^\infty(\Omega \times [0, \infty))$ ,  $\nabla \cdot \phi_i = 0$ . We want to put  $\phi_1 = u$  and  $\phi_2 = v$ . Since we have for (5.11)

$$\left| \int_0^t b(u_1, u_2, u_3) ds \right| \leq C(\|u_1\|_{2, \infty, t} + \|\nabla u_1\|_{2, 2, t})$$

$$\times \|\nabla u_2\|_{2, 2, t} \|u_3\|_{p, q, t},$$

the trilinear form  $\int b(u_1, u_2, u_3)$  can be extended to  $V_\sigma \times V \times L^q(0, T; L^p)$  with  $p > n$ , where  $V = L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ ,  $V_\sigma = V \cap L_T^{2, \infty}$ . Moreover, by the standard density argument

$$\int_0^t b(u_1, u_2, u_3) ds = - \int_0^t b(u_1, u_3, u_2) ds$$

holds for  $u_1 \in V_\sigma$ ,  $u_2, u_3 \in V \cap L^q(0, T; L^p)$ . This shows that we may replace  $b(v, \phi_1, v)$  in (5.13) by  $-b(v, v, \phi_1)$  and that

$$- \int_0^t b(v, v, u_\varepsilon) ds \rightarrow - \int_0^t b(v, v, u) ds \quad \text{of } u_\varepsilon \rightarrow u \text{ in } L_T^{p, q} \quad (5.15)$$

$$\int_0^t b(u, v_\varepsilon, u) ds \rightarrow \int_0^t b(u, v, u) ds \quad \text{if } v_\varepsilon \rightarrow v \text{ in } V \quad (5.16)$$

Replace  $b(v, \phi_1, v)$  in (5.13) by  $-b(v, v, \phi_1)$  and call the new identity (5.13'). Since the set of divergence free vector fields with compact support is dense in  $F^p = H_{0,\sigma}^1 \cap L^p(\Omega)$  (see appendix), we see (5.13') holds for  $\phi \in C_c^\infty([0, \infty); F^p)$  where  $p > n$ . Here we have to handle  $(\nabla v, \nabla \phi_1)$  and  $b(v, v, \phi_1)$  simultaneously so we need a density proposition in  $F^p$ . It is not difficult to see (5.14) holds for  $\phi_2 \in C_c^\infty([0, \infty); H^1)$ . We now plug  $\phi_1 = u_\varepsilon$  and  $\phi_2 = v_\varepsilon$  in (5.13-14), where

$$u_\varepsilon = \rho_\varepsilon * \rho_\varepsilon * \tilde{u}, \quad v_\varepsilon = \rho_\varepsilon * \rho_\varepsilon * \tilde{v}$$

and  $\tilde{u}, \tilde{v}$  are zero extension of  $u$  and  $v$  outside  $[0, t]$ ; here  $\rho_\varepsilon(t) = \varepsilon^{-1} \rho(t/\varepsilon)$  and  $\rho \geq 0$  is a even smooth function with compact support and  $\int \rho ds = 1$ . This technique is due to Temam [26]. We see easily

$$\int_0^t ((v, u_{\varepsilon t}) + (u, v_{\varepsilon t})) ds \rightarrow (u(t), v(t)) - \|a\|_2^2 \quad \text{as } \varepsilon \rightarrow 0 \quad (t \geq 0),$$

because  $(u(t), v(t))$  is continuous in  $t \geq 0$  by (5.2) and (5.5). This together with (5.15-16) shows that adding (5.13) and (5.14) with  $\phi_1 = u_\varepsilon, \phi_2 = v_\varepsilon$  yields

$$- \int_0^t (2(\nabla u, \nabla v) + b(w, v, u)) ds = (u(t), v(t)) - \|a\|_2^2$$

by tending  $\varepsilon \rightarrow 0$ . Since  $\int b(w, u, u) ds$  makes sense and vanishes, this identity yields (5.10).

Remark. The proof given above is essentially found in [21, 26]. In [21] there is a restriction on space dimensions but as is seen above it can be removed. Recently, Sohr and von Wahl [23] gave a proof by using a different approximation. Moreover, they improve Proposition 2 itself. Instead of (5.9) they only assume  $u \in C([0, T]; E^n)$  and get the same conclusion.

A function  $v$  is called a turbulent solution if  $v$  satisfies (5.5-8). We shall prove regularity criteria for turbulent solutions. Let  $v$  be a turbulent solution. Philosophically, our results read: if  $\int_0^t \int |v|^p dx)^{q/p} dt$  having scaling dimension

$k = 2 - q + nq/p$  is finite, the Hausdorff dimension of possible time singularity of  $v$  is less than  $k/2$ . Here  $\Omega$  is a smoothly bounded domain of  $\mathbb{R}^n$  or  $\mathbb{R}^n$  itself.

THEOREM 5. (i) Let  $v$  be a turbulent solution of (4.5). If  $v$  is in  $L_T^{p,q}$  with  $k > 0$  and  $p > n$ , for some  $p, q \geq 1$ , then there is a closed set  $\Sigma$  of  $(0, T)$  whose  $k/2$ -dimensional Hausdorff measure vanishes and such that  $v$  is in  $C^\infty(\bar{\Omega} \times ((0, T) \setminus \Sigma))$ .  
 (ii) Let  $v$  be a solution satisfying (5.6-7). If  $v$  is in  $L_T^{p,q}$  with  $k \leq 0$ ,  $p > n$  for some  $p, q$ , then  $v$  is in  $C^\infty(\bar{\Omega} \times (0, T])$ .

Proof. If we admit Theorem 4, the proof is standard [5, 17].

However, we give it for completeness. Let  $\Sigma_0 = \{t; \|v\|_p(t) = \infty\}$ . Clearly,  $\Sigma_0$  has Lebesgue measure zero. We shall show that there is a closed set  $\Sigma \supset \Sigma_0$  such that  $v$  is smooth in  $\bar{\Omega} \times ((0, T) \setminus \Sigma)$  and  $\Sigma \setminus \Sigma_0$  has Lebesgue measure zero. For  $t \in (0, T) \setminus \Sigma$  Theorem 4 gives a regular solution  $u$  for initial data  $v(t) \in E^p$ ,  $p > n$ . Since  $v(t) \in E^2$ ,  $u$  satisfies the energy equality (5.2) by Proposition 1. We apply Proposition 2 and see  $v$  agrees with the regular solution  $u$  in  $(t, T(t)+t)$  for some  $T(t) > 0$ . Let

$$\Sigma = (0, T) \setminus \bigcup_{t \notin \Sigma_0} (t, T(t)+t).$$

Clearly  $\Sigma$  is closed and  $v$  is smooth in  $\bar{\Omega} \times ((0, T) \setminus \Sigma)$ . Since the points of  $\Sigma \setminus \Sigma_0$  consist of the left end points of one of the connected components of  $(0, T) \setminus \Sigma$ ,  $\Sigma \setminus \Sigma_0$  is countable and has Lebesgue measure zero.

We shall first prove (i). Let  $(r_i, s_i)$   $i \in I$  be the con-

nected component of  $(0, T) \setminus \Sigma$ . Since  $v \in L_T^{p, q}$ , just like Leray [17], (4.8) implies

$$\sum_{i \in I} (s_i - r_i)^{k/2} < \infty \quad (5.17)$$

In fact on  $(r_i, s_i)$ ,  $v$  is a regular solution, so (4.8) with  $r = p$  yields

$$\|v(t)\|_p^q \geq C/(s_i - t)^{1 - k/2}. \quad (5.18)$$

Integrating over  $(r_i, s_i)$  and adding all these inequalities for  $i \in I$  give

$$\int_0^T \|v(t)\|_p^q dt \geq C \sum_{i \in I} (s_i - r_i)^{k/2}.$$

Since  $v \in L_T^{p, q}$ , this implies (5.17). The result (i) now follows from Scheffer's argument [20] (cf. [5, 9]).

If  $k \leq 0$ , according to Prodi's result [21, Theorem 5], the assumptions for  $v$  in (ii) implies that  $v$  satisfies the energy equality (5.2) so  $v$  is a turbulent solution. As is seen in the preceding paragraphs, we have (5.18) with  $k \leq 0$ . Integrating on  $(r_i, s_i)$  yields

$$\int_{r_i}^{s_i} \|v(t)\|_p^q dt = \infty,$$

so  $s_i$  should be greater than  $T$ , since  $v \in L_T^{p, q}$ . In other words there is no time singularity of  $v$  in  $(0, T]$ . We thus have proved  $v \in C^\infty(\bar{\Omega} \times (0, T])$ .  $\square$

Theorem 5 (ii) improves results of Serrin (see [21]) and Kaniel and Shinbrot [13]. They discussed the case  $k < 0$ .

Recently, Sohr [22] proved (ii), however, the method seems different.

Theorem 5 is also useful to understand the difference between the cases  $n = 2$  and  $n \geq 3$ . If  $n = 2$ , (5.6) implies  $v \in L^{p,q}_T$  with  $k = 0$  and  $p > n$ , so every weak solution satisfying (5.6-7) is smooth. However, when  $n = 3$ , (5.6) just implies  $v \in L^{6,2}_T$  so  $k = 1$ . Theorem 5 (i) says that 1/2-dimensional Hausdorff measure of time singularity set vanishes. These results are previously known by many authors; see e.g. [5,17,20,21,26]. However, our results clarify the situation.

In [2] Caffarelli, Kohn and Nirenberg study space-time interior singularities of suitable weak solutions which satisfy a localized version of energy inequality. They have proved that for  $n = 3$  every suitable weak solution  $w$  is smooth w.r.t.  $x$  in  $\bar{\Omega} \times (0,T) \setminus F$  such that  $F$  is a closed set of  $\bar{\Omega} \times (0,T)$  whose Hausdorff dimension is less than one. Hence,  $v \in C^\infty(\bar{\Omega} \times ((0,T) \setminus E))$ , where the Hausdorff dimension of a closed set  $E$  is less than 1/2. This is different from the result for time singularity mentioned in the preceding paragraph because only the interior regularity is discussed.

Remark. It is not difficult to show the existence of a turbulent solution if  $\Omega$  is a bounded domain [12,26]. However, if  $\Omega = \mathbb{R}^n$ , as is pointed out by Kato [15], the existence is known only for  $n \leq 4$ .

Remark. In [9] there is an error in the definition of the Leray-Hopf solutions; the energy estimate (5.1) in [9] should be re-

placed by (5.8). In the proof of Lemma 7.4 in [9] the definition of singular sets should be changed; see the definition of  $\Sigma$  in the proof of Theorem 5 (i).

In Theorem 5 we assume  $p > n$ . We next discuss the marginal case  $p = n$ .

THEOREM 6. (i) Let  $v$  be a turbulent solution of (4.5). If  $v$  is in  $L_T^{n,q}$  then there is a closed set  $\Sigma$  of  $(0,T)$  whose Lebesgue measure vanishes and such that  $v$  is in  $C^\infty(\overline{\Omega} \times ((0,T) \setminus \Sigma))$ .  
 (ii) ([27]). Let  $v$  be a solution satisfying (5.6-7). If  $v$  is in  $C((0,T); E^n)$ , then  $v$  is in  $C^\infty(\overline{\Omega} \times (0,T))$ .

Proof. The proof of (i) is similar to the beginning part of the proof of Theorem 5 (i). The main difference is that we apply Theorem 4 with  $E^n$ -initial data  $v(t)$ . Even in this case (4.6) says the regular solution satisfies assumptions of Proposition 2 if  $v(t) \in E^2$ .

To prove (ii) we may assume that  $v$  is a turbulent solution as in the proof of Theorem 5 (ii). We also may assume that  $(0,T) \setminus \Sigma = \bigcup_{i \in I} (r_i, s_i)$ . For all  $t \in (r_i, s_i)$  we have a regular solution with initial data  $v(t)$  which agrees with  $v$  on  $(t, t+T(t))$ . If we go back to the proof of (3.13), we see  $T(t)$  depends continuously on  $v(t)$  in  $E^n$ -topology. The assumption  $v \in C((0,T); E^n)$  now shows that  $v$  is regular in  $(r_i, s_i + \epsilon)$ ,  $\epsilon > 0$  if  $s_i < T$ . This implies that  $\Sigma$  is empty so  $v \in C^\infty(\overline{\Omega} \times (0,T))$ .  $\square$

If  $n = 4$ , (5.8) implies  $v \in L_T^{4,2}$ , so Theorem 6 (i) says that possible time singularity set of 4-dimensional turbulent solutions has Lebesgue measure zero. This result is proved by

Kato [15] and Sohr and von Wahl [23]. Moreover, in [23] they proved Theorem 6 (i) for  $2 \leq q \leq \infty$  by using their improved version of uniqueness theorem; see the remark of Proposition 2. They do not use  $u \in L^q(0,T; L^p)$  in (4.6) to apply the uniqueness because they replace (5.9) of Proposition 2 by  $u \in C((0,T]; E^n)$ . This is different from our proofs. Of course, Theorem 6 (ii) also follows similarly if we use their uniqueness theorem, although the proof in [27] is different. In [23] they also proved the uniqueness of weak solutions satisfying (5.6-7) in  $L^\infty(0,T; E^n)$ .

Acknowledgments. I am grateful to Professor Folland for pointing out that the inequality in Lemma holds for  $q = r > 1$  if one appeals to the generalized Marcinkiewicz theorem.

## Appendix

Let  $\Omega$  be a smoothly bounded domain in  $\mathbb{R}^n$  or  $\mathbb{R}^n$  itself. Let  $C_{0,\sigma}^\infty$  denote the set of smooth divergence free vector fields with compact support in  $\Omega$ . Let  $H_{0,\sigma}^1$  be the closure of  $C_{0,\sigma}^\infty$  in  $H^1(\Omega)$ . We consider Banach space  $F^p = H_{0,\sigma}^1 \cap L^p(\Omega)$ ,  $1 \leq p < \infty$ , whose norm is defined by the sum of the norms of  $H_{0,\sigma}^1$  and  $L^p$ . As far as I know, the following proposition is not stated in the literature.

PROPOSITION: The set  $C_{0,\sigma}^\infty$  is dense in  $F^p$ .

Proof. If  $1/p \geq 1/2 - 1/n$ ,  $H_{0,\sigma}^1$  is continuously embedded in  $L^p$  by the Sobolev inequality. Hence  $F^p = H_{0,\sigma}^1$ , so obviously  $C_{0,\sigma}^\infty$  is dense in  $F^p$ . In particular we may assume  $p > 2$ .

Let  $A = A + M$ , where  $A$  is the Stokes operator and  $M$  is a positive constant. As is mentioned in [8,9],  $e^{-tA}$  is an analytic semigroup in  $E^p$  and  $E^2$ . For  $f \in F^p$  we see  $f_\delta = e^{-\delta A} f$  belongs to the domain of  $A^\alpha$  in  $E^p$ ,  $\alpha > 0$ ; in particular  $f_\delta \in D(A^{(1+\beta)/2})_p$ ,  $\beta < 1/p$ . A characterization of  $D(A^\alpha)$  (ref. [15] in [9]) implies that  $f_\delta$  is in  $H_{0,\sigma}^{1+\beta,p}$ , the closure of  $C_{0,\sigma}^\infty$  in  $H^{1+\beta,p}(\beta < 1/p)$ , where  $H^{s,p}$  is the space of Bessel potentials. If  $1/2 > (1 - 1/n)/p$ , for a choice of  $\beta < 1/p$ ,  $H_{0,\sigma}^{1+\beta,p}$  is continuously embedded in  $H_{0,\sigma}^1$  by the Sobolev inequality; if  $p > 2$  the assumption on  $p$  is automatically satisfied. Since  $C_{0,\sigma}^\infty$  is dense in  $H_{0,\sigma}^{1+\beta,p}$ , this implies that there is a sequence  $\{f_{\delta,j}\}$  such that  $f_{\delta,j} \rightarrow f_\delta$  in  $F^p$  as  $j \rightarrow \infty$ .

It remains to prove  $f_\delta \rightarrow f$  in  $F^P$  as  $\delta \rightarrow 0$ . Since  $e^{-tA}$  is a continuous semigroup in  $E^2$  and  $E^P$ , we see  $A^{1/2}f_\delta \rightarrow A^{1/2}f$  in  $L^2$  and  $f_\delta \rightarrow f$  in  $L^P$ . The first converges ~~is~~ exactly the same as the convergence of  $f_\delta$  in  $H_{0,\sigma}^1$  because  $D(A^{1/2})_2 = H_{0,\sigma}^1$ . Thus we have proved  $C_{0,\sigma}^\infty$  is dense in  $F^P$ .

Remark: The reason we consider  $A + M$  instead of  $A$  is to handle unbounded domains. In fact the proof works for exterior domains with no modifications. For  $\Omega = \mathbb{R}^n$  or bounded star-shaped domain Masuda [32] gave another proof.

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