NOTE ON THE ROCKING OF A COLUMN

BY

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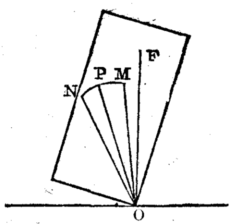
Let M be mass of column, K^2 M its moment of inertia about corner of base O. M N is circular arc traced out by centre of gravity in one half oscillation. Find the time of traversing M N. Let M O = NO = r when C. of G. is at P (let angle FOP be θ)



M g r sin θ = moment of couple causing motion

$\ddot{\theta}$ = Acceleration of angular velocity = $\dfrac{\text{couple}}{\text{moment of inertia}}$

$$\ddot{\theta} = \frac{M\,g\,r\sin\theta}{M\,K^2} \quad \ldots\ldots(1) \left\{ \ddot{\theta} \text{ is merely written for } \frac{d^2\theta}{dt^2} \right.$$

If the column is very slender then θ is always small and sin $\theta = \theta$ $\therefore$ for slender column $\ddot{\theta} = \dfrac{g\,r\,\theta}{K^2}$ The solution of this differential equation is

$$\theta = C_1\,\varepsilon^{\frac{t}{K}\sqrt{g\,r}} + C_2\,\varepsilon^{-\frac{t}{K}\sqrt{g\,r}} \quad \ldots\ldots\ldots\ldots(2)$$

Now when θ = M O F (call this angle a), $t = 0$

$a = C_1 + C_2$ Also at this time the ang. velocity is 0

$$\frac{d\,\theta}{d\,t} = \frac{\sqrt{g\,r}}{K}\,C_1\,\varepsilon^{\frac{t}{K}\sqrt{g\,r}} - \frac{\sqrt{g\,r}}{K}\,C_2\,\varepsilon^{-\frac{t}{K}\sqrt{g\,r}} \quad \ldots\ldots(3)$$

Put $t = 0$ & $0 = \dfrac{\sqrt{g\,r}}{K}C_1 - \dfrac{\sqrt{g\,r}}{K}C_2$ $\therefore$ $C_1 = C_2$

Hence $C_1 = \dfrac{a}{2} = C_2$ and we have (2) becoming

$$\theta = \frac{a}{2}\,\varepsilon^{\frac{t}{K}\sqrt{g\,r}} + \frac{a}{2}\,e^{-\frac{t}{K}\sqrt{g\,r}} \quad \ldots\ldots\ldots\ldots(4)$$

Let the angle N O F be called β then if t' is time of half oscillation

$$\frac{2\beta}{a} = \varepsilon^{mt'} + \varepsilon^{-mt'} \quad \left\{ \text{where } m \text{ is written for } \frac{\sqrt{g\,r}}{\mathrm{K}} \right.$$

Now if mt' or $\dfrac{t'\sqrt{g\,r}}{\mathrm{K}}$ is small $\left\{\begin{array}{l}\text{that is if the column is}\\ \text{slender or in any case}\\ \text{after vibr. get rather}\\ \text{quick}\end{array}\right\}$

ε^{mt} = approximately to $1 + mt' + \dfrac{m^2\,t'^2}{2}$ and

$$\frac{2\beta}{a} = 2 + m^2\,t'^2 \quad \therefore\ t' = \frac{\sqrt{\frac{2\beta}{a} - 2}}{m^2} = \frac{1}{m}\sqrt{\frac{2\beta}{a} - 2} \ldots(5)$$

So that we have as our conclusion —

The time of a very small oscillation of a slender column when the centre of gravity of the column traces the path M N is

$2\sqrt{2}\ \mathrm{K}\sqrt{\dfrac{1}{g\,r}\left(\dfrac{\mathrm{N\,O\,F}}{\mathrm{M\,O\,F}} - 1\right)}$ where K = radius of gyration about axis through O $\perp^s$ to paper and $r = \mathrm{M\,O}$ $g =$ 32·2 feet per sec. per sec.

Take case. cylindric. col. diameter of base d, length of column l, $\mathrm{K}^2 = \dfrac{5d^2}{16} + \dfrac{l^2}{3} = \dfrac{l^2}{3}$ very nearly if cyl. is slender

also $r = \dfrac{l}{2}$ very nearly Time of oscill. $= 2\sqrt{2}\ \dfrac{l}{\sqrt{3}}$

$$\sqrt{\frac{1}{g\frac{l}{2}}\ \&c} = \frac{4}{\sqrt{3}}\sqrt{\frac{l}{g}\left(\frac{\mathrm{N\,O\,F}}{\mathrm{M\,O\,F}} - 1\right)} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Of course this is all on the supposition that the time is really short; that is, that this time of complete oscillation $\times \frac{1}{2}$ $\sqrt{\dfrac{3\,g}{2\,l}}$ is something whose cube is insignificant compared with its square.

A solution of (4) is also

$$t = \frac{\log \frac{\theta \pm \sqrt{\theta^2 - a^2}}{a}}{\frac{\sqrt{g\,r}}{K} \log \theta}$$

Time of falling from α to β is

$$t' = \frac{\log \frac{\beta}{\alpha} \pm \sqrt{\frac{\beta^2}{\alpha} - 1}}{\frac{\sqrt{g\,r}}{K} \log \beta}$$

This is correct and ought to be used instead of (5)

Take case of a column being at rest and then getting an impulse.

When $t = 0$ let $\theta =$ N O F say & let $-\frac{d\,\theta}{d\,t} = v$

when $t = 0$ $\beta = C_1 + C_2$

Equation (2) becomes

$$\theta = \frac{m\,\beta - v}{2\,m} . \varepsilon^{mt} + \frac{m\,\beta + v}{2\,m} . \varepsilon^{-mt}$$

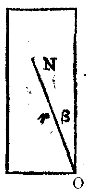
$$t = \frac{\log \left\{ \frac{\theta\,m}{m\,\beta - v} \pm \sqrt{\left(\frac{\theta\,m}{m\,\beta - v}\right)^2 - \frac{m\,\beta + v}{m\,\beta - v}} \right\}}{m} \ldots (7)$$

This gives time in which column reaches any value of angle θ. Evidently if $(m\,\beta + v)\,(m\,\beta - v)$ is negative the column will get negative values of θ. That is, everything depends on whether $v > = < m\,\beta$. If $v > m\,\beta$ col. will tumble. If $v = m\,\beta$ col. will just reach an unstable position resting on corner of base in an infinite time. If $v < m\,\beta$ col. will rock on its base. It is evident then, that if one sudden impulse is given to the col. its falling depends on v the velocity at first instant. If $v > m\,\beta$ then it is obvious that the time of reaching the position in which C. of G. is just above corner of base is

$$t' = \frac{1}{m} \log \frac{v + m\,\beta}{v - m\,\beta} \ldots\ldots\ldots\ldots (8)$$

For a given value of v this time is longer as $m\,\beta$ is greater

$\left(\text{or } \beta \dfrac{\sqrt{g\,r}}{K} \text{ greater}\right)$ $\quad \dfrac{d}{l} \cdot \dfrac{\sqrt{g \cdot \frac{l}{r} \cdot 3}}{l}$ is $m\,\beta$



The greater this is the more difficult is it to knock over the column.

The more nearly $\dfrac{d}{l^{\frac{3}{2}}} \sqrt{\dfrac{3\,g}{2}}$ is equal to v the longer will it take for col. to pass its unstable position.

$$t' = \sqrt{\frac{3\,l}{2\,g}} . \log \frac{v + \dfrac{d}{l^{\frac{3}{2}}}\sqrt{\dfrac{3\,g}{2}}}{v - \dfrac{d}{l^{\frac{3}{2}}}\sqrt{\dfrac{3\,g}{2}}}$$

For columns all of given length subjected to one *sudden impulse*, the improbability of falling is proportional to d the diameter

It is evident that we can apply our equation (1) to either $+^{ve}$ or $-^{ve}$ values of θ. Now suppose that O gets an oscillation $x = a \sin nt$

Here we have $\dfrac{d^2\,\theta}{d\,t^2} - \dfrac{g\,r}{K^2}\,\theta - a \sin nt = \theta$

Instead of an angle θ may be regarded as a measured distance when col is slender,—

The solution of which is, putting $\dfrac{\sqrt{g\,r}}{K} = m$

$$\theta = \frac{a}{2\,m}\left\{\frac{n \cos nt + m \sin nt}{m^2 - n^2} - \frac{m \sin nt - n \cos nt}{m^2 + n^2}\right\} + A\varepsilon^{mt} + B\varepsilon^{-mt}$$

This is general equation to motion and corresponds to equation at top of page 8 in our paper on a Neglected Principle read before the Asiatic Society of Japan, on the 23rd May, 1877.