Beam Polarization in High Energy Electron Storage Rings

高エネルギー電子貯蔵リングにおけるビームの偏極

横谷　馨

8010

Beam Polarization in High Energy Electron Storage Rings

Kaoru YOKOYA

National Laboratory for High Energy Physics
Oho-machi, Tsukuba-gun, Ibaraki-ken, 305, Japan

ABSTRACT

In electron storage rings the beam tends to polarize spontaneously along the guiding magnetic field owing to the spin-flip synchrotron radiation. This process has been confirmed by observations in many storage rings. However, we encounter several problems when we try to utilize this polarization in physics experiments. The degree of polarization is generally low because of various magnet errors. Further depolarization is caused by extra magnets to be installed in order to rotate the spin to the longitudinal direction, which is necessary for physics experiments in colliders. Moreover, in a very high energy storage ring the depolarization due to the beam energy spread is expected to become serious. These problems are investigated theoretically in the present paper. The explanations of various depolarization mechanisms are given and the methods of the design of storage rings free from depolarization and the correction procedures are proposed.

Contents

§1. Introduction

The most important phenomenon that makes electron storage rings different from proton rings is synchrotron radiation. Its major effects on the beam dynamics, such as the loss of the beam energy and the damping of the orbital oscillations, can be explained by the classical electro-magnetism. The corrections to these effects due to the quantum theory are of the order of the photon energy divided by the beam energy, which is less than 10^{-5} in the storage rings, existing or being planned. However, there are some effects, even in macroscopic phenomena, which require quantum theoretical treatments. The most remarkable among them is the finite emittance of the beam. If the energy loss of the beam were continuous as in the classical theory, the beam would shrink to a point by the radiation damping. In practice, the energy loss occurs in the form of photon emission with finite energy, and the equation of the particle motion acquires a diffusion term which causes the finite emittance of the beam. Hence, the well-known formula[56)] of the equilibrium emittance contains Planck's constant.

The second quantum effect is the spontaneous polarization of the beam, which is the theme of the present paper. The potential energy of an electron in the guiding magnetic field $\vec{B}$ of a storage ring has a term $-\vec{\mu}\cdot\vec{B}$, with $\vec{\mu}$ being the magnetic moment, which produces the energy level splitting between the spin up and down states. Electrons become polarized antiparallel to the magnetic field and positrons parallel due to the transition between these states. Because the transition is magnetic dipole in contrast to the electric dipole transition in the usual radiation, it is much rarer by more than ten orders of magnitude. This process was predicted by Ternov, Lokutov and Korovina in early

1960's[1]. Detailed calculation of the transition probability was done by Sokolov and Ternov[2]. The build-up time of the polarization is given by

$$\tau_p^{-1} = \frac{5\sqrt{3}}{8}\frac{e^2\gamma^5\hbar}{m^2c^2}\frac{1}{2\pi R}\oint\frac{ds}{|\rho(s)|^3} \quad . \qquad (1.1)$$

Here, e is the charge of electron, m the rest mass, c the velocity of light, $\hbar$ Planck's constant, γ the beam energy divided by mc^2, $2\pi R$ the ring circumference, s the coordinate length along the orbit and $\rho(s)$ the radius of curvature of the orbit. The integral over one revolution is denoted by $\oint$. Parameters of existing storage rings of the beam energy between several hundreds of MeV to several tens of GeV give the value of τ_p from several minutes to several hours. This spontaneous polarization was observed in early 1970's in storage rings such as ACO[3], VEPP-2[4] and SPEAR[5].

Since the probability of the opposite transition from the state with negative $-\vec{\mu}\cdot\vec{B}$ to positive is not zero because the orbit motion is coupled with the spin motion, the equilibrium polarization is not 100 % but is given by

$$P_\infty = 8/5\sqrt{3} = 92\ \% \quad . \qquad (1.2)$$

However, in actual storage rings the equilibrium polarization does not reach the ideal value 92 %. One reason is that the equilibrium spin direction may not be always parallel to the magnetic field because of magnet imperfections and misalignments. The other is that the electron spin is diffused by orbit oscillations excited by the ordinary (spin non-flip) synchrotron radiations. A general formula of the build-up of the polarization including these effects was first derived by Derbenev and Kondratenko[6]. The equation of the build-up of the polarization P is

$$\frac{dP}{dt} = -\frac{5\sqrt{3}}{8}\frac{e^2\gamma^5\hbar}{m^2c^2}(\alpha_+ P - \alpha_-) \tag{1.3}$$

with

$$\alpha_+ = \frac{1}{2\pi R}\oint \frac{ds}{|\rho(s)|^3}\left[1 - \frac{2}{9}(\vec{n}\cdot\vec{e}_v)^2 + \frac{11}{18}\left|\gamma\frac{\partial\vec{n}}{\partial\gamma}\right|^2\right] \tag{1.4}$$

and

$$\alpha_- = \frac{1}{2\pi R}\oint \frac{ds}{|\rho(s)|^3}\ \vec{e}_B\cdot\left(\vec{n} - \gamma\frac{\partial\vec{n}}{\partial\gamma}\right) \quad . \tag{1.5}$$

Here, $\vec{e}_v$ is the unit vector along the motion of the electron and $\vec{e}_B$ is the unit vector along the transverse component of the magnetic field, $\vec{B} - (\vec{e}_v\cdot\vec{B})\vec{e}_v$. If the synchrotron radiation is absent, the classical motion of the spin in each magnet is merely a rotation. Hence, the spin motion of an electron on the design orbit from s to s + $2\pi R$ can be expressed as a rotation, whose axis is denoted by $\vec{n}(s)$ in the expressions above. It coincides with the direction of the equilibrium polarization. The spin-orbit coupling vector $\gamma\partial\vec{n}/\partial\gamma$ (once called spin chromaticity) is the change of the spin direction through an orbit oscillation excited by one single (spin non-flip) radiation.

The degree of equilibrium polarization is given by

$$P_\infty = \frac{8}{5\sqrt{3}}\frac{\alpha_-}{\alpha_+} \quad . \tag{1.6}$$

In a planar storage ring without magnet errors (in this paper we call a ring planar if it does not contain vertical bending magnets), $\vec{n}$ and $\vec{e}_B$ are vertical everywhere around the ring and $\gamma\partial\vec{n}/\partial\gamma = 0$. Hence, $\alpha_+ = \alpha_- = 1$ and Eq. (1.6) reduces to Eq. (1.2).

The number of spin precession ν during one revolution is called spin tune. If it is close to a resonant value

$$\nu = n_1 + n_2\nu_x + n_3\nu_y + n_4\nu_s \quad , \tag{1.7}$$

where ν_x, ν_y and ν_s are the tunes of horizontal betatron oscillation, vertical betatron oscillation and synchrotron oscillation, respectively and n_1, n_2, n_3 and n_4 are integers, then $\gamma\partial\vec{n}/\partial\gamma$ and/or the difference between $\vec{n}$ and $\vec{e}_B$ become large and one loses polarization. The spin tune and the beam energy E are related in a very simple way

$$\nu = \gamma a = E/0.44 \text{ GeV} \tag{1.8}$$

in planar rings, where a = 0.00116 is the coefficient of the anomalous magnetic moment. Therefore, resonances exist every 0.44 GeV even if one restricts to the resonances with $n_2 = n_3 = n_4 = 0$. The degree of polarization is a very complicated function of the beam energy.

A. Chao[7)] gave an intuitive derivation of eq. (1.6) and developed a computer code SLIM[8)] which calculates P_∞ by a matrix formalism using linear approximations of betatron and synchrotron oscillations. It is now widely used in order to check analytical theories, to design storage rings and to study experimental results. J. Kewisch[9)] developed a computer tracking program SITROS which incorporated the effects of non-linear oscillations.

As the desire to make use of the radiative polarization for high energy physics experiments has increased, various difficulties have been discovered theoretically and the breakthroughs have been studied. Theoretical progress in this field during these several years is quite remarkable. Prominent problems are;

(a) Planar storage rings give vertical polarization only, although the polarization interesting from the point of view of high energy physics is longitudinal. Designs of storage rings with spin rotators which rotate the spin to the longitudinal direction at the collision points had been proposed by many authors[10,11,12,13]. Spin rotators, however make storage rings non-planar inevitably. It was pointed out that these designs gave large spin-orbit coupling vectors and did not give meaningful polarization even at energies far from spin resonances. This problem was solved by A. Chao and the present author[14] by the method now called "spin matching". They showed that the spin motion and the orbit motion can be decoupled if one imposes some linear constaints on the strengths of quadrupole magnets like the matching of the orbit optics.

(b) Depolarization occurs even in planar rings because of magnet errors. It becomes more severe, the higher the beam energy. The problem is whether it is possible to design storage rings insensitive to magnet errors and whether we can correct the polarization in a similar way to the orbit corrections done in most storage rings. K. Steffen[28] and the present author[29] independently solved this problem. Their methods are a little different from each other but they both take the way to control the Fourier harmonics of the perturbation force. The actual correction of polarization in machine operation was done by the PETRA group[31] successfully.

(c) Since the beam energy in storage rings has a spread, the spin tune has a spread, too, as can be seen from Eq. (1.8). It is proportional to $E^2/\sqrt{R}$. The ring radii are roughly proportional to E^2 in existing (or planned) electron storage rings. Hence, the spread in spin tune becomes large in high energy machines. It was pointed out by Möhl and Montague[32] that if the spread in the beam energy is a significant fraction of the resonance spacing 0.44 GeV, we cannot expect high degree of polarization. The expected polarization was calculated by Derbenev, Kondratenko and Skrinsky[33], Biscari and Buon[34,36], Kondratenko and Montague[35] and the present author.[37]. Cures for this problem have been investigated by several authors. Biscari, Buon and Montague[39] studied the usage of non-linear wigglers. The so-called "Siberian snake[40]" may be another candidate. The present author[46] proposed a method to eliminate the spin tune spread without changing the energy spread. These cures, however, have more or less defects for the application to the actual storage rings. Further extensive studies for the cures to this problems are required.

(d) Since the direction of polarization at a given point on the ring is uniquely determined once the layout of the ring is fixed, it is impossible to observe collisions with different combinations of the spin states by a single detector. A design was proposed for HERA by Steffen[48] in which the orbit is deflected in large aperture magnets to invert the direction of the polarization. This method, when applied to e^+e^- rings, has the problems that the beam energy is restricted within a very narrow range and that it is impossible to collide beams with helicities opposite to each other. The Novosivirsk group[54] made an experiment in VEPP-2M to invert the spin by crossing a spin resonance caused by an artificial perturbation with a successful result. This method makes feasible the collision with any combination of helicities, but they pointed out that the synchrotron radiation during the process of spin-flip may cause a depolarization at an energy much higher than that of VEPP-2M. The present author derived a formula to estimate this effect.[55)]

(e) The beam-beam interaction at the collision points may depolarize the beam. How serious is it ?

(f) Development of non-destructive rapid polarization monitors.

(g) Methods to polarize electron beams artificially which are necessary if the depolarization due to the spin tune spread is large.

(h) Beam diagnostics using polarization.

In this paper we discuss the four problems (a) through (d) in the following chapters. In each chapter more detailed review of the problem is presented and the theory developed by the present author is discussed. The basic equations of classical spin motion are derived in Appendix A. Appendix B is devoted to the derivation of the mathematical formulae used in §5.

§2. Spin Matching

2.1) Diffusive Depolarization

Although the natural direction of the radiative polarization is transverse with respect to the particle motion, longitudinal polarization is desired in experiments of high energy physics. In order to make collisions in eigenstates of helicity it is necessary to rotate the spin to the longitudinal direction before the beam comes to the collision point and to restore the spin direction after passing it. It can be achieved by "spin rotators" which are suitable combinations of horizontal and vertical bending magnets and/or solenoids. Several storage ring designs with spin rotators had been proposed in this course such as CHEEP[10)], LEP[11)], TRISTAN[12)] and HERA[13)].

However, it was soon pointed out theoretically that only poor polarizations could be expected from these designs of storage rings.

There are three mechanisms of the depolarization of stored electron beams, namely

(1) inverse spin-flip radiation,

(2) non-parallelism of the spin and the magnetic field, and

(3) spin non-flip radiation plus spin-orbit coupling.

The depolarization due to the first mechanism is not important, since at most it amounts to $1 - 8/5\sqrt{3} = 8$ % as stated in §1. If the magnetic field is not vertical somewhere in the ring due to magnet errors or spin rotators, the depolarization of the second mechanism occurs because spin-flip synchrotron radiations there tend to turn the spin towards the wrong direction. However, since such regions are not given much weight in the whole ring even if it has spin rotators, the depolarization due to the second mechanism is not serious. All the designs cited above take into account these two mechanisms only.

The depolarization due to the third mechanism proceeds as follows. In the absence of synchrotron radiation the spin motion is a simple precession. Suppose that an electron which had been circulating on the design orbit emitted an ordinary (spin non-flip) radiation at a point in the ring. The electron begins to undergo betatron and synchrotron oscillations. It experiences the magnetic field different from the one on the design orbit due to quadrupole magnets and the precession becomes very complicated. After a few radiation damping times of the order of milliseconds, the orbit oscillation dies away and the spin motion becomes a simple precession again. However, it is different from the extrapolation of the precession before the radiation by the amount proportional to the integral of the extra magnetic field experienced during the oscillation. Because the damping time is much shorter than the polarization time by several orders of magnitude, this process may be described as if the spin phase changed at the instant of radiation. The magnitude of the jump is the photon energy times a certain function of the location in the ring, called spin-orbit coupling vector $\gamma\partial\vec{n}/\partial\gamma$. This effect brings about a diffusion term in the equation of spin motion because the radiation is stochastic. From now on we call the depolarization of this type "diffusive depolarization". The notion of spin-orbit coupling vector was first introduced in the pioneering work of Ya.S. Derbenev and A.M. Kondratenko[6], but its importance was not recognized for a long time. One reason is that their definition and its physical implication were not clear. The other may be the lack of a simple formula to evaluate $\gamma\partial\vec{n}/\partial\gamma$.

Later it was shown by A. Chao and the present author[14] that the diffusive depolarization can be avoided if the strengths of the quadrupole magnets satisfy certain linear relations. Finding proper combinations of quadrupole strengths satisfying these conditions is now called "spin matching". In their paper the spin matching conditions are introduced in a somewhat heuristic manner but here we derive them in a more systematic way. The basic equation of the spin motion is summarized in the appendix A.

2.2) Definitions of the Spin-Orbit Coupling Vector

There are two currently used definitions of the spin-orbit coupling vector. The original definition by Derbenev and Kondratenko[6] is more general but the second definition by Chao[7] is physically more transparent, although restricted to the linear case. Both definitions lead to the same polarization formula. The latter is used in this chapter but the former is necessary for the higher order calculation in Chap. 4.

The original definition is as follows. The orbit motion of an electron can be described by six variables x, x', y, y', τ and ε, where x and y are the horizontal and vertical displacements from the design orbit, x' and y' their derivatives with respect to s, $\varepsilon = (E-E_0)/E_0$ is the relative energy deviation from the nominal energy E_0 and τ the variable cannonically conjugate to ε. The rigorous definitions of these quantities are given in the appendix A. We choose $\theta = s/R$, the generalized aximuth, as the indepedent variable. Then the equation of the classical spin motion without synchrotron radiation can be written in the form

$$\frac{d\vec{s}}{d\theta} = \vec{\Omega}(x,x',y,y',\tau,\varepsilon,\theta) \times \vec{s} \quad , \tag{2.1}$$

where $\vec{s}$ is the spin vector seen in the rest frame of the electron. The orbit variables may be written by action-angle variables I_x, ψ_x, I_y, ψ_y, I_s and ψ_s. Hence, $\vec{\Omega}$ can be considered as a function of I, ψ and θ, where I and ψ symbolically denote I_x, I_y, I_s and ψ_x, ψ_y, ψ_s. It is a periodic function of ψ_x, ψ_y, ψ_s and θ; i.e.,

$$\vec{\Omega}(I,\psi,\theta) = \vec{\Omega}(I,\psi+2\pi,\theta) = \vec{\Omega}(I,\psi,\theta+2\pi) \quad . \tag{2.2}$$

The solutions of the orbit motion are given by

$$\psi_i = \nu_i\theta + \psi_{io} \qquad (i = x,y,s) \quad , \tag{2.3}$$

and I_i = constant, where ψ_{io} is the initial values, ν_x and ν_y the horizontal and vertical oscillation tunes and ν_s the synchrotron oscillation tune. It was shown by Derbenev and Kondratenko that Eq. (2.1) has a unique solution $\vec{s} = \vec{n}(I,\psi,\theta)$ that satisfies the periodicity condition

$$\vec{n}(I,\psi,\theta) = \vec{n}(I,\psi+2\pi,\theta) = \vec{n}(I,\psi,\theta+2\pi) \quad , \tag{2.4}$$

unless ν's do not satisfy the resonance condition Eq. (1.7). The vector $\vec{n}$ may also be written as a function of x,x',y,y',τ and ε. Now the original definition of the spin-orbit coupling vector $\gamma\partial\vec{n}/\partial\gamma$ is merely a partial derivative of $\vec{n}$ with respect to ε; $\gamma\partial\vec{n}/\partial\gamma = \partial\vec{n}/\partial\varepsilon$. The partial derivative reflects the fact that at the instant of (spin nonflip) radiation only ε changes but all other variables remain constant.

A. Chao defined $\gamma\partial\vec{n}/\partial\gamma$ as follows. If an electron is circulating on the design orbit, its spin motion can be described by

$$\frac{d\vec{s}}{d\theta} = \vec{\Omega}_0(\theta) \times \vec{s}$$

with

$$\vec{\Omega}_0(\theta) = \vec{\Omega}(0,0,0,0,0,0,\theta) \quad . \tag{2.5}$$

Because Eq. (2.5) shows that the spin motion is a product of infinitesimal rotations, the effect of one revolution from θ to $\theta+2\pi$ may be expressed as a rotation whose axis we denote $\vec{n}_0(\theta)$. Since $\vec{\Omega}_0$ is periodic, $\vec{n}_0$ is periodic, too. It is a particular solution of Eq. (2.5). Let us denote two other solutions by $\vec{m}_0(\theta)$ and $\vec{\ell}_0(\theta)$. We can choose them so that $(\vec{n}_0,\vec{m}_0,\vec{\ell}_0)$ make right-handed orthonormal basis. (Note that Eq. (2.5) does not affect the orthonormality.) Any solution to Eq. (2.5) may be expressed as a linear combination of $\vec{n}_0$, $\vec{m}_0$ and $\vec{\ell}_0$ with constant coefficients. Now, suppose that an electron emitted a photon with the energy $-E_0\Delta\varepsilon$ $(\Delta\varepsilon < 0)$ at $\theta = \theta_0$ and the spin motion before the emission was $\vec{n}_0(\theta)$. The orbit motion is disturbed by the radiation and the spin motion is described by Eq. (2.1) instead of Eq. (2.5). After a time much longer than the orbit damping time, Eq. (2.5) becomes valid again but the solution is no longer $\vec{n}_0(\theta)$. It is a linear combination of $\vec{n}_0$, $\vec{m}_0$ and $\vec{\ell}_0$;

$$\begin{aligned} \vec{s}(\theta) &= (1+a_1)\vec{n}_0(\theta) + a_2\vec{m}_0(\theta) + a_3\vec{\ell}_0(\theta) \\ &\equiv \vec{n}_0(\theta) + \Delta\vec{s}(\theta) \quad . \end{aligned} \tag{2.6}$$

Because the orbit damping time is much shorter than the polarization time by several orders of magnitude, this process is equivalent to the sudden change of spin vector by $\Delta\vec{s}(\theta_0)$ at $\theta = \theta_0$. Because $\Delta\varepsilon$ is very small, $\Delta\vec{s}$ is proportional to $\Delta\varepsilon$. Then Chao's definition is

$$\left[\gamma \frac{\partial\vec{n}}{\partial\gamma}\right]_{\theta=\theta_0} = -\frac{\Delta s(\theta_0)}{\Delta\varepsilon} \quad . \tag{2.7}$$

Note the minus sign which comes from the fact that $\Delta\vec{s}$ is the jump of the solution but, in the notation of Derbenev and Kondratenko, $\partial\vec{n}/\partial\gamma$ is the change of the basis vector. It is not adequate to use the notation $\gamma\partial\vec{n}/\partial\gamma$ in this definition because the right hand side of Eq. (2.7) is not a differentiation. In Chao's sense $\gamma\partial\vec{n}/\partial\gamma$ is a function of θ only. However, it can be shown that $\left[\gamma\partial\vec{n}/\partial\gamma\right]_{x,x',y,y',\tau,\varepsilon,\theta}$ of the original definition agrees with Chao's definition in the limit of $x = x' = y = y' = \tau = \varepsilon = 0$.

2.3) Derivation of the Spin-Orbit Coupling Vector

In this section we derive $\gamma\partial\vec{n}/\partial\gamma$ using Chao's definition. Let us rewrite the equation of spin motion Eq. (2.1) in the presence of orbit oscillations as

$$\frac{d\vec{s}}{d\theta} = \left[\vec{\Omega}_0(\theta) + \vec{\omega}(x,x',y,y',\varepsilon,\theta)\right] \times \vec{s} \tag{2.8}$$

and retain the terms of $\vec{\omega}$ linear in x,x',y,y' and ε. Then

$$\vec{\Omega}_0 = -\gamma a\left(\frac{\vec{e}_y}{\rho_x} - \frac{\vec{e}_x}{\rho_y}\right) \cdot R \tag{2.9}$$

and

$$\vec{\omega} = \left[\varepsilon\left(\frac{\vec{e}_y}{\rho_x} - \frac{\vec{e}_x}{\rho_y}\right) - (\gamma a+1)(xG_x\vec{e}_y - yG_y\vec{e}_x)\right] \cdot R \quad , \tag{2.10}$$

where $\vec{e}_x$ and $\vec{e}_y$ are unit vectors along horizontal and vertical axes, $1/\rho_x$ and $1/\rho_y$ the curvatures of the orbit and G_x and G_y the focussing functions defined in the appendix A. Eq. (2.9) and (2.10) are derived in the appendix A. We have ignored the small terms proportional to x'

or y' in bending magnets.

Two of the orthonormal solutions to the unperturbed equation (2.5), $\vec{m}_0$ and $\vec{\ell}_0$, can be combined into a complex vector $\vec{k}_0 = \vec{m}_0 + i\vec{\ell}_0$. It has the quasi-periodicity

$$\vec{k}_0(\theta+2\pi) = e^{i\mu}\,\vec{k}_0(\theta) \quad , \tag{2.11}$$

where μ is 2π times the spin tune ν.

Now, let us expand the solution to Eq. (2.8) in terms of $\vec{n}_0$ and $\vec{k}_0$ and assume that the deviation from $\vec{n}_0$ is small. Then we have

$$\vec{s} = \vec{n}_0(\theta) + \mathrm{Re}(\zeta(\theta)\vec{k}_0^*(\theta)) \quad , \tag{2.11}$$

where $\zeta(\theta)$ is a complex function to be found and the asterisk denotes complex conjugate. Eq. (2.8) reduces to

$$\frac{d\zeta}{d\theta} = -\,i\vec{\omega}(x,x',y,y',\varepsilon,\theta)\cdot\vec{k}_0(\theta) \tag{2.12}$$

up to the terms linear in ζ.

Just before the photon emission at $\theta = \theta_0$, $\zeta = 0$. Hence,

$$\zeta = \zeta(\theta;\ \theta_0) = -\,i\int_{\theta_0}^{\theta} \vec{\omega}\cdot\vec{k}_0\, d\theta \quad . \tag{2.13}$$

After a time much longer than the damping time, ζ approaches to a constant $\zeta(\infty;\ \theta_0)$ and, therefore, $\Delta\vec{s}$ defined in Eq. (2.6) is given by

$$\Delta\vec{s}(\theta_0) = \mathrm{Re}(\zeta(\infty;\ \theta_0)\vec{k}_0^*(\theta_0)) \tag{2.14}$$

Now

$$\left[\gamma\,\frac{\partial\vec{n}}{\partial\gamma}\right]_\theta = -\,\frac{1}{\Delta\varepsilon}\,\mathrm{Re}(\zeta(\infty;\ \theta)k_0^*(\theta)) \quad , \tag{2.15}$$

Let us perform the integral in Eq. (2.13) with $\theta = \infty$. The orbit motion after the emission of a photon with the energy $-E_0\Delta\varepsilon$ at $\theta = \theta_0$ is given by

$$x = x_\beta + \varepsilon\eta_x \quad , \qquad y = y_\beta + \varepsilon\eta_y \quad , \tag{2.16}$$

$$\varepsilon = \frac{1}{2}\Delta\varepsilon \cdot e^{-\lambda_s(\theta-\theta_0)} \sum_{\pm} a_{\pm s}(\theta_0)\exp\left[\pm i\Psi_s(\theta)\right]$$

$$x_\beta = \frac{1}{2}\Delta\varepsilon \cdot e^{-\lambda_x(\theta-\theta_0)} \sqrt{\beta_x(\theta)} \sum_{\pm} a_{\pm x}(\theta_0)\exp\left[\pm i\Psi_x(\theta)\right] \tag{2.17}$$

$$y_\beta = \frac{1}{2}\Delta\varepsilon \cdot e^{-\lambda_y(\theta-\theta_0)} \sqrt{\beta_y(\theta)} \sum_{\pm} a_{\pm y}(\theta_0)\exp\left[\pm i\Psi_y(\theta)\right]$$

with

$$a_{\pm s}(\theta) = \exp\left[\mp i\Psi_s(\theta)\right] \quad ,$$

$$a_{\pm i}(\theta) = \frac{1}{\sqrt{\beta_x}} \exp\left[\mp i\Psi_i(\theta)\right]\cdot\left[-\eta_x \pm i(\eta_x'\beta_x + \eta_x\alpha_x)\right] \qquad (i = x,y) \quad . \tag{2.18}$$

Here the notations are:

$\beta_x,\beta_y,\alpha_x,\alpha_y$ = betatron functions,

$\eta_x,\eta_y,\eta_x',\eta_y'$ = dispersion functions and their derivatives with respect to s,

Ψ_x,Ψ_y = phase functions of betatron oscillation,

$\Psi_s(\theta) = \nu_s\theta$ = phase function of synchrotron oscillation,

$\lambda_x,\lambda_y,\lambda_s$ = damping constants of oscillations.

These equations can be derived from the expressions of betatron and synchrotron oscillations

$$x_\beta = e^{-\lambda_x\theta} \sum_{\pm} C_{\pm x} \exp\left[\pm i\Psi_x(\theta)\right] \quad ,$$

$$y_\beta = e^{-\lambda_y\theta} \sum_{\pm} C_{\pm y} \exp\left[\pm i\Psi_y(\theta)\right] \quad ,$$

$$\varepsilon = e^{-\lambda_s\theta} \sum_{\pm} C_{\pm s} \exp\left[\pm i\Psi_s(\theta)\right] \quad ,$$

by putting the initial conditions

$$x = x' = y = y' = d\varepsilon/d\theta = 0 \quad \text{and} \quad \varepsilon = \Delta\varepsilon$$

at $\theta = \theta_0$ to find the constants $C'_{\pm}$s, ignoring the derivatives of the damping factors. Putting (2.16) and (2.17) into (2.10), we find

$$\vec{\omega} = \frac{1}{2} \Delta\varepsilon \sum_j a_j(\theta_0)\vec{w}_j(\theta)e^{-\lambda_j(\theta-\theta_0)} \quad , \tag{2.19}$$

summed over $j = \pm x$, $\pm y$ and $\pm s$, with

$$\vec{w}_{\pm x} = - R(\gamma a+1)G_x\sqrt{\beta_x} \exp(\pm i\Psi_x)\vec{e}_y \quad ,$$

$$\vec{w}_{\pm y} = - R(\gamma a+1)G_y\sqrt{\beta_y} \exp(\pm i\Psi_y)\vec{e}_x \quad , \tag{2.20}$$

$$\vec{w}_{\pm s} = R\left[\frac{\vec{e}_y}{\rho_x} - \frac{\vec{e}_x}{\rho_y} - (\gamma a+1)(\eta_x G_x\vec{e}_y - \eta_y G_y\vec{e}_x)\right]\exp(\pm i\Psi_s) \quad .$$

These $\vec{w}_j$'s have the quasi-periodicity

$$\vec{w}_j(\theta+2\pi) = \exp(i\mu_j)\vec{w}_j(\theta) \quad (j = \pm x,\pm y,\pm s) \tag{2.21}$$

where $\mu_{\pm j} = \pm 2\pi\nu_j$ $(j = x,y,s)$.

Now, we can perform the integral in Eq. (2.13) and find $\gamma\partial\vec{n}/\partial\gamma$ as follows.

$$\left(\gamma \frac{\partial\vec{n}}{\partial\gamma}\right)_{\theta_0} = -\frac{1}{\Delta\varepsilon} \mathrm{Re}\left[\zeta(\infty;\ \theta_0)\vec{k}_0^*(\theta_0)\right]$$

$$= \frac{1}{\Delta\varepsilon} \mathrm{Re}\left[i\ \vec{k}_0^*(\theta_0) \int_{\theta_0}^{\infty} \vec{\omega}\cdot\vec{k}_0(\theta)d\theta\right]$$

$$= \frac{1}{2} \sum_{j=\pm x,\pm y,\pm s} \mathrm{Re}\ i\ \vec{k}_0^*(\theta_0)a_j(\theta_0) \int_{\theta_0}^{\infty} \vec{w}_j(\theta)\cdot\vec{k}_0(\theta)e^{-\lambda_j(\theta-\theta_0)}d\theta$$

$$= \frac{1}{2} \sum_j \mathrm{Re}\ i\ \vec{k}_0^*(\theta_0) a_j(\theta_0) \sum_{n=0}^{\infty} \int_{\theta_0}^{\theta_0+2\pi} \vec{w}_j(\theta+2\pi n) \cdot \vec{k}_0(\theta+2\pi n)$$

$$e^{-\lambda_j(\theta+2\pi n-\theta_0)} d\theta$$

$$= \frac{1}{2} \sum_j \mathrm{Re}\ i\ \vec{k}_0^*(\theta_0) a_j(\theta_0) \sum_{n=0}^{\infty} e^{i(\mu+\mu_j)n-2\pi\lambda_j n}$$

$$\times \int_{\theta_0}^{\theta_0+2\pi} \vec{w}_j(\theta) \cdot \vec{k}_0(\theta) e^{-\lambda_j(\theta-\theta_0)} d\theta$$

$$= \frac{1}{2} \sum_j \mathrm{Re} \frac{i\ \vec{k}_0^*(\theta_0) a_j(\theta_0)}{1 - \exp[i(\mu+\mu_j) - 2\pi\lambda_j]}$$

$$\times \int_{\theta_0}^{\theta_0+2\pi} \vec{w}_j(\theta) \cdot \vec{k}_0(\theta) e^{-\lambda_j(\theta-\theta_0)} d\theta \quad .$$

Because $\lambda_j << 1$, we may neglect them and find

$$\left(\gamma \frac{\partial \vec{n}}{\partial \gamma}\right)_\theta = \frac{1}{2} \sum_j \mathrm{Re} \frac{i\ k_0^*(\theta) a_j(\theta)}{1 - \exp\ i(\mu+\mu_j)} \int_\theta^{\theta+2\pi} \vec{w}_j \cdot \vec{k}_0\ d\theta \quad . \tag{2.22}$$

Substituting $\vec{w}_j$ and a_j with eqs. (2.18) and (2.20), we finally get

$$\gamma \frac{\partial \vec{n}}{\partial \gamma} = \frac{1}{2} \mathrm{Re} \left[\vec{k}_0^*(\theta) \cdot (D_x + D_{-x} + D_y + D_{-y} + D_s + D_{-s}) \right] \tag{2.23}$$

with

$$D_{\pm x}(\theta) = \frac{i}{1 - \exp\ i(\mu \pm \mu_x)} \frac{e^{\mp i\Psi_x(\theta)}}{\sqrt{\beta_x(\theta)}} \left[-\eta_x \pm i(\eta_x' \beta_x + \eta_x \alpha_x) \right]_\theta$$

$$\times \int_\theta^{\theta+2\pi} \left[-(\vec{k}_0 \cdot \vec{e}_y)(\gamma a+1) G_x \sqrt{\beta_x}\ e^{\pm i\Psi_x} \right]_{\theta'} R d\theta' \tag{2.24}$$

$$D_{\pm y}(\theta) = \frac{i}{1 - \exp\ i(\mu \pm \mu_y)} \frac{e^{\mp i\Psi_y(\theta)}}{\sqrt{\beta_y(\theta)}} \left[-\eta_y \pm i(\eta_y' \beta_y + \eta_y \alpha_y) \right]_\theta$$

$$\times \int_\theta^{\theta+2\pi} \left[(\vec{k}_0 \cdot \vec{e}_y)(\gamma a+1) G_y \sqrt{\beta_y}\ e^{\pm i\Psi_y} \right]_{\theta'} R d\theta' \tag{2.25}$$

$$D_{\pm s}(\theta) = \frac{i}{1 - \exp i(\mu \pm \mu_s)} e^{+i\nu_s\theta}$$

$$\times \int_{\theta}^{\theta+2\pi} \vec{k}_0(\theta')\cdot\left[\frac{\vec{e}_y}{\rho_x} - \frac{\vec{e}_x}{\rho_y} - (\gamma a+1)(\eta_x G_x \vec{e}_y - \eta_y G_y \vec{e}_x)\right]_{\theta'} e^{\pm i\nu_s\theta'} R d\theta' \tag{2.26}$$

One can easily see that $\gamma\partial\vec{n}/\partial\gamma$ is a periodic function of θ, although D's are not.

2.4) Spin Matching Conditions

If $\gamma\partial\vec{n}/\partial\gamma = 0$ all over the ring, or

$$D_j(\theta) = 0 \quad (j = \pm x, \pm y, \pm s) \quad \text{for every } \theta \tag{2.27}$$

then we are free from the diffusive depolarization.

Let us carefully look into the structure of D's defined in the previous section. They are all in the form

$$D_{\pm j}(\theta) = R_{\pm j} \cdot H_{\pm j}(\theta) \int_{\theta}^{\theta+2\pi} F_{\pm j}(\theta')d\theta' \quad . \tag{2.28}$$

with

$$R_{\pm j} = \frac{i}{1 - \exp i(\mu \pm \mu_j)} \quad , \qquad (j = x,y,s) \quad , \tag{2.29}$$

$$H_{\pm j}(\theta) = \frac{e^{\mp i\Psi_j(\theta)}}{\sqrt{\beta_j(\theta)}}\left[-\eta_j \pm i(\eta_j'\beta_j + \eta_j\alpha_j)\right]_{\theta} \quad (j = x,y) \quad , \tag{2.30}$$

$$H_{\pm s}(\theta) = e^{\mp i\nu_s\theta} \quad , \tag{2.31}$$

$$F_{\pm x}(\theta) = \int_{\theta}^{\theta+2\pi} \left[(-\vec{k}_0\cdot\vec{e}_y)(\gamma a+1)G_x\sqrt{\beta_x}\, e^{\pm i\Psi_x}\right]_{\theta'} R d\theta' \quad , \tag{2.32}$$

$$F_{\pm y}(\theta) = \int_{\theta}^{\theta+2\pi} \left[(+\vec{k}_0\cdot\vec{e}_x)(\gamma a+1)G_y\sqrt{\beta_y}\; e^{\pm i\Psi_y}\right]_{\theta'} Rd\theta' \quad , \tag{2.33}$$

$$F_{\pm s}(\theta) = \int_{\theta}^{\theta+2\pi} \vec{k}_0(\theta')\cdot\left[\frac{\vec{e}_y}{\rho_x} - \frac{\vec{e}_x}{\rho_y} - (\gamma a+1)(\eta_x G_x \vec{e}_y - \eta_y G_y \vec{e}_x)\right]_{\theta'}$$

$$\times\; e^{\pm i\nu_s\theta'}\; Rd\theta' \quad . \tag{2.34}$$

As is seen from the derivation, the argument θ, which had been written as θ_0 during the derivation, shows the location where the photon is emitted. The function $H_{\pm j}(\theta)$ expresses the amplitude of the orbit oscillation in the j-th degree of freedom excited by the radiation of unit energy. The probability of emission itself does not appear in the expressions of D's. In Derbenev and Kondratenko's formula, (1.4) and (1.5), $\gamma\partial\vec{n}/\partial\gamma$ is always accompanied by $1/|\rho|^3$, which is related to the probability of photon emission.

The factor with the integral is the sum of the effects of the j-th oscillation of unit amplitude on the spin until it is damped. The resonant factor $R_{\pm j}$, which becomes infinity when $\mu\pm\mu_j = 2\pi$ times integer, or $\nu\pm\nu_j$ = integer, comes from the process to reduce the integration range to $(\theta, \theta+2\pi)$.

Now we find that the diffusive depolarization may be avoided if either one of the following three conditions is satisfied at each point θ in the ring;

(1) no photon is emitted at θ.

(2) no oscillation is excited even if a photon is emitted.

(3) the sum of the effects of oscillation, if any, on the spin vanishes. (spin-orbit decoupling)

The first condition indicates that D's need not vanish except in the bending mangets. The second shows that $F_j(\theta)$ $(j = \pm x, \pm y)$ need not vanish if $\eta_j = \eta'_j = 0$ at θ.

The expressions for D's, (2.24), (2.25) and (2.26), can be simplified by taking into account the following facts which apply in high energy electron storage rings. First, the contributions of the bending magnets to the focussing functions G_x and G_y are relatively small compared with those of quadrupole magnets. Hence,

$$G_x = - G_y = G \equiv \frac{e}{p}\frac{\partial B_y}{\partial x} \quad .$$

Second, the terms containing ρ_x and ρ_y in Eq. (2.26) are negligible compared with the term proportional to $(\gamma a+1)$, and the synchrotron tune ν_s is much smaller than unity. Therefore, Eq. (2.26) may be approximated by

$$D_{\pm s}(\theta) = \frac{- i(\gamma a+1)}{1 - \exp i(\mu \pm \mu_s)} \int_\theta^{\theta+2\pi} \left[\vec{k}_0 \cdot (\eta_x \vec{e}_y + \eta_y \vec{e}_x) G\right]_{\theta'} R d\theta' \quad , \tag{2.35}$$

where we have retained $\pm\mu_s$ in the denominator because it shows the position of the resonance.

Owing to these approximations we find that the bending magnets do not contribute to the integrals in Eq. (2.28) in all three degrees of freedom. Therefore, if $D_j(\theta)$ vanishes at a point in a bending magnet, it also vanishes everywhere in the same magnet. As a result, infinite number of conditions (2.27) reduce to

$$\int_\theta^{\theta+2\pi} F_j(\theta')d\theta' = 0 \qquad (j = \pm x, \pm y, \pm s) \tag{2.36}$$

(at each bending magnet except $\eta_j = \eta'_j = 0$) .

Since F's are complex and $F_{+s} = F_{-s}$ owing to Eq. (2.35), Eq. (2.36) consists of 10 N_B conditions, N_B being the number of relevant bending magnets. This is still too many to be satisfied in practice. However, the storage rings actually have various symmetries and these 10 N_B conditions often degenerate into a few. A trivial example is a planar ring. There, the vertical oscillations are not excited because $n_y = 0$ everywhere. The horizontal oscillations do not couple to the spin because $\vec{n}_0$ is parallel to $\vec{e}_y$ and, therefore $\vec{k}_0 \cdot \vec{e}_y = 0$ everywhere. The synchrotron oscillations do not couple either because of these two reasons. Hence, planar rings without magnet errors are free from diffusive depolarization. A non-trivial example with spin rotators is given in the next section.

2.5) An Example of Spin Matching

Let us consider a ring with the simplest spin rotators whose plan view and side view are depicted schematically in Fig. 1. Some points on the ring are numbered from - 3 to + 3 for the identification. (The point - 3 and + 3 are the same point). The point 0 is the collision point. The boxes are vertical bending magnets, dotted ones being assumed to have weak fields. The arcs, from - 3 to - 2 and from 2 to 3, consist of horizontal bends and

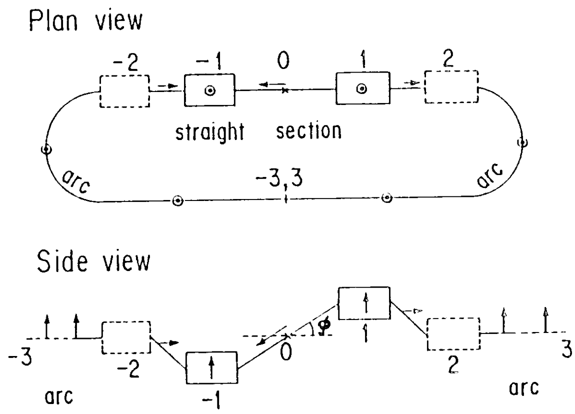


Fig.1.
Schematic layout of a ring with spin rotators.

quads. The straight section, from - 2 to + 2, consists of vertical bends and quads, the latter not shown. The direction of the equilibrium polarization (vector $\vec{n}_0$) is shown by arrows and ⊙. If the angle ϕ at the collision point is $(\pi/2)\cdot(0.44\ \mathrm{GeV}/E)$, the spin directs longitudinal. Because ϕ is given, the vertical bends at the points ± 1 can not be made weak. The betatron optics are chosen so that $\eta_x \equiv 0$ in the straight section and $\eta_y \equiv 0$ in the arcs, though these are not always necessary for the polarization.

First consider the horizontal betatron oscillation. It is excited only in the arcs because $\eta_x \equiv 0$ in the straight section. Hence, the condition for the spin-orbit decoupling is

$$\int_{\theta}^{\theta+2\pi} \vec{k}_0\cdot\vec{e}_y\, G\sqrt{\beta_x}\, e^{\pm i\Psi_x}\, d\theta' = 0 \quad \text{(at every } \theta \text{ in the arcs)} \quad .$$

Now, because the integrand is zero in the arcs due to vanishing $\vec{k}_0\cdot\vec{e}_y$, and because of the symmetry with respect to the collision point, it reduces to

$$\text{(a)} \qquad \int_0^2 \vec{k}_0\cdot\vec{e}_y\, G\sqrt{\beta_x}\, \cos\Psi_x\, d\theta = 0 \quad .$$

This is one single condition because $\vec{k}_0\cdot\vec{e}_y$ can be taken to be real in this section.

Next, consider the vertical oscillation. It is excited in the vertical bends. Ignoring the contributions of the weak bends, we get

$$\int_{\theta(-1)}^{\theta(-1)+2\pi} \vec{k}_0\cdot\vec{e}_x\, G\sqrt{\beta_y}\, \exp(\pm i\Psi_y)\, d\theta' = 0$$

and a similar equation with $\theta(-1)$ being replaced with $\theta(1)$. The two equations reduces to

$$\int_{-1}^{1} = 0 \qquad \text{and} \qquad \int_1^3 + \int_3^{-1} = 0 \quad .$$

The first of these can be written as

(b) $$\int_0^1 G\sqrt{\beta_y}\cos\Psi_y \, d\theta' = 0 \quad ,$$

if one takes into account the symmetry with respect to the collision point and the fact that $\vec{k}_0\cdot\vec{e}_x$ is a constnt in this region. The second equation reduces to the following two conditions by the symmetry.

(c) $$\int_1^3 G\sqrt{\beta_y} \cdot \mathrm{Re}\ \vec{k}_0\cdot\vec{e}_x \exp(\pm i\Psi_y)\, d\theta' = 0 \quad .$$

Note that the integration in the condition (b) can be done explicitly with the results

(b') $$\alpha_y = -\tan\Psi_y \qquad \text{at} \quad \theta = \theta(1) \quad ,$$

which was first given by J. Buon[16].

Finally, consider the synchrotron oscillation. The spin-orbit decoupling condition against the radiation at θ is

$$\int_\theta^{\theta+2\pi} G\ \vec{k}_0\cdot(\eta_x\vec{e}_y + \eta_y\vec{e}_x) d\theta' = 0 \quad .$$

The integrand vanishes if θ' is in the arc because $\eta_y = \vec{k}_0\cdot\vec{e}_y = 0$ there. Hence, if θ is in the arc, the condition above reduces to

$$\int_{-2}^{2} G\ \eta_y\ \vec{k}_0\cdot\vec{e}_x\, d\theta' = 0 \quad ,$$

which is trivially satisfied by the anti-symmetry with respect to the collision point. If θ is in the straight section, the condition can be written as

(d) $$\int_1^2 G\ \eta_y\cdot\vec{k}_0\cdot\vec{e}_x\, d\theta' = 0 \quad .$$

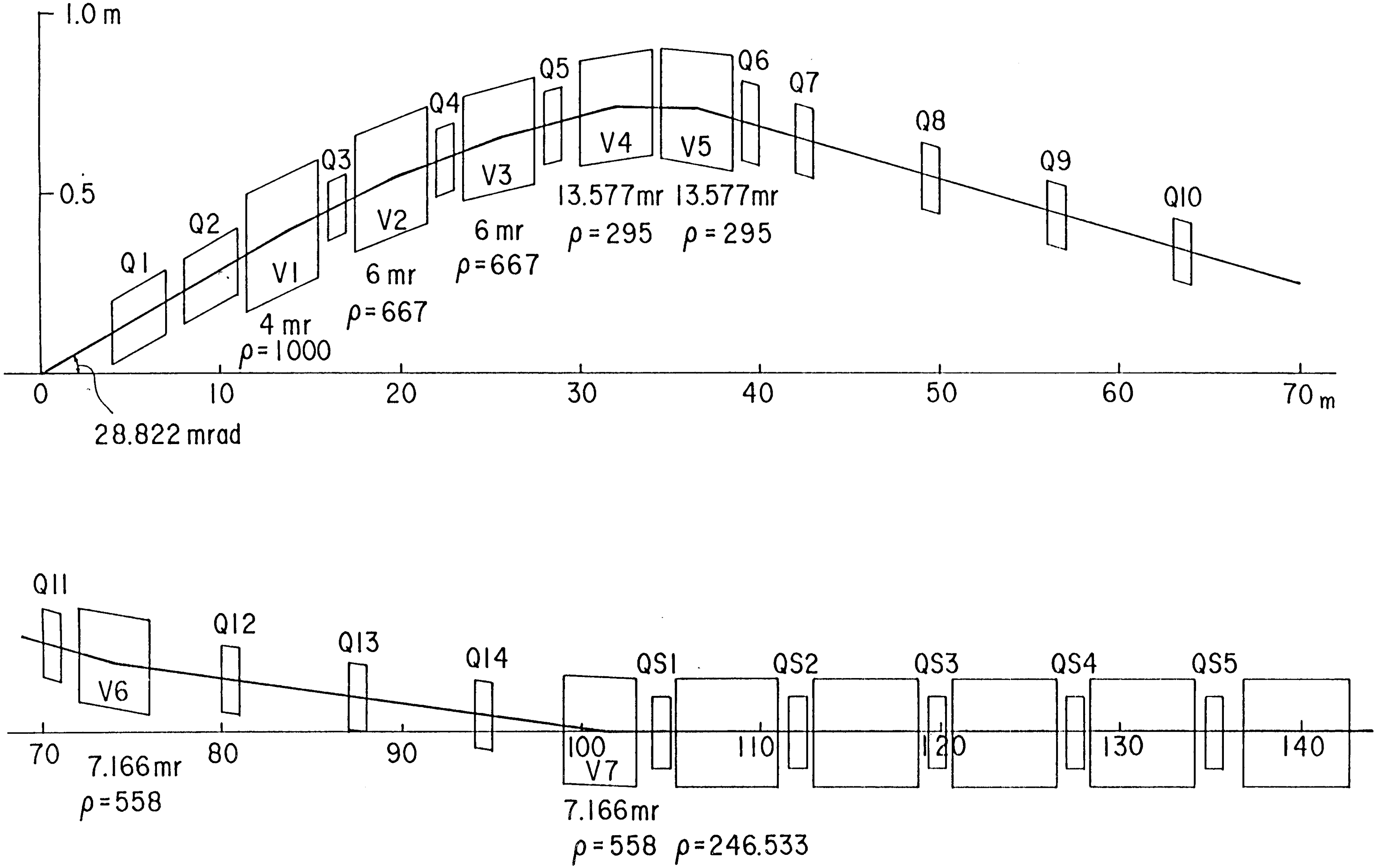


Fig.2. Layout of a spin rotator of vertical S-bend type for TRISTAN

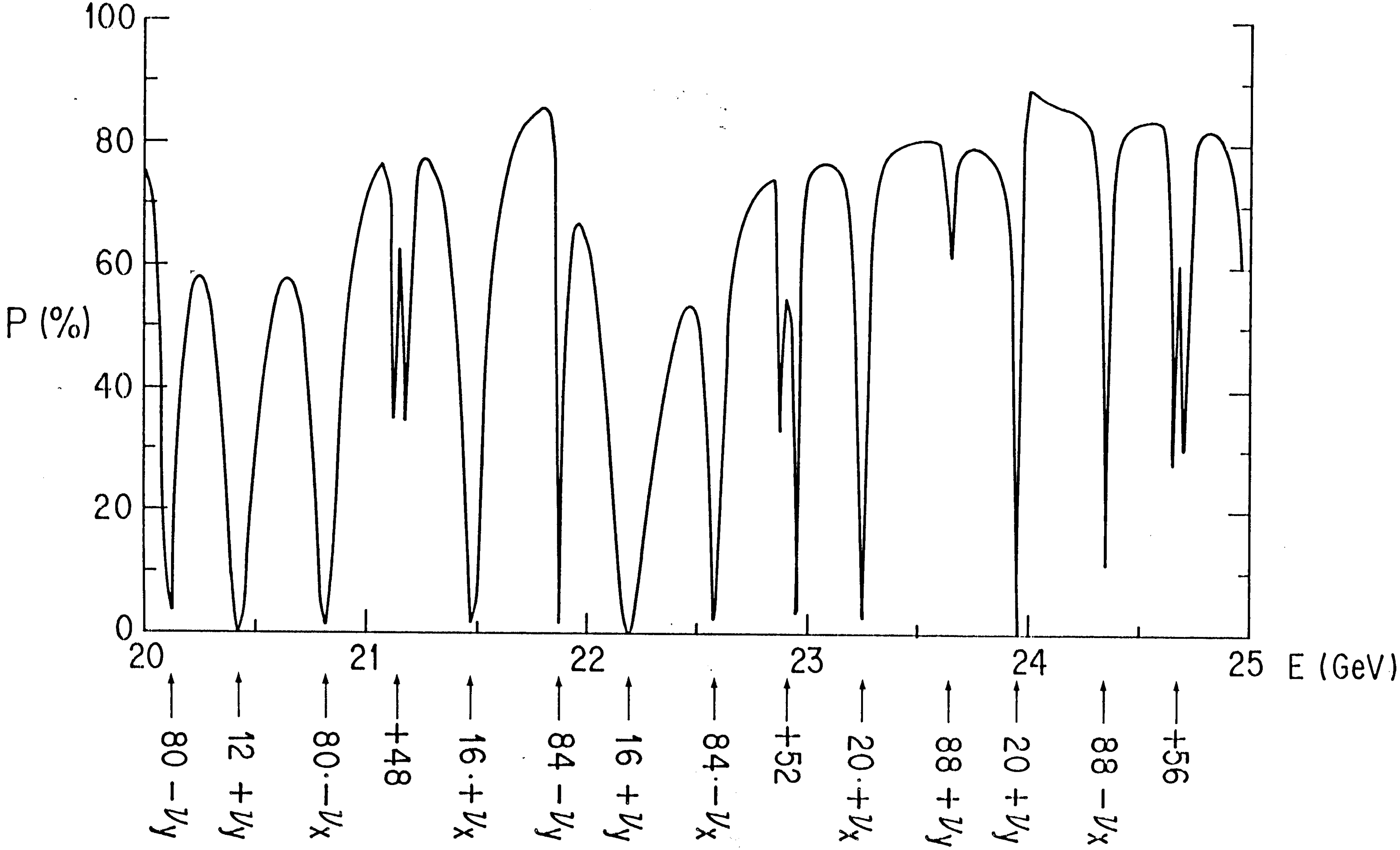


Fig.3. Expected polarization of the ring shown in Fig.2 without magnet errors. Calculated by the computer code SLIM. The arrows indicate the position of resonances.

Now we find that the 10 N_B spin matching conditions reduced to only five (real) conditions (a), (b), (c) and (d). If we choose the strengths of quadrupole magnets, G, so that these conditions are satisfied, we can avoid diffusive depolarization almost completely. An example of the layout of the straight section for the TRISTAN e^+e^- storage ring is given in Fig. 2. The equilibrium polarization is calculated by the computer code ECSLIM, the thick lens version of SLIM[8] rewritten by E.D. Courant[17], and is shown in Fig. 3. Since the conditions (a), (c) and (d) depend on the beam energy through the vector $\vec{k}_0(\theta)$, the spin matching must be done at each operation energy. However, we have done the matching only at 24 GeV and have used the same quadrupole strengths at every other energy. One can see a cusp at 24 GeV in Fig. 3, where the optimization is done, but even at other energies around 24 GeV the polarizations is relatively high because the condition (b) is independent of the beam energy and the conditions (a) and (d) are not sensitive to the energy owing to the short integration intervals.

The calculated polarization at 24 GeV is 88.86 %. On the other hand, if one takes into account only the first and the second depolarization mechanisms, the expected polarization is 89.84 %, which shows that the residual contribution of the diffusive depolarization caused by the various approximations is only about 1 %.

2.6) Extensions of the Spin Matching Conditions

Since the discovery of the spin matching conditions, a number of storage ring designs with spin rotators have ben proposed, such as

HERA[18], LEP[19], TRISTAN[20] and CHEER[21]. At the same time the concept of spin matching was extended and applied to other phenomena related to the polarization.

First of all, it was extended to the case with magnet errors, which will be discussed in the next chapter. Secondly, the original derivation does not take into account the vertical-horizontal coupling but the method of spin-orbit decoupling was applied to the rings with solenoid magnets by K. Steffen[22] and D.P. Barber et al.[23],[24]. Another attempt is to apply the spin matching conditions to the depolarization due to the beam-beam interaction. The betatron oscillations can be excited by the beam-beam interaction as well as by the synchrotron radiation. It was first suggested by E.D. Courant[25] that we can eliminate at least the linear effect of the beam-beam interaction and the conditions were derived explicitly by J. Buon[26]. It was also suggested by A. Chao[27] that the spin matching technique may be useful for proton rings, too.

§3. Spin Matching with Magnet Errors

In the previous chapter we discussed the diffusive depolarization in the absence of magnet errors but in practice the polarization does not reach the ideal value 92 % even in planar rings because of various errors of magnets.

In the presence of magnet errors the equilibrium orbit of the electron is no longer the design orbit. Even in such a case we can write down the spin matching conditions in the same manner as in the previous chapter by defining the coordinate system according to the distorted closed orbit. However, these conditions are not practical not only because the number of conditions does not reduce due to the lack of symmetry but also because we do not have the full information of the errors.

This problem had better be discussed in two stages. The first is whether it is possible or not to design a ring insensitive to magnet errors. The second is how we can correct the orbit in operation looking at the polarization monitors.

In the following we confine ourselves to planar rings. The problem is more complicated in non-planar rings because we can not consider horizontal and vertical orbit errors separately, though essentially the same idea may be applied.

3.1) Harmonic Spin Matching

The conditions of spin-orbit decoupling can be written in the form

$$\int_{\theta}^{\theta+2\pi} F_j(\theta')d\theta' = 0 \qquad (j = x,y,s) \quad , \qquad (3.1)$$

as discussed in the previous chapter. It consists of 10 N_B constraints. K. Steffen noted the following fact. Since $F_{\pm j}$ has the quasiperiodicity

$$F_{\pm j}(\theta+2\pi) = \exp i(\mu\pm\mu_j)\, F_{\pm j}(\theta) \quad , \tag{3.2}$$

it is a periodic function if $\mu\pm\mu_y$ = 2π times an integer or, equivalently, $\nu\pm\nu_y$ = integer. Then Eq. (3.1) reduces to only 10 conditions

$$\oint F_{\pm j}(\theta')d\theta' = 0 \quad . \tag{3.3}$$

Therefore the following recipe may lead to Eq. (3.1) in a good approximation. First, we make ν and ν_y integer close to their desired values, find an optics which satisfies Eq. (3.3) and then shift ν and ν_j slightly to the desired values. K.Steffen called this process "harmonic spin matching".

The present author independetly suggested a method to control the Fourier components of F_j[29]. These two methods are similar but the latter is more general in the sense that as many resonances as desired can be eliminated. In the following we discuss the latter.

Because of the quasi-periodicity (3.2), F_j can be expanded into Fourier series

$$F_{\pm j}(\theta) = \sum_{n=-\infty}^{\infty} e^{i(\nu\pm\nu_j+n)\theta}\, F_{\pm j,n} \quad , \tag{3.4}$$

with

$$F_{\pm j,n} = \frac{1}{2\pi}\oint F_{\pm j}(\theta)\, e^{-i(\nu\pm\nu_j+n)\theta}\, d\theta \quad . \tag{3.5}$$

Then Eq. (2.28) becomes

$$D_{\pm j}(\theta) = H_{\pm j}(\theta) \sum_{n=-\infty}^{\infty} \frac{-1}{\nu\pm\nu_j+n}\, F_{\pm j,n}\, e^{i(\nu\pm\nu_j+n)\theta} \quad , \tag{3.6}$$

which shows the locations and the strengths of the resonances.

Let us discuss the vertical betatron oscillation. The horizontal oscillation will be treated in the next section because it contribute to $\gamma\partial\vec{n}/\partial\gamma$ in a different way in planar rings. If the ring is free from magnet errors, the function $H_{\pm y}(\theta)$ in Eq. (3.6) vanishes because $\eta_y = \eta'_y = 0$ but $F_{\pm y,n}$ is not zero in general. It means that the vertical oscillation is not excited by the radiation in a complete planar ring, but if it were excited, it couples to the spin motion and causes the depolarization. In the ring with magnet errors $H_{\pm y}$ and $F_{\pm y,n}$ have different values from the ones in a complete ring but the change in $F_{\pm y,n}$'s is negligible because they have non-zero values in a complete ring. Therefore if we choose an integer m which makes $\nu\pm\nu_y+m$ (either plus or minus) close to zero and impose a constraint $F_{\pm y,m} = 0$, or

$$\oint(\vec{k}_0\cdot\vec{e}_x)G\sqrt{\beta_y}\, e^{\pm i\Psi_y - i(\nu\pm\nu_y+m)\theta}\, d\theta = 0 \quad , \qquad (3.7)$$

we are free from the diffusive depolarization due to the vertical betatron oscillation at least up to the first order of the amplitude of the oscillation. Since all the variables in Eq. (3.7) are known in the design stage, we can design a ring insensitive to the vertical oscillation. Eq. (3.7) can be satisfied, if necessary, for several values of m.

3.2) Polarization Correction in Planar Rings

In contrast to the vertical oscillation, the expression corresponding to Eq. (3.7) for the horizontal oscillation contains unknown functions involving magnet errors. Hence we cannot make $F_{\pm x,n}$ zero in the design stage but instead we can control them by exciting vertical steering magnet in the operation stage as shown below.

Coming back to Eq. (2.8), let us denote by $\Delta\vec{\Omega}$ the contribution of the error fields to the square brackets. It consists of two terms, namely the direct influence of the error fields to the spin and the indirect one through the closed orbit distortions x_{COD} and y_{COD}. It can be written explicity as

$$\Delta\vec{\Omega} = - R(\gamma a+1) \left[(x_{COD}\cdot G-t_x)\vec{e}_y + (Gy_{COD}+t_y)\vec{e}_x \right] \quad , \tag{3.8}$$

where t_x and t_y are the horizontal and vertical orbit curvature due to the error fields, which are defined so that they are positive if the error fields bend the orbit to the + x and + y direction, respectively. They come from the strength errors and tilts of the bending magnets and the misalignments of the quadrupole magnets. The tilts of the quadrupole magnets change $\vec{e}_x$ and $\vec{e}_y$ but they are not considered here. The direct effects of the errors in the strengths of quadrupole magnets are negligible. (The indirect effects through the change of the linear optics can be taken into account, if necessary, by using the perturbed values of $\beta_{x,y}$ and $\alpha_{x,y}$.) The closed orbit distortions are given by the standard formula[30)]

$$y_{COD}(\theta) = \frac{\sqrt{\beta_y(\theta)}}{2\sin\pi\nu_y} \oint \sqrt{\beta_y(\theta')}\, t_y(\theta')\cos\left[\pi\nu_y - |\Psi_y(\theta) - \Psi_y(\theta')|\right] Rd\theta' \quad , \tag{3.9}$$

and a similar fomula for x_{COD} with y replaced with x. Let us evaluate the perturbed orthonormal solutions $\vec{n}_0 + \Delta\vec{n}$ and $\vec{k}_0 + \Delta\vec{k}$ which satisfy

$$\frac{d\vec{s}}{d\theta} = \left[\vec{\Omega}_0(\theta) + \Delta\vec{\Omega}(\theta)\right] \times \vec{s} \quad . \tag{3.10}$$

First, consider $\Delta\vec{n}$. Because $\vec{n}_0$ and $\vec{n}_0 + \Delta\vec{n}$ are both unit vectors, $\Delta\vec{n}$ can be written up to the first order of $\Delta\vec{\Omega}$ as

$$\Delta\vec{n} = \mathrm{Re}\left[c_1(\theta)\vec{k}_0^*(\theta)\right] \quad , \tag{3.11}$$

where c_1 is a complex function to be determined. The equation for c_1 is

$$\frac{dc_1}{d\theta} = -i\,\Delta\vec{\Omega}\cdot\vec{k}_0 \quad . \tag{3.12}$$

and the solution is

$$c_1(\theta) = -i\int^{\theta}\Delta\vec{\Omega}(\theta')\cdot\vec{k}_0(\theta')d\theta' \quad ,$$

where the lower limit of the integration must be chosen so that $\Delta\vec{n}$ is periodic. This can be achieved by making the lower limit minus infinity as

$$\begin{aligned}
c_1(\theta) &= -i\int_{-\infty}^{\theta}\Delta\vec{\Omega}(\theta')\cdot\vec{k}_0(\theta')d\theta' \\
&= -i\sum_{n=0}^{\infty}\int_{\theta}^{\theta+2\pi}\Delta\vec{\Omega}(\theta'-2n\pi-2\pi)\vec{k}_0(\theta'-2n\pi-2\pi)d\theta' \\
&= -i\sum_{n=0}^{\infty}e^{-i(n+1)\mu}\int_{\theta}^{\theta+2\pi}\Delta\vec{\Omega}(\theta')\cdot\vec{k}_0(\theta')d\theta \\
&= \frac{-i}{e^{i\mu}-1}\int_{\theta}^{\theta+2\pi}\Delta\vec{\Omega}(\theta')\cdot\vec{k}_0(\theta')d\theta' \quad .
\end{aligned} \tag{3.13}$$

Hence, we get

$$\Delta\vec{n}(\theta) = \mathrm{Re}\,\frac{-i\,\vec{k}_0^*(\theta)}{e^{i\mu}-1}\int_{\theta}^{\theta+2\pi}\Delta\vec{\Omega}(\theta')\cdot\vec{k}_0(\theta')d\theta' \quad . \tag{3.14}$$

Next, consider the complex vector $\Delta\vec{k}(\theta)$. It can be expanded in terms of $\vec{n}_0$, $\vec{k}_0$ and $\vec{k}_0^*$. The coefficients of $\vec{n}_0$ and $\vec{k}_0^*$ are found to be $-c_1(\theta)$ and zero, respectively, by the orthonormality of $\vec{n}_0 + \Delta\vec{n}$ and $\vec{k}_0 + \vec{\Delta}k$. Therefore we can write

$$\Delta\vec{k} = -c_1(\theta)\vec{n}_0(\theta) + c_2(\theta)\vec{k}_0(\theta) \quad ,$$

where c_2 is a complex function. The equation for c_2 is

$$\frac{dc_2}{d\theta} = -i\,\Delta\vec{\Omega}\cdot\vec{n}_0$$

and the solution is

$$c_2 = -i\int^{\theta}\Delta\vec{\Omega}\cdot\vec{n}_0\,d\theta \quad ,$$

here the lower limit of integration is arbitrary, which corresponds to the definition of the origin of the spin phase. Now, $\Delta\vec{k}$ is found to be

$$\Delta\vec{k} = \frac{i\,\vec{n}_0}{e^{i\mu}-1}\int_{\theta}^{\theta+2\pi}\Delta\vec{\Omega}(\theta')\cdot\vec{k}_0(\theta')d\theta' - i\,\vec{k}_0\int^{\theta}\Delta\vec{\Omega}(\theta')\vec{n}_0(\theta')d\theta' \quad . \qquad (3.15)$$

Consider the horizontal betatron oscillation. As can be seen from Eq. (2.32), the relevant quantity is $(\vec{k}_0 + \Delta\vec{k})\cdot\vec{e}_y$. It vanishes in a complete planar ring because $\vec{k}_0$ is perpendicular to $\vec{e}_y$. Thus the horizontal oscillation does not affect the spin motion although it can be excited. In the presence of errors we get from Eq. (3.15) and (3.8) and from the fact $\vec{n}_0 = \vec{e}_y$ and $\vec{k}_0 \perp \vec{e}_y$

$$\Delta\vec{k}\cdot\vec{e}_y = -\frac{i(\gamma a+1)}{e^{i\mu}-1}\int_{\theta}^{\theta+2\pi}\left[\vec{k}_0\cdot\vec{e}_x(Gy_{COD}+t_y)\right]_{\theta'}Rd\theta' \quad . \qquad (3.16)$$

Now we can see that if the vertical closed orbit distorsion is present, horizontal betatron oscillation can couple to the spin motion and may excite the resonance $\nu \pm \nu_x$ = integer. This fact suggests that we can suppress the depolarization due to the horizontal oscillation by using vertical steering magnets.

Combining Eqs. (2.32), (3.5) and (3.16) we can write the Fourier coefficient $F_{\pm x,n}$ in terms of t_y and y_{COD} as

$$F_{\pm x,n} = \frac{1}{2\pi} \frac{i(\gamma a+1)^2}{e^{i\mu} - 1} \oint d\theta \; e^{-i(\nu \pm \nu_x + n)\theta} \left[G\sqrt{\beta_x} \; e^{\pm i \Psi_x}\right]_\theta$$

$$\times \int_\theta^{\theta+2\pi} \left[\vec{k}_0 \cdot \vec{e}_x (G y_{COD} + t_y)\right]_{\theta'} R d\theta' \quad . \tag{3.17}$$

Using Eq. (3.9) we get the change of $F_{\pm x,n}$ by a vertical steering magnet at $\theta = \theta_a$ with the kick angle Θ_a in the form

$$\frac{\partial F_{\pm x,n}}{\partial \Theta a} = \frac{1}{2\pi} \frac{i(\gamma a+1)^2}{e^{i\mu} - 1} \oint d\theta \; e^{-i(\nu \pm \nu_x + n)\theta} \left[G\sqrt{\beta_x} \; e^{\pm i \Psi_x}\right]_\theta$$

$$\times \int_\theta^{\theta+2\pi} d\theta' (k_0 \cdot e_x)_{\theta'} \; \{\delta(\theta' - \theta_a) + \frac{RG(\theta')}{2\sin\pi\nu_y} \sqrt{\beta_y(\theta')\beta_y(\theta_a)}$$

$$\cos\left[\pi\nu_y - |\Psi_y(\theta') - \Psi_y(\theta_a)|\right]\} \quad . \tag{3.18}$$

The present author proposed the following correction procedure. It is not possible to measure $F_{\pm x,n}$. However, the term t_y in Eq. (3.17) which is the direct contribution of the error field, is much smaller than the indirect term $G y_{COD}$. We can measure y_{COD} at the point of position monitors and can estimate $F_{\pm x,n}$ by interpolating y_{COD}. Let $F^{obs}_{\pm x,n}$ be the value estimated in this way. The right hand side of Eq. (3.18) can be calculated in the design stage. Hence by solving the linear simultaneous equation for Θ_a's

$$\sum_a \left(\frac{\partial F_{\pm x,n}}{\partial \Theta_a}\right) \cdot \Theta_a + F^{obs}_{\pm x,n} = 0 \tag{3.19}$$

for some values of n close to $-(\nu\pm\nu_x)$, we get the required strength of the steering magnets. If we have 2k vertical steerings for the polarization correction, we can control k harmonics because $F_{\pm x,y}$ is a complex quantity.

This algorithm is rather complicated. Instead, the PETRA group[31] chose a simpler and more practical way. They selected two combinations of vertical steering magnets so that each combination excites only real (or imaginary) part of a Fourier component. Then by adjusting the strengths of these combinations by trial and error, looking at the polarization monitor, they succeeded to raise the degree of polarization in the PETRA rings.

It can be said that the correction technique of the polarization in planar rings has already been established. It seems also possible to correct it in non-planar rings, though complicated due to the mixing of vertical and horizontal corrections.

§4. Depolarization due to the Beam Energy Spread

4.1) Depolarization Mechanism

The beam energy in electron storage rings has a finite spread due to the quantum effect of the synchrotron radiation. The distribution can be approximated as gaussian and the root-mean-squared energy spread σ_E (in GeV) is given by[56)]

$$\sigma_E = 0.86 \times 10^{-3} E^2/\sqrt{\rho} \quad , \tag{4.1}$$

where E is the beam energy in GeV and ρ the bending radius in meters. For example in LEP at 70 GeV it amounts to 70 MeV. Since the resonances are located every 440 MeV even if we take into account integer resonances only, the distance to the nearest resonance is shorter than 220 MeV. Hence at least the tail of the gaussian distribution at three standard deviations touches a resonance. The depolarization from the tail spreads over the whole beam by radiation damping and excitation. It was first pointed out by D. Möhl and B.W. Montague[32)] that one can not expect high polarization at more than 50 GeV in LEP because of this mechanism. A quantitative theory was developed by Ya.S. Derbenev, A.M. Kondratenko and A.N. Skrinsky[33)]. A different formalism was given by C. Biscari and J. Buon[34)]. Numerical calculations were done by A.M. Kondratenko and B.W. Montague[35)] for LEP and by J. Buon[36)] for HERA. The present author[37)] derived a depolarization formula which is general in the sense that it includes resonances with betatron oscillations and can be applid to non-planar rings and that the isolated resonance approximation is not employed. It turned out through these works that the first estimation by Möhl and Montague was too pessimistic. However, the depolarization enhancement factor, the ratio of the depolarization with and without the energy spread still amounts to 10 at 70 GeV in LEP, which necessitates elimination of resonances in a very high precision. The enhancement factor is a steep function of the beam energy. The depolarization of this mechanism has not yet been established experimentally in the existing rings, but it may be important not only in LEP but also at high energy regions of HERA and TRISTAN.

The above stated picture of the depolarization is too simplified because the electrons execute synchrotron oscillation. If the spin tune is close to a resonant value and an electron gets a large amplitude of synchrotron oscillation, it crosses the resonance several times before the oscillation amplitude damps away. The theories stated above assume that the diffusion of the spin phase by radiation during one synchrotron oscillation period is negligible and, therefore, the spin phase at each crossing is correlated. It was pointed out[33)] that if the radiation is extremely violent so as to destroy the spin phase during one oscillation period (uncorrelated crossing), then the depolarization is rather suppressed than enhanced. This may be in principle realized in LEP by installing very strong wiggler magnets but in practice it is impossible because a radiation loss of $\sim$ E/10 per turn by the wigglers is required. Hence in practice we may assume correlated crossings.

4.2) Formula of Depolarization due to the Energy Spread

Let us derive the depolarization formula according to ref. 37.

The depolarization treated in this chapter can be described in the framework of the diffusive depolarization or, in other words, in terms of the spin-orbit coupling vector $\gamma\partial\vec{n}/\partial\gamma$. However, Chao's definition of $\gamma\partial\vec{n}/\partial\gamma$ (see §2.2) can no longer be applied because the initial electron before radiation must be off energy. We must employ the original definition of Derbenev and Kondrotenko.

Let us return to Eq. (2.8) and (2.10). Picking up the betatron oscillation term x_β and y_β from x and y as

$$x = x_\beta + \eta_x\varepsilon \qquad \text{and} \qquad y = y_\beta + \eta_y\varepsilon \quad , \tag{4.2}$$

we can decompose $\vec{\omega}$ as

$$\vec{\omega} = \vec{\omega}_s(\theta)\varepsilon + \vec{\omega}_x(\theta)x_\beta + \vec{\omega}_y(\theta)y_\beta \tag{4.3}$$

with

$$\vec{\omega}_s = - R\left[(\gamma a+1)G_x\eta_x - \frac{1}{\rho_x}\right]\vec{e}_y + R\left[(\gamma a+1)G_y\eta_y - \frac{1}{\rho_y}\right]\vec{e}_x \tag{4.4}$$

$$\vec{\omega}_x = - R(\gamma a+1)G_x\vec{e}_y \quad , \tag{4.5}$$

$$\vec{\omega}_y = + R(\gamma a+1)G_y\vec{e}_x \quad . \tag{4.6}$$

In the following we omit the vertical oscillation since it has the same form as the horizontal oscillation and we do not treat the interference between these two oscillations. The effects of the vertical oscillation can easily be found by replacing x with y in the final formula. The variable x_β and ε can be written in terms of action-angle variables as

$$x_\beta = \sqrt{2 I_x \beta_x} \cos(\psi_x + \tilde{\Psi}_x(\theta)) \, , \tag{4.7}$$

$$\mathcal{E} = \sqrt{2 I_s} \cos \psi_s \tag{4.8}$$

where $\tilde{\Psi}_x(\theta)$ is the periodic part of the betatron phase function $\Psi_x(\theta)$; i.e.

$$\tilde{\Psi}_x(\theta) = \Psi_x(\theta) - \nu_x \theta \tag{4.9}$$

Note that $\tilde{\Psi}_x$ is independent of individual particles, in contrast to the angle variable ψ_x, and is an explicit function of θ. The solutions of the orbital motions are

$$\psi_j = \nu_j \theta + \psi_{jo} \qquad (j = x, s) \tag{4.10}$$

where ψ_{jo}'s are determined by the initial conditions. Let us find the solution $\vec{n}(I_x, \psi_x, I_s, \psi_s, \theta)$ to Eq. (2.8) which has the property Eq. (2.4). Since it is a unit vector, it can be written in terms of the unperturbed orthonormal solutions as

$$\vec{n}(I_x, \psi_x, I_s, \psi_s) = \sqrt{1 - |\zeta|^2}\, \vec{n}_0(\theta) + \mathrm{Re}\,(\vec{k}_0^*(\theta)\, \zeta)\ , \tag{4.11}$$

where ζ is a complex function to be determined. The equation for ζ is

$$\frac{d\zeta}{d\theta} = i\vec{\omega} \cdot (-\sqrt{1 - |\zeta|^2}\, \vec{k}_0 + \zeta\, \vec{n}_0)\ . \tag{4.12}$$

The first-order perturbation term is $\vec{\omega} \cdot \vec{k}_0$. We will ignore the third-order term involving $|\zeta|^2$ but we have to retain the second order term $\zeta \vec{\omega} \vec{n}_0$ because it gives the spin tune modulation which is essential for our problem. A general solution to Eq. (4.12) is given by

$$\zeta = e^{-i\chi} \left[-i \int^{\theta} e^{i\chi}\, \vec{\omega} \cdot \vec{k}_0\, d\theta' + \text{const.} \right] \tag{4.13}$$

with

$$\chi = - \int_{-\infty}^{\theta} \vec{\omega} \cdot \vec{n}_0\, d\theta'\ . \tag{4.14}$$

We have to choose the integration constant of Eq. (4.13) so that Eq. (2.4) is satisfied. This is accomplished by

$$\zeta = -i\, e^{-i\chi} \int_{-\infty}^{\theta} \vec{\omega}\cdot\vec{k}_0 \, d\theta' . \tag{4.15}$$

We will ignore the spin tune modulation due to betatron oscillations and set $\vec{\omega} = \vec{\omega}_s \varepsilon$ in Eq. (4.14). Integration of this expression gives

$$\chi = \sqrt{2 I_s}\, u_\varepsilon \sin(\psi_s + v_\varepsilon) \tag{4.16}$$

with real periodic functions u_ε and v_ε defined by

$$u_\varepsilon(\theta)\, e^{i v_\varepsilon(\theta)} = \frac{-i\, e^{-i\nu_s\theta}}{e^{i\mu_s} - 1} \int_{\theta}^{\theta+2\pi} \vec{\omega}_s\cdot\vec{n}_0\, e^{i\nu_s\theta'} d\theta' . \tag{4.16}$$

Since ν_s is usually very small, we take the limit $\nu_s \to 0$ in this expression, which leads to vanishing v_ε and constant u_ε; i.e.,

$$u_\varepsilon = -\frac{1}{2\pi\nu_s} \int_0^{2\pi} \vec{\omega}_s\cdot\vec{n}_0\, d\theta . \tag{4.17}$$

As is shown in the next section, Eq. (4.55), if ε is constant, the variation of the spin tune ν with ε is given by

$$\left(\frac{d\nu}{d\varepsilon}\right)_{\varepsilon\to 0} = -\frac{1}{2\pi} \oint \vec{\omega}_s\cdot\vec{n}_0\, d\theta , \tag{4.18}$$

which is equal to γa in a complete planar ring. Therefore, we can rewrite Eq. (4.17) as

$$u_\varepsilon = \frac{1}{\nu_s} \left(\frac{d\nu}{d\varepsilon}\right)_{\varepsilon\to 0} . \tag{4.19}$$

Now, using the Bessel function relations

$$\exp(i r \sin\theta) = \sum_{m=-\infty}^{\infty} J_m(r)\, e^{im\theta} \tag{4.20}$$

and

$$\exp(i r \sin\theta)\cos\theta = \frac{1}{r} \sum_{m=-\infty}^{\infty} m\, J_m(r)\, e^{im\theta} \tag{4.21}$$

the latter being derived by differentiating the former, we can expand $e^{i\chi}$ and $\varepsilon e^{i\chi}$ appearing in Eq. (4.15) into the Fourier series of ψ_s as

$$e^{i\chi} = \sum_{m=-\infty}^{\infty} J_m(\sqrt{2I_s}\, u_\varepsilon)\, e^{im\psi_s} \tag{4.22}$$

and

$$\varepsilon\, e^{i\chi} = \frac{1}{u_\varepsilon} \sum_{m=-\infty}^{\infty} m\, J_m(\sqrt{2I_s}\, u_\varepsilon)\, e^{im\psi_s} \quad . \tag{4.23}$$

Then, combining all expressions, we can evaluate the integral of Eq. (4.15) as

$$\begin{aligned}
\int_{-\infty}^{\theta} e^{i\chi}\, \vec{\omega}\cdot\vec{k}_0\, d\theta &= \int_{-\infty}^{\theta} e^{i\chi}\, (\vec{\omega}_s\cdot\varepsilon + \vec{\omega}_x\, x_\beta)\cdot\vec{k}_0\, d\theta \\
&= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\theta} \vec{k}_0\cdot\left[\vec{\omega}_\varepsilon\, \frac{m}{u_\varepsilon} + \vec{\omega}_x \sqrt{2I_x\beta_x}\cos(\psi_x + \tilde{\Psi}_x)\right] J_m(\sqrt{2I_s}\, u_\varepsilon)\, e^{im\psi_s}\, d\theta' \\
&= \sum_{m=-\infty}^{\infty} \Bigg\{ \frac{m/u_\varepsilon}{e^{i(\mu+m\mu_s)}-1} \int_{\theta}^{\theta+2\pi} \vec{\omega}_\varepsilon\cdot\vec{k}_0\, e^{im\psi_s} J_m(\sqrt{2I_s}\, u_\varepsilon)\, d\theta' \\
&\qquad + \sum_{\pm} \frac{\sqrt{2I_x}/2}{e^{i(\mu\pm\mu_x+m\mu_s)}-1} \int_{\theta}^{\theta+2\pi} \vec{\omega}_x\cdot\vec{k}_0 \sqrt{\beta_x} \\
&\qquad\qquad \times e^{i(\pm\psi_x \pm \tilde{\Psi}_x + m\psi_s)} J_m(\sqrt{2I_s}\, u_\varepsilon)\, d\theta' \Bigg\} .
\end{aligned} \tag{4.24}$$

Here we used the relation (2.11) in order to reduce the integral of infinite interval to a finite interval. We have to rewrite these integrands in terms of explicit functions of θ. For instance, the integral of the following form

$$\int_{\theta}^{\theta+2\pi} \vec{k}_0(\theta')\, f(\theta')\, \exp i(\psi_x + \tilde{\Psi}_x)\, d\theta' \quad ,$$

where $f(\theta)$ is a periodic function of θ, can be rewritten as

$$\begin{aligned}
&= \int_{\theta}^{\theta+2\pi} \vec{k}_0(\theta')\, f(\theta')\, \exp i(\nu_x\theta' + \psi_{x0} + \tilde{\Psi}_x(\theta'))\, d\theta' \\
&= e^{i\psi_{x0}} \int_{\theta}^{\theta+2\pi} \vec{k}_0(\theta')\, f(\theta')\, \exp i\tilde{\Psi}_x(\theta')\, d\theta' \\
&= e^{i(\psi_x - \nu_x\theta)} \int_{\theta}^{\theta+2\pi} (\vec{k}_0\, f\, \exp i\tilde{\Psi}_x)_{\theta'}\, d\theta' \quad .
\end{aligned}$$

Using this technique, we finally get

$$\zeta = e^{-i\chi} \sum_{m=-\infty}^{\infty} \left[\frac{m}{u_\varepsilon} A_m + \sum_{\pm} \sqrt{2I_x\beta_x}\, e^{\pm i(\psi_x+\tilde{\Psi}_x)} B_{m,\pm x} \right] e^{im\psi_s} J_m(\sqrt{2I_s}\, u_\varepsilon) \tag{4.25}$$

with

$$A_m(\theta) = \frac{-i\, e^{-im\nu_s\theta}}{e^{i(\mu+m\mu_s)}-1} \int_\theta^{\theta+2\pi} e^{im\nu_s\theta'} \vec{\omega}_\varepsilon \cdot \vec{k}_0 \, d\theta' \tag{4.26}$$

and

$$B_{m,\pm x}(\theta) = \frac{-i\, e^{\mp i\Psi_x - im\nu_s\theta}}{e^{i(\mu\pm\mu_x+m\mu_s)}-1} \frac{1}{2\sqrt{\beta_x}} \int_\theta^{\theta+2\pi} e^{\mp i\Psi_x + im\nu_s\theta'} \sqrt{\beta_x}\, \vec{\omega}_x \cdot \vec{k}_0 \, d\theta' \, . \tag{4.27}$$

The Fourier coefficients A_m and $B_{m,\pm x}$ have the quasi-periodicity

$$A_m(\theta+2\pi) = e^{i\mu} A_m(\theta) \quad \text{and} \quad B_{m,\pm x}(\theta+2\pi) = e^{i\mu} B_{m,\pm x}(\theta) \, . \tag{4.28}$$

They are almost independent of m, apart from the resonant factors $\exp i(\mu+m\mu_s) - 1$ and $\exp i(\mu\pm\mu_x+m\mu_s) - 1$, as long as $m\mu_s$ is small.

Now, let us evaluate the spin-orbit coupling vector

$$\gamma \frac{\partial \vec{n}}{\partial \gamma} = \frac{\partial \vec{n}}{\partial \varepsilon} = \mathrm{Re}\left(\vec{k}_0^{\,*} \frac{\partial \zeta}{\partial \varepsilon} \right) . \tag{4.29}$$

First we make sure that we have to carry out the differentiation with respect to ε fixing x, x' and τ rather than fixing x_β, x'_β and τ, which means

$$\frac{\partial}{\partial \varepsilon} = \frac{\partial I_x}{\partial \varepsilon}\frac{\partial}{\partial I_x} + \frac{\partial \psi_x}{\partial \varepsilon}\frac{\partial}{\partial \psi_x} + \frac{\partial I_s}{\partial \varepsilon}\frac{\partial}{\partial I_s} + \frac{\partial \psi_s}{\partial \varepsilon}\frac{\partial}{\partial \psi_s} \tag{4.30}$$

since ε and x couple each other through the eta function. One can easily verify

$$\frac{\partial}{\partial \varepsilon}\left[\sqrt{2I_x\beta_x}\, e^{\pm i(\psi_x+\tilde{\Psi}_x)} \right] = -\eta_x \pm (\eta'_x \beta_x + \alpha_x \eta_x) \tag{4.31}$$

and

$$\frac{\partial}{\partial \varepsilon}\left[J_m(\sqrt{2I_s}\, u_\varepsilon)\, e^{im\psi_s} \right] = \frac{u_\varepsilon}{2}\left[J_{m-1}(\sqrt{2I_s}\, u_\varepsilon)\, e^{i(m-1)\psi_s} - J_{m+1}(\sqrt{2I_s}\, u_\varepsilon)\, e^{i(m+1)\psi_s} \right] . \tag{4.32}$$

Since χ is a function of τ only, which is a cannonically conjugate variable to ε, $\partial\chi/\partial\varepsilon$ vanishes.

Then, introducing the terms involving the vertical oscillations, we get

$$\frac{\partial\zeta}{\partial\varepsilon} = e^{i\chi} \sum_{m=-\infty}^{\infty} C_m(\theta)\, J_m(\sqrt{2I_s}\, u_\varepsilon)\, e^{im\psi_s} \tag{4.33}$$

with

$$C_m(\theta) = \frac{1}{2}\left[(m+1)A_{m+1}(\theta) - (m-1)A_{m-1}(\theta)\right] + \sum_{\pm}\sum_{j=x,y}\left[-\eta_j \pm i(\eta_j'\beta_j + \eta_j\alpha_j)\right] B_{m,\pm j}(\theta), \tag{4.34}$$

where we have neglected the terms proportional to the betatron oscillation amplitudes after differentiation. The definition of $B_{m,\pm y}$ is given in Eq. (4.27) with x replaced with y.

The equations (4.29) and (4.33) give the desired spin-orbit coupling vector for our problem. If we let $I_s \rightarrow 0$, the only remaining terms are $A_{\pm 1}$, $B_{0,\pm x}$ and $B_{0,\pm y}$ which make up the linear spin-orbit coupling vector already given in Eq. (2.22). As can be seen from Derbenev and Kondratenko's formula (1.6) with (1.4) and (1.5), the quantities necessary to estimate the equilibrium polarization are $\langle|\partial\vec{n}/\partial\varepsilon|^2\rangle$ and $\langle\partial\vec{n}/\partial\varepsilon\rangle$, where the brackets $\langle\ \rangle$ denote the average over the distribution of I_s and ψ_s.

Assuming a gaussian distribution of ε, we get

$$\left\langle J_m^2(\sqrt{2I_s}\, u_\varepsilon)\right\rangle = \int_0^\infty \frac{dI_s}{\langle I_s\rangle}\, e^{-\frac{I_s}{\langle I_s\rangle}}\, J_m^2(\sqrt{2I_s}\, u_\varepsilon) = e^{-\sigma^2} I_m(\sigma^2) \tag{4.35}$$

with

$$\sigma = u_\varepsilon\sqrt{\langle I_s\rangle} = \sigma_\varepsilon \cdot \frac{d\nu}{d\varepsilon} \cdot \frac{1}{\nu_s} \ . \tag{4.36}$$

Here, $\sigma_\varepsilon = \sigma_E/E$ is the r.m.s. relative energy spread and I_m denotes the modified Bessel function. Thus, averaging over I_s and ψ_s gives

$$\left\langle\left|\frac{\partial n}{\partial\varepsilon}\right|^2\right\rangle = \left\langle\left|\frac{\partial\zeta}{\partial\varepsilon}\right|^2\right\rangle = \sum_{m=-\infty}^{\infty} |C_m(\theta)|^2\, e^{-\sigma^2} I_m(\sigma^2) \ . \tag{4.37}$$

Substituting $\exp(i\chi)$ in Eq. (4.33) with Eq. (4.22) and averaging over

I_s and ψ_s, we get

$$\left\langle \frac{\partial \vec{n}}{\partial \varepsilon} \right\rangle = \mathrm{Re}\left(\vec{k}_0^* \left\langle \frac{\partial \zeta}{\partial \varepsilon} \right\rangle \right) \tag{4.38}$$

with

$$\left\langle \frac{\partial \zeta}{\partial \varepsilon} \right\rangle = \sum_{m=-\infty}^{\infty} C_m(\theta)\, e^{-\sigma^2} I_m(\sigma^2) \quad . \tag{4.39}$$

These equations, (4.37) and (4.39) with (1.6), make up our final formulae of the depolarization due to the energy spread. The precise evaluation should be done by computers but here we roughly evaluate Eq. (4.37) by picking up one single resonance term. Eq. (4.39) is relatively unimportant because its contribution to Eq. (1.6) is usually small owing to the cancellation during the integration over the ring.

Expanding $\vec{\omega}_j \cdot \vec{k}_0$ (j = x,y,s) in Eqs. (4.26) and (4.27) into Fourier series, we get

$$\vec{\omega}_s \cdot \vec{k}_0 = \sum_{n=-\infty}^{\infty} b_{n,s}\, e^{i(n+\nu)\theta} \tag{4.40}$$

$$e^{\pm i \Psi_j} \sqrt{\beta_j}\, \vec{\omega}_j \cdot \vec{k}_0 = \sum_{n=-\infty}^{\infty} b_{n,\pm j}\, e^{i(\nu \pm \nu_j + n)\theta} \qquad (j = x, y) \tag{4.41}$$

and

$$A_m(\theta) = -\sum_n \frac{b_{n,s}}{\nu + m\nu_s + n} e^{i(\nu+n)\theta} \tag{4.42}$$

$$B_{m,\pm j}(\theta) = -\frac{1}{2} \sum_n \frac{b_{n,\pm j}}{\nu \pm \nu_j + m\nu_s + n} \frac{e^{\mp i \Psi_j}}{\sqrt{\beta_j}} e^{i(\nu \pm \nu_j + n)\theta} \qquad (j = x, y) \tag{4.43}$$

Then, we have

$$C_m(\theta) = \sum_n \frac{-(\nu+n)\, b_{n,s}\, e^{i(\nu+n)\theta}}{[\nu + n + (m+1)\nu_s][\nu + n + (m-1)\nu_s]}$$

$$-\frac{1}{2} \sum_{j=x,y} \sum_{\pm} \sum_n a_{\pm j}(\theta) \frac{b_{n,\pm j}}{\nu \pm \nu_j + m\nu_s + n} e^{i(\nu \pm \nu_j + n)\theta} \tag{4.44}$$

where $a_{\pm j}(\theta)$ is already defined in Eq. (2.17).

First, pick up side band resonances around an integer. When $\Delta\nu \equiv \nu_0 + n$ is small, C_m can be approximated by

$$C_m = -\frac{\Delta\nu \cdot b_{n,s}}{(\Delta\nu + m\nu_s)^2 - \nu_s^2} \tag{4.45}$$

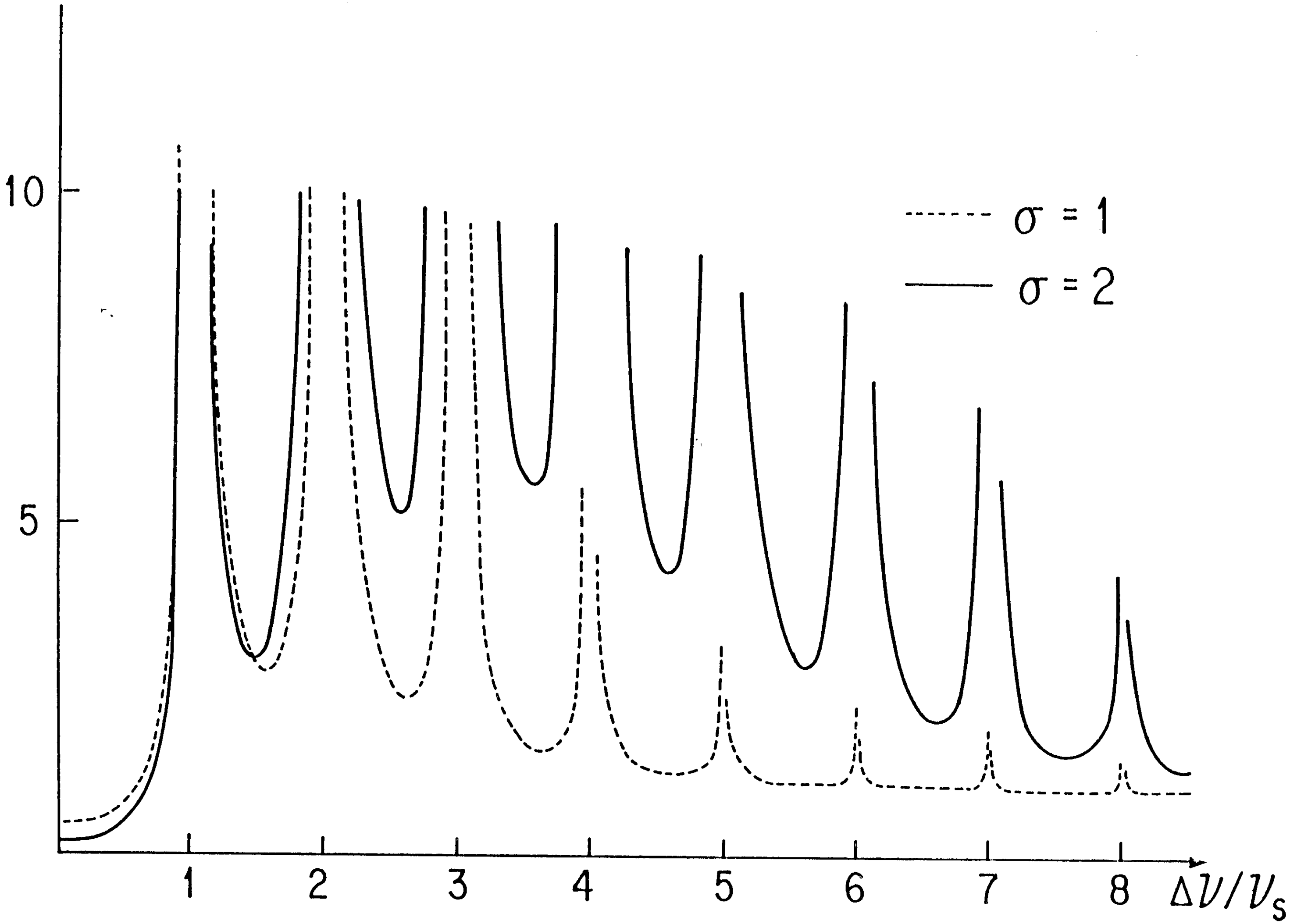


Fig.4. The enhancement factor of a betatron resonance given by Eq(4.49).

and $\langle|\partial\vec{n}/\partial\varepsilon|^2\rangle$ becomes

$$\left\langle\left|\frac{\partial\vec{n}}{\partial\varepsilon}\right|^2\right\rangle = |b_{n,s}|^2 \sum_{m=-\infty}^{\infty}\left[\frac{\Delta\nu}{(\Delta\nu+m\nu_s)^2-\nu_s^2}\right]^2 e^{-\sigma^2} I_m(\sigma^2) \quad . \tag{4.46}$$

When the energy spread is small and, therefore σ is small, only the term with m = 0 dominates and we get

$$\left\langle\left|\frac{\partial\vec{n}}{\partial\varepsilon}\right|^2\right\rangle = |b_{n,s}|^2 \frac{\Delta\nu^2}{\Delta\nu^2-\nu_s^2} \quad . \tag{4.47}$$

The ratio of (4.46) to (4.47) is the enhancement factor of the depolarization.

Similarly, for side band resonances of betatron oscillations we get

$$\left\langle\left|\frac{\partial\vec{n}}{\partial\varepsilon}\right|^2\right\rangle = \frac{1}{4}\,|a_{\pm j}|^2\cdot|b_{n,\pm j}|^2 \sum_{m=-\infty}^{\infty}\frac{1}{(\Delta\nu+m\nu_s)^2} e^{-\sigma^2} I_m(\sigma^2) \quad , \tag{4.48}$$

where $\Delta\nu = \nu_0 + n \pm \nu_j$ (j = x or y). Again comparing it with the case $\sigma_\varepsilon = 0$, we get the enhancement factor near the betatron resonances as

$$\frac{\Delta P}{\Delta P(\sigma_\varepsilon=0)} = \sum_{m=-\infty}^{\infty}\left(\frac{\Delta\nu}{\Delta\nu+m\nu_s}\right)^2 e^{-\sigma^2} I_m(\sigma^2) \quad . \tag{4.49}$$

Eqs. (4.48) and (4.49) have similar dependences on $\Delta\nu$ and ν_s. They have synchrotron oscillation sidebands at $\Delta\nu + m\nu_s = 0$ which are strong for $|m| \lesssim \sigma^2$. Fig. 4 shows the enhancement factor (4.49) as a function of $\Delta\nu/\nu_s$ for two fixed values of σ, 1 and 2. Even if the enhancement factor is large, we can reduce the strength of the resonance in principle by controlling the coefficients $b_{n,s}$ and $b_{n,\pm j}$, but since the enhancement factor is a rapidly increasing function of σ, a high polarization can not be expected at large σ in practice. For TRISTAN the upper limit may be around 30 GeV, which corresponds to $\sigma \sim 1$.

4.3) Cures for the Depolarization due to the Energy Spread

Some methods have been proposed to get rid of this kind of depolarization. One of them is to control the shape of the energy distribution function using non-linear wigglers[38] such as dipole-octupole wigglers and quadrupole-sextupole wigglers. According to the simple picture of the depolarization explained in §4.1 we may expect that the depolarization becomes weak if the tail of the distribution function is more strongly cut off than gaussian. An estimation of polarization was done by C. Biscari, J. Buon and B.W. Montague[39]. However, non-linear wigglers were first conceived in order to avoid beam instabilities which are in general strong at low energy operation of a given machine. In contrast, the usage at high energies is required for our purpose. Hence, the required length and field strength of the wigglers are enormous and look impractical.

The second method is the so-called Siberian snake,[40] which is a spin rotator to rotate the spin by 180° around the direction of motion. When it is inserted into a planar ring, the simple relation between the spin tune and the beam energy $\nu = \gamma a$ no longer holds and ν becomes 1/2 for particles of any energy. The depolarization is not enhanced since there is no spin tune spread $d\nu/d\varepsilon = 0$. The Siberian snake was originally proposed for the elimination of the spin resonaces in general rather than for the cure of the problem of the energy spread. It is best suited to the acceleration of polarized proton beams, but has two difficulties for the case of electrons. One of them is that the spin-orbit coupling vector $\gamma\partial\vec{n}/\partial\gamma$ is very large in the original version of th snake (first kind). This problem was solved by A. Turrin[41] by the double Siberian snake scheme, in which the second kind Siberian snake to rotate the spin by 180° around the x axis is inserted at the opposite side to the first kind Siberian snake. The second problem is that even if the electron beam is initially polarized, it will be depolarized by spin-flip radiation during the time of the order of τ_p defined in Eq. (1.1), because the periodic solution $\vec{n}_0(\theta)$ is on the horizontal plane in most portion of the ring in the single Siberian snake scheme. The double snake scheme cannot solve this problem because $\vec{n}_0 = + \vec{e}_y$ in one half of the ring and $\vec{n}_0 = - \vec{e}_y$ in the other half. The spontaneous polarization can not be expected in the both schemes due to the same reason. One solution to this problem is to irradiate a laser beam on the circulating electrons and polarize them more rapidly than τ_p[42], but it does not seem feasible. The other method is to make the ring asymetrical by placing the second kind Siberian snake near the first kind[43]. It is a mixture of the planar ring ($\nu = \gamma a$) and the Siberian snake ($\nu = 1/2$). A similar method is to make the field strength of the bending magnets in one half of the ring stronger than the other half. All of these are too complicated to be practical. Even if these problems are solved, the spin matching must still be done to get significant polarization. Attempts have been done by K. Steffen[44] and J. Buon[45] but still insufficient. It seems from the above considerations that the application of the Siberian snake to electron machines is not feasible.

The present author[46] proposed another method to avoid the depolarization due to the energy spread, in which the spin tune has no spread ($d\nu/d\varepsilon = 0$) and the spontaneous polarization may still be expected. The Siberian snake scheme has two special properties of the spin tune , i.e. $\nu_0 = 1/2$ for the synchronous particle and $d\nu/d\varepsilon = 0$. (In the rest of this chapter we denote by ν_0 the spin tune of the synchronous particle which has been written simply as ν.) The first property is important for proton rings but only the second is essential for our problem.

Let us find a general expression for $d\nu/d\varepsilon$. From now on let us call it "spin chromaticity", hoping that one may not confuse it with $\gamma\partial\vec{n}/\partial\gamma$, which had misleadingly been called spin chromaticity in old literatures. The orbit oscillations in all three degrees of freedom need not be considered and the relative energy deviation ε is a constant. The equation of spin motion is

$$\frac{d\vec{s}}{d\theta} = [\vec{\Omega}_0(\theta) + \varepsilon\,\vec{\omega}_s(\theta)] \times \vec{s} \qquad , \tag{4.50}$$

where $\vec{\Omega}_0$ is given in Eq. (2.9) and $\vec{\omega}_s$ in Eq. (4.4). They are periodic functions of θ. One can easily verify that Eq. (4.50) is equivalent to Hamilton's equation of motion derived from the Hamiltonian

$$H(J_0, \varphi_0, \theta) = \nu_0 J_0 - \varepsilon\,\vec{\omega}_s(\theta)\cdot\vec{s}(J_0, \varphi_0, \theta) \tag{4.51}$$

with

$$\vec{s}(J_0, \varphi_0, \theta) = J_0\,\vec{n}_0 + \sqrt{1-J_0^2}\;\mathrm{Re}\,[\vec{k}_0^{*}\,e^{-i(\varphi_0-\nu_0\theta)}] , \tag{4.52}$$

where J_0 and φ_0 is the action and angle variables of the unperturbed system $\varepsilon = 0$. Let J be the new action variable. The new Hamiltonian Hnew (J) up to the first order perturbation is given by averaging the perturbation over θ and φ_0 as

$$\begin{aligned} H_{new}(J) &= \nu_0 J - \varepsilon\cdot(2\pi)^{-2}\oint d\theta \oint d\varphi_0\;\vec{\omega}_s(\theta)\cdot\vec{s}(J,\varphi_0,\theta) \\ &= \nu_0 J - \varepsilon J/2\pi\cdot\oint \vec{\omega}_s\cdot\vec{n}_0\,d\theta \;. \end{aligned} \tag{4.53}$$

Hence, the spin tune $\nu(\varepsilon)$ is given by

$$\nu(\varepsilon) = \frac{\partial H_{new}}{\partial J} = \nu_0 - \frac{\varepsilon}{2\pi}\oint \vec{\omega}_s\cdot\vec{n}_0\,d\theta \quad , \tag{4.54}$$

Fig.5.

One superperiod of a ring of superperiodicity N.

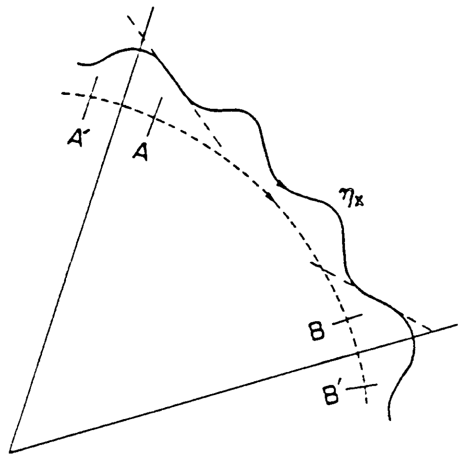


Fig.6.

Side view of a section, vertical component of $\vec{n}_0$ and eta function.

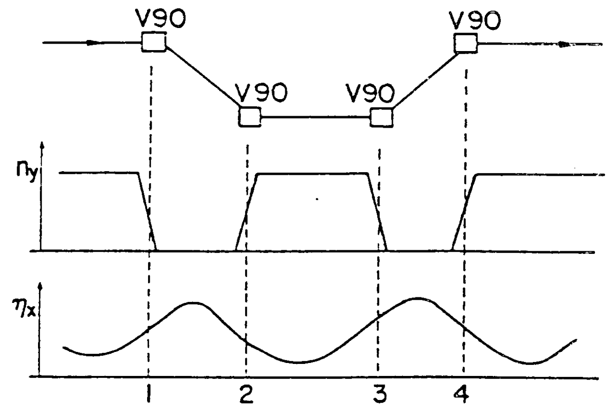
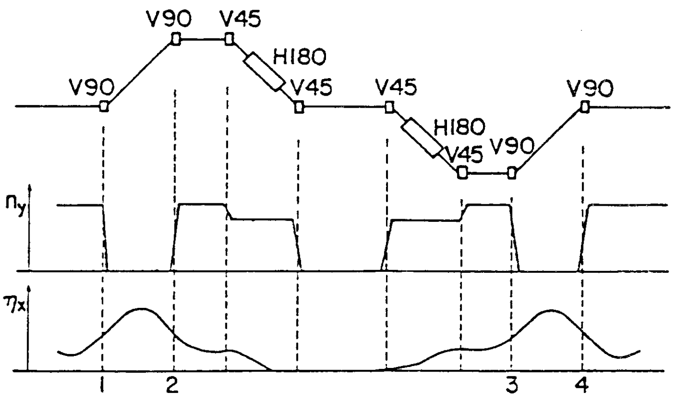


Fig.7.

Spin chromaticity corrector imbedded in a HERA type rotator.



which yields

$$\frac{d\nu}{d\varepsilon} = -\frac{1}{2\pi} \oint \vec{\omega}_s \cdot \vec{n}_0 \, d\theta \quad . \tag{4.55}$$

Using the explicit form Eq. (4.4) of $\vec{\omega}_s$, we have

$$\frac{d\nu}{d\varepsilon} = \frac{1}{2\pi} \oint \left\{ \left[(\gamma a+1) G_x \eta_x - \frac{1}{\rho_x} \right] \vec{n}_0 \cdot \vec{e}_y - \left[(\gamma a+1) G_y \eta_y - \frac{1}{\rho_y} \right] \vec{n}_0 \cdot \vec{e}_x \right\} R \, d\theta , \tag{4.56}$$

or, using the equation for η_x and η_y

$$\eta_j'' + G_j \eta_j = \frac{1}{\rho_j} \qquad (j = x, y) \; , \tag{4.57}$$

we get an expression of the spin chromaticity

$$\frac{d\nu}{d\varepsilon} = \frac{1}{2\pi} \oint \left\{ \left[(\gamma a+1) \eta_x'' + \frac{\gamma a}{\rho_x} \right] \vec{n}_0 \cdot \vec{e}_y - \left[(\gamma a+1) \eta_y'' + \frac{\gamma a}{\rho_y} \right] \vec{n}_0 \cdot \vec{e}_x \right\} R \, d\theta \tag{4.58}$$

In planar rings this gives $d\nu/d\varepsilon = \gamma a$, which is the contribution of the term $\gamma a/\rho_x$, since $\vec{n}_0 = \vec{e}_y$ and the integration of η''_x vanishes due to the periodicity of η_x. In the double Siberian snake scheme, the contribution of the term $\gamma a/\rho_x$ is cancelled because $\vec{n}_0 = \vec{e}_y$ in one half of the ring and $-\vec{e}_y$ in the other half.

The idea is to eliminate the spin chromaticity by making use of the term η''_x and/or η''_y in Eq. (4.58), keeping $n_y \equiv \vec{n}_0 \cdot \vec{e}_y = 1$ in most part of the ring, thus ensuring the spontaneous polarization.

Let us consider a ring of superperiodicity N, whose one superperiod is shown in Fig. 5. The dotted line is the design orbit and the full line is η_x drawn with the same scale as the design orbit. Suppose $n_y = 0$ in the intervals AA' and BB' and $n_y = 1$ in all the other parts. Then we find that if the full line at the point A and that at B are mutually parallel, which means $\eta'_B - \eta'_A = 2\pi/N$, the spin chromaticity is almost zero. This can easily be seen from Eq. (4.58) as

$$\frac{d\nu}{d\varepsilon} \doteq \frac{1}{2\pi} \left[N(\gamma a+1)(\eta'_B - \eta'_A) \right] + \gamma a = 1 \; ,$$

where the contributions of the terms of η''_y and ρ_y are ignored. (Note that unity is much smaller than γa, the value of $d\nu/d\varepsilon$ in planar rings.) One finds from Eq. (4.58) that in general the spin chromaticity can be reduced by changing n_y from unity to zero at the point $\eta'_x >$ 0 and by changing n_y from zero to unity where $\eta'_x < 0$.

A more realistic example is shown in Fig. 6. The upper drawing is the side view of a section which should be inserted to a planar ring where $n_y = 1$. "V90" is a vertical bend which rotates the spin by 90 degrees. Since $\vec{n}_0 \cdot \vec{e}_x = 0$ throughout this section, the contribution of η_y need not be considered. If we make η_x as in the lower drawing, we have a reduction of the spin chromaticity by the amount of about $\gamma a \times 4\eta'_0/2\pi$, assuming, for simplicity, $\eta'_1 = -\eta'_2 = \eta'_3 = -\eta'_4 \equiv \eta'_0$, where η'_1 is the value of η'_x at location 1, etc.

From the practical view point, these devices had better work also as spin rotators which rotate the spin to the longitudinal direction. Such an example is shown in Fig. 7, which is based on the spin rotator of HERA[47], with the orders of some magnets changed for our purpose. In this figure, "V_x" and "H_x" are vertical and horizontal bends which rotate the spin by x degrees. Assuming again for simplicity $\eta'_1 = -\eta'_2 = \eta'_3 = -\eta'_4 \equiv \eta'_0$ and neglecting the variation of n_y near "V45", we get a reduction of the spin chromaticity by $\gamma a \times 4\eta'_0/2\pi$. Unfortunately, if the insertion is completely antisymmetric with respect to the collision point, this method cannot be used, because the contribution of one half of the insertion cancells that of the other half.

This method is much simpler than the Siberian snake scheme because the spontaneous polarization can be expected. However, one practical problem was pointed out in the original paper and by J. Buon[34] that the required value of η'_0 is a little bit large.

In conclusion, the mechanism of the depolarization due to the energy spread has already been understood fully, but no satisfactory cures have been invented yet.

§5. Artificial Spin-Flip

5.1) Methods of Inverting the Spin

We have discussed in §2 about what is required to get a high polarization in the storage rings in which the spin is in an eigenstate of helicity at the colliding point. From the standpoint of experimental physicists it is desired that the experiments with various combinations of the helicities of e^+ and e^- beam can be performed by the same detectors. Unfortunately, however, the direction of the spontaneous polarization at a given point in the ring is uniquely determined by the design orbit and only slightly fluctuates due to magnet errors. It is possible in principle but not practical to invert the spin direction by moving the magnets in the spin rotators.

A design which contains vertical bends rotating the spin by 45 degrees has been proposed for the HERA electron ring[47]. At first it was planned to move vertically the magnets in the rotators in order to get the opposite helicity but later an improved rotator scheme[48], called mini-rotator, was proposed in which the spin can be inverted by distorting the closed orbit in large aperture magnets without moving magnets. It is now being studied extensively. A defect of these rotators is that a high polarization may be expected only when the beam energy is in a narrow interval of about ±1 % around a fixed value[49]. A correction method[50] was suggested which extends the allowed interval to ±10 %. This problem is relatively unimportant in electron-proton colliders such as HERA because one can vary the center-of-mass energy by adjusting the energy of the proton beam, but it is fatal for e^+e^- colliders.

It was shown theoretically by M. Froissart and R. Stora[51)] that the spin of the proton beam is flipped by a resonance crossing during acceleration if the resonance is strong enough or the acceleration is slow enough (adiabatic crossing). This phenomena has experimentally confirmed in ZGS[52)] and SATURN[53)]. The principle applies to electron rings, too. In stead of crossing the resonance by the change of spin tune or spin preccesion frequency during acceleration, the same phenomena occurs when the frequency of external perturbation fields changes and crosses the spin preccession frequency. An experiment in electron storage ring was done by Novosivirsk group[54)] using VEPP-2M at 650 MeV. They chose an oscillating longitudinal magnetic field as the perturbation in order to minimize the disturbance on the orbit motion. The spin was successfully inverted. The depolarization during the flip was within 5 % which is the accuracy of the polarization monitor. In high energy electron rings the transverse fields are preferred to longitudinal because the spin motion is much more sensitive to the former than to the latter. One can choose the combination of transverse field which does not affect the orbit motion.

If this method can be applied to high energy colliders, one can easily invert the spin during operation and it greatly relaxes the constraints on the designs of spin rotators. Moreover, one can in principle invert the spin of either electron or positron.

However, a problem in applying this method to high energy electron rings has also been pointed out by the same group[54)]. During the process of the spin-flip, the coherence of the spin phase will be lost by the stochastic change of the particle energy due to (spin-nonflip) synchrotron radiations and the beam may be depolarized. In order to avoid this phenomenon, the spin must be flipped fast enough, which, however, may prevent the complete spin flip. Hence a very large perturbation is required to keep the adiabaticity of the resonance crossing.

Their arguments may be summarized as follows. The variation of the spin precession phase Φ of an electron with the relative energy deviation ε can be expressed as

$$\frac{d\Phi}{d\theta} = \gamma (1 + \varepsilon) a \quad , \tag{5.1}$$

where γ is the energy of the synchronous particle divided by the rest energy. Let $\Delta\varepsilon_i$ be the change of ε due to a synchrotron radiation at $\theta = \theta_i$. Then the spin phase will be shifted by the amount

$$\gamma a \, \Delta\varepsilon_i \int_{\theta_i}^{\theta} \cos \nu_s (\theta - \theta_i) \, d\theta = \frac{\gamma a}{\nu_s} \Delta\varepsilon_i \sin \nu_s (\theta - \theta_i)$$

at the azimuth θ after the radiation. Summing up all the effects of radiations between $\theta = 0$ and $\theta = \theta$, we have the total change of the phase

$$\Delta\Phi(\theta) = \frac{\gamma a}{\nu_s} \sum_{0 < \theta_i < \theta} \Delta\varepsilon_i \sin \nu_s (\theta - \theta_i) \; . \tag{5.2}$$

Averaging $\Delta\Phi^2(\theta)$ over a long time, we get

$$\langle \Delta\Phi^2(\theta) \rangle = \left(\frac{\gamma a}{\nu_s}\right)^2 \langle \Delta\varepsilon_i^2 \rangle \frac{dN}{d\theta} \int_0^{\theta} d\theta' \sin^2 \nu_s (\theta - \theta') \; . \tag{5.3}$$

Here $dN/d\theta$ is the mean number of emitted photons during the excursion of one radian of θ. Ignoring the periodic term after integration, we get

$$\langle \Delta\Phi^2(\theta) \rangle = \left(\frac{\gamma a}{\nu_s}\right)^2 \langle \Delta\varepsilon_i^2 \rangle \frac{dN}{d\theta} \cdot \frac{\theta}{2} \; . \tag{5.4}$$

Using the relation[56)]

$$\langle \Delta\varepsilon_i^2 \rangle \frac{dN}{d\theta} = \frac{11}{9} \frac{T_0}{\tau_p} \frac{1}{2\pi} \quad , \tag{5.5}$$

where T_0 is the revolution period and τ_p the polarization time defined by Eq. (1.1), we find the expectation value of the square of the spin phase shift during n(= $\theta/2\pi$) revolutions

$$\langle \Delta\Phi^2 \rangle = \frac{11}{9} \left(\frac{\gamma a}{\nu_s}\right)^2 \frac{T_0}{\tau_p} n \tag{5.6}$$

The right hand side of this equation has a strong dependence on the beam energy (approximately the seventh power) for a given ring. Hence at high energies n must be very small in order to keep $\langle \Delta\Phi^2 \rangle$ below unity, i.e., the resonance must be crossed very quickly.

Obviously a more quantitative discussion is required for the actual design of the spin-flipper. It was done by the present author[55)] and is explained in the rest of this chapter. An elegant scheme of machine operation was proposed by J.M. Jowett and R.D. Ruth[57)] in which the collision of alternating helicities can be made by successively exciting the spin flipper.

5.2) Influence of Synchrotron Oscillations on the Spin-Flip

In this section the effects of synchrotron oscillations during the resonance crossing without radiations are discussed. Hence, the results can be applied to the spin-flip of proton beams if the relevant parameters lie in the range allowed by the approximations employed here.

The equation of the spin motion can be written as

$$\frac{d\vec{S}}{d\theta} = \left[\vec{\Omega}_0(\theta) + \varepsilon\, \vec{\omega}_s(\theta) + \vec{\Omega}_F(\theta)\right] \times \vec{S} \ . \tag{5.7}$$

Here $\vec{\Omega}_0$ and $\vec{\omega}_s$ are periodic functions of θ, defined already, and $\vec{\Omega}_F(\theta)$ is the perturbation by the spin-flipper. In this chapter, for the notational convenience, we denote by $\vec{n}_i$(i = 1,2,3) the orthonormal solutions of the unperturbed equation (2.5) which have been denoted by $\vec{n}_0$ and $\vec{k}_0 = \vec{m}_0 + i\vec{\ell}_0$ up to now, i.e.,

$$\vec{n}_1 = \vec{m}_0 \quad , \quad \vec{n}_2 = \vec{\ell}_0 \quad \text{and} \quad \vec{n}_3 = \vec{n}_0 \quad . \tag{5.8}$$

In planar rings $\vec{n}_3$ is directed to the vertical axis.

Let us rewrite Eq. (5.7) using spinor representation. We define two-component spinor $\Psi(\theta)$ by

$$\vec{S} = \sum_{j=1}^{3} \Psi^* \sigma_j \Psi \cdot \vec{n}_j \quad , \tag{5.9}$$

where σ's are Pauli matrices and the asterisk denotes Hermitian conjugate on spinors and matrices and complex conjugate on scalars. Using the fact that $\vec{n}_j$'s satisfy the unperturbed equation (2.5), one can easily verify that the equation

$$\frac{d\Psi}{d\theta} = -\frac{i}{2} \sum_{j=1}^{3} (\varepsilon \vec{\omega}_s + \vec{\Omega}_F) \cdot \vec{n}_j \, \sigma_j \Psi \tag{5.10}$$

is equivalent to Eq. (5.7). This can be written using the representation of Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.11}$$

as

$$\frac{d\Psi}{d\theta} = -\frac{i}{2} \begin{pmatrix} (\varepsilon\vec{\omega}_s + \vec{\Omega}_F)\cdot\vec{n}_3 \; , & (\varepsilon\vec{\omega}_s + \vec{\Omega}_F)\cdot(\vec{n}_1 - i\vec{n}_2) \\ (\varepsilon\vec{\omega}_s + \vec{\Omega}_F)\cdot(\vec{n}_1 + i\vec{n}_2) \; , & -(\varepsilon\vec{\omega}_s + \vec{\Omega}_F)\cdot\vec{n}_3 \end{pmatrix} \Psi \; . \tag{5.12}$$

We make the following assumptions for $\vec{\omega}_s$. Since $\vec{\omega}_s \cdot \vec{n}_1$ and $\vec{\omega}_s \cdot \vec{n}_2$ oscillate rapidly, we will neglect them. The integral of the term $\vec{\omega}_s \cdot \vec{n}_3$ gives the dependence of the spin tune on ε (spin chromaticity) as is seen in Eq. (4.55). Neglecting rapidly oscillating components we assume

$$\vec{\omega}_s \cdot \vec{n}_3 = -\left(\frac{d\nu}{d\varepsilon}\right)_{\varepsilon=0} \quad . \tag{5.13}$$

In actual rings except the Siberian snake the expression

$$\left(\frac{d\nu}{d\varepsilon} \right)_{\varepsilon=0} = \gamma a \tag{5.14}$$

is a very good approximation. Hence we use this expression, although the results can easily be extended to a general case.

Next, we make assumptions for $\vec{\Omega}_F$. We will ignore $\vec{\Omega}_F \cdot \vec{n}_3$, since it gives merely a weak modulation of the spin phase and does not cause spin-flip. As for the term $\vec{\Omega}_F \cdot (\vec{n}_1 + i\vec{n}_2)$, we will keep the resonating terms only

$$\vec{\Omega}_F \cdot (\vec{n}_1 + i\vec{n}_2) = -\nu_1 \, e^{i\,[(\nu+n)\theta + \phi_F(\theta)]} \tag{5.15}$$

as is usually done. Here ν_1 is a real positive constant, n an integer and $\phi_F(\theta)$ is the phase of the spin-flipper, which varies as

$$\phi_F(\theta) = \phi_{F0} + f_0\,\theta - \frac{1}{2}\alpha\theta^2 , \tag{5.16}$$

where $f_0 = -\nu - n$ is the frequency (in the unit of the revolution frequency) of the flipper at the very instant of resonance crossing and α is a constant that gives the speed of crossing. (If the resonance crossing is caused by the acceleration, $\alpha = (a/2\pi) \times$ (increment of γ during one revolution).) We will assume that α is positive, but the results for negative α can be obtained simply by replacing α with $|\alpha|$ in the final expressions of depolarization.

Under the above assumptions, Eq. (5.12) can be written as

$$\frac{d\Psi}{d\theta} = -\frac{i}{2} \begin{bmatrix} -\gamma a \varepsilon , & -\nu_1 \exp(+\frac{i}{2}\alpha\theta^2 - i\phi_{F0}) \\ -\nu_1 \exp(-\frac{i}{2}\alpha\theta^2 + i\phi_{F0}), & \gamma a \varepsilon \end{bmatrix} \Psi . \tag{5.17}$$

We have derived this equation in a general way since the ring we treat here is not planar in general, but the same equation can be obtained for planar rings. (See, for example, ref. 58.)

Replacing the independent variable θ with t, defined by $t = \theta\sqrt{\alpha}$

and rotating the axis around $\vec{n}_3$ as

$$\Psi = e^{i\left(\frac{1}{4}\alpha\theta^2 - \frac{1}{2}\phi_{F0}\right)\sigma_3}\,\psi\,, \tag{5.18}$$

we get an equation for ψ

$$\frac{d\psi}{dt} = \frac{i}{2}\left(H_0 + \Delta H\right)\psi \tag{5.19}$$

with

$$H_0 = \begin{pmatrix} -t & b \\ b & t \end{pmatrix}, \tag{5.20}$$

$$\Delta H = \begin{pmatrix} \gamma a \varepsilon/\sqrt{\alpha} & 0 \\ 0 & -\gamma a \varepsilon/\sqrt{\alpha} \end{pmatrix} \tag{5.21}$$

and

$$b = \frac{\nu_1}{\sqrt{\alpha}}\,. \tag{5.22}$$

Using the explicit form of the synchrotron oscillation

$$\varepsilon = \varepsilon_{max}\cos(\nu_s\theta + \phi_{so}) = \varepsilon_{max}\cos(\lambda_s t + \phi_{so})\,, \tag{5.23}$$

where ε_{max} is the amplitude, ϕ_{so} the phase at the moment of resonance crossing and λ_s is defined by

$$\lambda_s = \frac{\nu_s}{\sqrt{\alpha}}\,, \tag{5.24}$$

we obtain

$$\Delta H = u\cos(\lambda_s t + \phi_{so})\,\sigma_3 \tag{5.25}$$

with

$$u = \frac{\gamma a}{\sqrt{\alpha}}\,\varepsilon_{max}\,. \tag{5.26}$$

From now on we call the term of synchrotron oscillation ΔH perturbation. The unperturbed equation

$$\frac{d\psi}{dt} = \frac{i}{2}H_0\psi \tag{5.27}$$

contains the effect of the spin flipper and it was solved exactly by Froissart and Stora[50]. If the beam is completely polarized along the direction of $\vec{n}_3$ at $t = -\infty$, the polarization at $t = +\infty$ is given by the Froissart-Stora's formula

$$P_{FS} = 2\, e^{-\frac{\pi}{2} b^2} - 1 \ . \tag{5.28}$$

If b is large, $P_{FS} = -1$, i.e. complete spin-flip. In order to solve the perturbed equation (5.19) we make two approximations. First we assume $b \gtrsim 2$ which indicates that in the absence of synchrotron oscillation the spin is almost completely flipped. The opposite case $b \ll 1$, i.e., crossing of weak resonance, was solved by R.D. Ruth[59], but we are interested in the spin-flip case only. Second approximation is that u is small enough to allow the first-order perturbation. We may consider that this condition is satisfied if the resulting polarization is close to P_{FS} or, equivalently, the depolarization is small.

Now, let us denote the two independent solutions to the unperturbed equation (5.27) by $\psi_1(t)$ and $\psi_2(t)$ which correspond to the complete polarization in the direction $+\vec{n}_3$ and $-\vec{n}_3$, respectively, at $t = -\infty$. At $t = +\infty$ these solutions show almost complete polarization in the opposite direction due to the assumption $b \gtrsim 2$. They satisfy the orthogonality relation

$$\psi_i^*(t)\ \psi_j(t) = \delta_{ij} \qquad (i, j = 1, 2) \tag{5.29}$$

at any t, where δ_{ij} is Kronecker's delta. In addition it can easily be verified that the spin vectors for these solutions are always opposite, i.e.,

$$\psi_1^*(t)\ \sigma_j\ \psi_1(t) + \psi_2^*(t)\ \sigma_j\ \psi_2(t) = 0 \ . \quad (j = 1, 2, 3) \tag{5.30}$$

Let us expand the solution $\psi(t)$ to the perturbed equation (5.19) in terms of the unperturbed solutions as

$$\psi(t) = C_1(t)\,\psi_1(t) + C_2(t)\,\psi_2(t) \tag{5.31}$$

with

$$|C_1(t)|^2 + |C_2(t)|^2 = 1 . \tag{5.32}$$

We impose the initial condition so that the spin is directed to $+\vec{n}_3$ at $t = -\infty$;

$$C_1(-\infty) = 1 \quad \text{and} \quad C_2(-\infty) = 0 . \tag{5.33}$$

Because of the synchrotron oscillation, C_1 and C_2 will gradually move away from these values. The spin component along $\vec{n}_3$ at t is given by

$$\begin{aligned}\vec{S}\cdot\vec{n}_3 &= \psi(t)^* \sigma_3\, \psi(t) \\ &= |C_1|^2 \psi_1^* \sigma_3 \psi_1 + |C_2|^2 \psi_2^* \sigma_3 \psi_2 + 2\,\mathrm{Re}(C_1^* C_2\, \psi_1^* \sigma_3 \psi_2) .\end{aligned} \tag{5.34}$$

We may omit the last term by averaging many particles. Then using Eqs. (5.30) and (5.32), we get the beam polarization at $t = +\infty$ as

$$P = (1 - \Delta P)\, \psi_1^* \sigma_3 \psi_1 \tag{5.35}$$

with

$$\Delta P = 2 \lim_{t \to +\infty} \langle |C_2(t)|^2 \rangle , \tag{5.36}$$

where the brackets indicate the average over particles, or over the phase ϕ_{so} and the amplitude ε_{max} of the synchrotron oscillation. The factor $\psi^*_1 \sigma_3 \psi_1$ is just the final polarization of the unperturbed case, P_{FS}. Hence, we have

$$P = (1 - \Delta P)\, P_{FS} . \tag{5.37}$$

Since we consider only the case $P_{FS} = -1$, we may think that the depolarization due to the synchrotron oscillation is given by ΔP. Now what we have to know is $C_2(\infty)$.

The equation that C_1 and C_2 must satisfy is

$$\frac{d}{dt}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{i}{2}\begin{bmatrix} \psi_1^* \Delta H \psi_1 & , & \psi_1^* \Delta H \psi_2 \\ \psi_2^* \Delta H \psi_1 & , & \psi_2^* \Delta H \psi_2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} . \tag{5.38}$$

Taking the first-order perturbation of ΔH and substituting C_1 and C_2 on the right hand side with the unperturbed solutions $C_1(t) = 1$ and $C_2(t) = 0$, we get

$$\begin{aligned} C_2(\infty) &= \frac{i}{2}\int_{-\infty}^{\infty} \psi_2^*(t)\, \Delta H(t)\, \psi_1(t)\, dt \\ &= \frac{i}{2} u \int_{-\infty}^{\infty} \cos(\lambda_s t + \phi_{so})\, (\psi_2^* \sigma_3 \psi_1)\, dt . \end{aligned} \tag{5.39}$$

Now let us introduce the Fourier transform $G(\omega,b)$ of $\psi^*_2\sigma_3\psi_1$ defined by

$$G(\omega,b) = \frac{e^{-ic_0}}{2\pi}\int_{-\infty}^{\infty} \psi_2^*(t)\, \sigma_3\, \psi_1(t)\, e^{-i\omega t}\, dt , \tag{5.40}$$

where c_0 is a real constant added for convenience. We can choose c_0 so that $G(\omega,b)$ is real when $b \gg 1$. By using $G(\omega,b)$ we can rewrite Eq. (5.39) as

$$C_2(\infty) = i\,\frac{\pi u}{2}\, e^{ic_0}\left[e^{i\phi_{so}} G(\lambda_s, b) + e^{-i\phi_{so}} G(-\lambda_s, b) \right] . \tag{5.41}$$

Averaging $|C_2(\infty)|^2$ over the initial phase of the synchrotron oscillation ϕ_{so}, we get

$$\Delta P = \frac{(\pi u)^2}{2}\left(|G(\lambda_s,b)|^2 + |G(-\lambda_s,b)|^2 \right) . \tag{5.42}$$

Since $G(\omega,b)$ is very small for $\omega < 0$ as stated below, the following expression holds in practice

$$\Delta P = \frac{(\pi u)^2}{2}\left|G(\lambda_s,b)\right|^2 = \frac{(\pi \gamma a\, \varepsilon_{max})^2}{2\alpha}\left|G\left(\frac{\nu_s}{\sqrt{\alpha}}, b\right)\right|^2 . \tag{5.43}$$

If we average this expression over the distribution of ε_{max}, we obtain finally

$$\Delta P = \frac{(\pi \gamma a\, \sigma_\varepsilon)^2}{\alpha}\left|G\left(\frac{\nu_s}{\sqrt{\alpha}}, b\right)\right|^2 , \tag{5.44}$$

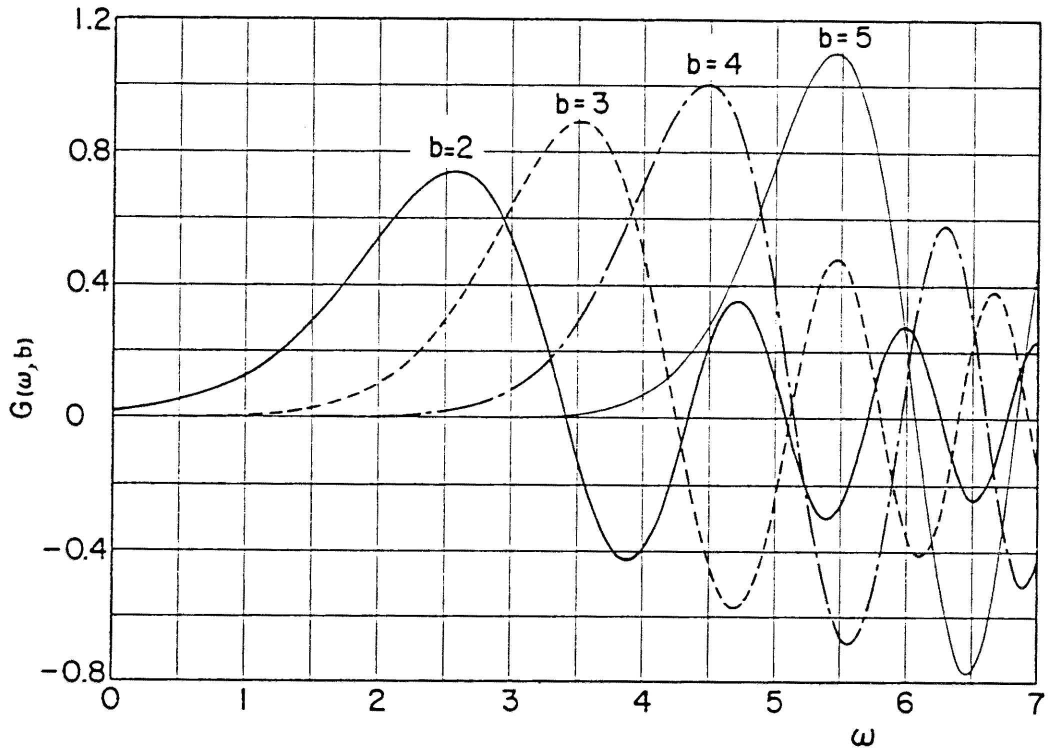


Fig.8. G(ω,b) as a function of ω for several fixed values of b.

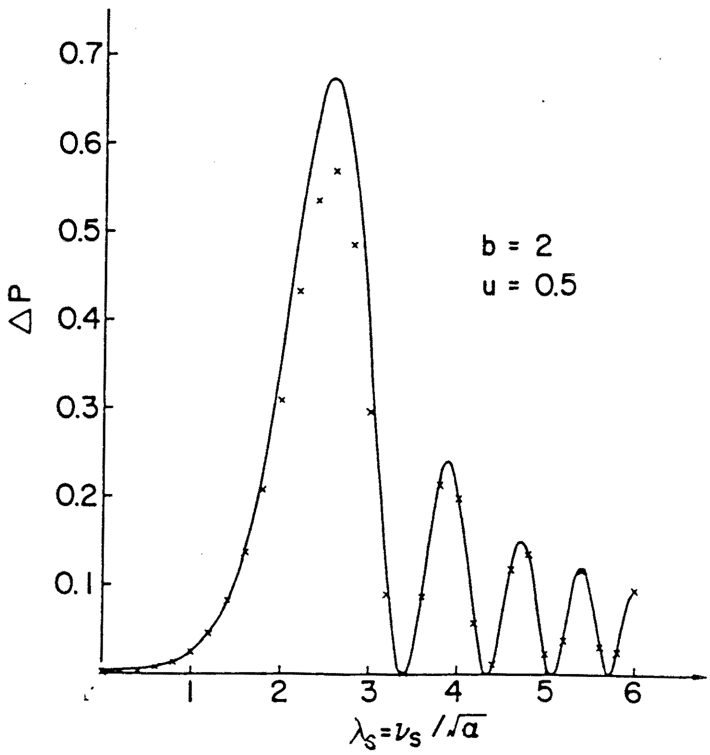


Fig.9. An example of depolarization due to synchrotron oscillations. The full line is plotted using Eq(5,43). The crosses show the results of numerical integration of differential equation Eq(5.19).

where σ_ε is the r.m.s. relative energy spread.

The evaluation of $G(\omega,b)$ is given in the appendix B. A simple expression for $G(\omega,b)$ for $\omega \gtrsim 1/b$ is Eq. (B.17) with (B.18). It is plotted in Fig. 8 for several values of b. When b is large, $G(\omega,b)$ for $\omega \le 0$ is very small. (The value $G(0,b)$ is of the order of $b^{1/3}\exp(-\pi b^2/4)$.) It increases rapidly with ω when $\omega > 0$ and reaches a maximum

$$G_{max} = 0.6749\ b^{1/3}\ (1 + O(b^{-4/3})) \tag{5.45}$$

at

$$\omega = \omega_{max} = b + 0.8086\ b^{-1/3} + O(b^{-5/3})\ . \tag{5.46}$$

For $\omega > \omega_{max}$, $G(\omega,b)$ oscillates with the amplitude decreasing as $1/\omega$.

An example is shown in Fig. 9 where ΔP calculated by Eq. (5.43) is plotted in a full line against λ_s with u fixed to 0.5. The crosses show the results of computer simulation or, in more proper words, numerical integration of the differential equation (5.19). They agree with each other quite well except around λ_s = 2 to 3 where ΔP is very large and, therefore, our perturbation approximation is not very good.

A remarkable fact seen from this figure is that the depolarization ΔP does not monotonically increase with synchrotron-oscillation frequency. This bebavior of ΔP can be understood as a kind of resonance between the synchrotron oscillation and the spin precession during the spin-flip. The first peak corresponds to $\lambda_s \approx b$ or, equivalently, $\nu_s \approx \nu_1$. This is due to the fact that the spin precesses around $\vec{n}_1$ (or $\vec{n}_2$) at frequency ν_1 at the very moment of the resonance crossing. At arbitrary time, however, the instantaneous spin precession frequency $\omega(\theta)$ (in units of the revolution frequency) is equal to $(\nu_1^2 + \alpha^2\theta^2)^{1/2}$, since the spin rotates at the frequency $\alpha\theta$ around $\vec{n}_3$ and at ν_1 around $\vec{n}_1$ (or $\vec{n}_2$). This spin motion contains frequency components higher than

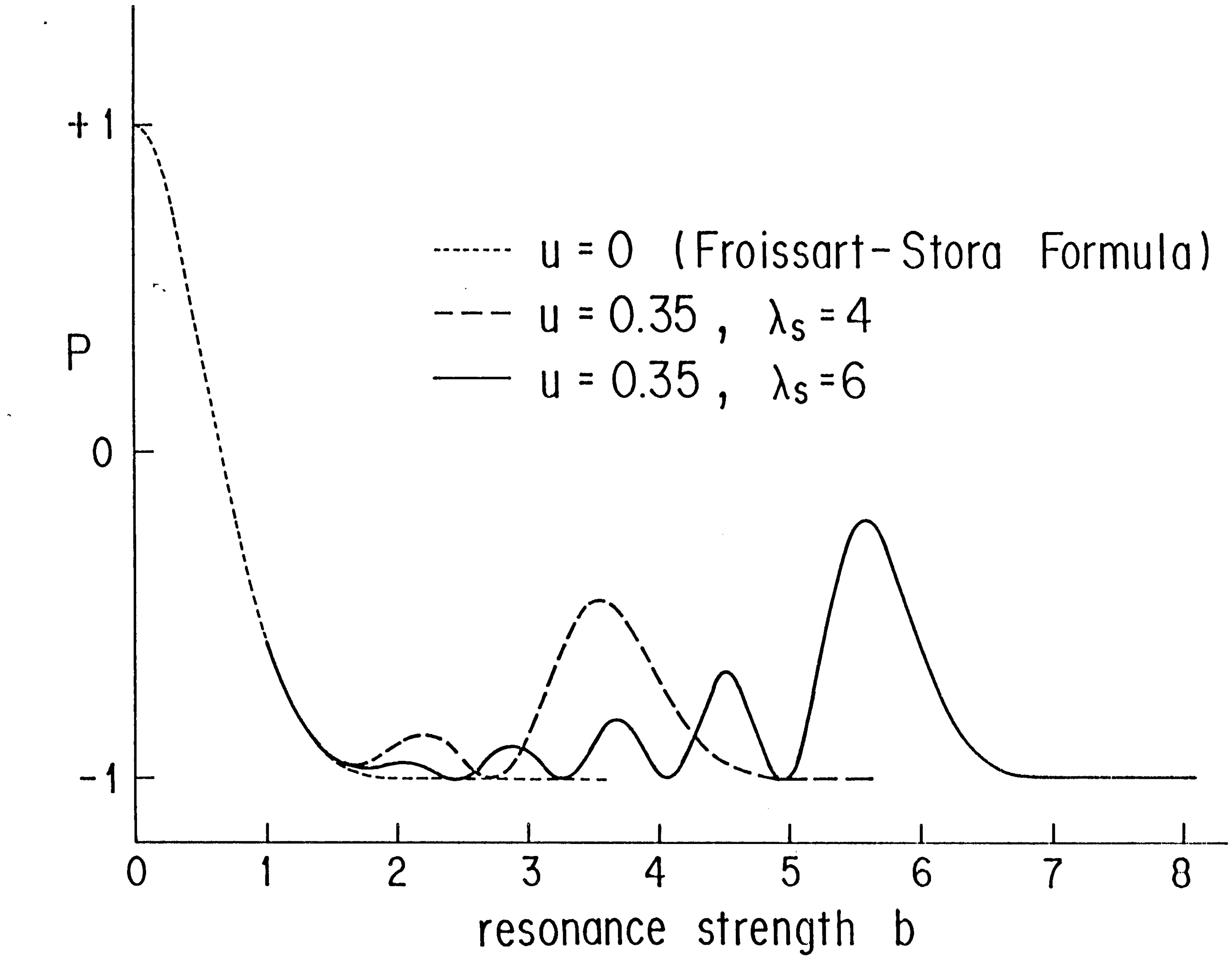


Fig.10. An example of the final polarization as a function of the resonance strength b with λ_s and u fixed.

ν_1, which explains the complicated behavior for $\lambda_s \gtrsim b$. As one can see in the appendix B, $G(\omega,b)$ is essentially the Fourier transform of exp $(i \int \omega(\theta) d\theta)$. A similar resonance-like behavior has been found by R.D. Ruth[59] in the case of weak resonances, i.e., spin-nonflip.

Fig. 10 shows an example of the final polarization given by Eq. (5.37) with Eq. (5.43) as a function of the resonance strength b with fixed λ_s and u. The dotted line shows the formula of Froissart and Stora. Two cases with $\lambda_s = 4$ and 6 are shown by the dashed and the full lines, respectively. In the absence of the synchrotron oscillation the spin flips almost completely if $b \gtrsim 1.5$ (dotted line). The synchrotron oscillation can cause a depolarization even if b is very large. The location of the right-most depolarization peak is given by inverting Eq. (5.46) as

$$b \approx \frac{\lambda_s}{1 + 0.8086\ \lambda_s^{-4/3}} \quad . \tag{5.47}$$

The same behavior of P as a function of the resonance strength was found by T. Aniel et al.[61] by computer trackings.

5.3) Influence of Diffusion by Radiations

Now, let us discuss the effects of the synchrotron radiation. The starting point is Eq. (5.19) with Eqs. (5.20) and (5.21). In the present case, ε in Eq. (5.21) can be written as

$$\varepsilon(t) = \sum_j \Delta\varepsilon_j \cos\lambda_s(t-t_j)\ \Theta(t-t_j)\ e^{-\lambda_D(t-t_j)} \quad . \tag{5.48}$$

Here $\Delta\varepsilon_j$ is the change of ε due to the radiation at $t = t_j$, Θ the unit step function and λ_D the damping constant, which is related to the longitudinal damping time τ_ε by

$$\tau_\varepsilon = \frac{T_0}{2\pi \lambda_D \sqrt{\alpha}} \quad . \tag{5.49}$$

In a manner similar to that of the previous section, we find the perturbation of $C_2(\infty)$ up to the first order of ΔH as

$$C_2(\infty) = \frac{i\gamma a}{2\sqrt{\alpha}} \sum_j \Delta\varepsilon_j \int_{-\infty}^{\infty} \Theta(t-t_j) \cos\lambda_s (t-t_j)\, e^{-\lambda_D (t-t_j)} \psi_2^*(t)\, \sigma_3\, \psi_1(t)\, dt \quad . \tag{5.50}$$

Making use of a Fourier-transformation formula

$$\cos\lambda_s t \; e^{-\lambda_D t}\, \Theta(t) = \int_{-\infty}^{\infty} V(\omega)\, e^{i\omega t}\, d\omega \, , \tag{5.51}$$

with

$$V(\omega) = \frac{1}{2\pi} \; \frac{-i\omega}{\omega^2 - \lambda_s^2 - 2i\lambda_D \omega} \quad , \tag{5.52}$$

we can rewrite Eq. (5.50) as

$$C_2(\infty) = i\pi \frac{\gamma a}{\sqrt{\alpha}} \sum_j \Delta\varepsilon_j \int_{-\infty}^{\infty} d\omega\, V(-\omega)\, G(\omega,b)\, e^{i\omega t} \quad . \tag{5.53}$$

Since there is no correlation among different radiations, the absolute square of the expression (5.53) can be averaged as

$$\begin{aligned} \langle |C_2(\infty)|^2 \rangle &= \frac{(\pi\gamma a)^2}{\alpha} \Big\langle \sum_j \Delta\varepsilon_j^2 \Big| \int_{-\infty}^{\infty} d\omega\, V(-\omega)\, G(\omega,b)\, e^{i\omega t_j} \Big|^2 \Big\rangle \\ &= \frac{(\pi\gamma a)^2}{\alpha} \langle \Delta\varepsilon_j^2 \rangle \frac{dN}{d\theta} \frac{d\theta}{dt} \\ &\quad \times \int_{-\infty}^{\infty} dt_j \iint d\omega\, d\omega'\, V(-\omega)\, G(\omega,b)\, V^*(-\omega')\, G^*(\omega',b)\, e^{i(\omega-\omega')t_j} \\ &= \frac{(\pi\gamma a)^2}{\alpha} \langle \Delta\varepsilon_j^2 \rangle \frac{dN}{d\theta} \frac{2\pi}{\sqrt{\alpha}} \int_{-\infty}^{\infty} d\omega\, |V(-\omega)\, G(\omega,b)|^2 \quad . \end{aligned} \tag{5.54}$$

Hence, using Eqs. (5.5), (5.36) and (5.52), we get

$$\Delta P = \frac{11}{18} \frac{T_0}{\tau_p} \frac{(\gamma a)^2}{\alpha\sqrt{\alpha}} \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(\omega^2 - \lambda_s^2)^2 + 4\lambda_D^2 \omega^2} \left| G(\omega,b) \right|^2 \quad . \tag{5.55}$$

Since $G(\omega,b)$ is very small for negative ω, as stated in the previous section, the lower limit of this integration can be set to zero.

One sees that the integrand of Eq. (5.55) has a sharp peak near $\omega = \lambda_s$ with a width of the order of λ_D. First, let us consider the physical meaning of this peak. Since λ_D is much smaller than λ_s, we may approximate $G(\omega, b)$ by $G(\lambda_s, b)$ around the peak. Then the contribution of this peak ΔP_1 is calculated to be

$$\Delta P_1 = \frac{11}{18}\frac{T_0}{\tau_p}\frac{(\gamma a)^2}{\alpha/\alpha}\frac{\pi}{4\lambda_D}\,|G(\lambda_s, b)|^2 \quad ,$$

which can be rewritten by using Eq. (3.2) as

$$\Delta P_1 = \frac{11\pi^2}{36}\frac{\tau_\varepsilon}{\tau_p}\frac{(\gamma a)^2}{\alpha}\,|G(\lambda_s, b)|^2 \quad . \tag{5.56}$$

Using the expression of the equilibrium energy spread in electron storage rings[56)]

$$\sigma_\varepsilon = \frac{11}{36}\frac{\tau_\varepsilon}{\tau_p} \quad , \tag{5.57}$$

one finds that Eq. (5.56) exactly coincides with Eq. (5.44). Hence the peak at $\omega = \lambda_s$ is the contribution of the energy spread that the beam had before the spin-flip. Indeed, one finds that such a spread has already been included in the expression (5.48) where the summation over j runs to $t_j = -\infty$.

Next, let us consider the integral (5.55) as a whole. In general, we have to carry out numerical integration, but we may integrate approximately under the assumption $\lambda_s \ll b$ which seems to be a practically important region because of the following reason. For rings with beam energy higher than about 20 GeV, the factor $(\pi\gamma a\varepsilon_{max})^2$ in Eq. (2.35) is at least about 0.02. Therefore, if $G(\lambda_s, b)$ is of the order of unity, i.e., if $\lambda_s \sim b$, sufficient spin-flip cannot be obtained unless α is more than 0.5 or so. On the other hand, b must be larger than about 2 in order to suppress the depolarization expressed by Froissart and Stora's formula. Hence one has to make $\nu_1 = (\alpha)^{1/2} b \gtrsim 1$. But this means that the spin-flipper has a strength that is enough to rotate the spin by more than 360 degrees during one single passage through the flipper. This seems to be very difficult in practice. Therefore, in order to reduce the value of the expression (5.44), one has to make $G(\lambda_s, b) \ll 1$ which is achieved by making λ_s considerably smaller than b. (As one can see from Fig. 8, $G(\lambda_s, b)$ has dips where $\lambda_s > b$. This point will be discussed later.)

Now, under this assumption, i.e., $\lambda_s \ll b$, $G(\lambda_s, b)$ is very small and the integrand of Eq. (5.55) consists of two portions which can clearly be distinguished. They are the sharp peak around $\omega \sim \lambda_s$ and the broad bump around $\omega \sim b$. The former is the contribution of the continuous synchrotron oscillation discussed above and the latter is that of the diffusion, which we denote by ΔP_2. Since near $\omega \sim b$ the factor in front of $|G(\omega,b)|^2$ can be approximated by ω^{-2}, we have

$$\Delta P_2 = \frac{11}{18} \frac{T_0}{\tau_p} \frac{(\gamma a)^2}{\alpha\sqrt{\alpha}} K(b) \tag{5.58}$$

with

$$K(b) = \int_{1/b}^{\infty} d\omega \, \frac{1}{\omega^2} \, |G(\omega, b)|^2 \, . \tag{5.59}$$

Here we take the lower limit of integration as 1/b, because the integrand diverges at $\omega = 0$. This value 1/b is near the minimum of the integrand. Since $|G|^2$ is a small quantity of the order of $\exp(-\pi b^2/2)$ near the origin, K(b) is insensitive to the value of the lower integration limit. When b is large, evaluation of this integral gives

$$K(b) \doteqdot \frac{1}{4b} \, , \tag{5.60}$$

as shown in the appendix B. This is a very good approximation. The error is less than 10 percent even at b = 2.

Then finally we get

$$\Delta P_2 = 0.15 \frac{T_0}{\tau_p} \frac{(\gamma a)^2}{\alpha\sqrt{\alpha}} \frac{1}{b} = 0.15 \frac{T_0}{\tau_p} \frac{(\gamma a)^2}{\alpha \nu_1} , \tag{5.61}$$

for $\lambda_s \ll b$ or, equivalently, $\nu_s \ll \nu_1$.

A comment should be added on the fact that this formula does not depend on ν_s as long as $\nu_s \ll \nu_1$. The value of $\langle \Delta\Phi^2 \rangle$ estimated in §5.1 is proportional to ν_s^{-2}, which means that a slow synchrotron oscillation gives large diffusion of the spin phase. We have to note, however, that we have dropped the periodic term between Eqs. (5.3) and (5.4) in the course of the estimation of $\langle \Delta\Phi^2 \rangle$. This is allowed only if the relavant time interval is long enough to satisfy $\nu_s\theta \gg 2\pi$. In the opposite case, $\nu_s\theta \ll 2\pi$, $\langle \Delta\Phi^2 \rangle$ does not depend on ν_s. In the actual case, θ should be taken to be the typical time scale of the resonance crossing.

Now, let us consider the possibility of making use of the dips of $G(\omega,b)^2$. When λ_s is exactly equal to the n-th zero ω_n of $G(\omega,b)$, ΔP_1 vanishes and by using Eq. (5.55), we get

$$\Delta P_2 = \frac{11}{18} \frac{T_0}{\tau_p} \frac{(\gamma a)^2}{\alpha\sqrt{\alpha}} Q_n(b) \qquad (\lambda_s = \omega_n) , \tag{5.62}$$

with

$$Q_n(b) = \int_{-\infty}^{\infty} \left(\frac{\omega}{\omega^2 - \omega_n^2} \right)^2 |G(\omega,b)|^2 \, d\omega . \tag{5.63}$$

In particular, for the first zero, the following formulae derived in the appendix B give very good approximations.

$$\omega_1 = b + 1.866\, b^{-1/3} \tag{5.64}$$

$$Q_1(b) = \frac{b}{2} \frac{1}{1 + 1.866\, b^{-4/3}} \tag{5.65}$$

Now, let us summarize how to estimate the optimum values of the parameters of the flipper, i.e., the strength ν_1 and the speed of frequency change α, which give only a small depolarization during the spin-flip process.

First, the depolarization given by Froissart-Stora's formula must be small enough, of course, which gives

$$b \equiv \frac{\nu_1}{\sqrt{\alpha}} \gtrsim 1.5 \quad . \tag{5.66}$$

Secondly, the depolarization due to the continuous synchrotron oscillation is given by Eq. (5.44);

$$\Delta P_1 = \frac{(\pi \gamma a \sigma_\varepsilon)^2}{\alpha} \left| G\left(\frac{\nu_s}{\sqrt{\alpha}}, \frac{\nu_1}{\sqrt{\alpha}}\right) \right|^2 \quad . \tag{5.67}$$

Finally, the depolarization by the diffusion is given by eq. (5.61)

$$\Delta P_2 = 0.15 \frac{T_0}{\tau_p} \frac{(\gamma a)^2}{\alpha \nu_1} \qquad \text{for} \quad \nu_1 \gtrsim \nu_s \; . \tag{5.68}$$

or, if one makes use of the first dip of $G(\omega,b)$, then ΔP_2 is given by Eq. (5.62) with Eq. (5.65);

$$\Delta P_2 = \frac{11}{36} \frac{T_0}{\tau_p} \frac{(\gamma a)^2}{\alpha \sqrt{\alpha}} \frac{b}{1 + 1.866\, b^{-4/3}} \quad . \tag{5.69}$$

In the latter choice, α must satisfy Eq. (5.64);

$$\frac{\nu_s}{\sqrt{\alpha}} = b + 1.866\, b^{-1/3} \quad . \tag{5.70}$$

Both ΔP_1 and ΔP_2 must be smaller than, say, 10 %. Numerical estimations for TRISTAN[62] give the following results. In the case of

$\nu_1 \gtrsim \nu_s$ a stronger flipper field ν_1 is required in order to suppress ΔP_1 than to suppress ΔP_2, i.e. the effect of the continuous synchrotron oscillation is more serious than that of diffusion. A large value of $\nu_1 \sim 0.2$ is necessary to give the depolarization less than ten percent at 25 GeV. In the case of the first dip of $G(\omega,b)$, the required ν_1 is about 0.06 but the resulting depolarization is

$$\Delta P_2 \sim 160\% \times \left(\frac{E}{25\,\mathrm{GeV}}\right)^7 .$$

Therefore, this idea is not practical unless the beam energy is less than 15 GeV.

§6. Conclusions

We have discussed several problems concerning the radiative polarization in electron storage rings.

Among them the first problem is how to avoid diffusive depolarization caused by spin rotators. We have shown that this can be realized by the method, called spin matching, to decouple the spin motion and the orbit oscillation. The spin matching conditions can be satisfied by a suitable choice of the strengths of the quadrupole magnets.

The diffusive depolarization occurs by magnet errors, too. We have shown that it is possible to design a planar storage ring insensitive to magnet errors. It is free from depolarization up to the first order of the errors. Moreover, it is possible to correct the polarization in the stage of machine operation by adjusting the steering magnets so as to control some Fourier components of the perturbation on the spin motion. This latter point has been established experimentally already.

Another problem is the depolarization due to the spread in the beam energy which is serious, for instance, in LEP and in high energy regions of HERA and TRISTAN. We have derived formulae to estimate the depolarization of this type. The cures for this problem have been conceived, such as the non-linear wigglers and the Siberian Snake scheme. The present author has suggested another scheme in which the relation between the spin tune and the particle energy is modified so that the spin tune has no spread. These cures, however, have some problems more or less if we put them into practice.

The last problem we have discussed is the depolarization during the process of spin-flip due to the diffusion by radiations. The result indicates that the artificial spin-flip may be feasible at an energy less than that of LEP, although a very strong spin-flipper is required. On the way we have derived a formula of depolarization due to synchrotron oscillation during the resonance crossing. It can be applied to proton synchrotrons, too.

A long time has passed since the problems on spin became a part of accelerator physics. Nowadays, we think that the technique of spin manipulation in proton synchrotrons during the acceleration has already been established well both theoretically and experimentally.

As for the electron beams, although pioneering works were done by Russian group in early 1960's, even the theoretical progress had been very slow because of the complications due to synchrotron radiations. The experimental data are poor still now. In particular, the spin motion in colliders with spin rotators, which are necessary for the application to physics experiments, had not been known well. The discovery of the spin matching condition made it possible to design such colliders. Now it is being studied extensively, especially for HERA, which might be the first electron storage ring equipped with spin rotators.

There are still other problems to be studied in the future such as the depolarization due to beam-beam interaction and artificial polarizers. The cure for the depolarization due to the energyspread will become an important problem if the electron-proton collision between LEP and SPS would set in. In order to solve these problems it is desired to get more experimental data in existing planar rings and in storage ring with spin rotators to be constructed in the near future.

Acknowledgement

The author wishes to thank Prof. Y. Kimura for his interest in this work and continual encouragement.

References

1) I.M. Ternov, Yu.M. Lokutov and K.L. Korovina, Sov. Phys. JETP, 14, 921 (1962).

2) A.A. Sokolov and I.M. Ternov, Sov. Phys. Doklady 8, 1203 (1964).

3) D. Potaux, Proc. 8th Int. Conf. High Energy Accelerators, CERN, (1971), p.127.

4) V.N. Baier, Sov. Phys. Usp. 14, 695 (1972).

5) U.Camerini et al., Phys. Rev. D12, 1855 (1975).

6) Ya.S. Derbenev and A.M. Kondratenko, Sov. Phys. JETP, 35, 230 (1972), 37, 968 (1973).

7) A. Chao, SLAC-PUB-2781, July 1981.

8) A. Chao, PEP Note 263, April 1978. SLAC-PUB-2564, July, 1980.

9) J. Kewisch, DESY 83/032, April 1983.

10) B.W. Montague, CERN-ISR-TH/77-34, May 1977.

11) B.W. Montague, CERN-ISR-th/80-39, Aug. 1980.

12) T. Suzuki and K. Yokoya, KEK 80-10, Oct. 1980.

13) K. Steffen, DESY HERA 81/05, Mar. 1981.

14) A. Chao and K.Yokoya, KEK 81-7, July, 1981.

15) V. Bargmann, K. Michel and V.L. Telegdi, Phys. Rev. Lett. 2, 435 (1959).

16) J. Buon, NI/07-81, Orsay, Mar. 1981.

17) E.D. Courant, private communication, 1981.

18) K. Steffen, DESY HERA 83-16, Aug. 1983, 84-09, May 1984.
J. Buon, LAL/RT/83-10, Jul. 1983, DESY HERA 84-11, May 1984.

19) B.W. Montague, Nucl. Instrum. Methods 192, 165 (1982).

20) K. Yokoya, in "Polarized Electron Acceleration and Storage", work shop at Hamburg, Mar. 1982. DESY M82/09, Apr. 1982, page T1.

21) A. Skuja, ibid page T10.

22) K. Steffen, DESY HERA 82/01, Jan. 1982.

23) D.P. Barber et al., DESY 82/076, Nov. 1982.

24) D.P. Barber et al., DESY 84/102, Oct. 1984.

25) E.D. Courant, in "Polarized Electron Acceleration and Storage", work shop at Hamburg, Mar.1982. DESY M82/09, Apr. 1982, page P30.

26) J. Buon, LAL/RT/83-04, Mar. 1983, LAL/RT/83-12, July 1983.

27) A. Chao, "High Energy Spin Physics-1982" at BNL, AIP Conference Proceedings No.95, ed. G.M. Bunce, page 458 and SLAC-PUB-2979, Sept. 1982.

28) K. Steffen, CBN 81-29, Newman Lab., Cornell, 1981, DESY HERA 82/02, Jan. 1982.

29) K. Yokoya, KEK 81-19, Mar. 1982.
Revised version: in "Polarized Electron Acceleration and Storage", workshop at Hamburg, Mar. 1982, DESY M82/09, Apr. 1982, page R19.

30) E.D Courant and H.S. Snyder, Annals of Physics, 3, 1 (1958).

31) H.D. Bremer, J.Kewish, H.C. Lewin, H.Mais, R. Rossmanith, R. Schmidt and D.P. Barber, DESY 82-026, May 1982.

32) D. Möhl and B.W. Montague, Nucl. Instr. Methods, 137, 423 (1976).

33) Ya.S. Derbenev, A.M. Kondratenko and A.N. Skrinsky, Part. Acc. 9, 247 (1979).

34) J. Buon, LAL/RT/83-02, Feb. 1983, Orsay.
C. Biscari and J. Buon, NI/40-81, Dec. 1981, Orsay, CERN-LEP-TH/83-8, Feb. 1983.
C. Biscari, CERN-LEP-TH/83-27, Jun. 1983.

35) A.M. Kondratenko and B.W. Montague, CERN-ISR-TH/80-38, 1980.

36) J. Buon, DESY HERA 83/14, Jun. 1983.

37) K.Yokoya, Part. Acc., 13, 85 (1983).

38) A. Hofmann, J.M. Jowett and S. Myers, IEEE Trans. Nucl. Sci. NS-28, 2393 (1981).
A. Hofmann and J.M. Jowett, CERN-ISR-TH/81-23, Apr. 1981, CERN-ISR-TH/81-24, May 1981.

39) C. Biscari, J.Buon and B.W. Montague, CERN-LEP-TH/83-8, Feb. 1983.

40) Ya.S. Derbenev, A.M. Kondratenko, Proc. 10th Int. Conf. on High Energy Accelerators, Protvino, Vol. 2, page 70 (1977).
Ya.S. Berbenev, A.M. Kondratenko, S.I. Serednyakov, A.N. Skrinsky, G.M. Tumaikin and Yu.M. Shatunov, Part. Acc. 8, 115 (1978).

41) A. Turrin, LNF-79/28, May 1979, Frascati,
LNF-79/44, Jul. 1979, Frascati.

42) Ya.S. Berbenev and A.M. Kondratenko, Nucl. Instr. Methods 165, 15 (1979), 165, 201 (1979).
R. Rossmanith and R. Schmidt, DESY M80/02, Feb. 1980.

43) E.D. Courant, in "Polarized Electron Acceleration and Storage", workshop at Hamburg, Mar. 1982, DESY M82/09, Apr. 1982, page P40.

44) K. Steffen, DESY HERA 82/06, June 1982.

45) J.Buon, LAL/RT/84-05, Nov. 1984.

46) K. Yokoya, Part. Acc. 13, 95 (1983).

47) K. Steffen, DESY HERA 81/05, Mar. 1981.

48) K. Steffen, DESY HERA 83/06, May 1983, 83/09, May 1983, 83/16, Aug. 1983, 83/19, Sept. 1983, 84/09, May 1984.
R. Brinkmann and J. Rossbach, DESY HERA 84/17, Aug. 1984.
J. Buon, DESY HERA 84/11, May 1984.

49) K. Steffen, DESY HERA 81/16, Sept. 1981.

50) J. Boun, NI/35-81, Dec. 1981, Orsay.

51) M. Froissart and R. Stora, Nucl. Instr. Methods 7, 297 (1960).

52) T. Khoe, et al., Part Acc. 6, 213 (1975).

53) E. Grorud, J.-L. Laclare, G. Leleux, A. Nakach and A. Ropert, in "High Energy Spin Physics-1982", BNL, AIP Conf. Proc. No.95, ed. G.M. Bunce, page 407.

54) Ju. Shatunov, in "Polarized Electron Acceleration and Storage", workshop at Hamburg, Mar. 1982, DESY M82/09, Apr. 1982, page 01.

55) K. Yokoya, Part Acc. 14, 39 (1983).

56) M. Sands, "The Physics of Electron Storage Rings. An Introduction", in "Physics with Intersecting Storage Rings, Proc. Int. School of Physics, Enrico Fermi, Course XLVI" ed. by B. Touschek, (Academic Press, 1971).

57) J.M. Jowett and R.D. Ruth, CERN-LEP-TH/83-26, May 1983.

58) E.D. Courant and R.D. Ruth, BNL 51270, UC-28, ISA 80-5, Upton, Sept. 1980.

59) R.D. Ruth, CERN-ISR-TH/82-21, 1982.

60) M. Abramowitz and I.A. Stegun, "Handbook of Mathematical Functions", Dover pub., Inc., New York.

61) T. Aniel, J.L. Laclare, G. Leleux, A. Nakach and A. Ropert, Proc. 6-th Int. Symp. High Energy Spin Phys. Sept. 1984, Marseille, France.

62) "Abridged Description of TRISTAN Electron-Positron Colliding-Beam Machine", KEK, Tsukuba, Oct. 1981.

APPENDIX A

In this appendix we derive the equation of classical spin motion in the absence of synchrotron radiations in circular accelerators.

First, let us define the coordinate system. A rigorous definition is necessary because spin rotators contain vertical bends. Suppose the design orbit is given in the parametrized form $\vec{r} = \vec{r}_0(s)$, s being the length along the design orbit from a fixed point 0 on this orbit. Then the unit tangent vector $\vec{e}_z(s)$ at s is given by

$$\vec{e}_z(s) = \frac{d\vec{r}_0(s)}{ds} \quad . \tag{A.1}$$

Let

$$\vec{\Omega}_{D0}(s) \equiv \vec{e}_z(s) \times \frac{d\vec{e}_z}{ds} \quad , \tag{A.2}$$

then, since $d\vec{e}_z/ds$ is perpendicular to $\vec{e}_z$, we have

$$\frac{d\vec{e}_z}{ds} = \vec{\Omega}_{D0}(s) \times \vec{e}_z(s) \quad . \tag{A.3}$$

$|\vec{\Omega}_{D0}|$ is the curvature of the design orbit.

We want to define right-handed orthonormal basis vectors ($\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$) at each point on the design orbit. Since $\vec{e}_z$ is already defined, the problem is how to define $\vec{e}_x$ and $\vec{e}_y$. If the ring is planar, then we can define them so that $\vec{e}_x$ lies on the horizontal plane, directs outwards and is perpendicular to $\vec{e}_z$, and $\vec{e}_y = \vec{e}_z \times \vec{e}_x$. However, we cannot do so in general. Eq. (A.3) states that $\vec{e}_z(s+ds)$ is given by rotating $\vec{e}_z(s)$ by the angle $|\vec{\Omega}_{D0}(s)ds|$ around the axis $\vec{\Omega}_{D0}(s)$. Thus, we define $\vec{e}_x$ and $\vec{e}_y$ so that all three axes acquire the same rotation; i.e.

$$\frac{d\vec{e}_i}{ds} = \vec{\Omega}_{D0}(s) \times \vec{e}_i(s) \qquad (i = x, y, z) \quad . \tag{A.4}$$

If we define right-handed orthonormal basis ($\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$) at the point

0, then Eq. (A.4) gives the basis at all the points on the design orbit, preserving the orthonormality. In the case of planar rings this definition is equivalent to the previous one if we define $\vec{e}_x$ to be horizontal at the point 0. Since even in the rings with spin rotators $\vec{e}_x$ is almost horizontal, hereafter we often call $\vec{e}_x$ and $\vec{e}_y$ "horizontal" and "vertical" axes.

Since $\vec{\Omega}_{DO}$ is perpendicular to $\vec{e}_z$, it can be expressed as a linear combination of $\vec{e}_x$ and $\vec{e}_y$;

$$\vec{\Omega}_{DO} = -\frac{\vec{e}_y}{\rho_x^0} + \frac{\vec{e}_x}{\rho_y^0} \quad , \tag{A.5}$$

where $1/\rho^\circ_x$ and $1/\rho^\circ_y$ is the horizontal and vertical curvature of the design orbit.

We define the coordinate (x,y,s) of a point $\vec{r}(P)$ near the design orbit as follows. Let Q be the foot of the perpendicular from P to the design orbit and its position vector be $\vec{r}_0(s)$. Then $\vec{r}(P) - \vec{r}_0(s)$, which is perpendicular to $\vec{e}_z(s)$, is a linear combination of $\vec{e}_x(s)$ and $\vec{e}_y(s)$. We define its coefficients to be x and y; i.e.

$$\vec{r}(P) = \vec{r}_0(s) + x\,\vec{e}_x(s) + y\,\vec{e}_y(s) \quad . \tag{A.6}$$

This definition of the coordinate system, which is widely used in the accelerator beam dynamics at least implicitly, has one defect. It is that, after one revolution over the ring, $\vec{e}_x$ and $\vec{e}_y$ may not coincide with their first definition at the point 0. However, since the difference is extremely small in practice even in the rings with complicated spin rotators, we ignore this problem and consider that $\vec{e}$'s are periodic.

Now, the displacement vector between the two points P(x,y,s) and P'(x+ds, y+dy, s+ds) may be written as

$$d\vec{r} = \vec{r}_0(s+ds) - \vec{r}_0(s) + (x+dx)\,\vec{e}_x(s+ds) - x\,\vec{e}_x(s)$$
$$+ (y+dy)\,\vec{e}_y(s+ds) - y\,\vec{e}_y(s)$$
$$= \vec{e}_z(s)\,ds + \vec{e}_x(s)\,dx + \vec{e}_y(s)\,dy + x\,\frac{d\vec{e}_x}{ds}\,ds + y\,\frac{d\vec{e}_y}{ds}\,ds$$
$$= g(x,y,s)\,\vec{e}_z\,ds + \vec{e}_x\,dx + \vec{e}_y\,dy \qquad \text{(A.7)}$$

with

$$g(x,y,s) = 1 + \frac{x}{\rho_x^0} + \frac{y}{\rho_y^0} \quad . \qquad \text{(A.8)}$$

Hence, the line element ds_1 is given by

$$ds_1^2 = g^2\,ds^2 + dx^2 + dy^2 \quad , \qquad \text{(A.9)}$$

from which one can derive various formulae concerning vector calculus, consulting textbooks on differential geometry.

The velocity vector of a particle which runs from P to P' in a time dt is given by

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt}\,(\,g\,\vec{e}_z + x'\,\vec{e}_x + y'\,\vec{e}_y\,) \qquad \text{(A.10)}$$

with $x' = dx/ds$ and $y' = dy/ds$. Let $v = |d\vec{r}/dt|$, then

$$\frac{d\vec{r}}{dt} = v\,\vec{e}_v \qquad \text{(A.11)}$$

with

$$e_v = \frac{e_z + (x'e_x + y'e_y)/g}{[\,1 + (x'^2 + y'^2)/g^2\,]^{1/2}} \qquad \text{(A.12)}$$

and

$$\frac{ds}{dt} = v\,\left(g^2 + x'^2 + y'^2\right)^{-1/2} \quad . \qquad \text{(A.13)}$$

The classical motion of the spin in the magnetic field $\vec{B}$ and electric field $\vec{E}$ can be described by the following equation, called BMT equation after Bargmann, Michel and Telegdi[15];

$$\frac{d\vec{s}}{dt} = -\frac{e}{m\gamma}\left[(\gamma a+1)\vec{B}_T + (a+1)\vec{B}_L - \gamma\left(a + \frac{1}{\gamma+1}\right)\beta\,\vec{e}_v \times \frac{\vec{E}}{c}\right] \times \vec{s} \quad , \qquad \text{(A.14)}$$

with

$$\vec{B}_T = \vec{e}_V \times (\vec{B} \times \vec{e}_V) \qquad and \qquad \vec{B}_L = \vec{e}_V \, (\vec{B} \cdot \vec{e}_V) \quad .$$

Here, $\vec{s}$ is the spin vector seen in the rest frame of the particle, m the rest mass of the particle, e the charge, γ the energy divided by mc^2, a the coefficient of the anomalous magnetic moment (= 0.00116 for electron) and $\beta = v/c$. Let us rewrite Eq. (A.14), choosing s as the independent variable and using our coordinate system. Let s_i be the components of $\vec{s}$, then

$$\frac{d\vec{s}}{ds} = \frac{d}{ds} \sum_i s_i \vec{e}_i = \sum_i \frac{ds_i}{ds} \vec{e}_i + \sum_i s_i \frac{d\vec{e}_i}{ds}$$

$$= \frac{d'\vec{s}}{ds} + \vec{\Omega}_{DO} \times \vec{s} \quad . \tag{A.15}$$

Here, d'/ds is the derivative seen in our coordinate system. From now on we simply write d/ds for d'/ds. Hence, $d\vec{s}/ds$ means to differentiate each component separately. Combining (A.13), (A.14) and (A.15), we get

$$\frac{d\vec{s}}{ds} = \vec{\Omega} \times \vec{s} \tag{A.16}$$

with

$$\vec{\Omega} = -\vec{\Omega}_{DO} - \frac{e}{p}\sqrt{g^2 + x'^2 + y'^2} \cdot \left[\begin{array}{l} (\gamma a + 1)\, \vec{B}_T + (a+1)\vec{B}_L \\ -\gamma(a + \frac{1}{\gamma+1})\, \beta \vec{e}_V \times \vec{E}/c \end{array} \right] \quad . \tag{A.17}$$

In this formula the electromagnetic field must be evaluated at (x,y,s).

Since we are interested in linear betatron oscillations, we expand Eq. (A.17) up to the linear terms in x, x', y and y'. After a laborious calculation, we find

$$\vec{\Omega} = -\vec{\Omega}_{DO} - \frac{e}{p}(\gamma a + 1) \left\{ \begin{array}{l} B_x \vec{e}_x + B_y \vec{e}_y \\ + x \left[\left(\frac{\partial B_x}{\partial x} + \frac{B_x}{\rho_x^0} \right) \vec{e}_x + \left(\frac{\partial B_y}{\partial x} + \frac{B_y}{\rho_x^0} \right) \vec{e}_y \right] \\ + y \left[\left(\frac{\partial B_x}{\partial y} + \frac{B_x}{\rho_y^0} \right) \vec{e}_x + \left(\frac{\partial B_y}{\partial y} + \frac{B_y}{\rho_y^0} \right) \vec{e}_y \right] \end{array} \right\}$$

$$
-\frac{e}{p}\left\{\begin{array}{l}
(a+1)\vec{e}_z\left[B_z + x\left(\frac{\partial B_z}{\partial x}+\frac{B_z}{\rho_x^0}\right)+y\left(\frac{\partial B_z}{\partial y}+\frac{B_z}{\rho_y^0}\right)\right] \\
-a(\gamma-1)\left[\vec{e}_z(x'B_x+y'B_y)+B_z(x'\vec{e}_x+y'\vec{e}_y)\right] \\
-\gamma\left(a+\frac{1}{\gamma-1}\right)\beta\,\vec{e}_z\times\frac{\vec{E}}{c}
\end{array}\right\} \quad \text{(A.18)}
$$

Here, p is the kinetic momentum and we have retained only the zero-th order terms in the parts involving the electric field. The quantities concerning the electromagnetic field in this equation is evaluated at (0,0,s) on the design orbit.

Let ε be the relative energy deviation $(\gamma-\gamma_0)/\gamma_0$ from the nominal energy γ_0 and expand the right-band side of Eq. (A.18) in terms of ε up to the linear terms. We get

$$
\begin{aligned}
\vec{\Omega} = & -(\gamma a+1)\left(\frac{\vec{e}_y}{\rho_x}-\frac{\vec{e}_x}{\rho_y}\right)+\frac{\vec{e}_y}{\rho_x^0}-\frac{\vec{e}_x}{\rho_y^0}+\frac{\gamma(\gamma+a)}{\gamma^2-1}\varepsilon\left(\frac{\vec{e}_y}{\rho_x}-\frac{\vec{e}_x}{\rho_y}\right) \\
& -(\gamma a+1)\left\{\begin{array}{l}
x\left[\left(\frac{e}{p}\frac{\partial B_x}{\partial x}-\frac{1}{\rho_x^0\rho_y^0}\right)\vec{e}_x+\left(\frac{e}{p}\frac{\partial B_y}{\partial x}+\frac{1}{\rho_x^0\rho_x}\right)\vec{e}_y\right] \\
+y\left[\left(\frac{e}{p}\frac{\partial B_x}{\partial y}-\frac{1}{\rho_y\rho_y^0}\right)\vec{e}_x+\left(\frac{e}{p}\frac{\partial B_y}{\partial y}+\frac{1}{\rho_x\rho_y^0}\right)\vec{e}_y\right]
\end{array}\right\} \\
& -(a+1)\vec{e}_z\cdot\frac{e}{p}\left[B_z+x\left(\frac{\partial B_z}{\partial x}+\frac{B_z}{\rho_x^0}\right)+y\left(\frac{\partial B_z}{\partial y}+\frac{B_z}{\rho_y^0}\right)\right] \qquad \text{(A.19)} \\
& +a(\gamma-1)\left[\vec{e}_z\left(-\frac{x'}{\rho_y}+\frac{y'}{\rho_x}\right)+\frac{e}{p}B_z(x'\vec{e}_x+y'\vec{e}_y)\right] \\
& +\gamma\left(a+\frac{1}{\gamma-1}\right)\beta\,\vec{e}_z\times\frac{e\vec{E}}{cp} \quad ,
\end{aligned}
$$

where

$$
\frac{1}{\rho_x}=\frac{e}{p}B_y \qquad \text{and} \qquad \frac{1}{\rho_y}=-\frac{e}{p}B_x . \qquad \text{(A.20)}
$$

Here and hereafter we denote the nominal quantities p_0 and γ_0 simply by p and γ. The differences between ρ_x, ρ_y and ρ°_x, ρ°_y come from the difference between the design bending field and the actual field and/or alignment errors of quadrupole magnets.

Eq. (A.19) contains all the terms except the nonlinear terms of betatron and synchrotron oscillations. Hence, it is valid in the presence of magnet errors, vertical-horizontal coupling, edge effects of the bending magnets and so on.

In the following we retain only the terms important for our discussion. We ignore the magnet errors, then we need not distinguish between ρ_x, ρ_y and ρ°_x, ρ°_y. We assume that the electric field, skew quadrupole field and longitudinal field are absent and that there is no inclined bending magnet; i.e. $1/\rho_x\rho_y = 0$. Moreover, since we are interested in electrons, we put $\gamma \gg 1$. With these asumptions we have

$$\vec{\Omega} = -\gamma a\left(\frac{\vec{e}_y}{\rho_x} - \frac{\vec{e}_x}{\rho_y}\right) + \varepsilon\left(\frac{\vec{e}_y}{\rho_x} - \frac{\vec{e}_x}{\rho_y}\right)$$
$$-(\gamma a+1)(xG_x\vec{e}_y - yG_y\vec{e}_x) + a(\gamma-1)\left(-\frac{x'}{\rho_y} + \frac{y'}{\rho_x}\right)\vec{e}_z \quad , \qquad (A.21)$$

where

$$G_x = \frac{e}{p}\frac{\partial B_y}{\partial x} + \frac{1}{\rho_x^2} \quad \text{and} \quad G_y = -\frac{e}{p}\frac{\partial B_x}{\partial y} + \frac{1}{\rho_y^2} \qquad (A.22)$$

are the focusing functions of the well-knwon equation of motions of betatron oscillations

$$\frac{d^2x}{ds^2} + G_x(s)\,x = 0 \quad \text{and} \quad \frac{d^2y}{ds^2} + G_y(s)\,y = 0 \quad . \qquad (A.23)$$

Note that in the text $\vec{\Omega}$ is multiplied by R, the ring circumference divided by 2π, because $\theta = s/R$ is used there as the independent variable instead of s.

APPENDIX B

This is an appendix to §5. Here, we investigate the properties of Froissart and Stora's solutions[51)], ψ_1 and ψ_2, and the Fourier transform $G(\omega,b)$ defined in Eq. (5.40).

Among the solutions to Eq. (5.27) with (5.20), the solutions which show the polarization in the direction $\pm\vec{n}_3$ at $t = +\infty$ are given by

$$\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -g^*(t) \\ f^*(t) \end{pmatrix} ,$$

respectively. Here, f(t) and g(t) are defined by using the parabolic cylinder function U as

$$f(t) = e^{-\frac{\pi}{16}b^2}\, U(-\tfrac{1}{2} + \tfrac{i}{4}b^2,\, t\, e^{\frac{\pi}{4}i})$$

and

$$g(t) = -\frac{b}{2}\, e^{-\frac{\pi}{16}b^2 + \frac{\pi}{4}i}\, U(+\tfrac{1}{2} + \tfrac{i}{4}b^2,\, t\, e^{\frac{\pi}{4}i}) \quad . \tag{B.1}$$

See Ref. 60 for the definition of U. The same functions f(t) and g(t) can be used to express the solutions which show polarization in the direction $\pm\vec{n}_3$ at $t = -\infty$;

$$\psi_1(t) = \begin{pmatrix} f(-t) \\ -g(-t) \end{pmatrix} , \quad \psi_2(t) = \begin{pmatrix} g^*(-t) \\ f^*(-t) \end{pmatrix} . \tag{B.2}$$

These functions have the asymptotic forms at $t \to +\infty$

$$\begin{aligned} f(t) &\sim \exp\left[-\frac{i}{4}(t^2 + b^2 \log t)\right] \times (1 + O(t^{-2})) \\ g(t) &\sim -\frac{1}{2t} \exp\left[-\frac{i}{4}(t^2 + b^2 \log t)\right] \times (1 + O(t^{-2})) \quad . \end{aligned} \tag{B.3}$$

The time-inverted functions f(-t) and g(-t) can be expressed by linear combinations of f(t), g(t) and their complex conjugates as

$$\begin{aligned} f(-t) &= C_{11} f(t) + C_{12}\, g^*(t) \\ g(-t) &= C_{21} f^*(t) + C_{22}\, g(t) \end{aligned} \tag{B.4}$$

with

$$C_{11} = -C_{22} = e^{-\frac{\pi}{4}b^2}$$

and

$$C_{21} = C_{12} = -i\,\frac{2\sqrt{2\pi}}{b\,\Gamma(ib^2/4)}\,e^{-\frac{\pi}{8}b^2+\frac{\pi}{4}i}$$

This coefficient c_{11} gives the formula of Froissart and Stora[51];

$$P_{FS} = 2|C_{11}|^2 - 1 \quad .$$

The asymptotic forms of f(t) and g(t) at t = $-\infty$ can easily be derived from (B.3) and (B.4).

So, far, all formulae are exact. But since we are interested in the case of almost complete spin-flip only, let us derive approximate formulae which are easier to handle. Substituting ψ = (f(t), g(t)) in Eq. (5.27) with (5.20), we get

$$\frac{d}{dt}\begin{pmatrix} f \\ g \end{pmatrix} = \frac{i}{2}\begin{pmatrix} -t & b \\ b & t \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \quad . \tag{B.5}$$

Elimination of g from this equation gives

$$\frac{d^2f}{dt^2} + \left(\frac{i}{2} + \frac{b^2}{4} + \frac{t^2}{4}\right) f = 0 \quad . \tag{B.6}$$

Now, under the assumption b $\gtrsim$ 1 which means almost complete spin-flip, we employ the WKB approximation which is, in our case, an asymptotic expansion uniform in t for large b. Then two independent solutions to Eq. (B.6) are

$$\left(\frac{i}{2} + \frac{b^2}{4} + \frac{t^2}{4}\right)^{-1/4} \exp\left[\pm \int_0^t \left(\frac{i}{2} + \frac{b^2}{4} + \frac{t^2}{4}\right)^{1/2} dt\right] \quad .$$

Multiplying a constant factor by this expression so that it has the same behavior for t = ∞ as Eq. (B.3), we find

$$f(t) = \left(\frac{t+\sqrt{t^2+b^2}}{b}\right)^{1/2} \exp\left[\frac{i}{8}b^2\left(1-\log\frac{b^2}{4}\right)\right] \frac{F^*(t)}{\sqrt{2}}\left(1+O(b^{-2})\right) \tag{B.7}$$

with

$$F(t) = \left(\frac{b^2}{t^2+b^2}\right)^{1/4} \exp\left[\frac{i}{2}\int_0^t \sqrt{t^2+b^2}\,dt\right]$$

$$= \left(\frac{b^2}{t^2+b^2}\right)^{1/4} \exp\left[\frac{i}{4}\left(t\sqrt{t^2+b^2} + b^2\sinh^{-1}\frac{t}{b}\right)\right] , \qquad \text{(B.8)}$$

which satisfies

$$F^*(t) = F(-t) .$$

Similarly, for g(t) we get

$$g(t) = -\left(\frac{b}{t+\sqrt{t^2+b^2}}\right)^{1/2} \exp\left[\frac{i}{8}b^2\left(1-\log\frac{b^2}{4}\right)\right]\frac{F^*(t)}{\sqrt{2}}\left(1+O(t^{-2})\right) \qquad \text{(B.9)}$$

Now, what we want to know is $G(\omega,b)$. Using the relation

$$\psi_2^* \sigma_3 \psi_1 = 2f(-t)\,g(-t) = e^{ic_0}\,[F(t)]^2$$

with

$$c_0 = \frac{b^2}{4}\left(1-\log\frac{b^2}{4}\right) + \pi ,$$

we have

$$G(\omega,b) = \frac{e^{-ic_0}}{2\pi}\int_{-\infty}^{\infty} \psi_2^* \sigma_3 \psi_1\, e^{-i\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [F(t)]^2 e^{-i\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{b^2}{t^2+b^2}\right)^{1/2}\exp\left[\frac{i}{2}\left(t\sqrt{t^2+b^2}+b^2\sinh^{-1}\frac{t}{b}\right)-i\omega t\right]dt . \qquad \text{(B.10)}$$

This expression is not suitable for numerical evaluation because the phase of the integrand varies rapidly with t. Hence, we deform the integration contour as shown in Fig. B1.

$$\int_{-\infty}^{\infty} = \int_{A_1} + \int_{A_2} + \int_{A_3}$$

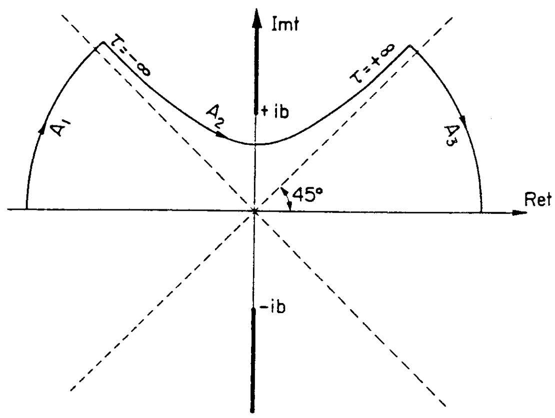


Fig.B1.

The integration contour of Eq(B.10). The path A_2 is defined by Eq(B.11).

The path A2 approaches arg(t) = $\pi/4$ and $3\pi/4$ at large |t| and crosses the imaginary axis between the two branch points $\pm ib$ of the integrand. Since the contributions of the arcs A1 and A3 vanish if the radius of the circle is infinitely large, we have

$$\int_{-\infty}^{\infty} = \int_{A_2} .$$

The expression becomes simple when we choose A2 as

$$t = e^{\frac{\pi}{2}i}\left[\frac{b^2}{2}(1-i\tau)\right]^{1/2}, \qquad -\infty < \tau < +\infty . \tag{B.11}$$

Here and hereafter, the branches of the square roots are chosen so that their arguments are larger than $-\pi/2$ and smaller than or equal to $\pi/2$. Then we have

$$G(\omega,b) = \frac{b}{4\pi}\int_{-\infty}^{\infty}\frac{d\tau}{\sqrt{1+\tau^2}}\exp\left(\frac{\omega b}{\sqrt{2}}\sqrt{1-i\tau} - \frac{\pi b^2}{8} - \frac{b^2}{4}\sqrt{1+\tau^2} + \frac{ib^2}{4}\sinh^{-1}\tau\right). \tag{B.12}$$

Results of the numerical integration of this expression are plotted in Fig. 8 for several values of b.

The general shape of G(ω,b) can be divided into three regions. In the first region $\omega \gtrsim b$, G(ω,b) oscillates with ω rapidly. Estimation of Eq. (B.12) by the saddle-point method gives

$$G(\omega,b) \approx \frac{\sqrt{2}\, b}{\sqrt{\pi\omega\sqrt{\omega^2-b^2}}}\cos\frac{1}{2}\left(\omega\sqrt{\omega^2-b^2} - b^2\cosh^{-1}\frac{\omega}{b} - \frac{\pi}{2}\right) \quad \text{for } \omega \gg b. \tag{B.13}$$

In the second regions $0 \lesssim \omega \lesssim b$, G(ω,b) increases exponentially with ω from a small value to a value of order unity. Again the saddle-point method gives

$$G(\omega,b) \approx \frac{b}{\sqrt{2\pi\omega\sqrt{b^2-\omega^2}}}\exp\frac{1}{2}\left(\omega\sqrt{b^2-\omega^2} - b^2\cos^{-1}\frac{\omega}{b}\right) \quad \text{for } 0 \ll \omega \ll b . \tag{B.14}$$

In the third region $\omega \lesssim 0$, G(ω,b) is very small as long as $b \gtrsim 1$. In the vicinity of the origin it can be approximated by

$$G(\omega,b) \approx Ai(0)\, b^{1/3}\cdot e^{-\frac{\pi}{4}b^2 + b\omega} \times \left(1 + O(b^{-4/3})\right) \quad \text{for } |\omega| \lesssim 1/b . \tag{B.15}$$

Here Ai(0) is the value of the Airy function at the origin given by

$$Ai(0) = \frac{3^{-2/3}}{\Gamma(2/3)} = 0.35503 \quad .$$

In particular, since G(0,b) is given by

$$G(0,b) \approx Ai(0)\, b^{1/3}\, e^{-\frac{\pi}{4} b^2} \quad , \tag{B.16}$$

$|G(0,b)|^2$ is of the order of the depolarization of Froissart and Stora's formula. In the present approximation, namely the asymptotic expansion for large b, this value should be taken to be zero.

Here, it may be necessary to make a comment on the physical meaning of G(0,b). As one can see from the results of §2 , Eq. (5.42) , G(0,b) is related to the depolarization of an off-energy particle with infinitely slow synchrotron oscillation. However, such a particle does not experience any depolarization because its effect is merely to shift the time of resonance crossing. Indeed, one finds that the differential Eq. (5.19) with $\lambda_s = 0$ is equivalent to the unperturbed Eq. (5.27), by shifting the origin of t. In spite of this property of our starting equation, the resulting G(0,b) is not exactly zero. The cause is not the WKB approximation. In fact, we can express G(ω,b) exactly from the expressions (B.1), using confluent hypergeometric functions of the second kind (those which have logarithmic singularity at the origin). But in that expression not only G(0,b) is non-zero but G(ω,b) diverges at the origin as 1/ω, although the residue is very small. What is wrong in our approach is the perturbation expansion in terms of ΔH. If we make the synchrotron oscillation slower with its amplitude fixed, the shift of the spin phase due to energy deviation during the half period of synchrotron oscillation becomes larger, and the solution goes away from the unperturbed one. Hence, in order to treat such a limiting case, we have to change the decomposition into H_0 and ΔH. However, in that case the depolarization is small any way as long as u is small, which we have already assumed. Therefore, in our perturbation expansion, the absolute error of the resulting depolarization formula is very small even though the relative error might be large. Moreover, $G(\omega,b)$ of WKB approximation is easier to handle than the exact expression using a confluent hypergeometric function not only because the former is simpler mathematically, but also becaue it does not diverge at the origin.

Now, among the three regions stated above, the first and the second regions together with their transition region are covered by the single formula

$$G(\omega,b) = 2b\left[\frac{\xi}{4\omega^2(\omega^2-b^2)}\right]^{1/4}\cdot\left[\mathrm{Ai}(-\xi)+O(b^{-4/3})\right] \quad \text{for } \omega \gtrsim 1/b\,. \tag{B.17}$$

Here, ξ is defined by

$$\begin{aligned}\xi &= \left[\frac{3}{2}\int_b^{\omega}\sqrt{\omega^2-b^2}\,d\omega\right]^{2/3}\\ &= \left[\frac{3}{4}\left(\omega\sqrt{\omega^2-b^2}-b^2\cosh^{-1}\frac{\omega}{b}\right)\right]^{2/3} \qquad (\omega>b)\\ &= -\left[\frac{3}{2}\int_{\omega}^{b}\sqrt{b^2-\omega^2}\,d\omega\right]^{2/3}\\ &= -\left[\frac{3}{4}\left(-\omega\sqrt{b^2-\omega^2}+b^2\cos^{-1}\frac{\omega}{b}\right)\right]^{2/3} \qquad (\omega<b)\,,\end{aligned} \tag{B.18}$$

and the factor $\xi/(\omega^2 - b^2)$ is always positive. The function Ai is the Airy function, which is related to the Bessel functions by

$$\begin{aligned}\mathrm{Ai}(-\xi) &= \frac{\sqrt{\xi}}{3}\left[J_{1/3}\left(\tfrac{2}{3}\xi^{3/2}\right)+J_{-1/3}\left(\tfrac{2}{3}\xi^{3/2}\right)\right] \qquad (\xi>0)\\ &= \frac{1}{\pi}\sqrt{-\frac{\xi}{3}}\,K_{1/3}\left(\tfrac{2}{3}|\xi|^{3/2}\right)\,, \qquad (\xi<0)\end{aligned}$$

and is tabulated, for instance, in Ref. 60. In practice one need not integrate Eq. (B.12) numerically. Eq. (B.17) reproduces the results of numerical integration fairly well except $\omega \lesssim 1/b$, where we may put $G(\omega,b) = 0$. From the expression (B.17) one finds ω_{max} which gives the maximum of $G(\omega,b)$;

$$\omega_{max} = b + (2b)^{-1/3} a_0 + O(b^{-5/3}),$$

$$G(\omega_{max}, b) = (2b)^{1/3} [Ai(-a_0) + O(b^{-4/3})], \tag{B.19}$$

where $-a_0$ is the maximum point of Ai and its value is[60)]

$$a_0 = 1.0188 \quad ,$$

$$Ai(-a_0) = 0.53566 \quad . \tag{B.20}$$

In addition, the n-th zero of $G(\omega,b)$ is given by

$$\omega_n = b + (2b)^{-1/3} a_n + O(b^{-5/3}), \tag{B.21}$$

where $-a_n$ is the n-th zero of Ai;

$$a_1 = 2.3381, \quad a_2 = 4.0879 \ldots.. \tag{B.22}$$

Next, let us consider some integrals appearing in the text. First, we evaluate K(b) defined in Eq. (5.58). The range of the integration can be divided into two regions, (1/b,b) and (b,∞). One finds that the contribution of the first region is of the order of $b^{-5/3}$ for large b, by using the approximate expression (B.14) and may be neglected. If we use (B.13) for the integration of the second region, we obtain

$$\int_b^\infty d\omega \frac{|G(\omega,b)|^2}{\omega^2} = \int_b^\infty d\omega \frac{2b^2}{\pi\omega^3\sqrt{\omega^2-b^2}}$$

$$\times \cos^2\left[\frac{1}{2}(\omega\sqrt{\omega^2-b^2} - b^2\cosh^{-1}\frac{\omega}{b} - \frac{\pi}{2}\right] .$$

When b is large, the square of the cosine oscillates very rapidly and can be replaced by the average value 1/2. Hence we have

$$\int_b^\infty d\omega \, \frac{|G(\omega,b)|^2}{\omega^2} \approx \frac{1}{4b}$$

and

$$K(b) \approx \frac{1}{4b} \quad . \tag{B.23}$$

Though the derivations is very rough, this expression gives a fairly good approximation of K(b). Comparing with the results of numerical integration using Eq. (B.12), one finds that the error of Eq. (B.23) is only 9 percent even for b = 2.

Next, let us estimate $Q_n(b)$ defined in Eq. (5.62). Using the representation (B.17) of $G(\omega,b)$ and choosing ξ as the integration variable instead of ω, we have

$$Q_n(b) = 2b^2 \int_{\xi_0}^\infty \frac{\omega}{(\omega^2-\omega_n^2)^2} \, \frac{\xi}{\omega^2-b^2} \, Ai^2(-\xi)\, d\xi \, .$$

Since the integrand is very small for negative ω, the lower limit of integration ξ_0, which corresponds to $\omega = 0$, can be replaced with $-\infty$. By using Eq. (B.18) we can expand ω as

$$\omega = b + (2b)^{-1/3}\xi - \frac{1}{5}(2b)^{-5/3}\xi^2 + \ldots\ldots$$

Then we get

$$Q_n(b) = \frac{b}{2}\int_{-\infty}^{\infty} d\xi \left(\frac{Ai(-\xi)}{\xi - a_n}\right)^2 \left[1 - 2(2b)^{-4/3}a_n - \frac{2}{5}(2b)^{-4/3}(\xi - a_n) + O(b^{-8/3})\right],$$

where a_n is the n-th zero of $Ai(-\xi)$. With the help of the formulae

$$\int_{-\infty}^\infty \frac{Ai^2(-\xi)}{(\xi-a_n)^2}\, d\xi = 1 \quad \text{and} \quad \int_{-\infty}^\infty \frac{Ai^2(-\xi)}{\xi-a_n}\, d\xi = 0 \quad , \tag{B.24}$$

which will be proved below, we get

$$Q_n(b) = \frac{b}{2}\left[1 - 2a_n(2b)^{-4/3} + \ldots\ldots\right] \quad .$$

Comparing with the results of numerical integration, one sees that the following formula gives much better approximation than the above one for not so large values of b, although it is still poor for large n.

$$Q_n(b) = \frac{b}{2} \frac{1}{1 + 2a_n (2b)^{-4/3}} \quad . \tag{B.25}$$

The relations (B.24) can be proved as follows. First, let us consider a function M(y) of a complex argument y defined by

$$M(y) = \int_M \frac{Ai^2(x)}{x-y} dx \, , \tag{B.26}$$

where the integration contour denoted by M runs from $-\infty$ to $+\infty$, passing below the pole at y which is the only singularity of the integrand. By using the differential equation for Ai(x)

$$Ai''(x) - x\,Ai(x) = 0 \tag{B.27}$$

and by the repeated use of partial integration, one can easily show that M(y) obeys the third order differential equation

$$M'''(y) - 4yM'(y) - 2M(y) = 0 \quad . \tag{B.28}$$

But this is exactly the same equation which is satisfied by the products of any two solutions to the Airy's equation (B.27). Hence, M(y) is a linear combination of $Ai^2(y)$, Ai(y)Bi(y) and $Bi^2(y)$, where Bi(y) is the other Airy function, which increases exponentially as $y \to +\infty$. The coefficients of the combination are common in the whole complex y-plane because M(y) is an entire function owing to the choice of the contour. One sees from the definition of M(y) that M(y) is bounded under $|y| \to \infty$ in the upper half plane because in this case the contour can be taken to be the real axis. But the only combination of Ai^2 and AiBi which has this property is Ai(Ai + iBi) times a constant, as can be found by using the asymptotic forms of Ai(y) and Bi(y) for $|y| \to \infty$ [60]. Hence we have only to find the overall factor. When y is real, one finds

$$\Im m\, M(y) = \Im m \int_{-\infty}^{\infty} \frac{Ai^2(x)}{x-y-i0}\, dx = \pi Ai^2(y)\ , \tag{B.29}$$

which fixes the unknown factor to be πi. Hence we find

$$\int_M \frac{Ai^2(x)}{x-y}\, dx = \pi \left[\, i Ai^2(y) - Ai(y) Bi(y) \,\right] \ . \tag{B.30}$$

When y is equal to a zero $x_n (= -a_n)$ of Ai(x), the contour can be taken to be the real axis because in this case the integrand is free from singularities. Then we have

$$\int_{-\infty}^{\infty} \frac{Ai^2(x)}{x-x_n}\, dx = 0\ .$$

Similarly, putting $y = x_n$ after differentiation with respect to y, we obtain

$$\int_{-\infty}^{\infty} \frac{Ai^2(x)}{(x-x_n)^2}\, dx = -\pi Ai'(x_n) Bi(x_n)\ .$$

Using the Wronskian relation[60)]

$$Ai(y) Bi'(y) - Ai'(y) Bi(y) = \frac{1}{\pi} \tag{B.31}$$

we get

$$\int_{-\infty}^{\infty} \frac{Ai^2(x)}{(x-x_n)^2}\, dx = 1\ .$$

This ends the proof.