General Fixed Point Formula for an Algebraic Surface
and
The Theory of Swan Representations for Two-Dimensional Local Rings

代数曲面に対する一般固定点定理と２次元局所環に対するSwan表現の理論

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General fixed point formula for an algebraic surface and the theory of Swan representations for two-dimensional local rings

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0. Introduction

Let k be an algebraically closed field of any characteristic, and X be a proper normal surface over k. Let Aut(X/k) be the set of all automorphisms of X over k. For each $\sigma \in \mathrm{Aut}(X/k)$, let σ^* be the automorphism which σ induces on each cohomology group

$$H^i(X) := H^i(X_{et}, \mathbb{Q}_\ell) \quad (\ell \neq ch(k))$$

(or if $k=\mathbb{C}$, you can take as $H^i(X)$ the usual rational singular cohomology group $H^i(X(\mathbb{C}), \mathbb{Q})$). It is known that $Tr(\sigma^*)|H^i(X)$ is an integer which is independent of ℓ (cf.[2]). We put

$$Tr(\sigma^*)|H^*(X) = \sum_{i=0}^{4} (-1)^i Tr(\sigma^*)|H^i(X).$$

The main purpose of this paper is to give the following formula for any non-trivial $\sigma \in \mathrm{Aut}(X/k)$,

$$Tr(\sigma^*)|H^*(X) = \sum_{y \in X^\sigma} \lambda_y \cdot \nu_y(\sigma),$$

where X^σ denotes the set of all scheme-theoretic points of X which are fixed by σ (in fact the right hand side will turn out to be a finite sum), and λ_y is an invariant attached to the closure of y in X which is independent of $\sigma \in \mathrm{Aut}(X/k)$, and lastly, $\nu_y(\sigma)$ is a rational number which is defined in a purely locally manner at y, namely it is determined by the action of σ on the completion of X at y. To be more precise, for $y \in X^\sigma$, σ induces an automorphism of the completion A_y of the local ring of X at y, which we denote by σ_y. Let $I_{A_y}(\sigma)$ be the ideal of A_y generated by all elements of the form $\sigma_y(a)-a$ with $a \in A_y$. In case that X^σ consists of isolated regular closed points, the classical fixed point formula says

$$(0.1) \qquad \mathrm{Tr}(\sigma^*)|H^*(X) = \sum_{x\in X^\sigma} \nu_x(\sigma),$$

where $\qquad \nu_x(\sigma) = \dim_k(A_x/I_{A_x}(\sigma)).$

In general, X^σ is divided into two disjoint subsets X_0^σ and X_1^σ, where for i=0 or 1, X_i^σ denotes the set of the points of X^σ whose closure in X has the dimension i. Moreover, for each $\mathfrak{p}\in X_1^\sigma$, σ will be defined to be either type I or II at $\mathfrak{p}$ by observing how σ acts on the stalk of $\Omega^1_{X/k}$ at $\mathfrak{p}$ (cf.(1.11)). Thus, we define X_{I}^σ (resp. X_{II}^σ) to be the set of all $\mathfrak{p}\in X_1^\sigma$ such that σ is type I (resp. II) at $\mathfrak{p}$. Then, the formula in the general case will be

$$(0.2) \qquad \mathrm{Tr}(\sigma^*)|H^*(X) = \sum_{\mathfrak{p}\in X_{\mathrm{I}}^\sigma} \chi_{\mathfrak{p}}\cdot\nu_{\mathfrak{p}}(\sigma) + \sum_{\mathfrak{p}\in X_{\mathrm{II}}^\sigma} \tau_{\mathfrak{p}}\cdot\nu_{\mathfrak{p}}(\sigma) + \sum_{x\in X_0^\sigma} \nu_x(\sigma).$$

Here, for $\mathfrak{p}\in X_1^\sigma$ and its closure $F_{\mathfrak{p}}$ in X, we put

$$(0.3) \qquad \nu_{\mathfrak{p}}(\sigma) = \mathrm{length}_{A_{\mathfrak{p}}}\left(A_{\mathfrak{p}}/I_{A_{\mathfrak{p}}}(\sigma)\right),$$

$\chi_{\mathfrak{p}} = 2-2g_{\mathfrak{p}}$, where $g_{\mathfrak{p}}$ is the genus of the normalization of $F_{\mathfrak{p}}$,

$\tau_{\mathfrak{p}}$ = the self-intersection number of $F_{\mathfrak{p}}$ on X,

where the intersection theory on a normal surface is defined by Mumford [9] ($\tau_{\mathfrak{p}}$ is a rational number, not an integer in general). Lastly, for $x\in X_0^\sigma$, $\nu_x(\sigma)$ is a rational number defined in a purely local manner at x; for a two-dimensional noetherian normal complete local domain A over k and its non-trivial automorphism φ, we will define its "multiplicity" $\nu_A(\varphi)\in\mathbb{Q}$ in §2. Then, $\nu_x(\sigma)$ is defined to be $\nu_{A_x}(\sigma_x)$. An explicit calculation of $\nu_x(\sigma)$ will be given in §4. In particular, we will see that (0.2) gives the formula (0.1) in the special case. Moreover, one would see that in case that X is smooth over k, the whole argument works well for any non-trivial *endomorphism* σ of X and we have the same formula (0.2).

In §6 and §7, we will develop the two-dimensional version of the classical Swan representation for a discrete valuation ring, and will give a character formula of a finite group acting on an algebraic surface which we can view as the two-dimensional version of the Weil formula for an algebraic curve (cf.SGA5X(5.1)). Moreover, applying the method in [6] to our result, we obtain a result which is considered the two-dimensional version of the Grothendieck-Ogg-Shafarevich formula for an algebraic curve (cf. SGA5X(7.1)). Such a formula was first discovered in [7] by a different method, which stimulated the author to make these researches.

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Notations

A ring means a commutative ring with unit.

For a homomorphism of rings $k \longrightarrow R$, $\mathrm{End}(R/k)$ (resp. $\mathrm{Aut}(R/k)$) denotes the set of all endomorphisms (resp. automorphisms) of R over k, and $\mathrm{End}(R) = \mathrm{End}(R/\mathbf{Z})$ and $\mathrm{Aut}(R) = \mathrm{Aut}(R/\mathbf{Z})$.

For a ring R and $\sigma \in \mathrm{End}(R)$, $I_R(\sigma)$ denotes the ideal of R generated by all elements of the form $\sigma(a)-a$ $(a \in R)$.

For a scheme Z and an integer $i \geqq 0$, $Z_i=\{z \in Z \mid \dim\overline{\{z\}}=i\}$.

For a morphism of schemes $Z \longrightarrow S$, $\mathrm{End}(Z/S)$ (resp. $\mathrm{Aut}(Z/S)$) denotes the set of all endomorphisms (automorphisms) of Z over S, and $\mathrm{End}(Z)=\mathrm{End}(Z/\mathrm{Spec}(\mathbf{Z}))$ and $\mathrm{Aut}(Z)=\mathrm{Aut}(Z/\mathrm{Spec}(\mathbf{Z}))$.

For a scheme Z and $\sigma \in \mathrm{End}(Z)$, Z^{σ} denotes the set of all points of Z fixed by σ, and $\mathcal{I}_Z(\sigma)$ denotes the unique ideal of $\mathcal{O}_Z$ whose stalk $\mathcal{I}_Z(\sigma)_z$ at each point $z \in Z$ is as follows: If $z \notin Z^{\sigma}$, $\mathcal{I}_Z(\sigma)_z= \mathcal{O}_Z$ and if $z \in Z^{\sigma}$, $\mathcal{I}_Z(\sigma)_z=I_{R_z}(\sigma_z)$, where R_z is the local ring of Z at z and σ_z is endomorphism of R_z induced by σ.

For Σ=End or Aut, Σ' denotes the subset of all non-trivial (namely, not the identity) elements.

1. Preliminaries

(1.1) Let k be an algebraically closed field and let

$$A = k[[X,Y]]$$

be the ring of formal power series of two variables over k. We fix a prime ideal $\mathfrak{p}$ of height one in A, and denote by $\varkappa[\mathfrak{p}]$ the normalization of $A/\mathfrak{p}$. By definition, $\varkappa[\mathfrak{p}]$ is a complete discrete valuation ring with the residue field k, so for a prime element t, we have an isomorphism $\varkappa[\mathfrak{p}] \simeq k[[t]]$. Let R be the localization of A at $\mathfrak{p}$ and put $\varkappa(\mathfrak{p}) = R/\mathfrak{p}R$, which is the quotient field of $\varkappa[\mathfrak{p}]$. Furthermore, we fix an integer $n>0$, and put

$$\overline{A} = A/\mathfrak{p}^n \quad \text{and} \quad \overline{R} = R/(\mathfrak{p}R)^n.$$

Let M be a free $\overline{R}$-module of rank one, and for each integer $i \geqq 0$,

$$F^iM = (\mathfrak{p}R)^iM \quad \text{and} \quad Gr^iM = F^iM/F^{i+1}M.$$

By definition, $Gr^iM = 0$ for $i \geqq n$, and for $0 \leqq i \leqq n-1$, it is a one-dimensional vector space over $\varkappa(\mathfrak{p})$.

Definition(1.2). M is filterwisely based if there is given a set $\phi=(\phi_i)_{0 \leqq i \leqq n-1}$ of isomorphisms

$$\phi_i : (\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1} \simeq Gr^iM$$

determined up to the multiplication by $\varkappa[\mathfrak{p}]^\times$. We define $Gr^i_\phi M$ to be the image under ϕ_i of

$$\mathfrak{p}^i/\mathfrak{p}^{i+1} \otimes_A \varkappa[\mathfrak{p}] \subset (\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1}.$$

This is well-defined free $\varkappa[\mathfrak{p}]$-module of rank one.

Definition(1.3). A sub-$\overline{A}$-module L of M is an $\overline{A}$-lattice if L is finitely generated over $\overline{A}$, and generates M over $\overline{R}$.

Put

$$F^iL = F^iM \cap L \quad \text{and} \quad Gr^iL = F^iL/F^{i+1}L.$$

Then, Gr^iL is a sub-$(A/\mathfrak{p})$-module of Gr^iM which is finitely generated over $A/\mathfrak{p}$ and generates Gr^iM over $\kappa(\mathfrak{p})$.

When M is filterwisely based as (1.2), for an $\bar{A}$-lattice L of M, we define its index by

$$(1.4) \qquad [M:L] \ (= [(M,\phi):L]) = \sum_{i=0}^{n-1} \mu_i,$$

where $\mu_i = \dim_k\left(Gr^i_\phi M/(Gr^i_\phi M \cap Gr^iL)\right) - \dim_k\left(Gr^iL/(Gr^i_\phi M \cap Gr^iL)\right)$.

(1.5) Let the notations be as (1.1). Fix a non-trivial endomorphism σ of A over k, and n be the maximum integer for which σ acts trivially on $A/\mathfrak{p}^n$. We assume $n \geqq 1$ and define $\bar{A}$ and $\bar{R}$ as (1.1). Let $\hat{\Omega}^1_{A/k}$ be the module of formal differentials of A over k. For convenience of readers, we recall its definition. First, let

$$C = A \hat{\otimes}_k A := \varprojlim_{i,j} A/\mathfrak{m}^i \otimes_k A/\mathfrak{m}^j,$$

where $\mathfrak{m}$ is the maximal ideal of A. Let $I_C(\Delta)$ be the ideal of C generated by all elements of the form $a\hat{\otimes}1 - 1\hat{\otimes}a$ $(a \in A)$. By definition, we have an isomorphism

$$(1.6) \qquad C/I_C(\Delta) \xrightarrow{\sim} A \ ; \ a\hat{\otimes}b \longrightarrow ab.$$

Then, we define

$$\hat{\Omega}^1_{A/k} = I_C(\Delta)/I_C(\Delta)^2.$$

We have a derivation of A over k

$$d_A : A \longrightarrow \hat{\Omega}^1_{A/k} \ ; \ a \longrightarrow a\hat{\otimes}1 - 1\hat{\otimes}a \mod I_C(\Delta)^2.$$

Let $\hat{\Omega}^1_{R/k} = \hat{\Omega}^1_{A/k} \otimes_A R$ and $d : R \longrightarrow \hat{\Omega}^1_{R/k}$ be the unique derivation of R over k whose restriction to A is d_A. Let $\hat{\Omega}^1_{\kappa(\mathfrak{p})/k}$ be the module of formal differentials of $\kappa(\mathfrak{p})$ over k which is defined as follows: First, we define

$$\hat{\Omega}^1_{\kappa[\mathfrak{p}]/k} = \varprojlim_n \Omega^1_{\kappa[\mathfrak{p}]_n/k},$$

where for each integer n>0, we put $\kappa[\mathfrak{p}]_n=\kappa[\mathfrak{p}]/(t^n)$ with a prime element t of $\kappa[\mathfrak{p}]$. We see that $\hat{\Omega}^1_{\kappa[\mathfrak{p}]/k}$ is a free $\kappa[\mathfrak{p}]$-module of rank one. Then, we put

$$\hat{\Omega}^1_{\kappa(\mathfrak{p})/k} = \hat{\Omega}^1_{\kappa[\mathfrak{p}]/k}\otimes_{\kappa[\mathfrak{p}]}\kappa(\mathfrak{p}).$$

There is a well-known exact sequence

$$(1.7) \qquad 0 \longrightarrow \mathfrak{p}R/(\mathfrak{p}R)^2 \xrightarrow{\alpha} \hat{\Omega}^1_{R/k}\otimes_R\kappa(\mathfrak{p}) \xrightarrow{\beta} \hat{\Omega}^1_{\kappa(\mathfrak{p})/k} \longrightarrow 0,$$

where the maps α and β are defined as follows:

$$\alpha(b \bmod (\mathfrak{p}R)^2) = db\otimes 1 \quad (b\in\mathfrak{p}R),$$

$$\beta(adb\otimes 1) = \bar{a}d\bar{b} \quad (a,b\in R \text{ and } \bar{a},\bar{b} \text{ are its images in } \kappa(\mathfrak{p})).$$

We see that $\hat{\Omega}^1_{R/k}$ is a free R-module of rank two. In fact, we can take as a basis $(d\pi, du)$, where π is a prime element of R and u is a unit of R such that du has a non-trivial image in $\hat{\Omega}^1_{\kappa(\mathfrak{p})/k}$.

Consider the following map

$$R \longrightarrow (\mathfrak{p}R)^n/(\mathfrak{p}R)^{2n} \ ; \ a \longrightarrow \sigma(a)-a \bmod (\mathfrak{p}R)^{2n}.$$

It is a continuous derivation of R over k, so we get a homomorphism

$$(1.8) \quad \varphi_\sigma : \hat{\Omega}^1_{R/k}\otimes_R\bar{R} \longrightarrow (\mathfrak{p}R)^n/(\mathfrak{p}R)^{2n} \ ; \ adb\otimes 1 \longrightarrow a(\sigma(b)-b).$$

Since φ_σ is surjective and $\hat{\Omega}^1_{R/k}\otimes_R\bar{R}$ (resp. $(\mathfrak{p}R)^n/(\mathfrak{p}R)^{2n}$) is a free $\bar{R}$-module of rank 2 (resp 1), $\mathrm{Ker}(\varphi_\sigma)$ is a free $\bar{R}$-module of rank 1.

Definition(1.9). σ is type I at $\mathfrak{p}$ if the image of $\mathrm{Ker}(\varphi_\sigma)$ under

$$\hat{\Omega}^1_{R/k}\otimes_R\overline{R} \longrightarrow \hat{\Omega}^1_{R/k}\otimes_R\varkappa(\mathfrak{p}) \xrightarrow{\alpha} \hat{\Omega}^1_{\varkappa(\mathfrak{p})/k}$$

is non-trivial. Otherwise, σ is type II at $\mathfrak{p}$.

Remark(1.10). The argument so far works well in more general context where A is a two-dimensional noetherian normal complete local domain over k such that $A/\mathfrak{m} \simeq k$ ($\mathfrak{m}$ is the maximal ideal of A).

Now, we define a natural structure of a filterwisely-based module over $\overline{R}$ on $\mathrm{Ker}(\varphi_\sigma)$. If σ is type I at $\mathfrak{p}$, (1.7) gives a canonical isomorphism for each i ($0\leqq i\leqq n-1$),

$$\mathrm{Gr}^i\mathrm{Ker}(\varphi_\sigma) \simeq \hat{\Omega}^1_{\varkappa(\mathfrak{p})/k}\otimes_R(\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1}.$$

So, for a prime element t of $\varkappa[\mathfrak{p}]$, the map

$$(\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1} \longrightarrow \hat{\Omega}^1_{\varkappa(\mathfrak{p})/k}\otimes(\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1} \ ; \ a \bmod (\mathfrak{p}R)^{i+1} \longrightarrow dt\otimes a$$

gives the desired structure on $\mathrm{Ker}(\varphi_\sigma)$. If σ is type II at $\mathfrak{p}$, (1.7) gives a canonical isomorphism for each i ($0\leqq i\leqq n-1$),

$$\mathfrak{p}R/(\mathfrak{p}R)^2\otimes_R(\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1} \simeq \mathrm{Gr}^i\mathrm{Ker}(\varphi_\sigma).$$

By the regularity of A, we can find an element $\pi\in A$ which is a generator of $\mathfrak{p}$. Then, the homomorphism

$$(\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1} \longrightarrow \mathfrak{p}R/(\mathfrak{p}R)^2\otimes_R(\mathfrak{p}R)^i/(\mathfrak{p}R)^{i+1} \ ; \ a \bmod (\mathfrak{p}R)^{i+1} \longrightarrow \pi\otimes a$$

gives the desired structure on $\mathrm{Ker}(\varphi_\sigma)$.

(1.11) Let k be an algebraically closed field. Let R be an excellent discrete valuation ring over k with the maximal ideal $\mathfrak{p}$ and the residue field $\varkappa(\mathfrak{p})$, and assume that $\Omega^1_{\varkappa(\mathfrak{p})/k}$ is a one-dimensional $\varkappa(\mathfrak{p})$-vector space. Put $R_i = R/\mathfrak{p}^i$ for each $i \geqq 0$, and

$$\hat{R} = \varprojlim_i R_i \quad \text{and} \quad \hat{\Omega}^1_{R/k} = \varprojlim_i \Omega^1_{R_i/k}.$$

Then, $\hat{\Omega}^1_{R/k}$ is a free $\hat{R}$-module of rank 2. Fix a non-trivial endomorphism σ of R, and put $n = \mathrm{length}_R(R/I_R(\sigma))$ and $\bar{R} = R_n$. Then, in the same argument as before, we have an $\bar{R}$-homomorphism

$$\varphi_\sigma : \hat{\Omega}^1_{R/k} \otimes_R \bar{R} \longrightarrow \mathfrak{p}^n/\mathfrak{p}^{2n} \ ; \ adb \otimes 1 \longrightarrow a(\sigma(b)-b),$$

and $\mathrm{Ker}(\varphi_\sigma)$ is a free $\bar{R}$-module of rank one. Thus, we define σ is type I or II at $\mathfrak{p}$ according to the image of $\mathrm{Ker}(\varphi_\sigma)$ in $\hat{\Omega}^1_{\varkappa(\mathfrak{p})/k}$ is non-trivial or trivial. Defining

$$\mathrm{Gr}^i\mathrm{Ker}(\varphi_\sigma) = \mathfrak{p}^i\mathrm{Ker}(\varphi_\sigma)/\mathfrak{p}^{i+1}\mathrm{Ker}(\varphi_\sigma),$$

we see isomorphisms

$$\mathrm{Gr}^i\mathrm{Ker}(\varphi_\sigma) = 0 \quad \text{for } i \geqq n, \text{ and}$$

$$\mathrm{Gr}^i\mathrm{Ker}(\varphi_\sigma) \simeq \begin{cases} \Omega^1_{\varkappa(\mathfrak{p})/k} \otimes_R (\mathfrak{p}^i/\mathfrak{p}^{i+1}) & \text{if } \sigma \text{ is type I at } \mathfrak{p}, \\ (\mathfrak{p}/\mathfrak{p}^2) \otimes_R (\mathfrak{p}^i/\mathfrak{p}^{i+1}) & \text{if } \sigma \text{ is type II at } \mathfrak{p}. \end{cases}$$

In particular, for X/k, $\sigma \in \mathrm{End}(X/k)'$ and $\mathfrak{p} \in X_1^\sigma$ as Introduction, we put R the local ring of X at $\mathfrak{p}$, and obtain the definition for σ to be type I or II at $\mathfrak{p}$. By definition, we see that the following conditions are equivalent for $\sigma \in \mathrm{End}(X/k)$ and $\mathfrak{p} \in X_1^\sigma$.

(1.11.1) σ is type I (resp. II) at $\mathfrak{p}$.

(1.11.2) There exist a closed point $x \in F_\mathfrak{p}$ and a prime ideal $\mathfrak{p}_x$ of A_x lying over $\mathfrak{p}$ such that $\sigma_x \in \mathrm{End}(A_x/k)$ is type I (resp. II) at $\mathfrak{p}_x$.

(1.11.3) The same condition as (1.11.2) holds for any $x \in F_\mathfrak{p}$ and $\mathfrak{p}_x$.

2. The definition of the multiplicity of an automorphism of a two-dimensional local ring (regular case)

Let k be an algebraically closed field and A a two-dimensional noetherian normal complete local ring over k such that $A/\mathfrak{m} \simeq k$, where $\mathfrak{m}$ is the maximal ideal of A. Let P be the set of all prime ideals of height one in A. For $\mathfrak{p} \in P$, let $R_{\mathfrak{p}}$ be the localization of A at $\mathfrak{p}$, and $\kappa(\mathfrak{p})$ its residue field.

In the following two sections, we define, for each $\sigma \in \mathrm{Aut}(A/k)'$, its "multiplicity" $\nu_A(\sigma) \in \mathbb{Q}$, which gives the desired invariant $\nu_X(\sigma)$ in the formula (0.2) as explained in Introduction.

Definition(2.1). For $\sigma \in \mathrm{End}(A/k)'$ and $\mathfrak{p} \in P$, we define

$$\nu_{\mathfrak{p}}(\sigma) = \mathrm{length}_{R_{\mathfrak{p}}}\left(R_{\mathfrak{p}}/I_A(\sigma)R_{\mathfrak{p}}\right).$$

Let P^{σ} be the set of all $\mathfrak{p}$ such that $\nu_{\mathfrak{p}}(\sigma)>0$. Following (1.9) and (1.10), σ is defined to be either type I at $\mathfrak{p}$ or type II at $\mathfrak{p}$ for each $\mathfrak{p} \in P^{\sigma}$. We define P^{σ}_{I} (resp. P^{σ}_{II}) to be the subset of P^{σ} consisting of all $\mathfrak{p}$ such that σ is type I (resp. II) at $\mathfrak{p}$.

In the following part of this section, we assume that A is regular, so that $A=k[[X,Y]]$. In this case, $\nu_A(\sigma)$ will be defined for any $\sigma \in \mathrm{End}(A/k)'$. Thus, we fix $\sigma \in \mathrm{End}(A/k)'$ in what follows. Let C and $I_C(\Delta)$ be as defined in (1.5), and let $I_C(\Gamma)$ be the ideal of C generated by all elements of the form $\sigma(a)\hat{\otimes}b - a\hat{\otimes}\sigma(b)$ with $a,b \in A$. For each integer $r \geqq 0$, we put

$$T^r_A(\sigma) = \mathrm{Tor}^C_r(C/I_C(\Delta), C/I_C(\Gamma)). \tag{2.2}$$

By definition, $T^r_A(\sigma)$ is a C-module of finite type and $I_C(\Delta)\cdot T^r_A(\sigma)=0$ so that $T^r_A(\sigma)$ is viewed as an A-module of finite type through the isomorphism (1.6).

Lemma(2.3). (1) *We have isomorphisms*

$T^0_A(\sigma) \simeq A/I_A(\sigma)$ *and* $T^1_A(\sigma) \simeq \Big(I_C(\Delta) \cap I_C(\Gamma)\Big)/\Big(I_C(\Delta)\cdot I_C(\Gamma)\Big)$.

(2) $T^r_A(\sigma) = 0$ *for* $r \geqq 2$.

Proof. (1) is an easy exercise in homological algebra and (2) follows from the computation of Tor_r using the Koszul complex associated to a regular sequence generating $I_C(\Delta)$ in C.

For each $\mathfrak{p}\in P$ and each r, we put

$$(2.4)\qquad T^r_A(\sigma)_{\mathfrak{p}} = T^r_A(\sigma)\otimes_A R_{\mathfrak{p}} ,$$

$$S^r_A(\sigma)_{\mathfrak{p}} = \text{the image of } T^r_A(\sigma) \text{ in } T^r_A(\sigma)_{\mathfrak{p}} .$$

For each r, let

$$\gamma^r : T^r_A(\sigma) \longrightarrow \bigoplus_{\mathfrak{p}\in P^\sigma} S^r_A(\sigma)_{\mathfrak{p}}$$

be the natural map. Then, $\mathrm{Ker}(\gamma^r)$ and $\mathrm{Coker}(\gamma^r)$ has a finite dimension over k. We define

$$(2.5)\qquad \delta_A(\sigma) = \sum_{r=0}^{1}(-1)^r\Big(\dim_k \mathrm{Ker}(\gamma^r) - \dim_k \mathrm{Coker}(\gamma^r)\Big).$$

Now we fix $\mathfrak{p}\in P^\sigma$ and put $R=R_{\mathfrak{p}}$ and $n=\nu_{\mathfrak{p}}(\sigma)$. Recall the notations in §1. By definition, we have

$$T^0_A(\sigma)_{\mathfrak{p}} = \overline{R} := R/(\mathfrak{p}R)^n \quad \text{and} \quad S^0_A(\sigma)_{\mathfrak{p}} = \overline{A} := A/\mathfrak{p}^n,$$

so that $T^0_A(\sigma)_{\mathfrak{p}}$ has a natural structure of a filterwisely-based $\overline{R}$-module and $S^0_A(\sigma)_{\mathfrak{p}}$ is an $\overline{A}$-lattice of $T^0_A(\sigma)_{\mathfrak{p}}$. Thus, we put

$$(2.6)\qquad \nu^0_{A.\mathfrak{p}}(\sigma) = -[T^0_A(\sigma)_{\mathfrak{p}}:S^0_A(\sigma)_{\mathfrak{p}}].$$

As for $T^1_A(\sigma)_{\mathfrak{p}}$, we have the following.

Lemma(2.7). *There is a canonical isomorphism*

$$T^1_A(\sigma)_{\mathfrak{p}} \simeq \mathrm{Ker}(\varphi_\sigma),$$

where $\varphi_\sigma : \hat{\Omega}^1_{R/k}\otimes_R\bar{R} \longrightarrow (\mathfrak{p}R)^n/(\mathfrak{p}R)^{2n}$ *is the map (1.8).*

The proof of (2.7) will be given later.

By (2.7) and the result in §1, $T^1_A(\sigma)_{\mathfrak{p}}$ is a free $\bar{R}$-module of rank one, which is canonically filterwisely based, and $S^1_A(\sigma)_{\mathfrak{p}}$ is an $\bar{A}$-lattice of $T^1_A(\sigma)_{\mathfrak{p}}$. Thus, we define

(2.8) $$\nu^1_{A.\mathfrak{p}}(\sigma) = -[T^1_A(\sigma)_{\mathfrak{p}}:S^1_A(\sigma)_{\mathfrak{p}}].$$

Lastly, we define

(2.9) $$\nu_A(\sigma) = \delta_A(\sigma) + \sum_{\mathfrak{p}\in P^\sigma}\left(\nu^0_{A.\mathfrak{p}}(\sigma) - \nu^1_{A.\mathfrak{p}}(\sigma)\right).$$

The proof of (2.7). Let C, $I_C(\Delta)$ and $I_C(\Gamma)$ be as defined before. Let $\tilde{\mathfrak{p}}$ be the inverse image of $\mathfrak{p}$ in C under (1.6) and $C_{\mathfrak{p}}$ be the localization of C at $\tilde{\mathfrak{p}}$. Put

$$I_C(\Delta)_{\mathfrak{p}} = I_C(\Delta)C_{\mathfrak{p}} \quad\text{and}\quad I_C(\Gamma)_{\mathfrak{p}} = I_C(\Gamma)C_{\mathfrak{p}}.$$

Noting the isomorphisms

$$C_{\mathfrak{p}}/\left(I_C(\Delta)_{\mathfrak{p}}+I_C(\Gamma)_{\mathfrak{p}}\right) \simeq R/(\mathfrak{p}R)^n = \bar{R},$$

$$I_C(\Delta)_{\mathfrak{p}}/I_C(\Delta)_{\mathfrak{p}}^{\,2} \simeq \hat{\Omega}^1_{R/k},$$

$$T^1_A(\sigma)_{\mathfrak{p}} \simeq \left(I_C(\Delta)_{\mathfrak{p}} \cap I_C(\Gamma)_{\mathfrak{p}}\right)/\left(I_C(\Delta)_{\mathfrak{p}}\cdot I_C(\Gamma)_{\mathfrak{p}}\right),$$

we see that $I_C(\Delta)_{\mathfrak{p}} \cap I_C(\Gamma)_{\mathfrak{p}} \subsetneq I_C(\Delta)_{\mathfrak{p}}$ induces a map

$$\theta_{\mathfrak{p}} : T^1_{\bar{A}}(\sigma)_{\mathfrak{p}} \longrightarrow I_C(\Delta)_{\mathfrak{p}}/I_C(\Delta)_{\mathfrak{p}}^{\,2} = \hat{\Omega}^1_{\bar{R}/k}\otimes_{\bar{R}}\bar{R}.$$

To prove that $\theta_{\mathfrak{p}}$ induces an isomorphism $T^1_A(\sigma)_{\mathfrak{p}} \simeq \mathrm{Ker}(\varphi_\sigma)$, we first note the isomorphism

$$T^1_A(\sigma)_{\mathfrak{p}} \simeq \mathrm{Tor}_1^{C_{\mathfrak{p}}}(C_{\mathfrak{p}}/I_C(\Delta)_{\mathfrak{p}}, C_{\mathfrak{p}}/I_C(\Gamma)_{\mathfrak{p}})$$

On the other hand, $C_{\mathfrak{p}}$ is a regular local ring of dimension 3, and that $C_{\mathfrak{p}}/I_C(\Delta)_{\mathfrak{p}}$ and $C_{\#}/I_C(\Gamma)_{\mathfrak{p}}$ are discrete valuation rings (in fact, both are isomorphic to R), so that we can find a regular sequence (a,b,c) in $C_{\mathfrak{p}}$ such that $I_C(\Delta)_{\mathfrak{p}}$ (resp. $I_C(\Gamma)_{\mathfrak{p}}$) is generated by (a,b) (resp. (a,c)). Now, our assertion follows from the computation of Tor_1 by using the Koszul complexes associated to the regular sequences (a,b) and (a,c) (cf. SGA6,VIII(2.5)).

3. The reduction to the regular case

Let A be as in the first part of this section. In this section, we reduce the definition of ν_A to the regular case which we treated in the previous section. First, we consider a resolution of Spec(A)

$$f : \mathfrak{X} \longrightarrow \mathrm{Spec}(A)$$

by which we mean a regular two-dimensional scheme $\mathfrak{X}$ and a proper birational morphism f such that f induces an isomorphism

$$\mathfrak{X} \setminus E \simeq \mathrm{Spec}(A)-x,$$

where x is the closed point of Spec(A) and $E = (f^{-1}(x))_{red}$ which is a one-dimensional proper scheme over k. For the existence of such an $\mathfrak{X}$, we refer the readers to [5]. For $\eta\in E_1$, let E_η be the closure of η in E. Note that $\mathfrak{p}\in P$ defines a closed point of Spec(A)-x, which we denote again by $\mathfrak{p}$, and we put $F_\mathfrak{p}$ the closure of $\mathfrak{p}$ in $\mathfrak{X}$. For a given $\sigma\in\mathrm{Aut}(A/k)'$, we choose a resolution $\mathfrak{X}$ of Spec(A) such that σ extends an automorphism of $\mathfrak{X}$ over k. If such an extension exists, it is unique and we denote it again by σ. The existence of such an $\mathfrak{X}$ is guaranteed by the theory of the minimal resolution of Spec(A) (cf.[8]). Let E_0^σ (resp. E_1^σ) be the set of $x\in E_0$ (resp. $\eta\in E_1$) which are fixed by σ. By (1.11), for $\eta\in E_1^\sigma$, σ is defined to be type I or II at η and we denote by E_{I}^σ (resp. E_{II}^σ) the set of all $\eta\in E_1^\sigma$ such that σ is type I (resp. II) at η. On the other hand, for $x\in E_0^\sigma$, we define

$$\nu_x(\sigma) = \nu_{A_x}(\sigma_x) \quad (\text{cf. §2}),$$

where A_x is the completion of the local ring of $\mathfrak{X}$ at x and σ_x is the automorphism of A_x induced by σ. Then, we put

$$(3.1) \qquad \nu_{\mathfrak{X}}(\sigma) = \sum_{x\in E_0^\sigma} \nu_x(\sigma) + \sum_{\eta\in E_{\mathrm{I}}^\sigma} \chi_\eta\cdot\nu_\eta(\sigma) + \sum_{\eta\in E_{\mathrm{II}}^\sigma} \tau_\eta\cdot\nu_\eta(\sigma),$$

where $\nu_\eta(\sigma)$ and χ_η is defined in the same way as (0.3), and τ_η is the self-intersection number of E_η on $\mathfrak{X}$ (cf.[14]). Lastly, we define

$$(3.2) \qquad \nu_A(\sigma) = \nu_{\mathfrak{X}}(\sigma) - \sum_{\mathfrak{p}\in P_{\mathrm{II}}^{\sigma}} \nu_{\mathfrak{p}}(\sigma)\cdot\varepsilon_{\mathfrak{X}.\mathfrak{p}} + 1 - \mathrm{Tr}(\sigma^*)|H^*(E),$$

where $\qquad \mathrm{Tr}(\sigma^*)|H^*(E) = \sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\sigma^*)|H^i(E_{et},\mathbb{Q}_\ell) \quad (\ell\neq \mathrm{ch}(k)),$

and $\varepsilon_{\mathfrak{X}.\mathfrak{p}}\in\mathbb{Q}$ ($\mathfrak{p}\in P$) is defined as follows. Let

$$F'_{\mathfrak{p}} = F_{\mathfrak{p}} + \sum_{\eta\in E_1} r_\eta\cdot E_\eta \quad (r_\eta\in\mathbb{Q})$$

be the total transform of $\mathfrak{p}$ on $\mathfrak{X}$ in the sense of [9], namely, r_η ($\eta\in E_1$) is determined uniquely by the equalities

$$(F'_{\mathfrak{p}},E_\eta) = 0 \quad \text{for any } \eta\in E_1.$$

Then, we put

$$\varepsilon_{\mathfrak{X}.\mathfrak{p}} = \sum_{\eta\in E_1} r_\eta\cdot(E_\eta,F_{\mathfrak{p}}).$$

Proposition(3.3). *The definition (3.2) of $\nu_A(\sigma)$ does not depend on the choice of $\mathfrak{X}$.*

In case that there exist X/k, $\tilde{\sigma}\in\mathrm{Aut}(X/k)$ and $x\in X^{\tilde{\sigma}}$ as in Introduction such that A is obtained by completing X at x and that σ is induced by $\tilde{\sigma}$, (3.3) follows from the global formula (0.2) proved in §4. The rest of this section will be devoted to a purely local proof of (3.3).

Before starting the proof of (3.3), we first introduce some definitions. Let R be a noetherian local ring and U=Spec(R)-x, where x is the unique closed point of Spec(R). Let

$$f : \mathfrak{X} \longrightarrow \mathrm{Spec}(R)$$

be a proper morphism such that f induces an isomorphism $\mathfrak{X} \setminus E \simeq U$, where $E = (f^{-1}(x))_{\mathrm{red}}$. Let $D_c^b(\mathfrak{X}) = D_{\mathrm{Coh}(\mathfrak{X})}^b(\mathfrak{X})$ be as defined in [4], and let Σ be the set of all triple $(\mathcal{M},\mathcal{N},\psi)$, where $\mathcal{M},\mathcal{N} \in \mathrm{Ob}(D_c^b(\mathfrak{X}))$ and $\psi \in \mathrm{Isom}_U(\mathcal{M},\mathcal{N})$, namely, an isomorphism

$$\psi : \mathcal{M}_{|U} \xrightarrow{\sim} \mathcal{N}_{|U},$$

where $\mathcal{M}_{|U}$ (resp. $\mathcal{N}_{|U}$) is the restriction of $\mathcal{M}$ (resp. $\mathcal{N}$) to U. Then, we can see that there exists a unique function

(3.4) $$\chi_R : \Sigma \longrightarrow \mathbf{Z}$$

which satisfies the following conditions (3.4.1) and (3.4.2).

(3.4.1) For $\mathcal{M},\mathcal{N},\mathcal{L} \in \mathrm{Ob}(D_c^b(\mathfrak{X}))$, $\psi \in \mathrm{Isom}_U(\mathcal{M}.\mathcal{N})$ and $\varphi \in \mathrm{Isom}_U(\mathcal{N},\mathcal{L})$,

$$\chi_R(\mathcal{M},\mathcal{L},\varphi\cdot\psi) = \chi_R(\mathcal{M},\mathcal{N},\psi) + \chi_R(\mathcal{N},\mathcal{L},\varphi).$$

(3.4.2) For $(\mathcal{M},\mathcal{N},\psi)\in\Sigma$, suppose that there exists $\tilde{\psi}\in\mathrm{Hom}(\mathcal{M},\mathcal{N})$ such that $\tilde{\psi}_{|U}= \psi$, and let $\mathcal{L}\in\mathrm{Ob}(D_c^b(\mathfrak{X}))$ be the mapping cone of $\tilde{\psi}$. Then,

$$\chi_R(\mathcal{M},\mathcal{N},\psi) = \chi_R([\mathcal{L}]),$$

where for $\mathcal{K}\in\mathrm{Ob}(D_c^b(\mathfrak{X}))$ supported in E, $[\mathcal{K}]$ denotes the elements of the Grothendieck group $K_0(E)$ of coherent sheaves on E determined by $\mathcal{K}$, and $\chi_R : K_0(E) \longrightarrow \mathbf{Z}$ is the unique homomorphism such that

$$\chi_R([\mathcal{K}]) = \sum_{i\in\mathbf{Z}} (-1)^i \mathrm{length}_R H^i(\mathfrak{X},\mathcal{K}).$$

(By EGAIII(3.2.3), $H^i(\mathfrak{X},\mathcal{K})$ is an R-module of finite length for any $i\in\mathbf{Z}$, and it is trivial except for a finite number of i's.)

Moreover, we have the following properties of χ_R.

(3.4.3) If $(\mathcal{M},\mathcal{N},\psi)\in\Sigma$, and if $\mathcal{M}$ and $\mathcal{N}$ are supported in E,

$$\chi_R(\mathcal{M},\mathcal{N},\psi) = \chi_R([\mathcal{M}]) - \chi_R([\mathcal{N}]).$$

(3.4.4) Let $g : \mathfrak{X}' \longrightarrow \mathfrak{X}$ be a proper morphism such that g induces an isomorphism $\mathfrak{X}'\setminus E' \simeq \mathfrak{X} \setminus E$, where $E'= (g^{-1}(E))_{red}$. Define Σ' and χ_R' for $\mathfrak{X}'$ as before. Then, for $(\mathcal{M}',\mathcal{N}',\psi')\in\Sigma'$,

$$\chi_R'(\mathcal{M}',\mathcal{N}',\psi') = \chi_R(Rg_*(\mathcal{M}'),Rg_*(\mathcal{N}'),\psi').$$

Now, let A and $\mathfrak{X}$ be as in the first part of this section. Let $\hat{\mathfrak{X}}$ be the formal completion of $\mathfrak{X}$ along E, and $\hat{\sigma}$ be the automorphism of $\hat{\mathfrak{X}}$ induced by σ. Let $\mathfrak{Z} = \hat{\mathfrak{X}} \times_k \hat{\mathfrak{X}}$ be the fiber product of formal schemes over k. Let Δ (resp. Γ) be the diagonal (resp. the graph of $\hat{\sigma}$) in $\mathfrak{Z}$. They are formal closed subschemes of $\mathfrak{Z}$. Let $\mathcal{J}_{\mathfrak{Z}}(\Delta)$ (resp. $\mathcal{J}_{\mathfrak{Z}}(\Gamma)$) be the ideal of definition of Δ (resp. Γ) in $\mathcal{O}_{\mathfrak{Z}}$, and for each integer $r\geqq 0$, put

$$\hat{\mathcal{T}}^r_{\mathfrak{X}}(\sigma) = \mathcal{T}or_r^{\mathcal{O}_{\mathfrak{Z}}}(\mathcal{O}_{\mathfrak{Z}}/\mathcal{J}_{\mathfrak{Z}}(\Delta),\mathcal{O}_{\mathfrak{Z}}/\mathcal{J}_{\mathfrak{Z}}(\Gamma)). \tag{3.5}$$

By definition, $\hat{\mathcal{T}}^r_{\mathfrak{X}}(\sigma)$ is a coherent $\mathcal{O}_{\mathfrak{Z}}$-module and $\mathcal{J}_{\mathfrak{Z}}(\Delta)\cdot\hat{\mathcal{T}}^r_{\mathfrak{X}}(\sigma)=0$ so that $\hat{\mathcal{T}}^r_{\mathfrak{X}}(\sigma)$ is viewed as a coherent $\mathcal{O}_{\hat{\mathfrak{X}}}$-module through the natural isomorphism $\mathcal{O}_{\mathfrak{Z}}/\mathcal{J}_{\mathfrak{Z}}(\Delta) \simeq \mathcal{O}_{\hat{\mathfrak{X}}}$. By EGAIII(3.1.6), there is a unique coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{T}^r_{\mathfrak{X}}(\sigma)$ whose formal completion along E is $\hat{\mathcal{T}}^r_{\mathfrak{X}}(\sigma)$. Moreover, as (2.3), we can see $\mathcal{T}^r_{\mathfrak{X}}(\sigma) = 0$ for $r\geqq 2$, and

$$\mathcal{F}^0_{\mathfrak{X}}(\sigma) \simeq \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_{\mathfrak{X}}(\sigma) \quad \text{and} \quad \hat{\mathcal{F}}^1_{\mathfrak{X}}(\sigma) \simeq \Big(\mathcal{I}_{\mathfrak{X}}(\Delta) \cap \mathcal{I}_{\mathfrak{X}}(\Gamma)\Big)/\Big(\mathcal{I}_{\mathfrak{X}}(\Delta)\cdot\mathcal{I}_{\mathfrak{X}}(\Gamma)\Big).$$

Now, for each $\mathfrak{p}\in P^{\sigma}$, choose $a_{\mathfrak{p}}\in A_{\mathfrak{p}}$ which satisfies the condition:

(3.6) If $\sigma\in P^{\sigma}_{I}$, $d\bar{a}_{\mathfrak{p}}$ is non-zero in $\hat{\Omega}^1_{\kappa(\mathfrak{p})/k}$, where $\bar{a}_{\mathfrak{p}}$ is the image of $a_{\mathfrak{p}}$ in $\kappa(\mathfrak{p})$. If $\sigma\in P^{\sigma}_{II}$, $a_{\mathfrak{p}}$ is a prime element of $A_{\mathfrak{p}}$ (cf. §2).

For $a_{\mathfrak{p}}$ chosen as above, we define $\theta_{a_{\mathfrak{p}}} \in \mathbf{Z}$ as follows: For $\mathfrak{p}\in P^{\sigma}_{I}$,

$$\theta_{a_{\mathfrak{p}}} = \text{the order of } d\bar{a}_{\mathfrak{p}}, \tag{3.7}$$

namely, writing $d\bar{a}_{\mathfrak{p}} = \alpha_{\mathfrak{p}}dt_{\mathfrak{p}}$ for some prime element $t_{\mathfrak{p}}$ of $\kappa(\mathfrak{p})$ and $\alpha_{\mathfrak{p}}\in\kappa(\mathfrak{p})$,

$$\theta_{a_{\mathfrak{p}}} = \mathrm{ord}_{\kappa(\mathfrak{p})}(\alpha_{\mathfrak{p}}) = \max\{n \mid t^n_{\mathfrak{p}} \text{ divides } \alpha_{\mathfrak{p}}\}.$$

For $\mathfrak{p}\in P^{\sigma}_{II}$, we define

$$\theta_{a_{\mathfrak{p}}} = (F_{\mathfrak{p}}, (a_{\mathfrak{p}})-F_{\mathfrak{p}}), \tag{3.8}$$

where $F_{\mathfrak{p}}$ is the closure of $\mathfrak{p}$ in $\mathfrak{X}$, $(a_{\mathfrak{p}})$ is the divisor of $a_{\mathfrak{p}}$ on $\mathfrak{X}$ and (,) denotes the intersection number of divisors on $\mathfrak{X}$.

For $\mathfrak{p}\in P$, let $\mathcal{F}^r_{\mathfrak{X}}(\sigma)_{\mathfrak{p}}$ be the stalk of $\mathcal{F}^r_{\mathfrak{X}}(\sigma)$ at $\mathfrak{p}$. By definition, $\mathcal{F}^r_{\mathfrak{X}}(\sigma)_{\mathfrak{p}}=0$ unless $\mathfrak{p}\in P^{\sigma}$. On the other hand, for $\mathfrak{p}\in P^{\sigma}$, we can see from the proof of (2.6) that there is an isomorphism

$$\psi_{a_{\mathfrak{p}}} : \mathcal{F}^0_{\mathfrak{X}}(\sigma)_{\mathfrak{p}} \xrightarrow{\sim} \mathcal{F}^1_{\mathfrak{X}}(\sigma)_{\mathfrak{p}};\ 1 \longrightarrow a_{\mathfrak{p}}\hat{\otimes}1 - 1\hat{\otimes}a_{\mathfrak{p}} \mod \mathcal{I}_{\mathfrak{X}}(\Delta)\cdot\mathcal{I}_{\mathfrak{X}}(\Gamma). \tag{3.9}$$

Thus, for $\mathbf{a} = (a_{\mathfrak{p}})_{\mathfrak{p}\in P^{\sigma}}$ chosen as (3.6), we obtain an isomorphism

$$\psi_{\mathbf{a}} : \mathcal{F}^0_{\mathfrak{X}}(\sigma)_{|U} \xrightarrow{\sim} \mathcal{F}^1_{\mathfrak{X}}(\sigma)_{|U} \tag{3.10}$$

such that the stalk of $\psi_{\mathbf{a}}$ at $\mathfrak{p}\in P^{\sigma}$ coincides with $\psi_{a_{\mathfrak{p}}}$, where $\mathcal{F}^r_{\mathfrak{X}}(\sigma)_{|U}$ denotes the restriction of $\mathcal{F}^r_{\mathfrak{X}}(\sigma)$ to $U := \mathfrak{X} \setminus E$.

Lemma(3.11). *We have*

$$\nu_{\mathfrak{X}}(\sigma) = \chi_A(\mathcal{F}^0_{\mathfrak{X}}(\sigma), \mathcal{F}^1_{\mathfrak{X}}(\sigma), \psi_{\mathbf{a}}) - \sum_{\mathfrak{p}\in P^\sigma} \theta_{a_{\mathfrak{p}}} \cdot \nu_{\mathfrak{p}}(\sigma) \quad \text{(cf. (3.1))}.$$

(3.11) is an immediate consequence of the definition of $\nu_{\mathfrak{X}}(\sigma)$.

Now, for the proof of (3.3), it suffices to determine how $\nu_{\mathfrak{X}}(\sigma)$ changes after replacing $\mathfrak{X}$ by the blowing-up of $\mathfrak{X}$ at a closed point fixed by σ. Thus, we are reduced to the following case: Let D=Spec(A) with A=k[[X,Y]] and σ be a non-trivial automorphism of A over k. Let $f : \mathfrak{X} \longrightarrow D$ be the blowing-up of D at the unique closed point x of D, and put $E = f^{-1}(x)$. Then, f identifies $\mathfrak{X} \setminus E$ with U:= D-x. For each integer $r \geqq 0$, let $\mathcal{F}^r_D(\sigma)$ (resp. $\mathcal{F}^r_{\mathfrak{X}}(\sigma)$) be the coherent $\mathcal{O}_D$-module (resp. $\mathcal{O}_{\mathfrak{X}}$-module) whose formal completion along x (resp. E) is defined as (3.5). By definition, the restrictions of $\mathcal{F}^r_D(\sigma)$ and $\mathcal{F}^r_{\mathfrak{X}}(\sigma)$ to U is identified, which we denote by $\mathcal{F}^r_U(\sigma)$. For each $\mathfrak{p}\in P^\sigma$, choose $a_{\mathfrak{p}} \in A$ as (3.6) so that we have an isomorphism (3.10)

$$\psi_{\mathbf{a}} : \mathcal{F}^0_U(\sigma) \xrightarrow{\sim} \mathcal{F}^1_U(\sigma) \quad (\mathbf{a} = (a_{\mathfrak{p}})_{\mathfrak{p}\in P^\sigma})$$

whose stalk at each $\mathfrak{p}\in P^\sigma$ is described as (3.9). Then, (3.3) follows from the following formula

$$(3.12) \qquad \chi_A(\mathcal{F}^0_D(\sigma), \mathcal{F}^1_D(\sigma), \psi_{\mathbf{a}}) = \chi_A(\mathcal{F}^0_{\mathfrak{X}}(\sigma), \mathcal{F}^1_{\mathfrak{X}}(\sigma), \psi_{\mathbf{a}}) + 1.$$

Let $\hat{D}$=Spf(A) (resp. $\hat{\mathfrak{X}}$) be the formal completion of D along x (resp. $\mathfrak{X}$ along E), and let

$$\mathfrak{Z} = \hat{D} \times_k \hat{D} \quad \text{and} \quad \mathfrak{Y} = \hat{\mathfrak{X}} \times_k \hat{\mathfrak{X}}$$

be the fiber product as formal schemes over k. Let $\mathcal{I}_{\mathfrak{Z}}(\Delta)$ and $\mathcal{I}_{\mathfrak{Y}}(\Delta)$

(resp. $\mathcal{I}_{\mathfrak{Z}}(\Gamma)$ and $\mathcal{I}_{\mathfrak{W}}(\Gamma)$) be the ideals of definition of the diagonal (resp. the graph of σ) in $\mathcal{O}_{\mathfrak{Z}}$ and $\mathcal{O}_{\mathfrak{W}}$ respectively. In addition to $\mathcal{T}^{r}_{D}(\sigma)$ and $\mathcal{T}^{r}_{\mathfrak{X}}(\sigma)$, we introduce a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{T}^{r}_{\mathfrak{X}/D}$ whose formal completion along E is

$$\hat{\mathcal{T}}^{r}_{\mathfrak{X}/D}(\sigma) := \mathcal{T}or_{r}^{\mathcal{O}_Z}(\mathcal{O}_{\mathfrak{W}}/\mathcal{I}_{\mathfrak{W}}(\Delta), \mathcal{O}_{\mathfrak{Z}}/\mathcal{I}_{\mathfrak{Z}}(\Gamma)).$$

We can see that $\mathcal{T}^{r}_{\mathfrak{X}/D}(\sigma) = 0$ for $r \geqq 2$, and that its restriction to U coincides with $\mathcal{T}^{r}_{U}(\sigma)$. Now, we deduce (3.12) from the following.

Claim. (1) $\chi_A(\mathcal{T}^{0}_{\mathfrak{X}/D}(\sigma), \mathcal{T}^{1}_{\mathfrak{X}/D}(\sigma), \psi_{\mathbf{a}}) = \chi_A(\mathcal{T}^{0}_{D}(\sigma), \mathcal{T}^{1}_{D}(\sigma), \psi_{\mathbf{a}})$.

(2) $\chi_A(\mathcal{T}^{0}_{\mathfrak{X}/D}(\sigma), \mathcal{T}^{1}_{\mathfrak{X}/D}(\sigma), \psi_{\mathbf{a}}) = \chi_A(\mathcal{T}^{0}_{\mathfrak{X}}(\sigma), \mathcal{T}^{1}_{\mathfrak{X}}(\sigma), \psi_{\mathbf{a}}) + 1$.

Proof of (1). We have isomorphisms

$$R^{i}g_{*}(\mathcal{O}_{\mathfrak{W}}/\mathcal{I}_{\mathfrak{W}}(\Delta)) = R^{i}\hat{f}_{*}\mathcal{O}_{\hat{\mathfrak{X}}} = \begin{cases} \mathcal{O}_{\hat{D}} & \text{for } i=0, \\ 0 & \text{for } i>0, \end{cases}$$

where $g : \mathfrak{W} \longrightarrow \mathfrak{Z}$ and $\hat{f} : \hat{\mathfrak{X}} \longrightarrow \hat{D}$ are the natural morphisms. Hence, by EGAIII(6.9.8), we have a spectral sequence

$$\mathcal{E}_2^{p,q} = R^{-p}f_{*}(\mathcal{T}^{q}_{\mathfrak{X}/D}(\sigma)) = \mathcal{T}^{p+q}_{D}(\sigma).$$

Now, our assertion follows from this and (3.4.4).

Proof of (2). First, we compute $\mathcal{T}or_{p}^{\mathcal{O}_{\mathfrak{Z}}}(\mathcal{O}_{\mathfrak{W}}, \mathcal{O}_{\mathfrak{Z}}/\mathcal{I}_{\mathfrak{Z}}(\Gamma))$ by using the Koszul complex associated to a regular sequence in $\mathcal{O}_{\mathfrak{Z}}$ generating $\mathcal{I}_{\mathfrak{Z}}(\Delta)$, and we can see

$$\mathcal{T}or_{p}^{\mathcal{O}_{\mathfrak{Z}}}(\mathcal{O}_{\mathfrak{W}}, \mathcal{O}_{\mathfrak{Z}}/\mathcal{I}_{\mathfrak{Z}}(\Gamma)) = \begin{cases} 0 & \text{for } p>0, \\ \mathcal{O}_{\mathfrak{W}}/\Big(\mathcal{I}_{\mathfrak{W}}(\Gamma)\cdot\mathcal{I}_{\mathfrak{W}}(E\times E)\Big) & \text{for } p=0, \end{cases}$$

where $E\times E$ is the fiber of $g : \mathscr{W} \longrightarrow \mathscr{Z}$ over the unique closed point of $\mathscr{Z}$ and $\mathscr{I}_{\mathscr{W}}(E\times E)$ is its ideal of the definition. Hence, we have

$$\hat{\mathscr{T}}^{r}_{\mathscr{X}/D}(\sigma) \simeq \mathscr{T}or_r^{\mathcal{O}_{\mathscr{W}}}\Big(\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(\Delta),\mathcal{O}_{\mathscr{W}}/(\mathscr{I}_{\mathscr{W}}(\Gamma)\cdot\mathscr{I}_{\mathscr{W}}(E\times E))\Big).$$

Put

$$\mathscr{C} = \mathrm{Coker}\Big(\mathcal{O}_{\mathscr{W}}/(\mathscr{I}_{\mathscr{W}}(\Gamma)\cdot\mathscr{I}_{\mathscr{W}}(E\times E)) \longrightarrow \mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(\Gamma) \oplus \mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(E\times E)\Big).$$

The sheaves

$$\mathscr{T}or_r^{\mathcal{O}_{\mathscr{W}}}(\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(\Delta),\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(E\times E)) \quad \text{and} \quad \mathscr{T}or_r^{\mathcal{O}_{\mathscr{W}}}(\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(\Delta),\mathscr{C})$$

are supported in $E \subset \mathscr{X} \simeq \Delta_{\mathscr{X}} \subset \mathscr{W}$, so determine elements in $K_0(E)$. Set

$$\mathscr{F}_1 = \sum_{r=0}^{\infty}(-1)^r[\mathscr{T}or_r^{\mathcal{O}_{\mathscr{W}}}(\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(\Delta),\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(E\times E))],$$

$$\mathscr{F}_2 = \sum_{r=0}^{\infty}(-1)^r[\mathscr{T}or_r^{\mathcal{O}_{\mathscr{W}}}(\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(\Delta),\mathscr{C})]$$

in $K_0(E)$. Then, by (3.4.1) and (3.4.3), we have

$$\chi_A(\mathscr{T}^0_{\mathscr{X}/D}(\sigma),\mathscr{T}^1_{\mathscr{X}/D}(\sigma),\psi_a) = \chi_A(\mathscr{T}^0_{\mathscr{X}}(\sigma),\mathscr{T}^1_{\mathscr{X}}(\sigma),\psi_a) + \chi_A(\mathscr{F}_1) - \chi_A(\mathscr{F}_2).$$

Hence, we are reduced to prove that $\chi_A(\mathscr{F}_2)=0$ and $\chi_A(\mathscr{F}_1)=1$. Since $\mathrm{Supp}(\mathscr{C}) \subset E = \Delta_{\mathscr{X}} \cap (E\times E)$, the first assertion follows from [10] V,§B,n°3,Th.1. The second assertion follows from isomorhisms

$$\mathscr{T}or_r^{\mathcal{O}_{\mathscr{W}}}(\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(\Delta),\mathcal{O}_{\mathscr{W}}/\mathscr{I}_{\mathscr{W}}(E\times E)) \simeq \mathscr{T}or_r^{\mathcal{O}_{\mathscr{X}}}(\mathcal{O}_{\mathscr{X}}/\mathscr{I}_{\mathscr{X}}(E),\mathcal{O}_{\mathscr{X}}/\mathscr{I}_{\mathscr{X}}(E))$$

$$\simeq \begin{cases} \mathscr{I}_{\mathscr{X}}(E)/\mathscr{I}_{\mathscr{X}}(E)^2 & \text{for } r=0, \\ 0 & \text{for } r>0, \end{cases}$$

where $\mathscr{I}_{\mathscr{X}}(E)$ is the ideal of definition of E in $\mathcal{O}_{\mathscr{X}}$ and the first isomorphism follows from [10] V,§B,n°1.

4. The proof of the main formula (0.2)

Let X/k be as in **Introduction**. In this section, we prove (0.2) for $\sigma\in\mathrm{Aut}(X/k)'$. First, by definition, we are easily reduced to the case that X is smooth over k. Then, the following proof works well, only assuming $\sigma\in\mathrm{End}(X/k)'$. Put $Z = X \times_k X$ and let Δ (resp. Γ) be the diagonal (resp. the graph of σ) in Z.

Lemma(4.1). *We have*

$$\mathrm{Tr}(\sigma^*)|H^*(X) = \sum_{r=0}^{\infty}\chi\left(Z,\mathcal{T}or_r^{\mathcal{O}_Z}(\mathcal{O}_Z/\mathcal{I}_Z(\Delta),\mathcal{O}_Z/\mathcal{I}_Z(\Gamma))\right),$$

where $\mathcal{I}_Z(\Delta)$ (resp. $\mathcal{I}_Z(\Gamma)$) denotes the ideal of definition of Δ (resp. Γ), and

$$\chi(Z,*) = \sum_{q=0}^{\infty}(-1)^q\dim_k H^q(Z,*).$$

This follows from SGA4 1/2,cycle and [3](20.4).

In the following, we put for each integer $r\geqq 0$,

$$\mathcal{T}_X^r = \mathcal{T}or_r^{\mathcal{O}_Z}(\mathcal{O}_Z/\mathcal{I}_Z(\Delta),\mathcal{O}_Z/\mathcal{I}_Z(\Gamma)). \tag{4.2}$$

By definition, $\mathcal{T}_X^r$ is a coherent $\mathcal{O}_Z$-module which is annihilated by $\mathcal{I}_Z(\Delta)$, so it is viewed as a coherent $\mathcal{O}_X$-module through the natural isomorphism $\mathcal{O}_Z/\mathcal{I}_Z(\Delta) \simeq \mathcal{O}_X$. Moreover, as (2.3), we see that $\mathcal{T}_X^r = 0$ for $r\geqq 2$, and that there are canonical isomorphisms

$$\mathcal{T}_X^0 \simeq \mathcal{O}_X/I_X(\sigma) \quad\text{and}\quad \mathcal{T}_X^1 \simeq \left(\mathcal{I}_Z(\Delta)\cap\mathcal{I}_Z(\Gamma)\right)/\left(\mathcal{I}_Z(\Delta)\cdot\mathcal{I}_Z(\Gamma)\right).$$

For $x\in X_0^\sigma$, σ induces an element $\sigma_x\in\mathrm{End}(A_x/k)'$ and we see

$$\mathcal{T}_X^r\otimes_{\mathcal{O}_X}A_x = T_{A_x}^r(\sigma_x) \quad \text{(cf. (2.2) and (2.3))} \tag{4.3}$$

For each $\mathfrak{p}\in X_1^\sigma$ and each integer $r\geqq 0$, we put

$T^r_{\mathfrak{p}}$; the stalk of $\mathcal{T}^r_X$ at $\mathfrak{p}$ and $\mathcal{G}^r_{\mathfrak{p}}$; the image of $\mathcal{T}^r_X$ in $T^r_{\mathfrak{p}}$.

By definition, $\mathcal{G}^r_{\mathfrak{p}}$ is a coherent $\mathcal{O}_X$-module such that

$$\mathrm{Supp}(\mathcal{G}^r_{\mathfrak{p}}) \subset F_{\mathfrak{p}},$$

where $F_{\mathfrak{p}}$ is the closure of $\mathfrak{p}$ in X. Let

$$\gamma^r : \mathcal{T}^r_X \longrightarrow \bigoplus_{\mathfrak{p}\in X_1^\sigma} \mathcal{G}^r_{\mathfrak{p}}$$

be the natural homomorphism. Then, $\mathrm{Ker}(\gamma^r)$ and $\mathrm{Coker}(\gamma^r)$ has a punctual support and we can see (cf.(2.5))

$$(4.4) \qquad \sum_{r=0}^{1}\Big(\chi(X,\mathrm{Ker}(\gamma^r)) - \chi(X,\mathrm{Coker}(\gamma^r))\Big) = \sum_{x\in X_0^\sigma}\Big(\delta_{A_x}(\sigma_x) + \varepsilon_x\Big).$$

Here, ε_x is defined as follows: For $x\in X_0^\sigma$ and $\mathfrak{p}\in X_1^\sigma$, let $P_x(\mathfrak{p})$ be the set of all prime ideals of height one in A_x lying over $\mathfrak{p}$. Then, we have a natural map for each integer $r\geqq 0$,

$$\iota^r_x : \mathcal{G}^r_{\mathfrak{p}}\otimes_{\mathcal{O}_X} A_x \longrightarrow \bigoplus_{\mathfrak{p}_x\in P_x(\mathfrak{p})} S^r_{A_x}(\sigma_x)_{\mathfrak{p}_x} \quad \text{(cf.(2.4))}.$$

We can see that ι^r_x is injective, and put

$$\varepsilon^r_x = \dim_k(\mathrm{Coker}(\gamma^r_x)) \quad\text{and}\quad \varepsilon_x = \sum_{r=0}^{\infty}(-1)^r\varepsilon^r_x = \varepsilon^0_x - \varepsilon^1_x.$$

Now, fix $\mathfrak{p}\in X_1^\sigma$, and let F denote $F_{\mathfrak{p}}$ for simplicity. Let R be the local ring of X at $\mathfrak{p}$. We denote by the same letter $\mathfrak{p}$ the maximal ideal of R, and put $\kappa(\mathfrak{p})=R/\mathfrak{p}$ which is the function field of F. Let $n=\nu_{\mathfrak{p}}(\sigma)$ (cf. (0.2)). By definition, we have

$$(4.5) \qquad T^0_{\mathfrak{p}} = \bar{R} := R/\mathfrak{p}^n,$$

and in the same argument as the proof of (2.7), we have the following canonical isomorphism

(4.6) $$T^1_{\mathfrak{p}} \simeq \mathrm{Ker}(\varphi_\sigma),$$

where

$$\varphi_\sigma : \Omega^1_{R/k}\otimes_R \bar{R} \longrightarrow \mathfrak{p}^n/\mathfrak{p}^{2n} \; ; \; adb\otimes 1 \longrightarrow a(\sigma(b)-b) \quad (\text{cf.}(1.11)).$$

For each integer $i \geqq 0$ and for $r=0$ or 1, we put

$$\mathrm{Gr}^i T^r_{\mathfrak{p}} = \mathfrak{p}^i T^r_{\mathfrak{p}}/\mathfrak{p}^{i+1}T^r_{\mathfrak{p}}.$$

Then, $\mathrm{Gr}^i T^r_{\mathfrak{p}}=0$ for $i \geqq n$, and it is a one-dimensional $\kappa(\mathfrak{p})$-vector space for $0 \leqq i \leqq n-1$. Moreover, by (4.5),(4.6) and (1.11), we can see the following isomorphisms for $0 \leqq i \leqq n-1$,

$$\mathrm{Gr}^i T^0_{\mathfrak{p}} = \mathfrak{p}^i/\mathfrak{p}^{i+1},$$

$$\mathrm{Gr}^i T^1_{\mathfrak{p}} \simeq \begin{cases} \Omega^1_{\kappa(\mathfrak{p})/k}\otimes_R(\mathfrak{p}^i/\mathfrak{p}^{i+1}) & \text{if } \sigma \text{ is type I at } \mathfrak{p}, \\ (\mathfrak{p}/\mathfrak{p}^2)\otimes_R(\mathfrak{p}^i/\mathfrak{p}^{i+1}) & \text{if } \sigma \text{ is type II at } \mathfrak{p}. \end{cases}$$

Let $f : \tilde{F} \longrightarrow F$ be the normalization of F and put

$$\mathcal{G}_i = \mathcal{O}_X(-iF)\otimes_{\mathcal{O}_X}\mathcal{O}_F,$$

$$\mathcal{K}_{i.0} = f_*\mathcal{O}_{\tilde{F}}\otimes_{\mathcal{O}_F}\mathcal{G}_i,$$

$$\mathcal{K}_{i.1} = \begin{cases} f_*\Omega^1_{\tilde{F}/k}\otimes_{\mathcal{O}_F}\mathcal{G}_i & \text{if } \sigma \text{ is type I at } \mathfrak{p}, \\ f_*f^*\mathcal{G}_1\otimes_{\mathcal{O}_F}\mathcal{G}_i & \text{if } \sigma \text{ is type II at } \mathfrak{p}. \end{cases}$$

Then, $\mathscr{L}_i$ is an invertible $\mathcal{O}_F$-module, and $\mathscr{K}_{i.0}$ (resp. $\mathscr{K}_{i.1}$) is a coherent $\mathcal{O}_F$-module whose stalk at $\mathfrak{p}$ is canonically isomorphic to $\mathrm{Gr}^i T^0_{\mathfrak{p}}$ (resp. $\mathrm{Gr}^i T^1_{\mathfrak{p}}$). For r=0 and 1, we put

$$(4.7) \qquad \delta^r_{\mathfrak{p}} = \chi(X, \mathscr{T}^r_{\mathfrak{p}}) - \sum_{i=0}^{n-1} \chi(F, \mathscr{K}_{i.r}),$$

By (4.3), we see

$$(4.8) \qquad -\delta^r_{\mathfrak{p}} = \sum_{x \in X^\sigma_0} \left(\varepsilon^r_x + \sum_{\mathfrak{p}_x \in P_x(\mathfrak{p})} \nu^r_{A_x . \mathfrak{p}_x}(\sigma_x) \right) \quad \text{(cf. (2.6) and (2.8))}.$$

Lastly, putting

$$\lambda_i = \chi(F, \mathscr{K}_{i.0}) - \chi(F, \mathscr{K}_{i.1}) \quad (0 \leqq i \leqq n-1),$$

we have

$$\lambda_i = \begin{cases} \chi(\tilde{F}, f^*\mathscr{L}_i) - \chi(\tilde{F}, \Omega^1_{\tilde{F}/k} \otimes_{\mathcal{O}_{\tilde{F}}} f^*\mathscr{L}_i) & \text{if } \sigma \text{ is type I at } \mathfrak{p}, \\ \chi(\tilde{F}, f^*\mathscr{L}_i) - \chi(\tilde{F}, f^*\mathscr{L}_1 \otimes_{\mathcal{O}_{\tilde{F}}} f^*\mathscr{L}_i) & \text{if } \sigma \text{ is type II at } \mathfrak{p}. \end{cases}$$

Hence, by the Riemann-Roch theorem for the curve $\tilde{F}$, we have

$$(4.9) \qquad \lambda_i = \begin{cases} \chi_{\mathfrak{p}} = \text{the degree of } \Omega^1_{\tilde{F}/k} & \text{if } \sigma \text{ is type I at } \mathfrak{p}, \\ \tau_{\mathfrak{p}} = \text{the degree of } f^*\mathscr{L}_1 & \text{if } \sigma \text{ is type II at } \mathfrak{p}, \end{cases}$$

where $\chi_{\mathfrak{p}}$ and $\tau_{\mathfrak{p}}$ are defiend in (0.3).

Now, our assertion follows from (4.1),(4.4),(4.8) and (4.9), inview of (2.9).

5. An explicit calculation

Let $A=k[[X,Y]]$ be the ring of formal power series of two variables over k, and let $\mathfrak{m}$ be the maximal ideal of A. In this section, we will compute explicitly $\nu_A(\sigma)$ for a fixed $\sigma \in \mathrm{End}(A/k)'$ (cf. §2). First, we can write

$$(5.1) \qquad \begin{cases} \sigma(X) = X + \tilde{g} \\ \sigma(Y) = Y + \tilde{h} \end{cases} \qquad (\tilde{g},\ \tilde{h} \in \mathfrak{m}).$$

Let f be a greatest common divisor of $\tilde{g}$ and $\tilde{h}$, and let $g=\tilde{g}/f$ and $h=\tilde{h}/f$. We put

$$(5.2) \qquad \mathfrak{a}_\sigma = (f) \quad \text{and} \quad \mathfrak{b}_\sigma = (g,h).$$

By definition, $I_A(\sigma) = \mathfrak{a}_\sigma \cdot \mathfrak{b}_\sigma$, and we can see that $\mathfrak{a}_\sigma$ and $\mathfrak{b}_\sigma$ depend only on A and σ, neither on the choice of the coordinates X and Y nor on that of f. Since g and h are relatively prime, $A/\mathfrak{b}_\sigma$ has a finite length, and we put

$$(5.3) \qquad \delta = \dim_k(A/\mathfrak{b}_\sigma).$$

Let $\{\mathfrak{p}_\alpha\}_{\alpha \in S}$ be all distinct prime ideals of height one in A which divides $\mathfrak{a}_\sigma$. For each $\alpha \in S$, let $n_\alpha = \nu_{\mathfrak{p}_\alpha}(\sigma)$ (cf. (2.1)). By the definitions in §2, $P^\sigma = \{\mathfrak{p}_\alpha | \alpha \in S\}$. For each $\alpha \in S$, fix an element $\pi_\alpha \in A$ such that $\mathfrak{p}_\alpha = (\pi_\alpha)$ and a prime element t_α of the normalization $\varkappa[\mathfrak{p}_\alpha]$ of $A/\mathfrak{p}_\alpha$ so that $\varkappa[\mathfrak{p}_\alpha] \simeq k[[t_\alpha]]$. We put

$$(5.4) \qquad \omega = h \cdot dX - g \cdot dY \in \hat{\Omega}^1_{A/k},$$

and let ω_α $(\alpha \in S)$ be the image of ω in $\hat{\Omega}^1_{\varkappa(\mathfrak{p}_\alpha)/k}$.

Lemma(5.5). *If ω_α is non-trivial, σ is type* I *at $\mathfrak{p}_\alpha$, otherwise σ is type* II *at $\mathfrak{p}_\alpha$.*

The proof of (5.5) will be given later.

Let S_{I} (resp. S_{II}) be the subset of S consisting of such α that σ is type I (resp. II) at $\mathfrak{p}_\alpha$. In view of (1.7), for each $\alpha\in S_{I}$ (resp. S_{II}), we can find a non-zero $a_\alpha \in \kappa[\mathfrak{p}_\alpha]$ such that

$$\omega_\alpha = a_\alpha \cdot dt_\alpha \quad (\text{resp. } \omega \bmod \mathfrak{p}_\alpha \hat{\Omega}^1_{A/k} = a_\alpha \cdot d\pi_\alpha).$$

We define

$$(5.6) \qquad \mu_\alpha = \mathrm{ord}_{\kappa(\mathfrak{p}_\alpha)}(a_\alpha) = \max\{r \mid t_\alpha^r \text{ divides } a_\alpha\}.$$

Theorem(5.7). *We have*

$$\nu_A(\sigma) = \delta + \sum_{\alpha\in S} n_\alpha \cdot \mu_\alpha .$$

Define $C = A\hat{\otimes}_k A$, $I_C(\Delta)$ and $I_C(\Gamma)$ as §2. By the map

$$A \longrightarrow C \ ; \ a \longrightarrow a\hat{\otimes}1.$$

We consider A a subring of C, and fix an identification

$$C = A[[Z,W]] = k[[X,Y,Z,W]] \ ; \ 1\hat{\otimes}X \longrightarrow Z, \ 1\hat{\otimes}Y \longrightarrow W.$$

Then, $I_C(\Delta)$ (resp. $I_C(\Gamma)$) is identified with the ideal generated by $(X-Z, Y-W)$ (resp. $(\sigma(X)-Z, \sigma(Y)-W)$). For each integer $r \geqq 0$, let $T_A^r(\sigma)$ be as (2.2). By the equality

$$h(X-Z) - g(Y-W) = h(\sigma(X)-Z) - g(\sigma(Y)-W),$$

we can see

$$\Omega := h(X-Z) - g(Y-W) \in I_C(\Delta) \cap I_C(\Gamma).$$

Lemma(5.8). *The map*

$$A \longrightarrow T_A^1(\sigma) \ ; \ 1 \longrightarrow \Omega \bmod I_C(\Delta)\cdot I_C(\Gamma).$$

induces an isomorphism $A/\mathfrak{a}_\sigma \simeq T_A^1(\sigma)$ (cf.(2.3.1)).

Proof. To compute $T^1_A(\sigma) = \mathrm{Tor}^C_1(C/I_C(\Delta),C/I_C(\Gamma))$, we use the Koszul complex associated to the regular sequence $(\sigma(X)-Z,\sigma(Y)-W)$;

$$C\cdot e_1\wedge e_2 \xrightarrow{d_2} C\cdot e_1 \oplus C\cdot e_2 \xrightarrow{d_1} C,$$

where

$$d_1(e_1) = \sigma(X)-Z, \quad d_1(e_2) = \sigma(Y)-W \quad \text{and}$$

$$d_2(e_1\wedge e_2) = (\sigma(X)-Z)e_2 - (\sigma(Y)-W)e_1.$$

We get

$$(5.9) \qquad T^1_A(\sigma) \simeq \mathrm{Ker}(d_1\otimes A)/\mathrm{Im}(d_2\otimes A),$$

where

$$d_1\otimes A : A \oplus A \longrightarrow A \ ; \ (a,b) \longrightarrow -(a\tilde{g}+b\tilde{h}) = -f(ag+bh),$$

$$d_2\otimes A : A \longrightarrow A \oplus A \ ; \ a \longrightarrow (-a\tilde{h},a\tilde{g}).$$

Hence, we have

$$\mathrm{Ker}(d_1\otimes A) = A\cdot(h,-g) \quad \text{and} \quad \mathrm{Im}(d_2\otimes A) = A\cdot(fh,-fg).$$

So, we get an isomorphism $T^1_A(\sigma) \simeq A/\mathfrak{a}_\sigma$. There remains to prove that $(h,-g)$ is mapped to $(\Omega \bmod I_C(\Delta)\cdot I_C(\Gamma))$ under (5.10). In fact, this follows from the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & (A \oplus A)/\mathrm{Im}(d_2\otimes A) & \xrightarrow{d_1\otimes A} & A \\
 & & & & \downarrow \phi & & \downarrow f \\
0 \longrightarrow & \mathrm{Tor}^C_1(C/I_C(\Delta),C/I_C(\Gamma)) & \longrightarrow & & I_C(\Gamma)/(I_C(\Delta)\cdot I_C(\Gamma)) & \longrightarrow & C/I_C(\Delta),
\end{array}$$

where ϕ is defined by

$$\phi((a,b)) = a(\sigma(X)-Z) + b(\sigma(Y)-W) \bmod I_C(\Delta)\cdot I_C(\Gamma).$$

Now, noting an isomorphism

$$C/(I_C(\Delta)+I_C(\Gamma)) \simeq A/I_A(\sigma),$$

$I_C(\Delta) \cap I_C(\Gamma) \subsetneq I_C(\Delta)$ induces a homomorphism

$$(5.10) \qquad \theta_\sigma : T^1_A(\sigma) \longrightarrow \hat{\Omega}^1_{A/k}\otimes_A(A/I_A(\sigma)) \twoheadrightarrow \hat{\Omega}^1_{A/k}\otimes(A/\mathfrak{a}(\sigma)).$$

By (5.8), we see that θ_σ induces an isomorphism

$$(5.11) \qquad T^1_A(\sigma) \simeq (A/\mathfrak{a}(\sigma))\cdot\omega \subset \hat{\Omega}^1_{A/k}\otimes(A/\mathfrak{a}(\sigma)).$$

Now, (5.5) follows from this and the definitions.

Now, by (5.8), we can see

$$(5.12) \qquad \delta_A(\sigma) = \dim_k\mathrm{Ker}\Big(A/I_A(\sigma) \longrightarrow A/\mathfrak{a}(\sigma)\Big) = \delta \quad (\text{cf. } (2.4),(5.3)).$$

On the other hand, fixing $\alpha\in S$, we denote simply $\mathfrak{p}=\mathfrak{p}_\alpha$, $\pi=\pi_\alpha$, $n=n_\alpha$ and $\mu=\mu_\alpha$. Put $\bar{A} = A/\mathfrak{p}^n$ and $\bar{R} = \bar{A}\otimes_A R_\mathfrak{p}$, where $R_\mathfrak{p}$ is the localization of A at $\mathfrak{p}$. Then, we can see

$$S^0_A(\sigma)_\mathfrak{p} = \bar{A} \subset T^0_A(\sigma)_\mathfrak{p} = \bar{R},$$

$$S^1_A(\sigma)_\mathfrak{p} = \bar{A}\cdot\omega \subset T^1_A(\sigma)_\mathfrak{p} = \bar{R}\cdot\omega \subset \hat{\Omega}^1_{A/k}\otimes_A\bar{R}.$$

From these descriptions, we get

$$(5.13) \qquad \begin{aligned} \nu^0_{A.\mathfrak{p}}(\sigma) &:= -[T^0_A(\sigma)_\mathfrak{p}:S^0_A(\sigma)_\mathfrak{p}] = -n\xi \ , \\ \nu^1_{A.\mathfrak{p}}(\sigma) &:= -[T^1_A(\sigma)_\mathfrak{p}:S^1_A(\sigma)_\mathfrak{p}] = -n(\xi+\mu) \ , \end{aligned}$$

where $\xi = \dim_k\Big(\kappa[\mathfrak{p}]/(A/\mathfrak{p})\Big)$.

Now, (5.7) follows from (5.12) and (5.13) and the definition.

Lemma(5.14). *Let* A *be as in the first part of §2, and* $\sigma\in\mathrm{Aut}(A/k)'$ *have a finite order prime to* ch(k). *Then, we have*

$$\nu_A(\sigma) = 1 - \#P^{\sigma} \text{ (cf. §2).}$$

Proof. We begin with the following facts.

Lemma(5.15). *Let* $A=k[[X,Y]]$ *and* $\sigma\in\mathrm{Aut}(A/k)'$ *and assume that* σ *has a finite order* n *which is prime to* ch(k). *Then, there exists* $\tau\in\mathrm{Aut}(A/k)$ *such that if we put* $\sigma' = \tau^{-1}\sigma\tau$,

$$\sigma'(X) = \zeta_1\cdot X \quad \textit{and} \quad \sigma'(Y) = \zeta_2\cdot Y,$$

where ζ_i *(i=1 and 2) is an element of* k such that $\zeta_i^n = 1$.

Corollary(5.16). *Let* A *and* σ *be as (5.15). There are only two possibilities: One is that* P^{σ} *is empty and* $\nu_A(\sigma)=\dim_k\left(A/I_A(\sigma)\right)=1$. *The other is that* P^{σ} *consisits of one element* $\mathfrak{p}$ *such that* $A/\mathfrak{p}$ *is regular, and* $\nu_A(\sigma)=0$.

(5.16) is an easy cosequence of (5.15) and (5.7).

Proof of (5.15) (cf. [1] p.32). For $\varphi\in\mathrm{End}(A/k)$, there is a unique expression

$$\varphi(X) = aX + bY + f \quad \text{and} \quad \varphi(Y) = cX + dY + g$$

with $a,b,c,d \in k$ and $f,g \in \mathfrak{m}^2$. Then, we define $L(\varphi)\in\mathrm{End}(A/k)$ by

$$L(\varphi)(X) = aX + bY \quad \text{and} \quad L(\varphi)(Y) = cX + dY.$$

By definition,

$$L : \mathrm{End}(A/k) \longrightarrow \mathrm{End}(A/k) \ ; \ \varphi \longrightarrow L(\varphi)$$

is a ring homomorphism, and $\sigma\in\mathrm{Aut}(A/k)$ if and only if $L(\varphi)\in\mathrm{Aut}(A/k)$. In particular, if $\varphi\in\mathrm{Aut}(A/k)$, $L(\varphi\cdot L(\varphi)^{-1})$ is the identity. Now, for $\sigma \in \mathrm{Aut}(A/k)$ as (5.15), we define $\varphi_\sigma \in \mathrm{End}(A/k)$ by

$$\varphi_\sigma(a) = \frac{1}{n}\left(\sum_{i=0}^{n-1} \sigma^i \cdot L(\sigma^{-i})(a)\right) \quad \text{for } a \in A.$$

Then, we see that $L(\varphi_\sigma)$ is the identity so that $\varphi_\sigma \in \mathrm{Aut}(A/k)$ and

$$\begin{aligned}\sigma \cdot \varphi_\sigma(a) &= \frac{1}{n}\left(\sum_{i=0}^{n-1} \sigma^{i+1} \cdot L(\sigma^{-i})(a)\right) \\ &= \frac{1}{n}\left(\sum_{i=0}^{n-1} \sigma^{i+1} \cdot L(\sigma^{-(i+1)}) \cdot L(\sigma)(a)\right) \\ &= \varphi_\sigma(L(\sigma)(a)).\end{aligned}$$

This means $\varphi_\sigma^{-1} \cdot \sigma \cdot \varphi_\sigma = L(\sigma)$. Since $L(\sigma)^n$ is the identity by definition, (5.15) follows at once from this.

Now, we return to the proof of (5.14). Fix a resolution

$$f : \mathfrak{X} \longrightarrow \mathrm{Spec}(A),$$

such that σ extends to an automorphism of $\mathfrak{X}$. Then, we use the same notation as the last part of §2. We note that for each $\mathfrak{p} \in P$, $F_\mathfrak{p}$ intersects with E at a unique closed point of E which we put $x_\mathfrak{p}$. By definition, if $\mathfrak{p} \in P^\sigma$, $x_\mathfrak{p} \in E_0^\sigma$. Now, from (5.16), we can see the following facts.

(5.14.1) For any $\eta \in E_1^\sigma$, $\eta \in E_I^\sigma$ and E_η is regular.

(5.14.2) If $\mathfrak{p}$ and $\mathfrak{p}'$ are distinct elements of P^σ, $x_\mathfrak{p} \neq x_{\mathfrak{p}'}$.

(5.14.3) Let $\mathfrak{p} \in P$ and $\eta \in E_1$. If $F_\mathfrak{p}$ intersects with E_η and $\mathfrak{p} \in P^\sigma$, $E_\eta \notin E_1^\sigma$.

Now, (5.14) follows from these facts and the classical fixed point formula for the curve E in view of the definition of $\nu_A(\sigma)$ (cf.(3.1) and (3.2)).

6. The Swan functions for two-dimensional local rings

In this section, we will define a two-dimensional version of the classical theory of Swan representations for complete discrete valuation rings, which is expected to play an important role in the ramification theory of algebraic surfaces.

First, we review briefly the classical theory. In general, for a group G and a ring R, let Map(G,R) denote the set of all mappings $G \longrightarrow R$, and let C(G,R) denote the subset of Map(G,R) consisting of all elements f such that $f(\sigma)=f(\sigma')$ if σ and σ' are conjugate in G. Let K(G,R) be the Grothendieck group on the category of finitely generated projective R[G]-modules. We can consider K(G,R) a subset of C(G,R) by taking the traces of elements of K(G,R) (cf. SGA5X).

Let E be a complete discrete valuation field and $\mathcal{O}_E$ (resp. $\overline{E}$) the ring of integers (resp. the residue field) of E. Let F/E be a finite Galois extension and G=Gal(F/E). Let $\mathcal{O}_F$ (resp. $\overline{F}$) be the ring of integers (resp. the residue field) of F. We assume that

(6.1) the extension $\overline{F}/\overline{E}$ is separable.

Put $d=[\overline{F}:\overline{E}]$. For $\sigma\in G-\{e\}$ (e is the identity of G), we define

$$\nu_G(\sigma) = \text{length}_{\mathcal{O}_E}\left(\mathcal{O}_F/I_{\mathcal{O}_F}(\sigma)\right). \tag{6.2}$$

Then, we define $Sw_G \in Map(G,\mathbf{Z})$ as follows:

$$Sw_G(\sigma) = \begin{cases} d - \nu_G(\sigma) & \text{if } \sigma\neq e \text{ and } \nu_G(\sigma)>0, \\ 0 & \text{if } \sigma\neq e \text{ and } \nu_G(\sigma)=0, \\ -\sum\limits_{\sigma\neq e} Sw_G(\sigma) & \text{if } \sigma=e. \end{cases} \tag{6.3}$$

We can see easily $Sw_G \in C(G,\mathbf{Z})$. One of the main results in the classical ramification theory is that for any prime number ℓ, the image of Sw_G in $C(G,\mathbf{Q}_\ell)$ lies in $K(G,\mathbf{Q}_\ell)$ (cf.[11] and [12]) and that it plays an important role in the ramification theory of algebraic curves (cf. SGA5X).

Now, let A be as §2 and K be its quotient field. Let L/K be a finite Galois extension and G=Gal(L/K). Let B be the integral closure of A in L. We denote by P_A (resp. P_B) the set of all prime ideals of height one in A (resp. B). Let S be the subset of P_A consisting of all $\mathfrak{p}$ where the extension B/A ramifies. We assume

(6.4) for any $\mathfrak{p}\in P_A$ and $\tilde{\mathfrak{p}}\in P_B$ lying over $\mathfrak{p}$, $\kappa(\tilde{\mathfrak{p}})/\kappa(\mathfrak{p})$ is separable.

Under this assumption, we will define

$$\nu_G : G-\{e\} \longrightarrow \mathbf{Z} \quad (\text{resp. } Sw_G \in C(G,\mathbf{Z}))$$

which plays the role corresponding to (6.2) (resp. (6.3)) in our two-dimensional context. First, we put

$$(6.5) \qquad \nu_G(\sigma) = \nu_B(\sigma) - \sum_{\tilde{\mathfrak{p}}\in P_B} \nu_{\tilde{\mathfrak{p}}}(\sigma)\cdot\delta_{\tilde{\mathfrak{p}}} ,$$

Here $\nu_B(\sigma)$ is the multiplicity of $\sigma\in\mathrm{Aut}(B/k)$ defined in §3. For $\tilde{\mathfrak{p}}\in P_B$, $\nu_{\tilde{\mathfrak{p}}}(\sigma)$ is defined as (2.1), and if $\mathfrak{p}\in P_A$ lies under $\tilde{\mathfrak{p}}$, we put

$$(6.6) \qquad \delta_{\tilde{\mathfrak{p}}} = \dim_k \Omega^1_{\kappa[\tilde{\mathfrak{p}}]/\kappa[\mathfrak{p}]} ,$$

where $\kappa[\tilde{\mathfrak{p}}]$ (resp. $\kappa[\mathfrak{p}]$) is the normalization of $B/\tilde{\mathfrak{p}}$ (resp. $A/\mathfrak{p}$). In particular, if B is regular and $\dim_k\left(B/I_B(\sigma)\right)$ is finite, we have

$$(6.7) \qquad \nu_G(\sigma) = \dim_k\left(B/I_B(\sigma)\right).$$

Let $\mathfrak{p}\in P_A$ and choose $\tilde{\mathfrak{p}}\in P_B$ lying over $\mathfrak{p}$. Let $G_{\tilde{\mathfrak{p}}}$ (resp. $I_{\tilde{\mathfrak{p}}}$) be the decomposition (resp. inertia) subgroup of G at $\tilde{\mathfrak{p}}$, and put $\overline{G}_{\tilde{\mathfrak{p}}} = G_{\tilde{\mathfrak{p}}}/I_{\tilde{\mathfrak{p}}} \simeq \mathrm{Gal}(\kappa(\tilde{\mathfrak{p}})/\kappa(\mathfrak{p}))$. Note that $\kappa(\mathfrak{p})$ and $\kappa(\tilde{\mathfrak{p}})$ are complete discrete valuation field with residue field k. So we have defined $Sw_{\overline{G}_{\tilde{\mathfrak{p}}}} \in C(\overline{G}_{\tilde{\mathfrak{p}}},\mathbf{Z})$ (cf.(6.3)). Let $\mathbf{Z}[\overline{G}_{\tilde{\mathfrak{p}}}] \in C(\overline{G}_{\tilde{\mathfrak{p}}},\mathbf{Z})$ be the character of the regular representation of $\overline{G}_{\tilde{\mathfrak{p}}}$. Consider

$$\rho_{\overline{G}_{\tilde{\mathfrak{p}}}} := Sw_{\overline{G}_{\tilde{\mathfrak{p}}}} + \mathbf{Z}[\overline{G}_{\tilde{\mathfrak{p}}}] \in C(\overline{G}_{\tilde{\mathfrak{p}}},\mathbf{Z})$$

as an element of $C(G_{\tilde{\mathfrak{p}}},\mathbf{Z})$ through the natural map $G_{\tilde{\mathfrak{p}}} \longrightarrow \overline{G}_{\tilde{\mathfrak{p}}}$, and put

$$\rho_{G.\mathfrak{p}} = \mathrm{Ind}_{G_{\mathfrak{p}}/G}(\rho_{\overline{G}_{\tilde{\mathfrak{p}}}}) \in C(G,\mathbf{Z}).$$

This definition depends only on $\mathfrak{p}$, and not on the choice of $\tilde{\mathfrak{p}}$. Finally, for a finite (may be empty) subset R of P_A containing S, we define $Sw_{G.R} \in \mathrm{Map}(G,\mathbf{Z})$ by

$$(6.8)\qquad Sw_{G.R}(\sigma) = \begin{cases} \nu_G(\sigma) + \sum_{\mathfrak{p}\in R} \rho_{G.\mathfrak{p}}(\sigma) - 1 & \text{if } \sigma\in G-\{e\}. \\ \sum_{\sigma\neq e} (-Sw_G(\sigma)) & \text{if } \sigma=e. \end{cases}$$

We give some basic properties of $Sw_{G.R}$.

Lemma(6.9). (1) $Sw_{G.R}\in C(G,\mathbf{Z})$.

(2) *If* $\sigma\in G-\{e\}$ *has an order prime to* ch(k), *we have*

$$Sw_{G.R}(\sigma) = 0.$$

Proof. (1) follows immediately from the definition, and (2) follows from (5.14) and the following facts:

(6.9.1) $\sigma\in G_{\tilde{\mathfrak{p}}}$ lies in $I_{\tilde{\mathfrak{p}}}$ if and only if $\nu_{\tilde{\mathfrak{p}}}(\sigma)>0$.

(6.9.2) $\rho_{\overline{G}_{\tilde{\mathfrak{p}}}}(\overline{e}) = 1+\delta_{\tilde{\mathfrak{p}}}$, where $\overline{e}$ is the identity of $\overline{G}_{\tilde{\mathfrak{p}}}$ (cf. SGA5X).

(6.9.3) If $\sigma\in G-\{e\}$ has an order prime to $\mathrm{ch}(\kappa(\mathfrak{p}))$, $\nu_{\tilde{\mathfrak{p}}}(\sigma)$ = 0 or 1.

Example(6.10) Let k be an algebraically closed field of characteristic 2, $A=k[[u,v]]$ and K be the quotient field of A. Fix an integer $n>0$, let L be the splitting field of the equation

$$T^2 - uT + v^n = 0,$$

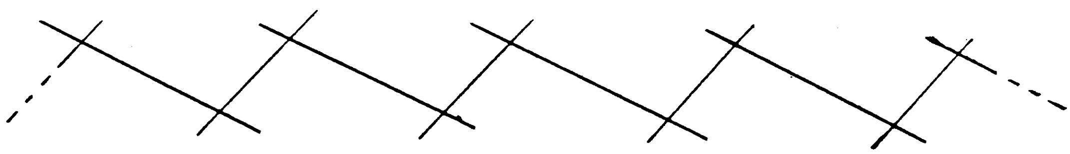
and let B be the normalization of A in L. Then, we see

$$B \simeq k[[t,s,v]]/(ts-v^n) \quad \text{with} \quad t+s=u.$$

The extension L/K is a cyclic extension of degree 2, and the non-trivial element σ of $G=\mathrm{Gal}(L/K)$ is determined by

$$t^\sigma = s \quad \text{and} \quad s^\sigma = t.$$

Then, we see that $\mathfrak{p}:=(u)\in P_A$ is the unique prime ramifying in the extension B/A and that (6.4) is satisfied if and only if n is even. So, we assume $n=2m$ for some integer $m>0$. Then, $\tilde{\mathfrak{p}}:=(t+s,t+v^m)\in P_B$ is the unique prime lying over $\mathfrak{p}$. Let $f : \mathfrak{X} \longrightarrow \mathrm{Spec}(B)$ be the minimal resolution. Then, we see that the special fiber E of f is a sequence of n-1 projective lines $E_1,\ldots,E_{n-1}$ as follows:



(Figure 1)

Moreover, E^σ consists of one point x which lies on the middle line E_m in the above chain and does not lie on any other line E_i $(i\neq m)$. The closure F of $\tilde{\mathfrak{p}}$ in $\mathfrak{X}$ intersects E at x and is a regular closed subscheme of $\mathfrak{X}$. Finally, σ acts on E as a symmetry with respect to the line F (cf. Figure 2). In a neighbourhood of x in $\mathfrak{X}$, v defines E_m and $w:= \frac{t}{v} + 1$ defines F, and the action of σ is described as follows:

$$\sigma(v)=v \quad \text{and} \quad \sigma(w)= w+w^2(1+w)^{-1}.$$

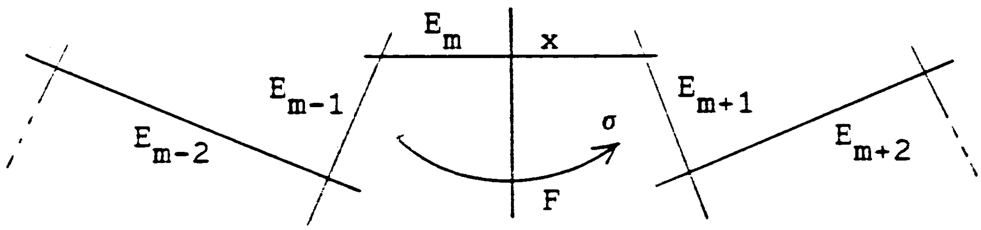
From these observations, we compute

$$\nu_x(\sigma) = 0 \text{ and } \operatorname{Tr}(\sigma^*)|H^*(E) = -2.$$

Consequently, we obtain

$$Sw_{G.S}(\sigma) = -2,$$

where $S=\{\mathfrak{p}\} \subset P_A$ (cf.(6.5)).



(Figure 2)

7. The Weil formula for an algebraic surface

Let k be an algebraically closed field and X be a proper normal surface over k with the function field K. Let L/K be a finite Galois extension with the Galois group G. Let Y be the integral closure of X in L. Fix a closed subscheme R in X such that the natural map $f : Y \longrightarrow X$ is etale over $U := X \backslash R$, and put $V = f^{-1}(U)$. We assume

(7.1) For any $\mathfrak{p} \in X_1$ and $\tilde{\mathfrak{p}}$ lying over $\mathfrak{p}$, $\kappa(\tilde{\mathfrak{p}})/\kappa(\mathfrak{p})$ is separable.

Fix $\mathfrak{p} \in X_1$ and choose $\tilde{\mathfrak{p}} \in Y_1$ lying over $\mathfrak{p}$. Let $G_{\tilde{\mathfrak{p}}} = \{\sigma \in G \mid \sigma(\tilde{\mathfrak{p}}) = \tilde{\mathfrak{p}}\}$. It is identified with $\mathrm{Gal}(L_{\tilde{\mathfrak{p}}}/K_{\mathfrak{p}})$, where $K_{\mathfrak{p}}$ (resp. $L_{\tilde{\mathfrak{p}}}$) is the quotient field of the completion of the local ring of X at $\mathfrak{p}$ (resp. Y at $\tilde{\mathfrak{p}}$). By (7.1), the condition (6.1) is satisfied for $L_{\tilde{\mathfrak{p}}}/K_{\mathfrak{p}}$ so that we have defined the Swan character $Sw_{G_{\tilde{\mathfrak{p}}}}$ for $G_{\tilde{\mathfrak{p}}}$ (cf.(6.3)). Then, we put

$$(7.2) \qquad Sw_{G.\mathfrak{p}} = \mathrm{Ind}_{G_{\tilde{\mathfrak{p}}}/G}(Sw_{G_{\tilde{\mathfrak{p}}}}).$$

Clearly, this definition does not depend on the choice of $\tilde{\mathfrak{p}}$.

Next, fix $x \in X_0$ and choose $y \in Y_0$ lying over x. Let $G_y = \{\sigma \in G \mid \sigma(y) = y\}$. It is identified with $\mathrm{Gal}(L_y/K_x)$, where K_x (resp. L_y) is the quotient field of the completion A_x (resp. B_y) of the local ring of X at x (resp. Y at y). Note that (7.1) implies that (6.4) is satisfied for B_y/A_x so that we have defined $Sw_{G_y, R_x} \in C(G_y, \mathbf{Z})$, where R_x is the set of all prime ideals in A_x lying over some element in R_1. Then, we put

$$(7.3) \qquad Sw_{G.x.R} = \mathrm{Ind}_{G_y/G}(Sw_{G_y.R_x}).$$

Clearly, this definition does not depend on the choice of y.

The following theorem, which is an immediate consequence of (0.2) and the definitions, is viewed as the two-dimensional version of the Weil formula for an algebraic curve (cf. SGA5X(5.1)).

Theorem(7.4). *For any prime number* $\ell \neq \mathrm{ch}(k)$, *we have the following equality in* $C(G, \mathbb{Q}_\ell)$,

$$\mathrm{Tr}_{H_c^*(V)} = \chi_c(U)\mathbb{Q}_\ell[G] - \sum_{\mathfrak{p}\in R_1} \chi_{\mathfrak{p}} \cdot \mathrm{Sw}_{G.\mathfrak{p}} + \sum_{x\in R_0} \mathrm{Sw}_{G.x.R} \quad ,$$

where $\mathbb{Q}[G] \in C(G, \mathbb{Q}_\ell)$ *comes from the regular representation of* G, *and*

$$\chi_c(U) = \sum_{i=0}^{4} (-1)^i \dim_{\mathbb{Q}_\ell} H_c^i(U_{et}, \mathbb{Q}_\ell),$$

$$\mathrm{Tr}_{H_c^*(V)}(\sigma) = \sum_{i=0}^{4} (-1)^i \mathrm{Tr}(\sigma^*) | H_c^i(V_{et}, \mathbb{Q}_\ell).$$

Corollary(7.5). *If a Sylow p-subgroup of* G *acts freely on* Y,

$$\mathrm{Tr}_{H_c^*(V)} = \chi_c(U)\mathbb{Q}_\ell[G].$$

This is an immediate consequence of (7.4) and (6.10.2).

Remark(7.6). The statement of (7.5) is proved by Deligne in case of arbitrary dimension (cf.[6]).

By (7.4), the image of $\sum_{x\in R_0} \mathrm{Sw}_{G.x.R}$ in $C(G, \mathbb{Q}_\ell)$ lies in the subset $K(G, \mathbb{Q}_\ell)$. So we propose the following.

Conjecture(7.7). Let B/A, G and $R \subset P$ be as §6. Then, for any prime number $\ell \neq \mathrm{ch}(k)$, the image of $\mathrm{Sw}_{G.R}$ in $C(G, \mathbb{Q}_\ell)$ lies in $K(G, \mathbb{Q}_\ell)$.

Remark(7.8). In view of (6.7), when B is regular and there is no $\tilde{\mathfrak{p}} \in P_B$ ramifying in B/A, this is conjectured by Serre [12].

Let X be as before and let U be a non-empty open subscheme of X and $R = X\backslash U$. Let $\mathcal{F}$ be a locally constant constructible sheaf of $\mathbb{F}_\ell$-vector space on U_{et} ($\ell \neq ch(k)$). Following [7], we introduce

Definition(7.9). $\mathcal{F}$ is weakly ramified along R if there exists a finite Galois etale covering $f : V \longrightarrow U$ such that $f^*\mathcal{F}$ is constant, and that $\tilde{f} : Y \longrightarrow X$ satisfies (7.1), where Y is the normalization of X in V and $\tilde{f}$ is the extension of f.

Now, assume that $\mathcal{F}$ is weakly ramified along R and fix a finite etale Galois covering $f : V \longrightarrow U$ as (7.9). Putting G=Aut(V/U), the stalk M of $\mathcal{F}$ at a geometric generic point of U is endowed with a canonical action ρ of G. Then, by [6], putting

$$\chi_c(U,\mathcal{F}) = \sum_{i=0}^{4}(-1)^i \dim_{\mathbb{F}_\ell} H_c^i(U_{et},\mathcal{F}),$$

we have the following formula

$$\chi_c(U,\mathcal{F}) = (1/\#G) \sum_{\sigma\in G_{\ell\text{-reg}}} \mathrm{Tr}_{H_c^*(V)}(\sigma)\cdot \mathrm{Tr}_M^{Br}(\sigma).$$

Here $G_{\ell\text{-reg}}$ is the subset of G consisting of all elements whose order is prime to ℓ, and for $\sigma \in G_{\ell\text{-reg}}$,

$$\mathrm{Tr}_M^{Br}(\sigma) = \sum_{\lambda} (\lambda,0,0,\ldots\ldots) \in W(\mathbb{F}_\ell),$$

where λ runs over all eigen values of $\rho(\sigma)$ in an algebraic closure of $\mathbb{F}_\ell$ and $W(\mathbb{F}_\ell)$ is the ring of Witt vectors with coefficient in $\mathbb{F}_\ell$. Combining this with (7.4), we obtain a formula

$$(7.10)\quad \chi_c(U,\mathcal{F}) = r_{\mathcal{F}}\cdot\chi_c(U) - \sum_{\mathfrak{p}\in R_1} \chi_{\mathfrak{p}}\cdot \mathrm{Sw}_{\mathfrak{p}}(\mathcal{F}) + \sum_{x\in R_0} \mathrm{Sw}_x^V(\mathcal{F}),$$

Here, $r_{\mathcal{F}}$ is the rank of $\mathcal{F}$, and for $\mathfrak{p}\in R_1$,

$$(7.11)\qquad Sw_{\mathfrak{p}}(\mathcal{F}) = (1/\#G)\sum_{\sigma\in G_{\ell\text{-reg}}} Sw_{G.\mathfrak{p}}(\sigma).Tr_M^{Br}(\sigma).$$

It is so called the Swan conductor of $\mathcal{F}$ at $\mathfrak{p}$, and it is known that $Sw_{\mathfrak{p}}(F)$ is an integer independent of V as (7.9) (cf.[7] and [13]). Finally, we put for $x\in R_0$,

$$(7.12)\quad Sw_x^V(\mathcal{F}) = (1/\#G)\sum_{\sigma\in G_{\ell\text{-reg}}} Sw_{G.x.R}(\sigma)\cdot Tr_M^{Br}(\sigma) \in \mathbb{Q}_\ell \text{ (cf.(7.3))}.$$

This formula is viewed as a two-dimensional version of the Grothendieck-Ogg-Shafarevich formula for an algebraic curve (cf. SGA5X(7.1)). Such a formula was first discovered in Laummon[7]: In the same situation, he defiend $Sw_x^{La}(\mathcal{F})\in\mathbf{Z}$ ($x\in R_0$) which depends only on the restriction of $\mathcal{F}$ to the henzelization of X at x, and he obtained the same formula as (7.10) except replacing $Sw_x^V(\mathcal{F})$ by $Sw_x^{La}(\mathcal{F})$. It is essentially defined as the total dimension of the space of vanishing cycles $R\psi_{\pi.x}(j_!\mathcal{F})$ ($j : U \longrightarrow X$ is the inclusion) for a "good" fibration $\pi : X \longrightarrow \mathbb{P}_k^1$, and eventually he proved that the definition does not depend on the choice of π.

In view of these facts, we propose the following

Conjecure(7.13). $Sw_x^V(\mathcal{F})$ does not depend on the choice of V as (7.9) and coincides with $Sw_x^{La}(\mathcal{F})$.

By the joint work with three other people (S.Bloch, K.Kato and T.Saito), the conjectures (7.7) and (7.13) have been solved affirmatively in the "algebraic case", namely, all objects involved come by completion from objects which are finite type over k. The proof will appear in the near future.

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