Homology Cobordism Group of Homology 3-spheres
ホモロジー３球面のなすホモロジー同境群

古 田 幹 雄

Homology Cobordism Group of Homology 3-spheres

Mikio Furuta

Department of Mathematics, University of Tokyo,

Hongo, Bunkyo-ku, Tokyo 113, Japan.

1. Introduction.

In this paper all (V-)manifolds are assumed to be (V-)smooth. R. Fintushel and R. Stern [8] proved that Θ_3^H, the integral cobordism group of oriented integral homology 3-sphere, has a subgroup isomorphic to $\mathbb{Z}\oplus(\mathbb{Z}/2k)$ for some $k\geq 0$. It is not difficult to see that Θ_3^H has a subgroup isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. In fact, since Poincaré homology sphere $\Sigma(2,3,5)$ is given as a quotiont space of S^3 divided by a free linear action of a group of order 120, the V-manifold version of Theorem 2.1 of [8] (see [11]) implies that $\Sigma(2,3,5)$ and $\Sigma(a_1,a_2,\cdots,a_n)$ are linearly independent over $\mathbb{Z}$ if $a_1a_2\cdots a_n>120/4=30$ and the invariant $R(a_1,a_2,\cdots,a_n)$ defined in [8 §1] (see below) is positive.

In this paper we show that Θ_3^H have a subgroup isomorphic to $\mathbb{Z}^\infty$. The framework of the proof depends on [8] and also C.H. Taubes' works on gauge theory on end-periodic 4-manifolds in [14]. We generalize the above argument to general homology sphere, not only $\Sigma(2,3,5)$. A subgroup isomorphic to $\mathbb{Z}^\infty$ is given by Seifert fibered homology 3-spheres, and it is mapped isomorphically to $\mathbb{Z}^\infty$ in the $\mathbb{Z}_2$-homology cobordism group of oriented $\mathbb{Z}_2$-homology 3-sphere. A subgroup isomorphic to $\mathbb{Z}^\infty$ in the $\mathbb{Z}_2$-homology cobordism group of $\mathbb{Z}_2$-homology 3-spheres is also given by lens spaces [4,5,6].

2. Statement of the results.

Let $a_1, a_2, \cdots, a_n$ be integers pairwise relatively prime to each other. Fintushel-Stern [8] define $R(a_1, a_2, \cdots, a_n)$ as virtual dimension of a certain moduli space of self-dual connections, which is calculated as:

$$R(a_1, \cdots, a_n) = \frac{2}{\alpha} - 3 + n + \sum_{i=1}^{n} \frac{2}{\alpha} \sum_{k=1}^{a_i - 1} \cot(\pi a k / a_i{}^2) \cot\left(\frac{\pi k}{a_i}\right) \sin^2\left(\frac{\pi k}{a_i}\right)$$

where $\alpha = a_1 a_2 \cdots a_n$. One of their results is that $\Sigma(a_1, \cdots, a_n)$, the Seifert fibered homology 3-shpere fibered over S^2 with exceptional fibers of orders $a_1, \cdots, a_n$, has infinite order in Θ_3^H, if $R(a_1, \cdots . a_n)$ is positive.

Our main result is stated below.

Theorem 2.1. *For any homology 3-sphere Σ, we can associate a positive number $\varepsilon(\Sigma)$ with the following significance: Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_m$ be homology 3-spheres and $\Sigma_0 = \Sigma(a_1, a_2, \cdots, a_n)$ a Seifert fibered homology 3-sphere. Suppose that $\alpha = a_1 a_2 \cdots a_n$ is larger than any of $\varepsilon(\Sigma_i)^{-1}$ for $i = 1, 2, \cdots, m$, and that $R(a_1, a_2, \cdots, a_n)$ is positive. Then*

$$(2.2) \qquad \mathbb{Z}[\Sigma_0] \cap (\mathbb{Z}[\Sigma_1]+\mathbb{Z}[\Sigma_2]+\cdots+\mathbb{Z}[\Sigma_m]) = \{0\} \qquad \text{in } \Theta_3^H.$$

For example, $R(2,3,6k-1)$ $(k\in\mathbb{N})$ is calculated to be 1 [8 §10]. Hence there exists a sequence $k_1<k_2<k_3<\cdots$ such that $[\Sigma(2,3,6k_i-1)]$ $(i=1,2,3,\cdots)$ are linearly independent over $\mathbb{Z}$.

More precisely we can estimate $\varepsilon(\Sigma)$ for Seifert homology 3-spheres.

Theorem 2.3. *We can take* $\varepsilon(\Sigma(a_1,a_2,\cdots,a_n))=(a_1a_2\cdots a_n)^{-1}$ *in Theorem 2.1.*

Corollary 2.4. *Suppose that a sequence* $\Sigma_i=\Sigma(a_1{}^i,a_2{}^i,\cdots,a_{n_i}{}^i)$ $(i=1,2,\cdots)$ *satisfies that* $R(a_1{}^i,a_2{}^i,\cdots,a_{n_i}{}^i)>0$ $(i=1,2,\cdots)$ *and* $\prod a_j{}^1<\prod a_j{}^2<\cdots$. *Then* $[\Sigma_i]$ $(i=1,2,\cdots)$ *are linealy independent over* $\mathbb{Z}$ *in* Θ_3^H.

For example, $[\Sigma(2,3,6k-1)]$ $(k=1,2,\cdots)$ are linearly independent over $\mathbb{Z}$ in Θ_3^H.

According to [8] and [7], we can generalize Theorem 2.1 to the case that cobordisms admit positive definite intersection form or certain torsions in their homology groups. But we state our theorem as above for simplicity. Here we only notice that Theorem 2.1 is valid when $\Sigma_1,\Sigma_2,\cdots,\Sigma_m$ are $\mathbb{Z}_2$-homology 3-spheres and (2.2) is regarded as a relation in the $\mathbb{Z}_2$-homology cobordism group of oriented $\mathbb{Z}_2$-homology 3-sphere. Hence Corollary 2.3 is also valid in the $\mathbb{Z}_2$-homology cobordism group.

In Section 3 we show, according to [14] and [8], that Theorem 2.1 follows from compactness of a certain moduli space of self-dual V-connections. The compactness is proved in Section 4, assuming a decay estimate of curvature of self-dual connection. A crucial point is that the class of product flat connection on the product bundle over a (rational) homology 3-sphere is isolated in the moduli space of flat connections. We give a proof of the decay estimate in Section 5. In Section 6, Theorem 2.3 is proved by using Chern-Simons invariant. In Section 7, we explain the details of "patching arguments" used in Section 4 and Section 6.

3. V-manifolds with product ends.

In this section we give a framework of the proof of Theorem 2.1 following [14] and [8], and see that if we assume compactness of a certain moduli space of self-dual V-connections, then the proof is completed.

Let $\Sigma_0=\Sigma(a_1,\cdots,a_n)$ be a Seifert fiberd homology 3-sphere, and $\Sigma_1,\cdots,\Sigma_m$ oriented homology 3-spheres. Let W be a compact oriented 4-manifold with (not necessarily connected) boundary. Suppose that W has the integral homology of punctured 4-sphere, and has boundary

$$\partial W = \bigcup_{k=0}^{m} \bigcup_{j=1}^{m_k} \Sigma_{k,j} \quad (m_0\geqq 1, \text{ and } m_k\geqq 0 \text{ for } k=1,\cdots,m)$$

where $\Sigma_{0,j}$ is a copy of Σ_0, and $\Sigma_{k,j}$ $(1\leqq k\leqq m)$ is a copy of Σ_k or $-\Sigma_k$ (the orientation reversed Σ_k), and the union is disjoint union.

Let Y be the mapping cylinder of the projection from Σ_0 to S^2. Then Y is a V-manifold with n branched points $x_1,\cdots,x_n$ of orders $a_1,\cdots,a_n$. If we introduce the V-orientation of Y so that $\partial Y=\Sigma_0$, then Y is negative definite. In fact, we have $H^2(Y\backslash\{x_1,\cdots.x_n\};\mathbb{Z})\simeq H^2(Y\backslash\{x_1,\cdots,x_n\},\Sigma_0;\mathbb{Z})\simeq\mathbb{Z}$ and its generator e satisfies $e^2[Y,\Sigma_0]=-1/\alpha$ $(\alpha=a_1a_2\cdots a_n)$, where $[Y,\Sigma_0]$ is the fundamental class of V-manifold Y, and we regard e as an element of $H^2(Y,\Sigma_0;\mathbb{Q})\simeq H^2(Y\backslash\{x_1,\cdots,x_n\},\Sigma_0;\mathbb{Q})$ (see [8 §2] and also [11]).

Now we define a V-manifold X with product ends to be:

$$X = W \cup \bigcup_{j=1}^{m_0}(-Y_j) \cup \bigcup_{k=1}^{m} \bigcup_{j=1}^{m_k}\Sigma_{k,j}\times[0,\infty),$$

where Y_j is a copy of Y. Let P_0 be the principal $SO(2)$-V-bundle over $(-Y_1)$ corresponding to a generator of $H^2(Y_1\setminus\{x_1,\cdots x_n\},\Sigma_{0,1};\mathbb{Z})\simeq\mathbb{Z}$, which is trivial over $\Sigma_{0,1}$. Fix a smooth connection θ_0 such that θ_0 is equal to the product connection over a neighborhood U of $\Sigma_{0,1}$ in Y_1 on which P_0 is a product bundle. Let P be the principal $SO(3)$-V-bundle over X defined to be:

$$P = P_0 \times_{SO(2)} SO(3) \cup (X\setminus(-Y_1))\times SO(3),$$

where the identification over $\Sigma_{0,1}$ is constructed by using the product structure of $P_0|U$. We extend θ_0 on P trivially and denote it by the same notation. We fix Riemannian metrics on Σ_k $(1\leqq k\leqq m)$. Choose and fix a V-metric on X which is the product metric of the given metric on $\Sigma_{k,1}\simeq\pm\Sigma_k$ and the standard metric on $[0,\infty)$ over each end of X. Let E be the $so(3)(=\mathbb{R}^3)$-V-bundle associated with P by the adjoint representation (or the standard representation.)

For a positive number δ, we consider the following classes of V-connections and V-gauge transformations on P.

$$C_\delta = \{\theta_0+a;\ a\in L^2_{3,loc}(\Omega^1(E)),\|a\|_\delta<\infty\},$$

$$G_\delta = \{g\in L^2_{4,loc}(P\times_{Ad}SO(3));\|\nabla_{\theta_0}g\|_\delta<\infty\}.$$

Here we fix a V-smooth map $\tau:X\to\mathbb{R}$ which is equal to the projection from $\Sigma_{k,j}\times[0,\infty)$ to $[0,\infty)$ over each end of X, and use the norm $\| \ \|_\delta$:

$$\|a\|_\delta=\left(\int_X e^{\tau\delta} \sum_{i=0}^{3} |\nabla_{\theta_0}{}^i a|^2 \right)^{1/2}.$$

$\Omega^1(E)$ is the set of E-valued 1-V-forms on X. Although X is an open V-manifold, the first Pontrjagin number of P is defined as follows.

$$p_1(P) = -\frac{1}{8\pi^2}\int_X \mathrm{tr}\, F(A)\Lambda F(A).$$

Here A is an element of C_δ, and F(A) is its curvature, which is an E-valued 2-V-form. $p_1(P)$ does not depend on the choice of A (see [12 §2]). If we take $A=\theta_0$, then we obtain $p_1(P)=1/\alpha$.

Let * be the Hodge's star operator acting on E-valued V-forms, and $F_-=\frac{1}{2}(F-{*F}):C_\delta\to L^2_{2,loc}(\Omega^2(E))$ be the anti-self-dual part of curvature. The moduli space M_δ which we will consider is now defined to be $M_\delta=F_-^{-1}(0)/G_\delta$. The topology of C_δ is defined by $\| \ \|_\delta$. The topology of G_δ is defined by using $\|\nabla_{\theta_0}g\|_\delta$ and the "limitting value" (see [14 §7]). Then we give M_δ the quotient topology.

In the next section we show the following compactness.

Proposition 3.1. *There exist positive numbers* $\varepsilon(\Sigma)$ *and* $\delta(\Sigma)$ *depending only on* Σ *(and the fixed metric on it) for each (rational) homology 3-sphere* Σ *with the following significance: Suppose* $p_1(P)<\varepsilon(\Sigma_k)$ *and* $\delta\leqq\delta(\Sigma_k)$ *for every* Σ_k $(1\leqq k\leqq n)$. *Then* M_δ *is compact.*

If we assume the above proposition, then Theorem 2.1 is shown along the line of [14 §2] and [8 §9] in the following way: Suppose $R=R(a_1,\cdots,a_n)$ is positive, and $\varepsilon(\Sigma_k)>p_1(P)=\alpha^{-1}$, $\delta(\Sigma_k)\geqq\delta$ for every Σ_k $(1\leqq k\leqq n)$. R turns to be an odd integer [8]. Then, according to Taubes, when we take sufficiently small δ, and perturb F_- if neccesary, M_δ has a natural structure of R-dimensional compact singular manifold with a single singular point of the form of the vertex of cone over $\mathbb{C}P^{(R-1)/2}$, which corresponds to reducible self-dual V-connections in C_δ (see [14 §8,9]). Fix a smooth point x_0 on X. Let Q be the principal SO(3)-bundle over $M_\delta\backslash$\{the singular point\} defined by $Q=(F_-^{-1}(0)\times P_{x_0})/G_\delta$. Then the restricted bundle $Q|\mathbb{C}P^{(R-1)/2}$ bounds $Q|(M_\delta\backslash$the cone over $\mathbb{C}P^{(R-1)/2})$, hence every characterisic number of $Q|\mathbb{C}P^{(R-1)/2}$ must vanish. On the other hand, if we consider the neighborhood of the singular point locally, we get (see [8 §9])

$$(w_2(Q)^{(R-1)/2},[\mathbb{C}P^{(R-1)/2}]) = 1 \in \mathbb{Z}_2.$$

This is a contradiction. Therefore Theorem 2.1 follows.

4. Compactness of the moduli space.

Under the situation in the previous section, Taubes [14 §10] proved that M_δ is compact if (1) $p_1(P)<4$, and (2) $\pi_1(\Sigma_k)$ has only the trivial reprensentation into SO(3) for $1\leqq k\leqq m$. We want to weaken the condition (2) about π_1, but instead we may strengthen the condition (1) about $p_1(P)$. In order to prove Proposition 3.1, we modify arguments of Taubes in [14 §10].

We need the lemma below.

Lemma 4.1. *Let Σ be a rational homology 3-sphere with a fixed metric,* $Q=\Sigma\times SO(3)$ *the product bundle over Σ, and θ_0' the product connection on* Q.

(1) Fix $1\leqq p\leqq\infty$ *and* $q=0,1,\cdots$ *such that* $p(q-1)>3$. *Let* $\{B_i\}$ *be a sequence of flat connections on* Q. *Then we can take a subsequence* $\{B_{i'}\}$ *and a sequence* $\{g_{i'}\}$ *of gauge transformations such that* $g_{i'}^* B_{i'}$ *is a* L^p_{q-1}*-connection, and* $\{g_{i'}^* B_{i'}\}$ *converges to some* L^p_{q-1}*-connection in* L^p_{q-1}*-toplogy.*

(2) There is a positive constant c *with the following significance: Suppose that* B *is a flat connection such that* $\|B-\theta_0'\|_{L^\infty}<c$. *Then* B *is gauge equivalent to* θ_0'.

Proof of (1). Locally any flat connection is gauge equivalent to θ_0'. Therefore the claim immediately follows from "patching of convergences". We explain detail of this argument in Section 7. See Proposition 7.5.

Proof of (2). Suppose a sequence $\{B_i\}$ of flat connections satisfies $\|B_i-\theta_0'\|_{L^\infty}\to 0$ $(i\to 0)$. Using (1), we can take a subsequence $\{B_{i'}\}$ and a sequence $\{g_{i'}\}$ such that $\{g_{i'}{}^*B_{i'}\}$ converges to some L_2^2-connectin B_∞ in L_2^2-topology. Then Sobolev embbedding $L_2^2\to L^\infty$ and the relation

$$dg_{i'}=(B_{i'}-\theta_0')g_{i'}-g_{i'}((g_{i'}{}^*B_{i'}-B_\infty)+(B_\infty-\theta_0'))$$

imply that $\{\|g_{i'}\|_{L_1^\infty}\}$ is bounded. (Note $\|g_{i'}\|_{L^\infty}\leqq 3$.) Take a subsequence $\{g_{i''}\}$ such that $\{g_{i''}\}$ converges to some g_∞ in L_1^2-topology. Then we can show that $g_\infty{}^*\theta_0'=B_\infty$, and hence g_∞ is a L_3^2-gauge transformation by bootstrapping. Then $\{(g_{i''}g_\infty{}^{-1})^*B_{i''}\}$ converges to θ_0' in L_2^2-topology. "Slice Theorem" implies that any connection close to θ_0' in L_2^2-norm is gauge equivalent to $\theta_0'+b$ by a L_3^2-gauge transformation such that $d^*b=0$ and $\|b\|_{L_2^2}$ is small (See, for example, [1], [9], or [3]). Therefore it suffices to show that the equations

$$F(\theta_0'+b)=db+[b\Lambda b]/2=0 \quad \text{and} \quad d^*b=0$$

imply $b=0$ for small $b\in L_2^2(\Omega^1(\Sigma,so(3)))$. Let $\lambda_0{}^2$ $(\lambda_0\geqq 0)$ be the first eigenvalue of Laplacian on 1-forms of Σ. Then $H^1(\Sigma,\mathbb{R})=0$ implies that λ_0 is strictly positive. We have

$$\lambda_0\|b\|_{L^2} \leqq \|(d+d^*)b\|_{L^2} = \|[b\Lambda b]/2\|_{L^2} \leqq c'\|b\|_{L^2}\|b\|_{L^\infty}.$$

Here c' is a positive constant independent of b. If $\|b\|_{L_2^2}$ is small, then $\|b\|_{L^\infty}<\lambda_0/c'$, by Sobolev embedding $L^2_2 \subset L^\infty$, and hence we obtain $\|b\|_{L^2}=0$, i.e., $b=0$.

Now we state a crucial lemma to show compactness of the modili space.

Lemma 4.2. *Let Σ be a rational homology 3-sphere with a fixed metric, $P_\Sigma=\Sigma\times[0,\infty)\times SO(3)$ the product bundle over $\Sigma\times[0,\infty)$, and θ_0 the product connection on P_Σ. There exist positive numbers $\varepsilon=\varepsilon(\Sigma)$ and $c_0=c_0(\Sigma)$, which depend only on Σ and its metric, with the following significance: Let A be a $L^2_{3,\mathrm{loc}}$-self-dual connection on P_Σ such that*

$$\|F(A)\|_{L^2} = \left(\int_{\Sigma\times[0,\infty)} |F(A)|^2 \right)^{1/2} < \varepsilon.$$

Suppose that there exists an integer $n_0 \geqq 0$ and a gauge transformation $h\in L^2_4(\Sigma\times[n_0+1,n_0+2],SO(3))$ of $P_\Sigma|[n_0+1,n_0+2]$ such that

$$(4.3) \qquad \int_{\Sigma\times[n_0+1,n_0+2]} \sum_{j=0}^{4} |\nabla_{\theta_0}{}^j (h^* A-\theta_0)|^2 \leqq \varepsilon^2.$$

Then there exists a gauge transformation $g\in L^2_{4,\mathrm{loc}}(\Sigma\times[1,\infty),SO(3))$ of P_Σ such that for any $n=0,1,\cdot\cdot$, we have

$$(4.4) \qquad \sup_{\Sigma\times[n+1,n+3]} \sum_{j=0}^{4} |\nabla_{\theta_0}{}^j (g^* A-\theta_0)|^2 \leqq c_0{}^2 \int_{\Sigma\times[n,n+4]} |F(A)|^2.$$

We require one more derivarive in (4.4) than in the definiton of C_δ and G_δ, in order to make use of compact embedding $L_4^\infty \subset L_3^\infty$ later in the proof of Proposition 4.8.

Proof. In the following we write $c_1, c_2, \cdots$ for positive constants depending only on Σ and its metric. Lemma 10.4 of [14] implies that there exist a gauge transformation $g_n \in L^2_{4,loc}(S\times[n,n+4],SO(3))$ and a smooth flat connection A_n on $P_\Sigma|\Sigma\times[n+1,n+3]$ for each integer n such that (4.4) holds with g_n, A_n and c_1 replacing g, θ_0 and c_0:

$$(4.5)\quad \sup_{\Sigma\times[n+1,n+3]} \sum_{j=0}^{4} |\nabla_{A_n}{}^j(g_n{}^*A-A_n)|^2 \leqq c_1{}^2 \int_{\Sigma\times[n,n+4]} |F(A)|^2.$$

Using parallel translation as in the proof of Corollary 2.3 of [13], we can arrange g_n and A_n so that $A_n-\theta_0$ has no component along the path $\{x\}\times[n+1,n+3]$ for any $x\in\Sigma$. Then there is a smooth flat connection B_n on $Q=\Sigma\times SO(3)$ such that $A_n=\pi_n{}^*B_n$, where $\pi_n:\Sigma\times[n,n+3]\to\Sigma$ is the projection. By Lemma 4.1 (1), we can arrange g_n and A_n so that $\|B_n-\theta_0'\|_{L_4^\infty}$ is bounded by a constant c_2, and hence the L_q^∞-norms on sections of $\Sigma\times so(3)$ defined by using $\nabla_{\theta_0'}$ and ∇_{B_n} $(n=0,1,\cdot\cdot)$ are uniformly equivalent for $0\leqq q\leqq 5$ up to a constant c_3. Here we have written $\|b\|_{L_q^\infty}$ for $\sum_{0\leqq j\leqq q} \sup_\Sigma |\nabla_{\theta_0'} b|$. If we restrict (4.5) for (g_n,A_n) and (g_{n+1},A_{n+1}) on $\Sigma\times\{n+2\}$, and set $g_n'=g_n|\Sigma\times\{n+2\}$, $g_{n+1}'=g_{n+1}|\Sigma\times\{n+2\}$ and $A'=A|\Sigma\times\{n+2\}$, then we get

$$\frac{1}{5}\|g_n'{}^*A'-B_n\|^2_{L_4^\infty} \leqq \sup_\Sigma \sum_{j=0}^{4} |\nabla_{\theta_0'}{}^j(g_n'{}^*A'-B_n)|^2$$

$$\leqq c_3^2 \sup_{\Sigma} \sum_{j=0}^{4} |\nabla_{B_n}{}^j (g_n{}'^{*}A'-B)|^2 \leqq c_3^2 c_1^2 \int_{\Sigma\times[n,n+3]} |F(A)|^2 \leqq c_3^2 c_1^2 \varepsilon^2,$$

Similarly we get

$$\frac{1}{5}\ \|g_{n+1}{}'^{*}A'-B_{n+1}\|^2_{L_4^\infty} \leqq c_3^2 c_1^2 \varepsilon^2.$$

Let k_n be $g_n{}'^{-1} g_{n+1}{}'$. Then from the relation and estimate below

$$d_{\theta_0{}'} k_n = k_n((g_{n+1}{}'^{*}A'-B_{n+1})+(B_{n+1}-\theta_0{}'))-((g_n{}'^{*}A'-B_n)+(B_n-\theta_0{}'))k_n,$$

$$\|k_n\|_{L^\infty_{q+1}} \leqq \|k_n\|_{L^\infty_q} + 5\|k_n\|_{L^\infty_q}(c_1c_3\varepsilon+c_2) + 5(c_1c_3\varepsilon+c_2)\|k_n\|_{L^\infty_q} \text{ for } 0\leqq q\leqq 4,$$

we get, by bootstrapping, that $\|k_n\|_{L_5^\infty}$ is bounded by a constant c_4. (Note that $k_n \in C^0(\Sigma, SO(3))$ implies $\|k_n\|_{L^\infty} \leqq 3$.) Then we have

$$(4.6)\qquad \|k_n{}^{*}B_n-B_{n+1}\|_{L_4^\infty} \leqq \|k_n{}^{*}(B_n-g_n{}'^{*}A')\|_{L_4^\infty} + \|g_{n+1}{}'A'-B_{n+1}\|_{L_4^\infty}$$

$$\leqq \|k_n{}^{-1}\|_{L_4^\infty}\|B_n-g_n{}'^{*}A'\|_{L_4^\infty}\|k_n\|_{L_4^\infty} + \|g_{n+1}{}'A'-B_{n+1}\|_{L_4^\infty}$$

$$\leqq 3c_4^2 c_1 c_3 \varepsilon + 3c_1c_3\varepsilon.$$

(Note that $k_n \in C^0(\Sigma, SO(3))$ implies $\|k_n{}^{-1}\|_{L_4^\infty} = \|k_n\|_{L_4^\infty}$.)

We set $h''=h|\Sigma\times\{n_0+2\}$ and $A''=A|\times\{n_0+2\}$. By (4.3)and Sobolev embedding $L_3^2 \subset L^\infty$, we have

$$\|h''^{*}A''-\theta_0{}'\|_{L^\infty} \leqq c_6\varepsilon,$$

$$\|(g_{n_0}'^{-1}h'')^{*}B_{n_0}-\theta_0'\|_{L^\infty}\leqq \|h''^{*}A''-\theta_0'\|_{L^\infty}$$

$$+\|(g_{n_0}'^{-1}h'')^{-1}\|_{L^\infty}\|B_{n_0}-g_{n_0}'^{*}A''\|_{L^\infty}\|g_{n_0}^{-1}h''\|_{L^\infty}$$

$$\leqq c_6\varepsilon+30c_3c_1\varepsilon.$$

Then, using Lemma 4.1 (2), we see that B_{n_0} is gauge equivalent to θ_0', if ε is sufficiently small. Similarly (4.6) implies inductively that inductively that all B_n's are gauge equivalent to θ_0', if ε is sufficiently small. Therefore there is a gauge transformatin $h_{n,n+1}$ on $P_\Sigma|\Sigma\times[n+2,n+3]$ such that $h_{n,n+1}{}^{*}A_n=A_{n+1}$ on $\Sigma\times[n+2,n+3]$. Difine a proncipal SO(3)-bundle P_Σ' over $\Sigma\times[1\times\infty)$ by using data $\{P_\Sigma|\Sigma\times[n+1,n+3],\ h_{n,n+1}\}_{n=0,1,\cdots}$ Then data $\{A_n\}$ is regarded as a flat connection on P_Σ', which is gauge equivalent to a product product connection. Let $h_n\in\mathrm{Iso}(P_\Sigma'|\Sigma\times[n+1,n+3],P_\Sigma\|\Sigma\times[n+1,n+3])$ be the given "trivialization" such that $h_{n,n+1}=h_nh_{n+1}^{-1}$. (4.5) implies that g_nh_n gives a local approximation between A and $\{A_n\}$. Then by patching argument as in the proof of Lemma 10.5 of [14], we get g satisfying (4.4). This completes the proof of Lemma 4.2. We explain detail of the patching argument ("patching of connection approximation") in Section 7. See Proposition 7.4 and Remark 7.7).

In the next section we prove the following lemma.

Lemma 4.7. *Let Σ, P_Σ and ε be as in Lemma 4.2. There exist $\gamma>0$ and $c_7>0$ depending only on Σ with the following significance: Suppose* A *is a self-dual connection satisfying the conditions in Lemma 4.2. Then the following decay estimate holds.*

$$|F(A)|^2 \leqq c_7 e^{-\tau\gamma}\|F(A)\|_{L^2}^2.$$

Such a decay estimate is obtained by Donaldson [3] and Freed-Uhlenbeck [9] when Σ is the standard 3-sphere. Donaldson's argument might be valid in our case, if we use Lemma 4.2. Then we could take γ to be essentially the first eigenvalue of Laplacian on 1-forms of Σ (see Appendix of [3]). In the next section we give a proof of a rough estimate using Lemma 4.2.

In the rest of this section we assume Lemma 4.5 and prove the compactness of moduli space of self-dual connections on P_Σ in the following form.

We set $C_\delta(P_\Sigma)$ and $G_\delta(P_\Sigma)$ as in the definition of C_δ and G_δ in Section 3. Note that, by definition, $C_\delta(P_\Sigma)$ satiefies (4.3) for sufficiently large n_0.

Proposition 4.8. *Let Σ, P_Σ, ε and γ be as in Lemma 4.2. Take $\delta>0$ smaller than γ. Suppose a sequence $\{A_i\}$ of self-dual connections in $C_\delta(P_\Sigma)$ satisfies $\|F(A_i)\|_{L^2}\leqq\varepsilon$. Then there is a subsequence $\{A_{i'}\}$,* *a sequence* $\{g_{i'}\}\subset L^2_{4,loc}(\Sigma\times[1,\infty),SO(3))$ *and a self-dual connection* $A_\infty\in C_\delta(P_\Sigma)$ *such that* $g_{i'}^{*}A_{i'}$ *converges to* A_∞ *in* $\| \ \|_\delta$*-norm.*

Note that, by Lemma 7.2 of [14], $g_{i'}$ is automatically in $G_\delta(P_\Sigma)$.

Proof of Proposition 4.8, given Lemma 4.7. By Lemma 4.2 there is g_i such that if we set $g_i^{*}A_i=\theta_0+a_i$, then

$$(4.9)\qquad \sup_{\Sigma\times[n+1,n+3]}\sum_{j=0}^{4}|\nabla_{\theta_0}{}^j a_i|^2 \leqq c_0^2\int_{\Sigma\times[n,n+4]}|F(A_i)|^2 \leqq c_0^2\varepsilon^2.$$

Since $L^\infty_4\subset L^\infty_3$ is a compact embedding, we can take a subsequence $A_{i'}$ such that $\{a_{i'}\}$ converges to some a_∞ in $L^\infty_{3,loc}$-topology. Set $A_\infty=\theta_0+a_\infty$. We claim that $\|a_{i'}-a_\infty\|_\delta\to 0$. Lemma 4.7 and (4.9) imply

$$\int_{\Sigma\times[n_0+1,\infty)}e^{\tau\delta}\sum_{j=0}^{3}|\nabla_{\theta_0}{}^j a_i|^2 \leqq \sum_{n=n_0}^{\infty}e^{(n+3)\delta}c_0^2\|F(A_i)\|^2_{L^2(\Sigma\times[n,n+4))}$$

$$\leqq \sum_{n=n_0}^{\infty}e^{(n+3)\delta}c_0^2c_7e^{-n\gamma}\varepsilon^2 4\mathrm{vol}(\Sigma) \leqq c_8e^{-n_0(\gamma-\delta)}.$$

Here we set $c_8=4(1-e^{-(\gamma-\delta)})^{-1}e^{3\delta}c_0^2c_7\varepsilon^2\cdot\mathrm{vol}(\Sigma)$.

We write $\|a_i\|^2_{\delta,[n_0+1,\infty)}$ for the left hand side.
Now take any positive number ε_1. For large n_0, we have

$$\|a_i\|_{\delta,[n_0+1,\infty)} \leqq \varepsilon_1. \qquad (i=1,2,\cdots).$$

Since $a_{i'}$ converges to a_∞ in $L^\infty_{3,loc}$-topology,

$$\|a_\infty\|^2_{\delta,[n_0+1,\infty)} \leqq \liminf_{i'\to\infty} \|a_{i'}\|^2_{\delta,[n_0+1,\infty)} \leqq \varepsilon_1^{\,2}.$$

This implies $A_\infty \in C_\delta(P_\Sigma)$. For sufficiently large i', we have $\|a_{i'}-a_\infty\|_{\delta,[0,n_0+1]} \leqq \varepsilon_1$. Then we can estimate as follows.

$$\|g_{i'}{}^* A_{i'} - A_\infty\|_\delta{}^2 = \|a_{i'}-a_\infty\|_{\delta,[0,\infty)}{}^2$$

$$\leqq \|a_{i'}-a_\infty\|_{\delta,[0,n_0+1]}{}^2 + 2\|a_{i'}\|_{\delta,[n_0+1,\infty)}{}^2 + 2\|a_\infty\|_{\delta,[n_0+1,\infty)}{}^2$$

$$\leqq 5\varepsilon_1{}^2.$$

This completes the proof of Proposition 4.6.

Now we can show Proposition 3.1.

Proof of Propositon 3.1, given Lemma 4.7. We write $\varepsilon(\Sigma)$ and $\delta(\Sigma)$ for the constants in Proposition 4.8 for Σ. Proposition 4.8 and a patching argument ("patching of convergences") imply that M_δ is compact. We explain the detail of the patching argument in Section 7. See Propsition 7.5.

5. Decay estimate.

In order to give a proof of Lemma 4.5 and complete the whole proof of Theorem 2.1, we prepare two lemmas. We fix a positive number ε as in Lemma 4.2.

Lemma 5.1. *There exists* $c_7>0$ *depending only on* Σ *with the following significance: Suppose* A *is a self-dual connection on* P_Σ *satisfying the conditions of Lemma 4.2. Then we have for any integers* $n_0<n_1$,

$$\|F(A)\|^2_{L^2(\Sigma\times[n_0,n_1])}$$
$$< c_7(\|F(A)\|^2_{L^2(\Sigma\times[n_0-1,n_0+2])}+\|F(A)\|^2_{L^2(\Sigma\times[n_1-2,n_1+1])})$$

Proof. By Lemma 4.2 we can assume $A=\theta_0+a$ and

$$(5.2)\qquad \sup_{\Sigma\times[n+1,n+2]}(|a|^2+|\nabla_{\theta_0}a|^2)\leqq c_0\|F(A)\|^2_{L^2(\Sigma\times[n,n+3])}\leqq c_0\varepsilon.$$

According to Appendix of [3] we set $T(a)=\mathrm{tr}(da\wedge a+\frac{2}{3}a\wedge a\wedge a)$ so that $dT(a)=\mathrm{tr}(F(A)\wedge F(A))$. Since $F(A)$ is self-dual, $|F(A)|^2=-\mathrm{tr}(F(A)\wedge F(A))$. Then for integers $n_0<n_1$, we have

$$(5.3)\qquad \|F(A)\|^2_{L^2(\Sigma\times[n_0,n_1])}=\int_{\Sigma\times n_0}T(a)-\int_{\Sigma\times n_1}T(a).$$

The claim follows from (5.2) and (5.3).

Lemma 5.4 *Let* $\{a_k\}_{k=1}^{\infty}$ *be a sequence of non negative numbers* *satisfying*

$$(5.5) \qquad \sum_{i=1}^{\infty} a_i < \infty.$$

Suppose there is an integer N *such that for any integers* $k_0<k_1$,

$$(5.6) \qquad \sum_{i=k_0}^{k_1} a \leqq N(a_{k_0}+a_{k_1}).$$

Then we obtain

$$a_k \leqq 4Na_1 2^{-(k/2N)}.$$

Proof. (5.5) implies $a_i \to 0$ $(i\to\infty)$. If we take $k_1\to\infty$ in (5.6), then we get

$$(5.7) \qquad \sum_{i=k}^{\infty} a_i \leqq Na_k.$$

Define k_j which satisfies

$$(5.8) \qquad 2Nj \leqq k_j < 2N(j+1) \quad \text{and} \quad (5.9) \qquad a_{k_j} \leqq a_1/2^j,$$

inductively as follows. Take $k_0=1$. Assume that we can take k_j satisfying (5.8) and (5.9). (5.7) implies

$$\sum_{i=2N(j+1)}^{2N(j+2)-1} a_i \leqq Na_{k_j}.$$

Therefore there is k_{j+1} satisfying (5.8), replaced j by j+1, such that

$$\alpha_{k_{j+1}} \leqq (N\alpha_{k_j})/(2N) \leqq \alpha_1/2^{j+1}.$$

For any $k>0$, we can take k_j so that $k_j \leqq k < 2N(j+2)$. Then we get

$$\alpha_k \leqq \sum_{i=k_j}^{\infty} \alpha_i \leqq N\alpha_{k_j} \leqq N\alpha_1/2^j \leqq 4N\alpha_1 2^{-(k/2N)}.$$

Proof of Lemma 4.5. By Lemma 5.1 and Lemma 5.4, there are $c_8>0$ and $\gamma>0$ depending only on Σ such that

$$\|F(A)\|^2_{L^2(\Sigma\times[n,n+3])} \leqq c_8 e^{-n\gamma}.$$

Then (5.2) implies a similar estimate for $|F(A)|^2$.

6. Chern-Simons invatiant.

In the previous sections we show the compactness of M_δ when L^2-norm of curvatures are bounded by a sufficiently small constant. In this section we estimate the constant using Chern-Simons invariant and calculate in the case of Seifert fibered homology 3-sphere to prove Theorem 2.3.

First we define Chern-Simons invariant for SO(3)-connection as follows (see [2]).

Definition 6.1. Let Z be an oriented closed 3-manifold, Q a principal SO(3)-bundle over Z with $w_2(Q)=0$ (i.e. $Q\simeq Z\times SO(3)$), and B a connection on Q. Take a compact oriented 4-manifold Y, a principal SO(3)-bundle $\mathfrak{Q}$ over Y with $w_2(\mathfrak{Q})=0$, and a connection A on $\mathfrak{Q}$ such that the boundary of Y is a disjoint union of Z and an oriented closed 3-manifold Z' and that the restriction of $(\mathfrak{Q},A)$ on Z is isomorphic to (Q',B), and the restriction of $(\mathfrak{Q},A)$ on Z' is isomorphic to ($Z'\times SO(3)$, product connection). Then we define $Tp_1(B)\in\mathbb{R}/4\mathbb{Z}$ by

$$Tp_1(B) = -\frac{1}{8\pi^2}\int_Y tr\, F(A)\Lambda F(A) \mod 4\mathbb{Z}.$$

When P is an SO(3)-bundle over an oriented closed 4-manifold with $w_2(P)=0$, then P is induced from an SU(2)=Spin(3)-bundle P', and $p_1(P)=-4c_2(P')\in 4\mathbb{Z}$. This implies that the above difinition is well-defined.

Similarly we define $Tc_2(B')\in\mathbb{R}/\mathbb{Z}$ for an SU(3)-connection B' over an oriented closed 3-manifold.

The following lemma is an immediate consequence of the definition.

Lemma 6.2. *(1) Let* $\tilde{Z}\to Z$ *be a finite regular covering with covering transformation group* G*, and* B' *a flat connection over* Z *corresponding a homomorphism from* G *to* $SU(3)$*. Then*

$$Tc_2(B')\in(\#G)^{-1}\mathbb{Z}/\mathbb{Z}.$$

(2) Let Y *be a compact oriented 4-manifold with* $\partial Y=\cup Z_i$ *(disjoint union), and* A *a flat connection over* Y*. Then*

$$\sum_i Tc_2(A|Z_i) = 0.$$

(3) Let B *be an* $SO(3)$*-connection on the product bundle over an oriented closed 3-manifold, and* B' *the* $SU(3)$*-connection induced from* B *by the inclusion* $SO(3)\subset SU(3)$*. Then*

$$Tc_2(B') = -Tp_1(B) \bmod \mathbb{Z} \quad \in \mathbb{R}/\mathbb{Z}.$$

Proposition 6.3. *Let* B *be a flat* $SO(3)$*-connection over* $\Sigma_0=\Sigma(a_1,a_2,\cdots,a_n)$*. Then we have*

$$Tp_1(B) \in (a_1a_2\cdots a_n)^{-1}\mathbb{Z}/\mathbb{Z}.$$

Proof. Let Y be the mapping cylinder of $\pi:\Sigma_0\to S^2$ introduced in

Section 3. Let $x_1, x_2, \cdots, x_n \in S^2 \subset Y$ be the singular points of Y. The link of x_i is isomorphic to a lens space with fundamental group $\mathbb{Z}/a_i$. Therefore Lemma 6.2 implies that it suffices to show that B can be extended to a flat connection over $Y\backslash\{x_1, \cdots x_n\}$. Let $\gamma \in \pi_1(\Sigma_0)$ be the element corresponding to general fiber of $\pi: \Sigma_0 \to S^2$. Since $Y\backslash\{x_1, \cdots x_n\}$ has a homotopy type of union of Σ_0 and the mapping cylinder of $\Sigma_0\backslash\{\text{singular fibers}\} \to S^2\backslash\{x_1, \cdots, x_n\}$, van Kampen's theorem implies that

$$\pi_1(Y\backslash\{x_1, \cdots x_n\}) = \pi_1(\Sigma_0)/\langle\gamma\rangle.$$

Suppose that B is corresponding to a homomorphism $\varphi: \pi_1(\Sigma_0) \to SO(3)$. Since γ is in the center of $\pi_1(\Sigma_0)$, the image $\varphi(\pi_1(\Sigma_0))$ is contained in the centralizer $C_{SO(3)}(\varphi(\gamma))$. Since $\pi_1(\Sigma_0)$ is perfect, the image $\varphi(\pi_1(\Sigma_0))$ is contained in the commutator subgroup of $C_{SO(3)}(\varphi(\gamma))$. Similarly $\varphi(\pi_1(\Sigma_0))$is contained in the twofold commutator subgroup of $C_{SO(3)}(\varphi(\gamma))$. Let g be an element of SO(3). When the order of g is equal to 2, we have $C_{SO(3)}(g) \simeq O(2)$ and $[O(2), O(2)] = S^1$, $[S^1, S^1] = 1$. When the order of g is neither equal to 1 nor 2, we have $C_{SO(3)}(g) \simeq S^1$ and $[S^1, S^1] = 1$. In either case the twofold commutator subgroup of $C_{SO(3)}(g)$ is trivial. Therefore $\varphi(\gamma)$ must be equal to 1, and hence φ can be reduced to a homomorphism from $\pi_1(Y\backslash\{x_1, \cdots, x_n\})$ to SO(3). This completes the proof.

For a homology 3-sphere Σ, we define $0 \leqq \varepsilon_1(\Sigma) \leqq 4$ as an estimate of the length from the product connection to other flat connections on $\Sigma\times SO(3)$.

Definition 6.4. Let $\pi:(0,4]\to\mathbb{R}/\mathbb{Z}$ be projection. We set

$$\varepsilon_1(\Sigma) = \inf\{\pi^{-1}(Tp_1(B)) \ ; \ B \text{ is a flat connection of } \Sigma\times SO(3)\}.$$

Proposition 6.3 implies the following estimate.

Proposition 6.5. $\varepsilon_1(\pm\Sigma(a_1,a_2,\cdots,a_n))\geqq(a_1a_2\cdots a_n)^{-1}$.

We show that we can take $\varepsilon_2(\Sigma)=\min\{\varepsilon(\Sigma),\varepsilon(-\Sigma)\}$ as the constant $\varepsilon(\Sigma)$ in Theorem 2.1, when $\varepsilon_2(\Sigma)>0$.

Theorem 6.6. *Let* $\Sigma_1,\Sigma_2,\cdots,\Sigma_m$ *be homology 3-spheres. Suppose* $\varepsilon_2(\Sigma_i)>0$ $(i=1,2,\cdots,m)$. *Let* $\Sigma_0=\Sigma(a_1,a_2,\cdots,a_n)$ *be a Seifert fibered homology 3-sphere such that* $R(a_1,a_2,\cdots,a_n)>0$ *and* $a_1a_2\cdots a_n>\varepsilon_2(\Sigma_i)^{-1}$ $(i=1,2,\cdots,m)$. *Then*

$$\mathbb{Z}[\Sigma_0]\cap\mathbb{Z}([\Sigma_1]+\mathbb{Z}[\Sigma_2]+\cdots\mathbb{Z}[\Sigma_m]) = \{0\} \text{ in } \Theta_3^H.$$

Proof of Theorem 2.3, given Theorem 6.6. Theorem 2.3 immediately follows from Proposition 6.5 and Theorem 6.6.

In the rest of this section we prove Theorem 6.6 by modifying the argument of [14,§10].

We use the notations in Section 4. The following lemma is an extension of Lemma 10.3 of [14].

Lemma 6.7. *Let* $\{A_i\}\subset C(P_\Sigma)$ *be a sequence of self-dual connections. Then either one of the followings is satisfied.*

(6.8) $$\lim_{n\to\infty}\limsup_{i\to\infty}\int_{\Sigma\times[n,\infty)}|F(A_i)|^2 = 0.$$

(6.9) $$\lim_{n\to\infty}\limsup_{i\to\infty}\int_{\Sigma\times[n,\infty)}|F(A_i)|^2 \geqq 8\pi^2\varepsilon_1(-\Sigma).$$

Proof. Suppose that (6.9) is not satisfied. There are n_0 and $c<\varepsilon_1(-\Sigma)$ such that

$$\int_{\Sigma\times[n_0,\infty)}|F(A_i)|^2 \leqq 8\pi^2 c < 8\pi^2\cdot 4 \qquad (i>>1).$$

Theorem 8.8 of [9] implies that there is a subsequence $\{A_{i'}\}$ and a sequence $\{g_{i'}\}$ of gauge transformations such that $\{g_{i'}{}^*A_{i'}\}$ converges to some A_∞ in C^2_{loc}-topology. Note that since a "bubble" of SO(3)-self-dual connection must carry away a positive integral multiple of $8\pi^2\cdot 4$ from the square of L^2-norm of curvature, there is no room for bubbles in this case. To prove (6.8), we argue as follows. As in the proof of Lemma 4.2, we can approximate $A_\infty|\Sigma\times[n_1,n_1+4]$ by a flat connection induced from a flat connection B_{n_1} on $\Sigma\times SO(3)$. We take $n_1(>n_0)$ sufficiently large so that the error is small. When i' is sufficiently large, it is also an approximation of $g_{i'}{}^*A_{i'}|\Sigma\times[n_1,n_1+4]$. Fix such large i'. When n_2 $(>n_1)$ is sufficiently large, we can approximate $g_{i'}{}^*A_{i'}|\Sigma\times[n_2,n_2+4]$ by a flat connection isomorphic to the product connection on

$\Sigma\times[n_2,n_2+4]\times SO(3)$ from the definition of $C_\delta(P_\Sigma)$. Therefore by considering cutting off at $[n_1,n_1+4]$ and $[n_2,n_2+4]$ (see [14 §10]), we can approximate the Chern-Simons invariant of B_{n_1} by

$$(6.10)\quad \int_{\Sigma\times[n_1+2,n_2+2]} p_1(A_{i'}) = \frac{1}{8\pi^2}\int_{\Sigma\times[n_1+2,n_2+2]} |F(A_{i'})|^2.$$

The error tends to zero when we take the threefold limit with respect to $n_2\to\infty$, $i'\to\infty$ and $n_1\to\infty$ in order. Since (6.10) is bounded by $c<\varepsilon_1(-\Sigma)$, this implies that $Tp_1(B_{n_1})\in[0,\varepsilon_1(-\Sigma))$ mod $\mathbb{Z}$ when n_1 is sufficiently large. Then the definition of $\varepsilon_1(-\Sigma)$ means $Tp_1(B_{n_1})=0$. This in turn implies that (6.10) is almost equal to zero, and the error tends to zero when we take the threefold limit with respect to $n_2\to\infty$, $i'\to\infty$ and $n_1\to\infty$ in order. This completes the proof.

Proof of Theorem 6.6. As in the proof of Theorem 2.1, it suffices to show Proposition 3.1 with $\varepsilon_2(\Sigma)$ replacing $\varepsilon(\Sigma)$. Take the same δ as in Proposition 3.1. Let $\{[A_i]\}$ be a sequence in M_δ. Then on each end $\Sigma\times[0,\infty)$, (6.9) is not satisfied from Proposition 6.3, and hence (6.8) is satisfied. Let $\varepsilon=\varepsilon(\Sigma)$ be the constant as in Proposition 4.8. Then for sufficiently large n, there is a subsequence $\{A_{i'}\}$ such that

$$\int_{\Sigma\times[n,\infty)} |F(A_{i'})|^2 < \varepsilon(\Sigma).$$

Apply Proposition 4.8 to take a subsubsequence which converges on $\Sigma\times[n,\infty)$ in $\|\ \|_\delta$-norm when we choose suitable gauges. Then, as in the proof of Proposition 3.1, "patching of covergence" (Proposition 7.5) implies that there is a subsubsequence of $\{A_{i'}\}$ which converges on the whole V-manifold in $\| \ \|_\delta$-norms. This implies the requied compactness of M_δ. This completes the proof.

7. Patching argument.

The arguments stated in this section are essentially given by K.K. Uhlenbeck in [15], and already used in [3] and [14]. But since there seems to be no articles which state these arguments in general context, we explain them here for convenience of readers. (We do not cover all patching arguments in [15]. We do not give "patching of gauges" used by Taubes to prove Lemma 10.4 of [14] quoted in Section 4.)

For simplicity we consider only principal SO(3)-bundle over smooth manifolds. (This assumption is used only in the proof of Lemma 7.1. Other arguments are valid for any bundles whose structure groups are compact Lie groups.) In this section we use only Sobolev embeddings as tools.

We use the following notations: We fix n, $1\leqq p\leqq\infty$ and $q=0,1,\cdots$ such that $p(q-1)>n$. Let M be an n-dimensional Riemannian manifold, and $\{U_i\}$ a finite open covering such that $U_i\cap U_j$ is relatively compact and admits Sobolev embedding $L^p_{q-1}\subset L^\infty$ and Sobolev multiplication laws $L^p_{q-1}\times L^p_k\to L^p_k$ $(0\leqq k\leqq q-1)$ as bounded maps. Set $V_i=U_i\cap\bigcup_{j\neq i}U_j$. Let P_0 be a principal SO(3)-bundle over M, and θ_0 a smooth connection on P. For a open set $U\subset M$ and a connection A on $P_0|U$, we define $\|\cdot\|_{L^p_k(A,U)}$ by

$$\|s\|_{L^p_k(A,U)}=\Big(\int_U\sum_{j=0}^{k}|\nabla_A{}^j s|^p\Big)^{1/p}\ \text{if } p<\infty,$$

$$=\sum_{j=0}^{k}\sup_U|\nabla_A{}^j s|\qquad\text{if } p=\infty,$$

for sections s of associated vector bundles with P_0. In this section we call A a $L^p_{q-1,loc}$-connection on an open set $U \subset M$ if $A=\theta_0+a$ over U and $\|a\|_{L^p_{q-1}(\theta_0, U\cap V)}<\infty$ for any relatively compact open set V. We call A a L^p_{q-1}-connection on U if $\|A-\theta_0\|_{L^p_{q-1}(\theta_0,U)}<\infty$. When V is a relatively compact open set admitting Sobolev multiplication laws $L^p_{q-1}\times L^p_k \to L^p_k$ $(0 \leqq k \leqq q-1)$, then $L^p_k(\theta_0,V)$-norm is equivalent to $L^p_k(A,V)$-norm $(0\leqq k \leqq q)$ for L^p_{q-1}-connection A on V. Hence, for example, the same Sobolev multiplication laws hold for $L^p_k(A,V)$-norm. We use this fact freely.

Lemma 7.1 (patching of bundle isomorphisms). *Let* A_0 *be a* $L^p_{q-1,loc}$*-connection on* P_0. *Then there exist* $\varepsilon>0$ *and* $c>0$ *with the following significence: Suppose* P_0' *is a principal* SO(3)*-bundle over* M *and* $\{g_i\}$ *is a set of bundle isomorphisms from* $P_0|U_i$ *to* $P_0'|U_i$ *, covering identity, such that*

$$(7.2) \qquad \|g_i^{-1}g_j-1\|_{L^p_q(A_0,U_i\cap U_j)} < \varepsilon$$

Then there is a bundle isomorphism $g:P_0\to P_0'$*, covering identity, such that for each* i_0*,* g *is equal to* g_{i_0} *over* $U_{i_0}\backslash V_{i_0}$*, and*

$$\|g_{i_0}^{-1}g-1\|_{L^p_q(A_0,V_{i_0})} \leqq c\sum_{i,j}\|g_i^{-1}g_j-1\|_{L^p_q(A_0,U_i\cap U_j)}.$$

Proof. First we give a proof for principal Sp(1)-bundles. Let P_0

and P_0' be Sp(1)-bundles satisfying the corresponding conditions. Let L and L' be the standard associated quaternionic line bundles with P_0 and P_0' respectively. We identify $\mathrm{Iso}((P_0)_x,(P_0')_x)$ with the unit vectors of $\mathrm{Hom}_{\mathbb{H}}(L_x,L'_x)$ for $x\in M$. Fix a partition of unity $\{\rho_i\}$ of $\{U_i\}$. Sobolev embedding $L^p_q\subset C^0$ and (7.2) imply that, if ε is sufficiently small, we can define $g\in\mathrm{Iso}(P_0,P_0')$ by

$$(7.3)\qquad g=\frac{\sum_i \rho_i g_i}{|\sum_i \rho_i g_i|}\in \mathrm{Hom}_{\mathbb{H}}(L,L').$$

Note that $|\sum_i\rho_i g_i|=|\sum_i\rho_i g_{i_0}{}^{-1}g_i|$. Then we can check the required estimate.

In the case of SO(3)-bundle, we can pull back the structure group of P_0 and P_0' to Sp(1) (the double covering of SO(3)) at least locally. If ε is small, we can lift $\{g_i\}$ to local isomorphsms between the Sp(1)-bundles uniquely so as to preserve the estimate (7.2). Then g is well defined by (7.3). This completes the proof.

In the proof of Lemma 4.1 (1) in Section 4, we used the following proposition.

Proposition 7.4 (patching of connection approximations). *Let* $\{g_{ij}\subset\mathrm{Aut}(P_0|U_i\cap U_j\}_{i,j}$ *be a set of gauge transformations of* $P_0|U_i\cap U_j$ *satisfying cocycle conditions:* $g_{ii}=1$, $g_{ij}g_{ji}=1$ *and* $g_{ij}g_{jk}g_{ki}=1$. *We call it a set of transition functions. Let* $\{A_i\}_i$ *be a set of* $L^p_{q-1,loc}$*-connections on* $P_0|U_i$ *such that* $g_{ij}{}^*A_i=A_j$ *over* $U_i\cap U_j$. *We write* P *for the bundle defined by patching* $\{P_0|U_i\}$ *by* $\{g_{ij}\}$. *We* *write* A *for the connection on* P *defined by* $\{A_i\}$. *Then there exist* $\varepsilon>0$ *and* $c>0$ *with the following significance: Suppose* P' *is a bundle represented by data* $\{g_{ij}'\in \mathrm{Aut}(P_0|U_i\cap U_j)\}$ *and* A' *is a connection on* P' *represented by data* $\{A_i'\}$ *such that*

$$\|g_{ij}^{-1}g_{ij}'-1\|_{L_q^p(A_j,U_i\cap U_j)} \leqq \varepsilon,$$

Then there is a bundle isomorphism $g:P\to P'$ *such that* $g=g_i$ *on* $U_i\backslash V_i$ *and*

$$\|A-g^*A'\|_{L_{q-1}^p(A,\bigcup_j V_j)}$$

$$\leqq c\Big(\sum_{i,j}\|g_{ij}^{-1}g_{ij}'-1\|_{L_q^p(A_j,U_i\cap U_j)}+\sum_i \|A_i'-A_i\|_{L_{q-1}^p(A_i,V_i)}\Big)$$

Proof. Define $g_i:P|U_i\to P'|U_i$ by using the given "trivialities" with $P_0|U_i$ and the identity map, then $g_i^{-1}g_j=g_{ij}^{-1}g_{ij}'$ on $P|U_i\cap U_j\subset P_0|U_j$. Lemma 7.1 implies that there is $g:P\to P'$ such that $g=g_{i_0}$ on $U_{i_0}\backslash V_{i_0}$, and

$$\|g_{i_0}^{-1}g-1\|_{L_q^p(A,V_{i_0})} \leqq c\sum_{i,j}\|g_i^{-1}g_j-1\|_{L_q^p(A,U_i\cap U_j)}.$$

Let $h_i\in \mathrm{Aut}(P_0|U_{i_0})$ be the representation of $g|U_{i_0}$ using the "triviality". Then $h_{i_0}=1$ on $U_{i_0}\backslash V_{i_0}$ and

$$\|h_{i_0}-1\|_{L_q^p(A_{i_0},V_{i_0})} \leqq c\sum_{i,j}\|g_i^{-1}g_j-1\|_{L_q^p(A,U_i\cap U_j)}.$$

Using Sobolev multiplication law $L_q^p\times L_{q-1}^p\to L_{q-1}^p$ on V_{i_0}, we obtain

$$\|A-g^{*}A'\|_{L^p_{q-1}(A,V_{i_0})}=\|A_{i_0}-h_{i_0}{}^{*}A_{i_0}{}'\|_{L^p_{q-1}(A_{i_0},V_{i_0})}$$

$$\leqq\|A_{i_0}-A_{i_0}{}'\|_{L^p_{q-1}(A_{i_0},V_{i_0})}+c\|h_{i_0}{}^{-1}\|_{L^p_q(A_{i_0},V_{i_0})}\|d_{A_{i_0}}h_{i_0}\|_{L^p_{q-1}(A_{i_0},V_{i_0})}$$

$$+\ c\|h_{i_0}\|^2_{L^p_q(A_{i_0},V_{i_0})}\|A_{i_0}-A_{i_0}{}'\|_{L^p_{q-1}(A_{i_0},V_{i_0})}.$$

Here $c>0$ is a constant. Using $d_{A_{i_0}}1=0$, we obtain the requied estimate.

In the proof of Lemma 4.1 (1), in the last step of the proof of Proposition 3.1 in Section 4 and in the proof of Theorem 6.6, we used the following proposition.

Proposition 7.5 (patching of convergences). *Suppose a sequence* $\{A^k\}$ *of connections on* P_0 *and gauge transformations* $\{g_i^k\in\mathrm{Aut}(P_0|U_i)\}_{i,k}$ *satisfy that* $\{g_i^{k*}A^k\}_k$ *converges to some* L^p_{q-1}*-connection* A_i^∞ *on* U_i *in* $L^p_{q-1}(\theta_0,U_0)$*-norm. If we take a subsequence* $\{A^k\}$ *(now relabeled), then there are* $\{g^k\in\mathrm{Aut}P_0\}$ *such that* $g^{k*}A^k$ *converges to some* L^p_{q-1}*-connection* A^∞ *in* $L^p_{q-1}(\theta_0,M)$*-norm. We can take* g^k *so that* $g^k=g_i^k$ *on* $U_i\backslash V_i$.

Proof. Set $g_{ij}^k=(g_i^k)^{-1}g_j^k$, then we have $g_{ij}^k{}^{*}g_i^{k*}A^k=g_j^{k*}A^k$, i.e.,

$$\nabla_{A_i^\infty}g_{ij}^k=d_{A_i^\infty}g_{ij}^k=g_{ij}^k(g_i^{k*}A^k-A_i^\infty)-(g_j^{k*}A^k-A_i^\infty)g_{ij}^k \text{ on } U_i\cap U_j.$$

By bootstrapping, this implies that $\{g_{ij}^k\}$ is L^p_q-bounded. We can take a subsequence $\{A^k\}$ (now relabeled) such that $\{g_{ij}^k\}$ converges to some g_{ij}^∞ on $U_i \cap U_j$ in L_{q-1}^p-topology. Then we have $g_{ij}^{\infty *} A_i^\infty = A_j^\infty$ and also

$$\nabla_{A_i^\infty}(g_{ij}^k(g_{ij}^\infty)^{-1}) = d_{A_i^\infty}(g_{ij}^k(g_{ij}^\infty)^{-1}) = g_{ij}^k(g_{ij}^\infty)^{-1}\cdot g_{ij}^\infty(g_j^{k*}A^k - A_j^\infty)(g_{ij}^\infty)^{-1}$$
$$-(g_i^{k*}A^k - A_i^\infty)\cdot g_{ij}^k(g_{ij}^\infty)^{-1}$$

By bootstrapping, this implies that

$$\|\nabla_{A_i^\infty}(g_{ij}^k(g_{ij}^\infty)^{-1})\|_{L_{q-1}^p(A_i^\infty, U_i \cap U_j)} \longrightarrow 0 \quad (k \longrightarrow 0).$$

Therefore, using $\nabla_{A_i^\infty} 1 = 0$, we obtain

$$\|g_{ij}^k(g_{ij}^\infty)^{-1} - 1\|_{L_q^p(A_i^\infty, U_i \cap U_j)} \longrightarrow 0 \quad (k \longrightarrow 0).$$

Let P^k be a bundle defined by transition functions $\{g_{ij}^k\}_{i,j}$, and P^∞ a bundle defined by transition functions $\{g_{ij}^\infty\}_{i,j}$. Then the data $\{g_i^{k*}A^k\}_i$ is regarded as a connection on P^k, and the data $\{A_i^\infty\}_i$ as a connection A^∞ on P^∞. Apply Proposition 7.4 to get a bundle isomorphism $g^k : P^\infty \to P^k$ for large k such that $g^k = g_i^k$ on $U_i \backslash V_i$ and

$$\|A^\infty - g^k\{g_i^{k*}A^k\}_i\|_{L_{q-1}^p(A^\infty, \cup V_j)} \longrightarrow 0 \quad (k \longrightarrow 0).$$

Since $(P^k, \{g_i^{k*}A^k\}_i)$ is isomorphic to (P, A^k) by definition, this implies the required result.

Remark 7.7. In this section we have assumed that $\{U_i\}$ is a finite covering. But if all $U_i \cap U_j$ are disjoint, for example, we can apply the propositions locally to get a certain estimate. The final step of the proof of Lemma 4.2 is such a case.

References.

[1] M.F. Atiyah, N. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. London A, 362 (1978), 425-461.

[2] S.S.Chern and J.Simons, Some cohomology classes in principal fiber bundles and their application to Riemannian geometry, Proc. Nat. Acad. Sci. U.S.A. 68 (1971) 791-794.

[3] S.K. Donoldson, An application of gauge theory to four dimensional topology, J. Diff. Geom. 18 (1983), 279-315.

[4] M. Furuta, On self-dual pseudo-connections on some orbifolds, preprint.

[5] ————, Euler number of moduli spaces of instanton, to appear in Proc. Japan Acad.

[6] ————, in preparation.

[7] R. Fintushel and T. Lawson, Compactness of moduli spaces for orbifold instantons, Topology Appl. 23 (1986) 305-312.

[8] R. Fintushel and R. Stern, Pseudofree orbifolds, Ann. Math. 122 (1985), 335-364.

[9] D. Freed and K.K. Uhlenbeck, Instantons and Four Manifolds, Springer-Verlag MSRI Series vol. 1, 1984.

[10] T. Kawasaki, The index of elliptic operators over V-manifolds, Nagoya Math. J. 84 (1981), 135-157.

[11] T. Lawson, Normal bundles for an embedded $\mathbb{R}P^2$ in a positive definite 4-manifold, J. Diff. Geom. 22 (1985) 215-231.

[12] R.B. Lockhart and R.C. McOwen, Elliptic differential operators

on non-compact manifolds, Ann. Scuola Norm. Sup. Pisa classe Sci. Ser IV Vol XII, n.3 (1985) 409-447.

[13] D. Ruberman, Rational homology cobordisms of rational space forms, preprint.

[14] C.H. Taubes, Gauge theory on asymptotically periodic 4-manifolds, J. Diff. Geom. 25 (1987), 363-430.

[15] K.K. Uhlenbeck, Connection with L^p-bounds on curvature, Comm. Math. Phys. 83 (1982), 31-42.