Exact Solutions of Nonlinear Localized Wave Modes in Unstable Systems

（不安定系における非線形局在モードの厳密解）

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THESIS

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CONTENTS

Chapter 1. Introduction

In recent years, study of nonlinear phenomena in various systems has been acquired greater importance. In linear systems, the physical variables obey regular developments and their analyses are quite simple. On the contrary, nonlinear systems often show unexpected behaviors from the analogies of linear systems or perturbations. Many physicists have intensively made various contributions. Among these works, one of the most significant achievments on the subjects is the theory of integrable systems and solitons. In the first place, let us summarize the developments of the study of integrable system.

In 1955, Fermi, Pasta and Ulam considered a one dimensional lattice, which has cubic nonlinearity in the potential, and studied numerically an approach to the equilibrium (Fermi, Pasta and Ulam 1955). Contrary to the prediction, they observed that not all the Fourier modes are excited, in spite of the coupling of inter-modes due to the anharmonicity of the potential. The system is not thermalized and periodically goes back to the initial state. This recurrence phenomenon has its origin in the nonlinearity and the dispersion of the system. Later, Kruskal and Zabusky studied this problem in the continuum approximation (Zabusky and Kruskal 1965). They obtained the Korteweg - de Vries (KdV) equation as the lowest perturbation term and analyzed it numerically using the periodic boundary condition and a sinusoidal starting profile. The analysis showed that the slope is steepened and that solitary pulses emerge. The pulses, which are stable in collisions, and behave as if particles and undergo elastic collisions, was named solitons. They also observed the recurrence properties of the solutions.

At the same time, many efforts have been made for studying the nonlinear phenomena analytically. The most remarkable development is the inverse scattering method. This was first done for the KdV equation by Gardner, Greene, Kruskal and Miura (1967), and later, Lax (1968) tried to generalize it. From that time on, many physicists tried to apply this method to other systems. Zakharov and Shabat (1972), Zakharov, Faddeev and Takhtadzyan (1972), Wadati (1972, 1973), Ablowitz, Kaup, Newell and Segur (1973), and Wadati, Konno and Ichikawa (1979) have shown that the inverse scattering method is applicable to nonlinear Schrödinger equation, sine-Gordon equation, MKdV equation, and other physically interesting equations. During the time, the method has become more sophisticated and generalization to quantum system has been achieved (Sklyanin and Faddeev 1978, Sklyanin 1979, Sklyanin, Faddeev and Takhtadzyan 1980).

The inverse scattering method can be summarized as follows. First, we consider an auxiliary linear problem where the potential is a functional of a sought function. Second, we study the direct scattering problem and the time developments of the scattering data.

Finally, we reconstruct the function from the scattering data (the inverse scaqttering problem) by using the Gel'fand - Levitan - Marchenko equation (Gel'fand and Levitan 1955, Marchenko 1955). It has produced much influence on the analyses of integrable systems.

The solved models have commonly the property that is called integrability. From the mathematical understandings of the method, the integrability of the system has close relation to the symmetry of the system — an infinite number of conserved quantities. All the exactly solved models, which are called soliton equations, have been recognized to have infinite symmetries.

In the meantime, studies on localized wave modes have also been done in various systems. Especially, the main fields of research are plasma physics and fluid mechanics. Zakharov considered an interaction between the electrons and the plasma wave (Zakharov 1967), and collapse of the nonlinear localized modes in Langmuir waves (Zakharov 1972). As a model system, the Toda lattice, an integrable lattice model with nonlinear (exponential) potential, is applied in many fields (Toda 1967a,b, Toda and Wadati 1973), especially in field theory (Leznov and Saveliev 1979, Bulgadev 1980, Babelon, de Vega and Viallet 1981, Mikhailov, Olshanetsky and Perelomov 1981, Olive and Turok 1985). In analyzing a slow variation in wave packets, the reductive perturbation method is applicable in many cases. It is an expansion method of the dependent variables with Gardner - Morikawa transformation. Taniuti, N. Yajima, Kakutani, Ichikawa and others developed the method and applied it to the soliton theories (Taniuti 1974, Ichikawa and Taniuti 1973, Taniuti and N. Yajima 1969, 1973, Asano, Taniuti and N. Yajima 1969). In 1969, Newell et al. considered the Rayleigh - Benard convection and got a nonlinear amplitude equation (Newell and Whitehead 1969, Newell, Lange and Aucoin 1970). This was the beginning of the studies on convection patterns by lowest amplitude equations. Kodama and Taniuti showed that the nonlinear Schrödinger equation is related to the sine-Gordon equation in small amplitude region (Kodama and Taniuti 1978). This shows a piece of relations between soliton equations.

Among these various pictures, it can commonly be said that the exactly solved system is stable in the sense that the zero field configuration is the minimum energy state. In the latter half of the 1980's, there has been an increasing interest in the nonlinear phenomena in unstable systems. The author and Wadati (Yajima and Wadati 1987) investigated the so-called unstable sine-Gordon (USG) equation, which describes unstable pendulii systems, and got soliton solutions. Tanaka and N. Yajima (1987) proposed a set of nonlinear equations in electron beam plasma. Iizuka and Wadati (1990) have derived the nonlinear Schrödinger equation for the Rayleigh-Taylor Problem.

One of the importances of analyzing unstable system is that the systems that are

thought to be unstable in the above meaning often appear in many circumstances. For example, one of the physical picture of the sine-Gordon model is two-level atom system where the atom is initially in the ground state (Lamb 1973). The USG equation comes from the situation that the atom is initially in the excited state. As another example, in the plasma system, there usually exists electron or ion beam. So the assumption that the zero field configuration is the minimum energy state may not depict the system adequetly, because an energy flux of the beam is poured into the system. In any unstable system, it is quite difficult to investigate the system because of its nonlinearity and instability. This will be understood in the section 2.1 for one example. Therefore, an exact treatment is desirable.

In this thesis, we consider unstable systems and analyze the unstable nonlinear Schrödinger (UNS, for short) equation as a model. In the aforesaid three examples, we can derive the UNS equation in some critical wave number region. This derivation shows that the UNS model properly describes propagations of envelopes of high frequency waves, and this situation often occurs in various physical fields. Then we can regard this model as one of the canonical equations describing time developments of nonlinear unstable systems. We solve the UNS model equation explicitly, and show that this equation is an integrable model. Among the exact solutions, we study soliton solution (localized wave mode). The analyses on properties of these modes show the characteric properties of wave propagations in unstable media. The existence of these modes is not always known, so we analyze some initial problems to find that solitons do exist in rather general conditions. From this point of view, we can understand the roles of the localized wave modes excited in unstable systems. In addition, we consider some circumstances in various physical fields and get the UNS equation to apply the results in this thesis.

The thesis is organized as follows. In the next section, we introduce the UNS model and make a perturbational analysis of a disturbance caused in the system. In §3, we solve the initial value problem for the equation exactly. In addition, an infinite number of conservation laws are presented. In §4, we derive soliton solutions and study properties of solitons created in unstable media. We also analyze some initial value problems and show that solitons can exist under general circumstances. In §5, some physical systems leading to the UNS equation are considered. The last section is devoted to the concluding remarks.

Chapter 2. The Unstable Nonlinear Schrödinger Model

§2.1 The Model

The model equation we are going to consider in this thesis is the unstable nonlinear Schrödinger (hereafter UNS) model (Yajima and Wadati 1990a):

$$iq_x + q_{tt} + 2\sigma|q|^2 q = 0, \tag{2.1}$$

where q is a complex field variable, and σ is a sign factor. This equation describes nonlinear effects in various (unstable) nonlinear systems. The examples are discussed in Chapter 5. By the word unstable, we mean that the system is not in the minimum energy state, even in the zero-field configuration. We can easily recognize this situation in the examples there.

In nonlinear optics, this equation is often referred as the nonlinear Schrödinger equation. To discuss the wave profile in the optical fiber, the initial condition for the equation is $q(0,t)$. Therefore, a role of x in the UNS equation is essentially a 'time' variable in nonlinear optics.

In this thesis, we shall consider the case of $\sigma > 0$. The other case $\sigma < 0$ has been solved very recently (Iizuka, Wadati and Yajima 1991). The Lagrangian density for (2.1) with $\sigma = +1$ is

$$\mathcal{L} = q_t^* q_t - \frac{1}{2i}(q_x^* q - q^* q_x) - q^* q^* q q, \tag{2.2}$$

where the symbol '$*$' denotes the complex conjugate. The stress-energy tensor is

$$\begin{aligned} T_{00} &= p^* p + \frac{1}{2i}(q_x^* q - q^* q_x) + q^* q^* q q \\ T_{01} &= \frac{-1}{2i}(q p^* - q^* p) \\ T_{10} &= p q_x + p^* q_x^* \\ T_{11} &= -p^* p + q^* q^* q q, \end{aligned} \tag{2.3}$$

where the value p is the canonical momentum density defined by

$$p = \frac{\partial \mathcal{L}}{\partial q_t} = q_t^*.$$

The Lagrangian does not depend explicitly on x and t, so these values satisfy current conservation, and belong to the family of the conserved quantities. The components $T_{\mu 0}$ corresponds to the conserved densities by definition of the stress-energy tensor. The integrals of them belong to the family of infinite number of the conserved quantities. This situation is to be discussed in Chapter 3.

§2.2 Time development of small disturbance

Before developing discussions in detail, let us consider the time development of a small disturbance caused in the system by perturbation analysis, and get some insights into the roles of the localized modes. Regarding the amplitude of $q(x,t)$ to be small, the third term in the equation (2.1) is negligible. Then, the equation is:

$$iq_{0,x} + q_{0,tt} = 0. \tag{2.4}$$

We consider a propagating wave solution, $\exp[i(kx - \omega t)]$, for this equation. The dispersion relation is

$$k + \omega^2 = 0. \tag{2.5}$$

There is a complex frequency for the case with positive k. When we use a positive value η, the solution is

$$\begin{aligned} &k = 4\eta^2, \qquad \omega = \pm 2i\eta, \qquad \eta > 0. \\ &q_0(x,t) = A\exp(4i\eta^2 x + 2\eta t). \end{aligned} \tag{2.6}$$

But, as time goes on, this solution gets large and the nonlinear term comes to play an important role. Let us write the solution of the UNS equation as $q(x,t)$, and expand it as

$$\begin{aligned} q(x,t) &= e^{4i\eta^2 x}[B_0(t) + \varepsilon B_1(t) + \varepsilon^2 B_2(t) + \cdots], \\ B_0(t) &\equiv Ae^{2\eta t}. \end{aligned} \tag{2.7}$$

Note that the zeroth order term of this expression is $q_0(x,t)$. The expansion parameter ε denotes 'small' deviation from the plane wave solution $q_0(x,t)$, and is finally set to be $\varepsilon = 1$. Substitution of (2.7) into (2.1) gives

$$\begin{aligned} &-4\eta^2 B_n(t) + \frac{d^2 B_n(t)}{dt^2} + 2\sum_{l,m=0}^{n-1} B_l^*(t) B_m(t) B_{n-l-m-1}(t) = 0, \\ &n = 1, 2, \cdots. \end{aligned} \tag{2.8}$$

We set σ in (2.1) to be ε because the nonlinear term does not contribute to the lowest order in the expansion. Equation (2.8) can be solved iteratively. This is a set of inhomogeneous second order differential equations with constant coefficients. The fundamental solutions of the corresponding homogeneous equation for B_n are $e^{2\eta t}$ and $e^{-2\eta t}$. The solution $e^{2\eta t}$ has the same form as $B_0(t)$, and the only contribution to the whole solution is to change the constant A of $B_0(t)$. The other solution $e^{-2\eta t}$ goes to zero in the limit $t \to \infty$, so it can be neglected compared with time growing part of the solution. These facts allow us to exclude the fundamental solutions of the homogeneous equation. A simple observation shows that the particular solution of (2.8) has the following form:

$$B_n(t) = C_n e^{\alpha_n t}, \qquad n = 1, 2, \cdots, \tag{2.9}$$

where C_n's and α_n's are constants. The set of particular solution for each order is

$$B_n(t) = \left(-\frac{|A|^2}{16\eta^2}\right)^n A\, e^{(4n+2)\eta t} \qquad n = 1, 2, \cdots. \tag{2.10}$$

Finally, by summing up contribution from all orders and setting $\varepsilon = 1$, we get

$$\begin{aligned} q(x,t) &= e^{4i\eta^2 x} e^{2\eta t} A \left[1 + \sum_{n=1}^{\infty} \left(-\frac{|A|^2}{16\eta^2}\right)^n e^{4n\eta t}\right] \\ &= -\frac{2i\eta\, e^{4i\eta^2 x + i\phi}}{\cosh(2\eta t + \rho)}, \end{aligned} \tag{2.11}$$

where, in terms of real constants ρ and ϕ, we have set

$$A = -4i\eta\, e^{\rho + i\phi}. \tag{2.12}$$

This shows that the instability will not grow forever. The exponential growth in the linear regime is suppressed by the nonlinearity and the system returns to the unstable equilibrium state. This behavior is reasonable since the UNS equation is invariant under the time reversal $t \to -t$.

Chapter 3. Exact Solutions of the UNS Model.

In this chapter, we shall get the exact solutions for the model equation by using the inverse scattering transform. In addition, the conservation law is discussed.

§3.1 Inverse Scattering Method

§§3.1.1 Jost functions and scattering data

We apply the inverse scattering method te solve initial value problem of the UNS equation (Yajima and Wadati 1990a). From a relation between the UNS equation and the nonlinear Schrödinger (NLS) equation, we use the auxiliary linear equations of the conventional NLS equation (Zakharov and Shabat 1972) interchanging space x and time t:

$$\frac{\partial \chi}{\partial x} = L\chi, \qquad L = \begin{pmatrix} i|q|^2 - 2i\zeta^2 & iq_t + 2\zeta q \\ iq_t^* - 2\zeta q^* & -i|q|^2 + 2i\zeta^2 \end{pmatrix}, \tag{3.1a}$$

$$\frac{\partial \chi}{\partial t} = M\chi, \qquad M = \begin{pmatrix} -i\zeta & q \\ -q^* & i\zeta \end{pmatrix} \tag{3.1b}$$

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{3.2}$$

The compatibility condition of (3.1a) and (3.1b) with $\zeta_t = 0$ yields the UNS equation. We fix t until §§3.1.3 and consider the eigenvalue problem (3.1a). The spectral parameter ζ is in general a complex number corresponding to the eigenvalue. Here, a boundary condition that $q(x)$ approaches to zero sufficiently fast as $|x| \to \infty$,

$$q(x) \to 0 \qquad \text{for} \quad |x| \to \infty. \tag{3.3}$$

is adopted.

Let us introduce Jost functions ϕ and ψ for real $\zeta \equiv \xi$, which have the following asymptotic forms:

$$\begin{aligned} \phi(x,\xi) &\to \begin{pmatrix} e^{-2i\xi^2 x} \\ 0 \end{pmatrix} & x \to -\infty, \\ \psi(x,\xi) &\to \begin{pmatrix} 0 \\ e^{2i\xi^2 x} \end{pmatrix} & x \to +\infty. \end{aligned} \tag{3.4a}$$

We also introduce the other Jost functions $\bar{\phi}$ and $\bar{\psi}$ with asymptotic forms (Lamb 1971):

$$\begin{aligned} \bar{\phi}(x,\xi) &\to \begin{pmatrix} 0 \\ -e^{2i\xi^2 x} \end{pmatrix} & x \to -\infty, \\ \bar{\psi}(x,\xi) &\to \begin{pmatrix} e^{-2i\xi^2 x} \\ 0 \end{pmatrix} & x \to +\infty. \end{aligned} \tag{3.4b}$$

We define Wronskian as $W[f,g] \equiv f_1 g_2 - f_2 g_1$. We take functions f and g to be solutions of (3.1a) with eigenvalue ζ_1 and ζ_2, respectively. We see from (3.1a) that the Wronskian satisfies

$$\begin{aligned}\frac{\partial}{\partial x}W[f,g] &\equiv \frac{\partial}{\partial x}(f_1 g_2 - f_2 g_1) \\ &= -2(\zeta_1 - \zeta_2)\{i(f_1 g_2 + f_2 g_1)(\zeta_1 + \zeta_2) - q^* f_1 g_2 - q f_2 g_1\}. \end{aligned} \tag{3.5}$$

The right hand side of (3.5) is zero for Jost functions with the same eigenvalue, and then the Wronskian does not depend on x. Considering this fact and the asymptotic forms (3.4), we have

$$W[\bar{\psi},\psi] = 1, \qquad W[\bar{\phi},\phi] = 1. \tag{3.6}$$

We can choose a pair of independent solutions using this property. Then, we find that each of the sets $\{\phi,\ \bar{\phi}\}$ and $\{\psi,\ \bar{\psi}\}$ consists a fundamental system of solutions, because each of them is linearly independent set. So the following relations among Jost functions are satisfied:

$$\begin{aligned}\phi(x,\xi) &= a(\xi)\bar{\psi}(x,\xi) + b(\xi)\psi(x,\xi), \\ \bar{\phi}(x,\xi) &= -\bar{a}(\xi)\psi(x,\xi) + \bar{b}(\xi)\bar{\psi}(x,\xi).\end{aligned} \tag{3.7}$$

We make a list of these relations in Table 3.1.

Table 3.1

The coefficients $a(\xi)$, $\bar{a}(\xi)$, $b(\xi)$ and $\bar{b}(\xi)$ correspond to scattering amplitudes (Lamb 1971), and it should be noticed that they do not depend on space variable x. From (3.6) and (3.7), we can express scatterting amplitudes in terms of the Wronskian:

$$\begin{aligned} a(\xi) &= W[\phi,\psi], & \bar{a}(\xi) &= W[\bar{\phi},\bar{\psi}], \\ b(\xi) &= -W[\phi,\bar{\psi}], & \bar{b}(\xi) &= W[\bar{\phi},\psi], \end{aligned} \tag{3.8a}$$

and can show that

$$a(\xi)\bar{a}(\xi) + b(\xi)\bar{b}(\xi) = 1. \tag{3.8b}$$

We make analytic continuations of the Jost functions into complex ζ-plane. Because of (3.1a) and (3.4), the functions ϕ and ψ can be continued analytically into a region

$\mathrm{Im}(\zeta^2)=2\xi\eta > 0$ and the functions $\bar{\phi}$ and $\bar{\psi}$ into $\xi\eta < 0$, where $\zeta = \xi + i\eta$. Then the expressions (3.8a) tell us that $a(\xi)$ is continued into $\xi\eta > 0$ and $\bar{a}(\xi)$ into $\xi\eta < 0$. On the other hand, the amplitudes $b(\xi)$ and $\bar{b}(\xi)$ cannot generally be continued into complex ζ-plane, but from the analytic continuation of the Jost functions and (3.8a), they are continued only into the imaginary axes on the ζ-plane. Furthermore, if $q(x)$ is defined on a compact support, $b(\xi)$ can be extended into $\xi\eta > 0$ and $\bar{b}(\xi)$ into $\xi\eta < 0$. Then the relations (3.8) hold in the region where the corresponding amplitudes are continued.

We can show that in the limit of $|\zeta| \to \infty$, these Jost functions satisfy

$$\begin{aligned} \phi(x,\zeta) &\to \begin{pmatrix} e^{-2i\zeta^2 x} \\ 0 \end{pmatrix}, & \psi(x,\zeta) &\to \begin{pmatrix} 0 \\ e^{2i\zeta^2 x} \end{pmatrix}, \\ \bar{\phi}(x,\zeta) &\to \begin{pmatrix} 0 \\ -e^{2i\zeta^2 x} \end{pmatrix}, & \bar{\psi}(x,\zeta) &\to \begin{pmatrix} e^{-2i\zeta^2 x} \\ 0 \end{pmatrix}. \end{aligned} \tag{3.9}$$

We have asymptotic forms for the scattering amplitudes from (3.8a) and (3.9):

$$\begin{aligned} a(\zeta) &\to 1, & \bar{a}(\zeta) &\to 1, \\ b(\zeta) &\to 0, & \bar{b}(\zeta) &\to 0, \qquad \text{for} \quad |\zeta| \to \infty. \end{aligned} \tag{3.10}$$

So far, no definite relation between the sets of functions $\{\phi,\ \psi\}$ and $\{\bar{\phi},\ \bar{\psi}\}$ are used. We use the following definition of the bar hereafter in our discussion:

$$\bar{f}(x,\zeta) = \begin{pmatrix} \bar{f}_1(x,\zeta) \\ \bar{f}_2(x,\zeta) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f_1^*(x,\zeta) \\ f_2^*(x,\zeta) \end{pmatrix} = \begin{pmatrix} f_2^* \\ -f_1^* \end{pmatrix}. \tag{3.11}$$

Then due to (3.7), we find that $\bar{a}(\zeta)$ and $\bar{b}(\zeta)$ are respectively related to $a(\zeta)$ and $b(\zeta)$ as

$$\bar{a}(\zeta) = a(\zeta)^*, \qquad \bar{b}(\zeta) = b(\zeta)^*. \tag{3.12}$$

We denote zeros of $a(\zeta)$ in the region $\xi\eta > 0$ by ζ_j, $(j = 1, 2, \cdots, N)$. At $\zeta = \zeta_j$, the functions $\phi(x,\zeta)$ and $\psi(x,\zeta)$ are linearly dependent $\phi(x,\zeta_j) = b_j\psi(x,\zeta_j)$. A bound state occurs there since $\phi(x,\zeta_j)$ is a square integrable function. A set of quantities $\{a(\zeta), b(\zeta), b_j, \zeta_j\}$ is called scattering data. We can develop a similar discussion for the other amplitudes $\bar{a}(\zeta)$ and $\bar{b}(\zeta)$. Time dependence of the scattering data, when $q(x,t)$ obeys the UNS equation, will be discussed in §§3.1.3.

§§3.1.2 Inverse problem

We consider the inverse problem, that is, construction of $q(x)$ from the scattering data. We introduce a function $\Phi(\zeta)$ defined as

$$\Phi(\zeta) = \begin{cases} \dfrac{1}{a(\zeta)}\phi(x,\zeta)e^{2i\zeta^2 x} & \mathrm{Im}(\zeta^2) > 0\,; \\ \bar{\psi}(x,\zeta)e^{2i\zeta^2 x} & \mathrm{Im}(\zeta^2) < 0\,. \end{cases} \tag{3.13}$$

We assume that all the zeros of $a(\zeta)$ located at $\zeta = \zeta_j,\ j = 1, 2, \cdots, N$ are simple. Integrating a function $\Phi(z) \cdot (z-\zeta)^{-1}$ along a contour in complex z-plane (Fig.3.1), we get

$$\begin{aligned}\Phi(\zeta) = &\begin{pmatrix}1\\0\end{pmatrix} + \sum_{k=1}^{N} \frac{\gamma_k e^{2i\zeta_k^2 x}}{\zeta-\zeta_k}\psi(x,\zeta_k)\\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\phi^{(1)}(\xi)}{\xi-\zeta}\,d\xi - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\phi^{(2)}(\eta)}{\eta+i\zeta}\,d\eta\,,\end{aligned} \tag{3.14}$$

Here, we define γ_k by

$$\gamma_k = \frac{b_k}{a'(\zeta_k)}\,, \tag{3.15}$$

and the functions $\phi^{(1)}(\xi)$ and $\phi^{(2)}(\eta)$ are the jumps of $\Phi(\zeta)$ across ξ and η axes, respectively.

Fig.3.1

We can get an expression of $q(x)$ by expanding $\psi(x,\zeta) \cdot e^{-2i\zeta^2 x}$ in powers of ζ^{-1}. Reminding (3.9), the asymptotic form of $\psi(x,\zeta)$ in the limit $|\zeta| \to \infty$, we have an expansion formula

$$\psi(x,\zeta)e^{-2i\zeta^2 x} = \begin{pmatrix}0\\1\end{pmatrix} + \sum_{k=1}^{\infty}\zeta^{-k}\begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix}, \tag{3.16}$$

Substitution of this expression into (3.1a) enables us to write $q(x)$ and its polynomials in terms of α_j's. We have a relation between the solution $q(x)$ and the expansion coefficients by comparing the terms in the order of ζ^{-1}. In addition, we have from (3.14) and (3.16):

$$\begin{aligned}q(x) =& 2i\alpha_1\\ =& -2i\sum_{k=1}^{N}\gamma_k^* e^{-2i\zeta_k^{*2}x}\,\psi_2^*(x,\zeta_k)\\ &-\frac{1}{\pi}\int_{-\infty}^{\infty} d\xi\,\phi_2^{(1)*}(\xi) - \frac{i}{\pi}\int_{-\infty}^{\infty} d\eta\,\phi_2^{(2)*}(\eta)\,.\end{aligned} \tag{3.17}$$

From the definition of $\Phi(z)$, the jumps $\phi^{(1)}(\xi)$ and $\phi^{(2)}(\eta)$ can be expressed in terms of

the scattering amplitudes:

$$\begin{aligned}\phi^{(1)}(\xi) &= \begin{cases}\dfrac{b(\xi)}{a(\xi)}\psi(x,\xi)e^{2i\xi^2 x} & \xi > 0\,,\\[2ex] -\dfrac{b(\xi)}{a(\xi)}\psi(x,\xi)e^{2i\xi^2 x} & \xi < 0\,,\end{cases}\\ \phi^{(2)}(\eta) &= \begin{cases}\dfrac{b(i\eta)}{a(i\eta)}\psi(x,i\eta)e^{-2i\eta^2 x} & \eta > 0\,,\\[2ex] -\dfrac{b(i\eta)}{a(i\eta)}\psi(x,i\eta)e^{-2i\eta^2 x} & \eta < 0\,.\end{cases}\end{aligned} \tag{3.18}$$

We introduce functions $K^{(j)}(x,y)$'s $(j = 1, 2)$ and express $\psi(x,\zeta)$ as

$$\psi(x,\zeta) = \begin{pmatrix}0\\1\end{pmatrix}e^{2i\zeta^2 x} + \int_x^{\infty} ds\,K^{(2)}(x,s)e^{2i\zeta^2 s} + \int_x^{\infty} ds\,\zeta K^{(1)}(x,s)e^{2i\zeta^2 s}. \tag{3.19}$$

A set of functions $F_j(x)$'s $(j = 1, 2, 3)$ in terms of the scattering data are defined as

$$\begin{aligned}F_j(x) =& 2i\sum_{k=1}^{N}\zeta_k^{j-1}\gamma_k e^{2i\zeta_k^2 x} - \frac{1}{\pi}\Big(\int_0^{\infty} - \int_{-\infty}^{0}\Big)d\xi\,\frac{b(\xi)}{a(\xi)}\xi^{j-1}e^{2i\xi^2 x}\\ &+ \frac{i}{\pi}\Big(\int_0^{\infty} - \int_{-\infty}^{0}\Big)d\eta\,\frac{b(i\eta)}{a(i\eta)}(i\eta)^{j-1}e^{-2i\eta^2 x}\end{aligned} \tag{3.20}$$

We can reduce (3.14) into a set of Fredholm integral equations whose unknown functions are $K^{(j)}(x,y)$'s by using the definition of $\Phi(\zeta)$ and (3.18-20):

$$\begin{aligned}K_1^{(j)}(x,y) &+ F_j^*(x+y) + \int_x^{\infty} ds\,K_2^{(2)*}(x,s)F_j^*(s+y)\\ &+ \int_x^{\infty} ds\,K_2^{(1)*}(x,s)F_{j+1}^*(s+y) = 0\,,\end{aligned} \tag{3.21a}$$

$$\begin{aligned}K_2^{(j)*}(x,y) &- \int_x^{\infty} ds\,K_1^{(2)}(x,s)F_j(s+y)\\ &- \int_x^{\infty} ds\,K_1^{(1)}(x,s)F_{j+1}(s+y) = 0\,.\end{aligned} \tag{3.21b}$$

A set of these equations are the Gel'fand-Levitan-Marchenko type (GLM) equation for the system (3.1a). Assuming that the kernels $K^{(j)}(x,s)$ vanish in the limit of $s \to \infty$, we have from (3.16), (3.17) and (3.19)

$$q(x) = -K_1^{(1)}(x,x)\,. \tag{3.22}$$

Given the scattering data, we know $F_j(x)$ and solve the GLM equation to get $K^{(j)}(x,y)$. Then, we find $q(x)$ by (3.22). Thus from the scattering data, we can construct the potential $q(x)$ in (3.1a).

§§3.1.3 Time dependence of scattering data

The time development of the solution $q(x,t)$ is considered through those of the scattering data. We evaluate their time dependences. We take the limit $x \to \infty$ in (3.1b) and use (3.4). In this limit, using the boundary condition (3.3) and the form of the Jost functions (3.4), we have a relation

$$a_t\bar{\psi}(\infty,\zeta) + b_t\psi(\infty,\zeta) = \begin{pmatrix} -i\zeta & 0 \\ 0 & i\zeta \end{pmatrix} \{a\bar{\psi}(\infty,\zeta) + b\psi(\infty,\zeta)\} \tag{3.23}$$

Comparison of each component in both sides yields a set of differential equations for the data. A simple calculation gives

$$\begin{aligned} &a(\zeta,t) = e^{-i\zeta t}a(\zeta)\,, \qquad b(\zeta,t) = e^{i\zeta t}b(\zeta)\,, \\ &b_j(t) = e^{i\zeta_j t}b_j\,, \qquad j = 1,2,\cdots,N, \end{aligned} \tag{3.24a}$$

Time dependences of the other amplitudes are similarly derived:

$$\begin{aligned} &\bar{a}(\zeta,t) = e^{i\zeta^* t}a(\zeta)^*\,, \qquad \bar{b}(\zeta,t) = e^{-i\zeta^* t}b(\zeta)^*\,, \\ &\bar{b}_j(t) = e^{-i\zeta_j^* t}\bar{b}_j^*\,, \qquad j = 1,2,\cdots,N. \end{aligned} \tag{3.24b}$$

Time dependences of γ_k's, which will be necessary in the following section, are expressed from (3.15) as

$$\gamma_k(t) = e^{2i\zeta t}\gamma_k. \tag{3.25}$$

Then, we can solve the initial value problem for the UNS equation as follows. For a given initial condition, $q(x,0)$ and $q_t(x,0)$, we can calculate the scattering data at $t=0$. The bound state eigenvalues ζ_j's are time-independent. Time dependence of the scattering data is given in (3.24), so the explicit forms of $F_j(x+y;t)$'s are known. The sought function $q(x,t)$ is obtained by solving the GLM equations (3.20) with $F_j(x+y;t)$'s and by substituting the result $K_1^{(1)}(x,x;t)$ into (3.22).

§3.2 Infinite number of conserved quantities

So far in this section, we solved an initial value problem for the UNS model. In this section, we shall study conservation laws for the system. There exist an infinite number of conservation laws and from this fact it has been found that the model equations are integrable.

As it was considered in the USG model (Yajima and Wadati 1987), there should exist an infinite number of conservation laws, considering that the model has the form which is gotten by interchanging x and t. We get their recursion relations to find the conserved quantities.

We take ψ as the 'Jost function matrix', whose columns are the Jost functions ϕ, $\bar{\phi}$, and which has an asymptotic form:

$$\psi_{jk} \to \sigma_j e^{-2i\sigma_j\zeta^2 x}\delta_{j,k}, \qquad x \to -\infty, \qquad j,k = 1,2 \tag{3.27a}$$

$$\sigma_j = (-1)^{j+1}. \tag{3.27b}$$

Of course, this matrix satisfies the equations:

$$\psi_x = L\psi, \qquad \psi_t = M\psi.$$

We get a set of conservation laws from relation

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\ln\psi_{jj} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}\ln\psi_{jj}. \tag{3.28}$$

The quantity $[\ln\psi_{jj}]_x$ is the conserved density and $[\ln\psi_{jj}]_t$ is flux. As in other cases, a set of infinite number of conserved quantities are given by expanding the conserved density in terms of the spectral parameter. From (3.1a) and (3.28), the conserved quantity is

$$C = [\ln\psi_{jj}]_x = \int dx\,\frac{1}{\psi_{jj}}\sum_k L_{jk}\psi_{kj}. \tag{3.29}$$

We write the off-diagonal part of the matrix L as $\tilde{L}$, then we have

$$\begin{aligned} &\frac{1}{\psi_{jj}}\psi_{jj,x} = -2i\sigma_j\zeta^2 + i\sigma_j|\phi|^2 + \sum_k \tilde{L}_{jk}\Gamma_{kj}\,, \\ &\Gamma_{jk} = \psi_{jk}/\psi_{jj}. \end{aligned} \tag{3.30}$$

Now we differentiate Γ. From the relation (3.1a), we find

$$\begin{aligned} \frac{\partial\Gamma_{kj}}{\partial x} &= \frac{\psi_{kj,x}}{\psi_{jj}} - \frac{\psi_{kj}\psi_{jj,x}}{\psi_{jj}^2} \\ &= (-2i\zeta^2 + i|q|^2)(\sigma_k - \sigma_j)\Gamma_{kj} + \sum_\ell \tilde{L}_{k\ell}\Gamma_{\ell j} - \sum_\ell \tilde{L}_{j\ell}\Gamma_{kj}\Gamma_{\ell j}. \end{aligned} \tag{3.31}$$

The function ψ_{jk} can be expanded in the power of ζ^{-1} from the first order as it will be shown in the Appendix A, so Γ can be expanded as

$$\Gamma_{jk} = \begin{cases} 1 & j = k, \\ \sum_{\ell=1}^{\infty} \zeta^{-\ell} \Gamma_{jk}^{(\ell)} & j \neq k. \end{cases} \tag{3.32}$$

We can get an infinite set of conserved densities by using the form of the matrix $\tilde{L}$ and comparing the coefficient for each order of ζ^{-1}. We have a set of recursion relations for $\Gamma_{jk}^{(n)}$'s, $(j \neq k)$ (see Appendix A):

$$\begin{aligned} 4i(-)^j \Gamma_{kj}^{(1)} &= C_{kj}, \\ 4i(-)^j \Gamma_{kj}^{(2)} &= B_{kj}, \\ 4i(-)^j \Gamma_{kj}^{(3)} &= -\frac{\partial \Gamma_{kj}^{(1)}}{\partial x} + 2i(-)^j |q|^2 \Gamma_{kj}^{(1)} - C_{jk} \Gamma_{kj}^{(1)} \Gamma_{kj}^{(1)}, \\ 4i(-)^j \Gamma_{kj}^{(n+2)} &= -\frac{\partial \Gamma_{kj}^{(n)}}{\partial x} + 2i(-)^j |q|^2 \Gamma_{kj}^{(n)} - C_{jk} \sum_{m=1}^{n} \Gamma_{kj}^{(m)} \Gamma_{kj}^{(n+1-m)} \\ &\quad - B_{jk} \sum_{m=1}^{n-1} \Gamma_{kj}^{(m)} \Gamma_{kj}^{(n-m)}, \qquad (n \geq 2). \end{aligned} \tag{3.33}$$

Here the quantity B and C are given in the (A.3) and we have written the sums over p in (A.4) explicitly.

We have derived the recursion relations for the 'generator' of the conserved densities. We are considering 2×2-matrix, so we can confine ourselves to the quantities Γ_{21} and Γ_{12}. First four coefficients for Γ_{21} are:

$$\Gamma_{21}^{(1)} = \frac{1}{2i} q^*, \tag{3.34a}$$

$$\Gamma_{21}^{(2)} = -\frac{1}{4} q_t^*, \tag{3.34b}$$

$$\Gamma_{21}^{(3)} = -\frac{1}{8} q_x^* + \frac{1}{8i} |q|^2 q^*, \tag{3.34c}$$

$$\Gamma_{21}^{(4)} = -\frac{1}{16i} q_{tx}^* - \frac{1}{16} q^{*2} q_t. \tag{3.34d}$$

The other element Γ_{12} gives us the corresponding results:

$$\Gamma_{12}^{(1)} = \frac{1}{2i} q, \tag{3.35a}$$

$$\Gamma_{12}^{(2)} = \frac{1}{4} q_t, \tag{3.35b}$$

$$\Gamma_{12}^{(3)} = \frac{1}{8} q_x + \frac{1}{8i} |q|^2 q, \tag{3.35c}$$

$$\Gamma_{12}^{(4)} = -\frac{1}{16i} q_{tx} + \frac{1}{16} q^2 q_t^*. \tag{3.35d}$$

It should be noted that between these sequences of the generators, there exist a set of relations

$$\Gamma_{12}^{(n)} = -\Gamma_{21}^{(n)*}. \tag{3.36}$$

This is known from the forms of the recursion relations. Taking complex conjugate of (3.33) and using the fact that the matrices B and C have properties $B_{jk} = -B_{kj}^*$ and $C_{jk} = -C_{kj}^*$, we can see that Γ_{jk}^*'s satisfy the same relations as $-\Gamma_{kj}$'s do.

The infinite numbers of conserved densities emerge as the coefficients of ζ^{-n} in (3.29). They are:

$$I_j^{(n)} = B_{jk} \Gamma_{kj}^{(n)} + C_{jk} \Gamma_{kj}^{(n+1)}, \tag{3.37}$$

The explicit ζ-dependence of the conserved quantities are:

$$C = -\int dx \sum_k 2i\zeta^2 A_{jk} \Gamma_{kj} + \int dx \sum_k i|q|^2 A_{jk} \Gamma_{kj} + \int dx \sum_k B_{jk} \Gamma_{kj} + \zeta \int dx \sum_k C_{jk} \Gamma_{kj}. \tag{3.38}$$

The first term of this relation is constant because $A_{jk} = \sigma_j \delta_{jk}$. Therefore it is negligible in considering the conservation laws. We get the conserved densities as (3.37) substituting (3.32) and comparing the both sides. The first four of the conserved densities are

$$\begin{aligned} I_1^{(1)} &= (q_t q^* - q q_t^*)/2, \\ I_2^{(1)} &= (q q_t^* - q_t q^*)/2, \end{aligned} \tag{3.39a}$$

$$\begin{aligned} I_1^{(2)} &= -(i q_t q_t^* + q q_x^* + i|q|^4)/4, \\ I_2^{(2)} &= (i q_t q_t^* - q^* q_x + i|q|^4)/4, \end{aligned} \tag{3.39b}$$

$$\begin{aligned} I_1^{(3)} &= i(q q_{tx}^* - q_t q_x^*)/8, \\ I_2^{(3)} &= i(q_t^* q_x - q^* q_{tx})/8, \end{aligned} \tag{3.39c}$$

We can get the corresponding flux densities quite similarly. The result is

$$J_j^{(n)} = \tilde{M}_{jk} \Gamma_{kj}^{(n)}, \tag{3.40}$$

where the matrix $\tilde{M}$ is the off-diagonal part of M. Again, we write the first four of them:

$$J_1^{(1)} = -i|q|^2/2, \qquad J_2^{(1)} = i|q|^2/2, \tag{3.41a}$$

$$J_1^{(2)} = -qq_t^*/4, \qquad J_2^{(2)} = -q^*q_t/4, \tag{3.41b}$$

$$J_1^{(3)} = -(qq_x^* + i|q|^4)/8, \qquad J_2^{(3)} = -(q^*q_x - i|q|^4)/8. \tag{3.41c}$$

From the symmetric property of B, C, $\tilde{M}$ and $\Gamma_{jk}^{(n)}$'s, we get symmetries for I's and J's as

$$I_2^{(n)} = I_1^{(n)*}, \qquad J_2^{(n)} = J_1^{(n)*}.$$

We get new conservation laws that have physical meaning by Hermitizing the conserved quantities and fluxes above. For example, using the quantities in (3.39) and (3.41), we have new conserved densities $A^{(j)}$'s and $B^{(j)}$'s:

$$A^{(1)} = (q_tq^* - qq_t^*)/2i, \qquad B^{(1)} = -|q|^2/2, \tag{3.42a}$$

$$A_1^{(2)} = -(|q|^2)_x/8, \qquad B_1^{(2)} = -(|q|^2)_t/8, \tag{3.42b}$$

$$A_2^{(2)} = -(q_tq_t^* + |q|^4)/4, \qquad B_2^{(2)} = -(qq_t^* - q^*q_t)/8i, \tag{3.42c}$$

$$A_1^{(3)} = i(qq_t^* - q^*q_t)_x/16, \qquad B_1^{(3)} = -(|q|^2)_x/16 \tag{3.42d}$$

$$A_2^{(3)} = (qq_{tx}^* + q^*q_{tx} - q_tq_x^* - q_t^*q_x)/16,$$

$$B_2^{(3)} = -(qq_x^* - q^*q_x)/16i - |q|^4/8. \tag{3.42e}$$

From $I^{(1)}$ and $J^{(1)}$ we can essentially get only one conservation law (3.42a). The law (3.42b) is trivial. Among other laws, (3.42c) and (3.42e) gives the relation given from the stress-energy tensor, considering the integrated terms and using (3.42d). The quantity $B_2^{(2)}$ is equal to T_{01}, up to a constant factor, and $A_2^{(2)}$ to T_{00}, where a term $qq_x^* - q^*q_x$ is added. This additional term goes to zero when the density is integrated and (3.42d) is used.

Chapter 4. Soliton Solutions

In this chapter, we shall consider soliton solutions. We obtain N-soliton solution from the general method described in the previous section. In addition, we solve eigenvalue problems for some typical initial conditions. The results are useful for discussing the properties and the roles of solitons in unstable media: the mechanisms of the stabilization of the systems.

§4.1 Soliton Solutions

We assume that $b(\zeta) = 0$ and $a(\zeta)$ has N simple zeros at $\zeta = \zeta_j$, $j = 1, 2, \cdots, N$ in the following. Then, equation (3.17) gives $q(x,t)$ as

$$q(x,t) = -2i\sum_{k=1}^{N}\left(\gamma_k(t)\psi_2(x,\zeta_k)e^{2i\zeta_k^2x}\right)^*. \tag{4.1}$$

We can get the expressions for the forms of the elements of the Jost functions, $\psi_2(x,\zeta_k)$. From (3.14) we see that at the zeros of $a(\zeta)$, $\zeta = \zeta_j$, $j = 1, 2, \cdots, N$,

$$\psi_1(x,\zeta_j)e^{-2i\zeta_j^2x} = -\sum_{k=1}^{N}\frac{\gamma_k^* e^{-2i\zeta_k^{*2}x}}{\zeta_j - \zeta_k^*}\psi_2^*(x,\zeta_k),$$

$$\psi_2^*(x,\zeta_j)e^{2i\zeta_j^{*2}x} = 1 + \sum_{k=1}^{N}\frac{\gamma_k e^{2i\zeta_k^2x}}{\zeta_j^* - \zeta_k}\psi_1(x,\zeta_k), \tag{4.2}$$

where the functions ϕ's that appear in the expression of $\Phi(\zeta)$ are rewritten by using (3.7). We define $\psi_j^{(k)}$ and μ_j, respectively, as

$$\psi_j^{(k)} = \sqrt{\gamma_k(t)}\,\psi_j(x,\zeta_k), \qquad \mu_j = \sqrt{\gamma_j(t)}\,e^{2i\zeta_j^2x}, \tag{4.3}$$

then (4.2) is rewritten as

$$\psi_1^{(j)} + \sum_{k=1}^{N}\frac{\mu_j\mu_k^*}{\zeta_j - \zeta_k^*}\psi_2^{(k)*} = 0,$$

$$\psi_2^{(j)*} - \sum_{k=1}^{N}\frac{\mu_j\mu_k^*}{\zeta_j^* - \zeta_k}\psi_1^{(k)} = \mu_j^*. \tag{4.4}$$

This is a set of simultaneous equations of first order, then we can easily get the solutions.

For example, and for the later purpose, we present one soliton solution. It is obtained when $a(\zeta)$ has one simple zero in $\mathrm{Im}\zeta^2 > 0$. In fact, denoting the zero of $a(\zeta)$ by $\zeta = \xi + i\eta$, we have

$$q(x,t) = -2i\eta\frac{\exp(-4i(\xi^2 - \eta^2)x - 2i\xi t + i\phi)}{\cosh(8\xi\eta x + 2\eta t + \rho)}, \tag{4.5}$$

where

$$\phi = -2\arg(\mu(t=0)), \qquad \rho = \log\left(\frac{2|\eta|}{|\mu(t=0)|^2}\right). \tag{4.6}$$

Generally, the simple zeros of $a(\zeta)$ in $\mathrm{Im}\zeta^2 > 0$ correspond to solitons.

A soliton — and as we see shortly, constituent of multiple solitons — has two parameters, ξ and η. The velocity and amplitude are $-(4\xi)^{-1}$ and 2η, respectively. Other characters of solitons, such as width of the envelope, wavelength of the carrier wave, are determined by these two parameters. The solution (4.6), when x and t are interchanged, is the same as that of the stable (conventional) nonlinear Schrödinger equation (Zakharov and Shabat 1972). We see that (2.11) is a special case $\xi \to 0$ of the one soliton solution (4.6).

It is known that solitons experience position shift due to their mutual collisions (Wadati and Toda 1973). We examine the asymptotic behaviors of soliton solutions in order to study the effect of soliton collisions in the unstable media.

We denote the distinct zeros of $a(\zeta)$ by $\zeta_j = \xi_j + i\eta_j$, $j = 1, 2, \cdots, N$. We assume that $\xi_1 < \xi_2 < \cdots < \xi_N$: each soliton has different velocity since the velocity of soliton is $-(4\xi_j)^{-1}$. A soliton with a smaller subscript has a larger velocity with this labeling.

The time and spatial dependences of $|\mu_j(x,t)|$'s are from (3.24) and (4.3)

$$|\mu_j(x,t)| = \sqrt{|\gamma_j(0)|} \cdot e^{-\xi_j\eta_j(x+t/4\xi_j)}. \tag{4.7}$$

We observe the N-soliton solution in a coordinate such that $x + t/4\xi_m$ is constant. In this moving-coordinate, we have

$$\begin{aligned} \mu_j &\to 0 \quad \text{for } j < m, \quad \text{as } t \to +\infty, \\ \mu_j &\to 0 \quad \text{for } j > m, \quad \text{as } t \to -\infty. \end{aligned} \tag{4.8}$$

We can get asymptotic forms of simultaneous equations (4.4) in the limit $t \to \pm\infty$. For $t \to \infty$, we find that

$$\begin{aligned} &\psi_1^{(m)} + \frac{|\mu_m^{(+)}|^2}{2i\eta_m}\psi_2^{(m)*} = 0, \\ &\frac{|\mu_m^{(+)}|^2}{2i\eta_m}\psi_1^{(m)} + \psi_2^{(m)*} = \mu_m^{(+)*}, \end{aligned} \tag{4.9a}$$

where

$$\mu_m^{(+)} = \mu_m \prod_{j=m+1}^{N} \frac{\zeta_m - \zeta_j}{\zeta_m - \zeta_j^*}. \tag{4.9b}$$

In the other limit $t \to -\infty$, we have a set of equations in the same form as (4.9a), but $\mu_m^{(+)}$ is replaced with $\mu_m^{(-)}$. Similarly to (4.9b), $\mu_m^{(-)}$ is

$$\mu_m^{(-)} = \mu_m \prod_{j=1}^{m-1} \frac{\zeta_m - \zeta_j}{\zeta_m - \zeta_j^*}. \tag{4.9c}$$

We see that in the limit of large $|x|$, the equation for each Jost function decomposes into individual soliton parts, when we compare (4.9) with (4.4). These equations ensure the stability of solitons and give a useful information on soliton collisions. There exists the same set of N solitons in the limit $t \to \pm\infty$. Collisions of solitons occur in pair. Each soliton experiences the shifts of position and phase resulting from the collisions. Comparing (4.9) with (4.4) and remembering the expression of one soliton solution (4.5), we see that the center and phase of the m-th soliton in the asymptotic regions are respectively given by

$$\begin{aligned} x_m &= \frac{-1}{8\xi_m\eta_m}\rho_m = \frac{1}{8\xi_m\eta_m}\log\frac{|\mu_m|^2}{2|\eta_m|}, \\ \phi_m &= -2\arg(\mu_m). \end{aligned} \tag{4.10}$$

We write x_m's and ϕ_m's for $t \to \infty$ as $x_m^{(+)}$'s and $\phi_m^{(+)}$'s, and for $t \to -\infty$ as $x_m^{(-)}$'s and $\phi_m^{(-)}$'s. That is, we have

$$\begin{aligned} q(x,t) &\to \sum_{m=1}^{N} -2i\eta_m \frac{\exp[-4i(\xi_m^2 - \eta_m^2)x - 2i\xi_m t + i\phi_m^{(+)}]}{\cosh[8\xi_m\eta_m(x - x_m^{(+)}) + 2\eta_m t]} \quad \text{as } t \to \infty, \\ &\to \sum_{m=1}^{N} -2i\eta_m \frac{\exp[-4i(\xi_m^2 - \eta_m^2)x - 2i\xi_m t + i\phi_m^{(-)}]}{\cosh[8\xi_m\eta_m(x - x_m^{(-)}) + 2\eta_m t]} \quad \text{as } t \to -\infty. \end{aligned} \tag{4.11}$$

We find from (4.9b,c) and (4.10) that the position shift and the phase shift of the m-th soliton are:

$$\begin{aligned} \Delta x_m &= x_m^{(+)} - x_m^{(-)} \\ &= \frac{1}{4\xi_m\eta_m}\left(\sum_{j=m+1}^{N}\log\left|\frac{\zeta_m - \zeta_j}{\zeta_m - \zeta_j^*}\right| - \sum_{j=1}^{m-1}\log\left|\frac{\zeta_m - \zeta_j}{\zeta_m - \zeta_j^*}\right|\right), \\ \Delta\phi_m &= \phi_m^{(+)} - \phi_m^{(-)} \\ &= 2\left[\sum_{j=1}^{m-1}\arg\left(\frac{\zeta_m - \zeta_j}{\zeta_m - \zeta_j^*}\right) - \sum_{j=m+1}^{N}\arg\left(\frac{\zeta_m - \zeta_j}{\zeta_m - \zeta_j^*}\right)\right]. \end{aligned} \tag{4.12}$$

We restrict our consideration to two solitons to make a discussion simple; one with a larger velocity $-(4\xi_1)^{-1}$ and the other with a smaller velocity $-(4\xi_2)^{-1}$. According to (4.12), the faster soliton has a negative position shift

$$\Delta x_1 = \frac{1}{4\xi_1\eta_1}\log\left|\frac{\zeta_1-\zeta_2}{\zeta_1-\zeta_2^*}\right| < 0\,, \tag{4.13a}$$

and the slower one has the positive position shift

$$\Delta x_2 = -\frac{1}{4\xi_2\eta_2}\log\left|\frac{\zeta_2-\zeta_1}{\zeta_2-\zeta_1^*}\right| > 0\,. \tag{4.13b}$$

That is, the faster soliton decelerates and the slower one accelerates during the collision. This property is common to the USG equation (Yajima and Wadati 1987). On the contrary, the faster soliton accelerates and the slower one decelerates during the collision in stable media. We conclude that, in the unstable cases, the interactions between solitons are attractive. In other words, solitons form virtual bound states during the collision.

§4.2 Initial value problems

So far, we have assumed the existence of discrete simple eigenvalues. We have not examined if this assumption is realizable, or not. In this section, we shall study some initial value problems, and have confidence that this assumption is very reasonable.

We consider the following initial value problems:

a) $$q(x,0)=0,\quad q_t(x,0)=\frac{A}{\cosh x},\quad A:\text{constant}. \tag{4.14a}$$

b) $$q(x,0)=\frac{A\exp(ikx+2iA^2\tanh x)}{\cosh x},\quad q_t(x,0)=0,\quad A,k:\text{constants}. \tag{4.14b}$$

c) $$q(x,0)=\begin{cases} Ve^{ikx}, & |x|<L \\ 0, & |x|>L \end{cases}\quad q_t(x,0)=0,\quad V,L,k:\text{constants}. \tag{4.14 c}$$

(1) Initial condition a)

This corresponds to the situation that an 'impulsive force' is applied to the system. In (3.1a) we eliminate the function v_2 and use a new independent variable $z=(1-\tanh x)/2$. Then we have for v_1 ;

$$z(1-z)\frac{d^2v_1}{dz^2}+(\frac{1}{2}-z)\frac{dv_1}{dz} + \{A^2+\frac{4\zeta^4+2i\zeta^2(1-2z)}{4z(1-z)}\}v_1=0\,. \tag{4.15}$$

This equation is the same as the one appeared in the initial value problem of the conventional nonlinear Schrödinger equation (Satsuma and N. Yajima), when ζ is replaced by $2\zeta^2$. We get the similar equation for the function v_2. Setting $\nu=1/2-2i\zeta^2$, we have two linearly independent solutions, $v^{(1)}$ and $v^{(2)}$:

$$\begin{aligned} v_1^{(1)} &= z^{i\zeta^2}(1-z)^{-i\zeta^2}F(-A,A;1-\nu;z),\\ v_2^{(1)} &= z^{-i\zeta^2}(1-z)^{i\zeta^2}F(-A,A;\nu;z), \end{aligned} \tag{4.16a}$$

$$\begin{aligned} v_1^{(2)} &= z^{\frac{1}{2}-i\zeta^2}(1-z)^{-i\zeta^2}F(\nu+A,\nu-A;1+\nu;z),\\ v_2^{(2)} &= z^{\frac{1}{2}+i\zeta^2}(1-z)^{i\zeta^2}F(1-\nu+A,1-\nu-A;2-\nu;z), \end{aligned} \tag{4.16b}$$

where $F(a,b;c;z)$ is the hypergeometric function (Abramobitz and Stegun 1964). The Jost functions, and the transmission and reflection amplitudes are

$$\psi=\begin{pmatrix} \dfrac{A\,v_1^{(2)}}{2(\xi^2-\eta^2)+i/2} \\ v_2^{(2)} \end{pmatrix},\qquad \overline{\psi}=\begin{pmatrix} v_2^{(1)*} \\ -\dfrac{A\,v_1^{(2)*}}{2(\xi^2-\eta^2)-i/2} \end{pmatrix},$$

$$a = \frac{[\Gamma(-2i(\xi^2-\eta^2)+1/2)]^2}{\Gamma(-2i(\xi^2-\eta^2)+A+1/2)\,\Gamma(-2i(\xi^2-\eta^2)-A+1/2)}$$

$$b = \frac{i|\Gamma(2i(\xi^2-\eta^2)+1/2)|^2}{\Gamma(A)\,\Gamma(1-A)}. \tag{4.17}$$

The transmission amplitude $a(\zeta)$ can be analytically continued into the region $\xi\eta > 0$,

$$a(\zeta) = \frac{[\Gamma(-2i\zeta^2+1/2)]^2}{\Gamma(-2i\zeta^2+A+1/2)\,\Gamma(-2i\zeta^2-A+1/2)}. \tag{4.18}$$

Thus, we find that zeros of $a(\zeta)$ in the ζ-plane are located on the line $\xi = \eta$ (Fig.4.1), and their values are

$$\xi_j = \eta_j = \pm\frac{1}{2}\sqrt{A-j+\frac{1}{2}}\,,$$
$$j = 1,2,\cdots, \qquad n < A+\frac{1}{2}. \tag{4.19}$$

Fig.4.1

We see from (4.19) that the number of solitons, n, increases as the impulsive force becomes larger. This suggests that the energy injected into the system is used to create solitons and transported in the form of solitons. It is interesting that if the impulsive force is not strong enough, $A \le 1/2$, no soliton emerges from the initial condition (4.14a).

(2) Initial condition b)

This corresponds to a localized disturbance placed still. The detailed calculations are shown in the Appendix B. The fundamental solutions for v_1 are expressed as

$$v_1^{(1)} = e^{iA^2(1-2z)}e^{ikx/2}z^{i\lambda}(1-z)^{-i\lambda}\,F(2\zeta A,-2\zeta A;\frac{1}{2}+2i\lambda;z)\,,$$
$$v_1^{(2)} = e^{iA^2(1-2z)}e^{ikx/2}z^{1/2-i\lambda}(1-z)^{-i\lambda}$$
$$\times\,F(\frac{1}{2}-2i\lambda+2\zeta A,\frac{1}{2}-2i\lambda-2\zeta A;\frac{3}{2}-2i\lambda;z)\,. \tag{4.20a}$$

The other function v_2 is given as the linear combination of

$$v_2^{(1)} = e^{iA^2(1-2z)}e^{-ikx/2}z^{-i\lambda}(1-z)^{i\lambda}\,F(2\zeta A,-2\zeta A;\frac{1}{2}-2i\lambda;z)\,,$$
$$v_2^{(2)} = e^{iA^2(1-2z)}e^{-ikx/2}z^{1/2+i\lambda}(1-z)^{i\lambda}$$
$$\times\,F(\frac{1}{2}+2i\lambda+2\zeta A,\frac{1}{2}+2i\lambda-2\zeta A;\frac{3}{2}+2i\lambda;z)\,. \tag{4.21b}$$

As in (1), the function F is the hypergeometric function. Thus, the Jost functions and the scattering amplitudes are found to be

$$\psi = \begin{pmatrix} \dfrac{2\zeta A\, v_1^{(2)}}{2\lambda+i/2} \\ v_2^{(2)} \end{pmatrix}, \qquad \bar{\psi} = \begin{pmatrix} v_2^{(1)*} \\ -\dfrac{2\zeta A\, v_1^{(2)*}}{2\lambda-i/2} \end{pmatrix},$$

$$a = \frac{\{\Gamma(1/2-2i\lambda)\}^2}{\Gamma(1/2-2i\lambda+2\zeta A)\,\Gamma(1/2-2i\lambda-2\zeta A)},$$

$$b = \frac{i|\Gamma(2i\lambda+1/2)|^2}{\Gamma(2\zeta A)\,\Gamma(1-2\zeta A)}. \tag{4.22}$$

We look for zeros of $a(\zeta)$ in the region $\xi\eta > 0$. We see from (4.22) that the zeros are determined by

$$\frac{1}{2}-2i\lambda\pm 2\zeta A$$
$$= \{4\xi(\eta\pm\frac{A}{2})+\frac{1}{2}\}+i\{-2\xi^2+2(\eta\pm\frac{A}{2})^2-\frac{k}{2}-\frac{A^2}{4}\} = 1-m\,,$$
$$m = 1,2,\cdots. \tag{4.23}$$

The solution which satisfies $\xi\eta > 0$ does not exist when $k > 0$. When $k < 0$, we get

$$\xi_j = \pm Z_j\,, \qquad \eta_j = \pm\left(\frac{A}{2}-\frac{2j-1}{8Z_j}\right),$$
$$Z_j = \left[\frac{(A^2+2k)^2}{8}\left(\sqrt{1+\frac{(2j-1)^2}{(A^2+2k)^2}}-1\right)\right]^{1/2},$$
$$j = 1,2,\cdots,n, \tag{4.24}$$

where n is the integer which satisfies

$$1 \le n < \frac{1}{2}+A^2\sqrt{2|k|}\,. \tag{4.25}$$

In this case, the number of solitons, n, depends not only on the initial amplitude A but also on the wave number k in (4.14b).

(3) Initial condition c)

This case is another example in which a localized still disturbance exists. In this case, we can derive Jost functions by connecting at $x = \pm L$. The results of Jost functions are tabulated in the Table 4.1 (Appendix B).

Table 4.1

Of course, the condition (3.11) remains valid. Then, we can easily find the scattering amplitudes:

$$a(\xi) = e^{i(k+4\xi^2)L}\left[\cos 2\omega L - i\frac{k/2 + 2\xi^2 - V^2}{\omega}\sin 2\omega L\right],$$
$$b(\xi) = -\frac{2\xi V}{\omega}\sin 2\omega L,$$
$$\omega \equiv \sqrt{(V^2 - 2\xi^2 - k/2)^2 + 4\xi^2 V^2}. \qquad (4.25)$$

We continue analytically the function $a(\xi)$ into the region $\mathrm{Im}(\zeta^2) > 0$ and find its zeros. The amplitude $a(\zeta)$ depends on ζ through a form of ζ^2, so we set $\lambda \equiv 2\zeta^2$ and find zeros in upper half of λ-plane. The equation which gives the zeros is

$$\sin^2 2\omega L = \frac{\omega^2}{2\lambda V^2}. \qquad (4.26)$$

In general, it is hard to solve this equation, but we can easily see at least one of the solutions, which comes from $\omega = 0$. It is:

$$\lambda = -\frac{k}{2} \pm \sqrt{V^2(k - V^2)}. \qquad (4.27)$$

There always exists complex λ when $k < V^2$. If we adopt a condition $k = 2\pi M/L$, where M is an integer, this condition is $2\pi M < LV^2$. This shows that at least one soliton exists in the system when a product of the expanse and the amplitude of the wave exceed a threshold. Both of them can indicate the initial energy of the disturbance, so we can say that a soliton is created if the energy of the disturbance is enough, for a sinusoidal wave localized in a square well form. This may be said to contradict the result in §2.2, but we can say that the localized disturbance has much less energy compared to the plane wave, which expands in the whole space, and the existence of a threshold is not an obstacle of the discussion. It is extremely interesting to examine existence of zeros in the other region, $k > V^2$, and others which come from other conditions. We, however, leave them as future problems.

§4.3 Periodic solutions

In this section, we consider periodic solutions for the UNS equation which describe the modulations of carrier wave. We set

$$q(x,t) = \phi(y)\exp[i(Kx - \Omega t)],$$
$$y = 2\Omega x + t, \qquad (4.28)$$

where ϕ, K and Ω are real. We have an equation for ϕ by substituting this into the UNS equation:

$$\phi_{yy} - (K + \Omega^2)\phi + 2\phi^3 = 0. \qquad (4.29)$$

Now we introduce a positive constant a:

$$a^2 = \frac{1}{2}\left[(K+\Omega^2) + \sqrt{(K+\Omega^2)^2 + C}\right],$$
$$C = \text{real constant}. \qquad (4.30)$$

As in the Appendix C, the periodic solutions which satisfy the equation have the following forms:

$$\phi(y) = A\,\mathrm{cn}(\alpha(y - y_0), k),$$
$$= A\,\mathrm{dn}(\alpha(y - y_0), k). \qquad (4.31)$$

The quantities A, α and y_0 are constants and k is the elliptic moduli. The solutions are

(1) $K + \Omega^2 > 0$, $-(K+\Omega^2)^2 < C < 0$

$$\phi(y) = a\,\mathrm{dn}(a(y - y_0), k),$$
$$k^2 = \frac{1}{a^2}\sqrt{(K+\Omega^2)^2 + C}, \qquad 0 < k < 1. \qquad (4.32a)$$

(2) $C > 0$

$$\phi(y) = a\,\mathrm{cn}(\frac{a}{k}(y - y_0), k),$$
$$k^2 = \frac{a^2}{\sqrt{(K+\Omega^2)^2 + C}}, \qquad 0 < k < 1. \qquad (4.32b)$$

The limit $k \to 1$ for these solutions lead to the one soliton solution.

Chapter 5. Applications to Physical Systems

In this chapter, we make some applications of the models. The model systems are mechanical model analogous to the sine-Gordon equation, electron beam plasma and the Rayleigh-Taylor system. In those systems, there is a critical wave number in low frequency region and the envelopes of the fields obey the UNS equation. The results show us physical meanings of the model equation.

§5.1 The sine-Gordon type model

The first one among the applications of the UNS equation is the reductive perturbation method for a model called the unstable sine-Gordon (USG) model (Yajima and Wadati 1987, 1990a). The model equation is:

$$\phi_{tt} - \phi_{xx} - m^2 \sin\phi = 0, \tag{5.1}$$

where m^2 is a constant. The mechanical expression for this is given in the Fig.5.1. A chain of pendulii is connected with each other with spring (Scott 1969). In this case, the coefficient m^2 corresponds to a moment of the gravity. Each pendulum oscillates around the stable equilibrium point, $\phi = \pi$ in (5.1). Another example of the model (5.1) is two-level atomic system (Lamb 1971). The conventional sine-Gordon system describes the time evolution of the two level atomic system where initially all the atoms are in their ground states. The system described by the model (5.1) is considered to express the similar system with a condition that all the atoms are initially in the excited state.

Fig.5.1

The equation (5.1) has a linear dispersion relation:

$$-\omega^2 + k^2 = m^2. \tag{5.2}$$

As in §2, we can see from this relation that the plane wave number that satisfies $k^2 < m^2$ will exponentially grow in time. Therefore the system described by (5.1) can be considered as an unstable system.

The equation (5.1) is derived from the Lagrangian density and the Hamiltonian density:

$$\mathcal{L} = \frac{1}{2}(\phi_t^2 - \phi_x^2) + m^2(1 - \cos\phi),$$

$$\begin{aligned}\mathcal{H} &= \pi\dot{\phi} - \mathcal{L}\\ &= \frac{1}{2}(\phi_t^2 + \phi_x^2) - m^2(1-\cos\phi),\\ \pi &= \frac{\delta\mathcal{L}}{\delta\phi_t} = \phi_t.\end{aligned} \tag{5.3}$$

These show that in the model equation, a particle moves in a periodic potential.

The USG equation reduces to the UNS equation through the reductive perturbation method under small amplitude condition, and this gives a conspectus on the system described by the UNS equation.

We expand the field variable ϕ as

$$\phi = \sum_{n=1}^{\infty} \varepsilon^n \sum_{l=-\infty}^{\infty} \phi_n^{(l)}(\xi,\tau)\exp\left[il(k_0 x - \omega_0 t)\right], \tag{5.4}$$

(Kodama and Taniuti 1978) and introduce the Gardner-Morikawa transformation to independent variables (Gardner and Morikawa 1960):

$$\xi = \varepsilon^2 x, \qquad \tau = \varepsilon(t - \eta_0 x). \tag{5.5}$$

The term $\sin\phi$ is approximated as $\sin\phi \cong \phi - \phi^3/6$ when the amplitude is small. Substituting this and (5.4-5) into the USG equation, and comparing terms with the same order in ε, we get an infinite set of equations for each order of ε. We find in the first three orders:

$$\begin{aligned}
\varepsilon^1: \quad & \phi_l^{(1)} = 0 & (l \neq \pm 1), \quad & (5.6a)\\
& k_0^2 - \omega_0^2 = m^2, & & (5.6b)\\
\varepsilon^2: \quad & \phi_l^{(2)} = 0 & (l \neq \pm 1), \quad & (5.7a)\\
& \eta_0 = \frac{\omega_0}{k_0}, & & (5.7b)\\
\varepsilon^3: \quad & \phi_l^{(3)} = -\frac{1}{48}\{\phi_1^{(1)}\}^3 & (l = \pm 3), \quad & (5.8a)\\
& \phi_l^{(3)} = 0 & (l \neq \pm 1, \pm 3), \quad & (5.8b)\\
& i\phi^{(1)}_{1\ \xi} - \frac{1-\eta_0^2}{2k_0}\phi^{(1)}_{1\ \tau\tau} - \frac{m^2}{4k_0}|\phi_1^{(1)}|^2\phi_1^{(1)} = 0. & & (5.8c)
\end{aligned}$$

In (5.8b), we have used a symmetry between $\phi_l^{(k)}$ and $\phi_{-l}^{(k)}$ that assures a reality of ϕ:

$$\phi_{-l}^{(k)} = \phi_l^{(k)*}. \tag{5.9}$$

Relations (5.6b) and (5.7b) show that the factor $(1-\eta_0^2)/2k_0$ is positive. Then, rescaling variables as

$$\xi = -\frac{8k_0}{m^2}x, \quad \tau = \frac{2}{\sqrt{1-\eta_0}}t, \tag{5.10}$$

we find that $\phi_1^{(1)} = q$ satisfies the UNS equation (2.1).

We see that the UNS equation is derived from the USG equation, which clearly describes time development of unstable system, when an amplitude of the USG system is very small.

§5.2 Electron Beam Plasma

As a second application of the UNS equation, we shall consider an electron beam plasma system (Tanaka and N. Yajima 1988), where an electron beam is injected into plasma under high frequency electric field. due to the continuity relation and the equation of motion, we have a set of equations describing dynamics of the density n and the velocity $\boldsymbol{u}$ of electrons in plasma and beam due to the continuity relation and the Bernoulli equation. The ions (with positive charge) are much heavier than electrons, so we neglect their motion. We assume that the electrons in the injected beam have sufficiently large velocities and then the temperature dependent term is negligible. We have

$$\frac{\partial n_p}{\partial t} + \nabla\cdot(n_p \boldsymbol{u}_p) = 0, \tag{5.11a}$$

$$\frac{\partial \boldsymbol{u}_p}{\partial t} + (\boldsymbol{u}_p\cdot\nabla)\boldsymbol{u}_p = -\frac{e}{m}\boldsymbol{E}^{(h)} - \frac{T_p}{mn_p}\nabla n_p, \tag{5.11b}$$

$$\frac{\partial n_b}{\partial t} + \nabla\cdot(n_b \boldsymbol{u}_b) = 0, \tag{5.11c}$$

$$\frac{\partial \boldsymbol{u}_b}{\partial t} + (\boldsymbol{u}_b\cdot\nabla)\boldsymbol{u}_b = -\frac{e}{m}\boldsymbol{E}^{(h)}, \tag{5.11d}$$

where subscript p is for plasma and b for beam. Here $\boldsymbol{E}^{(h)}$ is the high frequency electric field, T_p the temperature of electrons, m the electron mass and $-e$ the electron charge. We divide the densities and velocities into three parts: the average, the high frequency and the low frequency parts. We distinguish these parts by the superscripts 0, h, and l.

We handle the high frequency terms as perturbations. The higher order terms of the high frequency part, such as $n_p^{(h)}\boldsymbol{u}_p^{(h)}$, are considered to be small. Eliminating $\boldsymbol{u}_p$ and $\boldsymbol{u}_b$ except the average parts, we have

$$\left(\frac{\partial^2}{\partial t^2} - \frac{T_p}{m}\nabla^2\right)n_p^{(h)} - \frac{e}{m}\nabla\cdot\left[(n_p^{(0)} + n_p^{(l)})\boldsymbol{E}^{(h)}\right] = 0, \tag{5.12a}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}_0\cdot\nabla\right)^2 n_b^{(h)} - \frac{e}{m}\nabla\cdot\left[(n_b^{(0)} + n_b^{(l)})\boldsymbol{E}^{(h)}\right] = 0, \tag{5.12b}$$

where $\boldsymbol{u}_0$ is the average velocity of the beam. The average velocity of the plasma electron is set to be zero because their motion is random. From now on, the nabra ∇ operates on all the functions in the right. We have from Gauss's law,

$$\nabla\cdot\boldsymbol{E}^{(h)} = -4\pi e(n_p^{(h)} + n_b^{(h)}). \tag{5.13}$$

Substitution of (5.13) into (5.12) gives us

$$\nabla\cdot\left[\frac{\partial^2}{\partial t^2}-\frac{T_p}{m}\nabla^2+\omega_e^2(1+\frac{n_p^{(l)}}{n_p^{(0)}})\right]\boldsymbol{E}^{(h)}=-\left(\frac{\partial^2}{\partial t^2}-\frac{T_p}{m}\nabla^2\right)4\pi e n_b^{(h)}, \qquad (5.14a)$$

$$\left(\frac{\partial}{\partial t}+u_0\cdot\nabla\right)^2 n_b^{(h)}=\frac{\alpha\omega_e^2}{4\pi e}\nabla\cdot\left(1+\frac{n_b^{(l)}}{n_b^{(0)}}\right)\boldsymbol{E}^{(h)}, \qquad (5.14b)$$

where

$$\alpha=\frac{n_b^{(0)}}{n_p^{(0)}},\qquad \omega_e=\frac{4\pi e^2 n_p^{(0)}}{m}. \qquad (5.14c)$$

We call α beam constant and ω_e electric plasma frequency. The frequency of the electric field is well approximated by ω_e. We introduce complex variables $\boldsymbol{E}$ and ρ as

$$\boldsymbol{E}^{(h)}=\frac{1}{2}(\boldsymbol{E}+\boldsymbol{E}^*),\qquad 4\pi e n_b^{(h)}=\frac{1}{2}(\rho+\rho^*), \qquad (5.15)$$

whose time dependences have forms

$$\boldsymbol{E}=e^{-i\omega_e t}\hat{\boldsymbol{E}},\qquad \rho=e^{-i\omega_e t}\hat{\rho}. \qquad (5.16)$$

Then we interpret envelopes $\hat{\boldsymbol{E}}$ and $\hat{\rho}$ as slowly varying functions in time. We have

$$\frac{\partial^2}{\partial t^2}\boldsymbol{E}\cong\left(-2i\omega_e\frac{\partial}{\partial t}+\omega_e^2\right)\boldsymbol{E}, \qquad (5.17)$$

when we neglect the higher derivatives of $\hat{\boldsymbol{E}}$. Taking ρ to the leading order, when we consider ρ as a traveling wave with wave number vector $\boldsymbol{k}$, we have

$$\left(\frac{\partial^2}{\partial t^2}-\frac{T_p}{m}\nabla^2\right)\rho\cong-\omega_e^2\rho\left(1-\frac{T_p k^2}{m\omega_e^2}\right). \qquad (5.18)$$

The second term in the right hand side of (5.18) can be dropped because the 'velocity', ω_e/k, of the high frequency part is much larger than the thermal velocity $\sqrt{2T_p/3m}$. Using (5.17) and (5.18) in (5.14a), we get

$$\nabla\cdot\left(i\frac{\partial}{\partial t}-\omega_e+\frac{T_p}{2m\omega_e}\nabla^2-\frac{\omega_e n_p^{(l)}}{2n_p^{(0)}}\right)\boldsymbol{E}=-\frac{\omega_e}{2}\rho. \qquad (5.19)$$

The term $n_p^{(l)}/n_p^{(0)}$ comes from the ponderomotive force (Zakharov 1972) and is expressed as

$$\frac{n_p^{(l)}}{n_p^{(0)}}=-\frac{|\boldsymbol{E}|^2}{16\pi n_p^{(0)}T_p}. \qquad (5.20)$$

Then we have from (5.19) and (5.20)

$$\nabla\cdot\left(i\frac{\partial}{\partial t}-\omega_e+\frac{T_p}{2m\omega_e}\nabla^2-\frac{\omega_e|\boldsymbol{E}|^2}{32\pi n_p^{(0)}T_p}\right)\boldsymbol{E}=-\frac{\omega_e}{2}\rho. \qquad (5.21a)$$

We can drop the term $n_b^{(l)}/n_b^{(0)}$ in (5.14b), because $n_b^{(l)}$ is usually much smaller than $n_b^{(0)}$. Then, we have

$$\left(\frac{\partial}{\partial t}+u_0\cdot\nabla\right)^2\rho=\alpha\omega_e^2\nabla\cdot\boldsymbol{E}. \qquad (5.21b)$$

From now on, we restrict our discussion to one-dimensional problem of (5.21). Let a direction of the average beam velocity u_0 be x-axis. We consider longitudinal wave for ρ, so the non-zero element of the electric field $\boldsymbol{E}=(E_1,E_2,E_3)$ is only E_1. In terms of dimensionless variables,

$$\xi=x\sqrt{\frac{m\omega_e^2}{T_p}},\qquad \tau=\omega_e t,$$

$$f=\frac{E_1}{\sqrt{32\pi n_p^{(0)}T_p}},\qquad g=\int^x\frac{\rho}{\alpha\sqrt{32\pi n_p^{(0)}T_p}}\,dx,$$

$$V=u_0\sqrt{\frac{m}{T_p}},\qquad \kappa^3=\frac{27\alpha}{8}, \qquad (5.22)$$

a set of equations (5.21) is reduced to

$$\left(i\frac{\partial}{\partial\tau}-1+\frac{1}{2}\frac{\partial^2}{\partial\xi^2}+|f|^2\right)f=-\frac{4}{27}\kappa^3 g, \qquad (5.23a)$$

$$\left(\frac{\partial}{\partial\tau}+V\frac{\partial}{\partial\xi}\right)^2 g=f. \qquad (5.23b)$$

The dispersion relation of the linearized (5.23) yields a cubic equation for ω:

$$(\omega-kV)^2\left(\omega-1-\frac{k^2}{2}\right)=\frac{4\kappa^3}{27}. \qquad (5.24)$$

This is shown in Fig.5.2.

Fig.5.2

Let us consider the circumstances of the solutions for this relation. As it can soon be seen, there are two branches in the solution; a branch with one real frequency and the other branch with two frequencies which grows into complex in a certain wave number region. From the formula of the solutions of cubic equation, the condition that gives the equation (5.24) complex solutions is:

$$(k-V)^2+2(1+\kappa)-V^2>0. \tag{5.25}$$

When $V^2<2(1+\kappa)$, there always exists complex ω solution. When $V^2>2(1+\kappa)$, the condition for k which gives complex ω in small wave number region is

$$k<k_c\equiv V\left(1-\sqrt{1-\frac{2(1+\kappa)}{V^2}}\right). \tag{5.26}$$

The other critical wave number, $k=V(1+\sqrt{1-2(1+\kappa)/V^2})$, gives us a frequency much greater than ω_e, so it does not match to our discussion. (The wave having this frequency would suffer the Landau damping.) The real frequency mode gives a stable plane wave, and we consider the branch that gives complex solutions. We investigate the equation (5.23) the region near the 'critical point' k_c. We denote by ω_c the frequency at $k=k_c$ which becomes complex number for $k<k_c$:

$$\omega_c=k_cV-\frac{2}{3}\kappa. \tag{5.27}$$

We express the solution of (5.23) as

$$\begin{pmatrix} f \\ g \end{pmatrix}=\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} e^{i(k_c\xi-\omega_c\tau)}. \tag{5.28}$$

The envelopes ϕ_1 and ϕ_2 are slowly varying functions of ξ and τ around the critical point where the wave number and the frequency are nearly k_c and ω_c. Using this approximation in (5.23), we have

$$\phi_2=-\left[\frac{9}{4\kappa^2}+\frac{27i}{4\kappa^3}\left(\frac{\partial}{\partial\tau}+V\frac{\partial}{\partial\xi}\right)\right.$$
$$\left.-\frac{243}{16\kappa^4}\left(\frac{\partial}{\partial\tau}+V\frac{\partial}{\partial\xi}\right)^2-O\left(\left(\frac{\partial}{\partial\tau}\right)^3,\left(\frac{\partial}{\partial\xi}\right)^3\right)\right]\phi_1. \tag{5.29}$$

From (5.28), (5.29) and (5.23) and neglecting higher derivative terms in ξ, we obtain

$$-iV\frac{\partial\phi_1}{\partial\xi}+\frac{9}{4\kappa}\frac{\partial^2\phi_1}{\partial\tau^2}+|\phi_1|^2\phi_1=0. \tag{5.30}$$

A suitable transformation of variables in (5.30) gives the UNS equation.

In summary, the envelope of the high frequency electric field near the critical wave number k_c obeys the unstable nonlinear Schrödinger (UNS) equation, under the conditions that the beam velocity is sufficiently large and that the system is one-dimensional. The envelope is considered to realize low frequency modes, so the model equation can be said to describe the time developments of low frequency modes in unstable media. The nonlinearity comes from the ponderomotive force due to the high frequency electric field.

§5.3 Rayleigh-Taylor Instability

In this section we shall discuss the Rayleigh-Taylor instability problem after Iizuka and Wadati (Iizuka and Wadati 1990), and derive the UNS equation. The Rayleigh-Taylor instability problem deals with the stability of a system where a heavy fluid is supported by a light fluid under the gravity. We suppose that the system is isotropic and uniform in the horizontal plane and restrict ourselves in two-dimensional case — one vertical and one horizontal. The situation is that the heavy fluid is bounded from above by a rigid plane and the light fluid from below (Fig.5.3).

Fig.5.3

The interface between the two fluids is $y=\eta(x,t)$. When it is completely flat, $\eta=0$. The following three assumptions for the fluids are made.

1) The density of a light fluid is negligibly small compared with that of heavy fluid.
2) The fluids are invicid and incompressible.
3) The motion of the fluid is irrotational.
4) Between the two fluids there exists the surface tension.

The motion of the light fluid needs not to be considered due to the condition 1). Because of the condition 3), the velocity field of the heavy fluid is expressed in terms of the velocity potential $\psi(x,y,t)$.

The fundamental equations for the system are

·condition for incomplessible fluid:

$$\Delta\psi=0 \quad (\eta\le y\le h), \tag{5.31}$$

·rigid boundary condition at $y=h$:

$$\psi_y=0 \quad (y=h), \tag{5.32}$$

·free boundary condition at the interface:

$$\psi_y=\eta_t+\psi_x\eta_x \quad (y=\eta). \tag{5.33}$$

·the Bernoulli equation:

$$\psi_t+\frac{1}{2}|\nabla\psi|^2+gy+\frac{T}{\rho}\eta_{xx}(1+\eta_x^2)^{-3/2}=0 \quad (y=\eta), \tag{5.34}$$

where T is the coefficient of surface tension, ρ the density of heavy fluid, and g the acceleration constant of gravity. The surface tension has a stabilizing effect while the gravity causes an instability in the system.

Let us linearize equations (5.31)-(5.34), then we have

$$\psi = C\cosh k(h-y)\cos(kx-\omega t), \tag{5.35}$$

$$\eta = C(k/\omega)\sinh kh \sin(kx-\omega t), \tag{5.36}$$

$$\omega^2 = (\frac{T}{\rho}k^3 - kg)\tanh kh, \tag{5.37}$$

where C is a constant. Equation (5.37) is a dispersion relation for the system (Fig.5.4).

Fig.5.4

For small (large) k, ω^2 is negative (positive). There is a critical wave number k_c at which ω^2 changes its sign,

$$k_c = \sqrt{\frac{\rho g}{T}}. \tag{5.38}$$

We analyze nonlinear evolution of the interface by means of the reductive perturbation method (Taniuti 1974). We express the expansions of $\psi(x,y,t)$ and $\eta(x,t)$ in powers of the smallness parameter ε:

$$\eta = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \varepsilon^n E^m \eta^{(n,m)}, \tag{5.39}$$

$$\psi = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \varepsilon^n E^m \psi^{(n,m)}, \tag{5.40}$$

where

$$E = \exp i(kx-\omega t) \qquad (k \geq k_c). \tag{5.41}$$

Relations

$$\eta^{(n,-m)} = (\eta^{(n,m)})^*, \tag{5.42}$$

$$\psi^{(n,-m)} = (\psi^{(n,m)})^*, \tag{5.43}$$

should be satisfied since η and ψ are real. Here the asterisk indicates the complex conjugate. The transformation of independent variables (Gardner-Morikawa transformation (Gardner and Morikawa 1960)) is set to be

$$\xi = \varepsilon^2 x, \tag{5.44}$$

$$\tau = \varepsilon(t - Wx). \tag{5.45}$$

Substituting equations (5.39-45) into equations (5.31-34), and comparing each coefficient of $\varepsilon^n E^m$ cause us to get relations among $\eta^{(n,m)}(\xi,\tau)$ and $\psi^{(n,m)}(\xi,\tau)$. If we analyze the lowest-order, the quantity W in (5.45) is found to be equal to the inverse of the group velocity, $\partial k/\partial\omega$.

We obtain a closed evolution equation for $\eta^{(1,1)}(\xi,\tau)$ by collecting terms up to the order of $\varepsilon^3 E$:

$$i\eta_{\xi}^{(1,1)} - \frac{1}{2}\frac{\partial^2 k}{\partial\omega^2}\eta_{\tau\tau}^{(1,1)} + K|\eta^{(1,1)}|^2\eta^{(1,1)} = 0, \tag{5.46}$$

where

$$\begin{aligned} K = &-k\omega^3\frac{(2s_1s_2 + s_1^2/2 - 3/2)^2 W}{s_1[(g-4Tk^2/\rho) + 2\omega^2 s_2/k]} + \omega k^2(2s_1s_2-3)W \\ &- \frac{k}{2(V^2+gh)}\Big[4\omega^2 V(s_1^2-1) + \omega^3 h(s_1^3 - 2s_1 + \frac{1}{s_1}) - 4\omega g s_1\Big]W \\ &+ \frac{3k^5 T}{2\omega\rho s_1}W, \end{aligned} \tag{5.47}$$

$$s_1 = \coth kh, \qquad s_2 = \coth 2kh, \quad V = \frac{\partial\omega}{\partial k}. \tag{5.48}$$

We adopt the variable transformations,

$$X = -\frac{1}{2}\frac{\partial^2 k}{\partial\omega^2}\xi. \tag{5.49}$$

$$q = \sqrt{|K|\left(\frac{\partial^2 k}{\partial\omega^2}\right)^{-1}}\,\eta^{(1,1)}, \tag{5.50}$$

then equation (5.46) reduces to the unstable nonlinear Schrödinger equation

$$iq_X + q_{\tau\tau} - 2\mathrm{sgn}(K\frac{\partial^2 k}{\partial\omega^2})|q|^2 q = 0. \tag{5.51}$$

We consider the 'deep water' case $kh \gg 1$ since the expression of $s = \mathrm{sgn}(K\partial^2 k/\partial\omega^2)$ is complicated. In this limiting case, $s = -(+)1$ when $k > (<)\sqrt{1+2/\sqrt{3}}k_c$.

It is fair to mention that the unstable nonlinear Schrödinger (5.51) ($s = +1$) has been derived for capillary waves on the surface of liquid column (Kakutani, Inoue and Kan 1974).

When $k > k_c$ the linearized wave is stable and the nonlinear Schödinger equation is derived by using a different Gardner-Morikawa transformation (Iizuka and Wadati 1990). Other wave number regions: $k < k_c$ have been discussed (Iizuka and Wadati 1990). The linearized wave is unstable when the wave number k is smaller than k_c. They obtained the Ginzburg-Landau type nonlinear diffusion equation in this region. These equations can be considered to form a set of model equations of nonlinear evolutions in unstable media.

6. Concluding Remarks

In this thesis, we have discussed nonlinear waves in unstable systems and have derived the following results:

(1) The unstable nonlinear Schrödinger (UNS) equation describes propagations of localized modes in some region of the wave number. (§5)

(2) The inverse scattering method is also applicable to the unstable system. The Gel'fand - Levitan - Marchenko equation is different from that for the conventional nonlinear Schrödinger equation. (§3)

(3) Initial value problems for the model equations are exactly solved by applying the inverse scattering method. Solitons can be generated and propagate also in unstable systems. These results give a firm basis for the analyses of physical phenomena, remembering the corresponding physical models. (§3)

(4) The disturbance caused in the system does not grow forever, but is suppressed by the nonlinearity. Also, the investigation on the initial value problems tells us that the solitons are caused generally, and play important roles in carrying away surplus energy. (§2, §4)

(5) The position shift due to mutual collisions between solitons has the opposite sign compared to the stable case. (§5)

(6) In connection with the result (3), we derived the infinite number of conservation laws and the model equations are found to be integrable. (§3.2).

All of these results is original and (1)-(5) have been published. We can discuss properties and roles of solitons in unstable media, based on these results. The result (5) suggests that the interaction between solitons can be considered to be attractive. This property occurs from the instability of the system and should be detectable by experiments.

The localized wave modes are created from rather general initial conditions. The existence of solitons in unstable media is very important to keep the system not to explode. A disturbance caused in the system grows because of the instability, but is suppressed due to the nonlinearity, as seen in §2. This suggests that, in unstable media, a soliton is a nonlinear excitation that makes the system stable by carrying the surplus energy in an effective way. The results of initial value problems support quantitatively this picture, since the number of solitons created in the system increases as the initial amplitude gets large. In three dimensional Langmuir turbulence, the following is well known: For monochromatic Langmuir wave, 'caverns' of density and developments of automodulations of Langmuir wave take place in the plasma. They formally develop infinitely, and then suffer Langmuir collapse, which play a role of dissipation of the energy (Zakharov 1972). We can consider

that this process is the nonlinear effect on the development of instability in plasma. But, for the instability in this picture the existence of beam is not taken into account, and the result in this thesis is the first exact analysis including such effects. In the electron beam plasma system, we can conclude that the localized wave mode is an important process to stabilize the system, although it is unclear if the collapse occurs or not in higher dimensions.

Recently, Yamagiwa et al. (Yamagiwa, Tokuda and Mineo, 1990) studied a high frequency beam mode in electron beam plasma. The observation shows that the wave envelope $A(z)$, z being the distance of the propagation, is given by

$$A(z) = \gamma \operatorname{sech}[\gamma(z - z_0)], \tag{6.1}$$

where γ is the linear growth rate and z_0 is the position of maximum amplitude. An amazing fact is that this is exactly the same as the result (2.11), in a moving system. We can say that our result is supported experimentally.

In §5, we have seen that the UNS equation is obtained in systems under various physical circumstances. This suggests that, under some conditions, the UNS model is a suitable equation for describing nonlinear modulation of amplitude in unstable media and has a wide applicability. In such a situation, the equation comes as the amplitude modulations of lowest frequency modes (§5.1 and §5.3) or modulation of envelope of plasma wave modes near the critical frequency, that is approximately equal to the plasma frequency ω_e. We see from this fact that the UNS equation describes nonlinear evolutions of localized modes with long wavelength modulation in unstable systems.

Alternatively, we can select special regimes of parameters of the system in deriving the UNS equation. Then, in other conditions, say, other regimes of wave number of carrier wave, we can of course get other types of equations, such as nonlinear diffusion equations (Iizuka and Wadati 1990), which are valuable to investigate. We can consider that these equations constitute the canonical models for nonlinear dynamics in unstable media.

Related to the UNS model, there are many generalizations or applications stimulating our interests. First, we can try to find the higher order equations of the UNS equation, like the AKNS hierarchy. The answer for this subject is not obtained yet, because the auxiliary linear problems (3.1a) are complicated and the treatment does not have good perspective. It is a future problem.

Second, it is interesting to consider the higher dimensional case. The treatments in §§5.2 and §§5.3 are limited in the one dimensional case. For example, in the electron beam plasma system, we can consider an axial symmetric boundary condition, which gives a time development of radial component of density and the electric field. In this case, interaction between the radial mode and the x-directed mode may exist because of the form of the model equation (5.21), and it is expected that some interesting phenomena can be observed. In addition, in the case of §5.3, an anisotropic case may be considered, and this will give another viewpoint in higher dimensions.

Third, we can consider a quantization of the UNS model. Again for example in §§5.2, the model equation includes some components that consist the plasma system, *e. g.* plasma electrons, beam electrons and ions. The immediate suggestion is to apply the results in this thesis to some one dimensional system that is constituted from particles and quasi particles of many kinds. Zakharov considered an electron-plasmon interacting system in plasma turbulence system (Zakharov 1967). Considering the model equations (5.11), we can apply this model to such a system. An quantization of the UNS model would be a simplified but effective and interesting model for one dimensional quantum system.

In the last of the concluding section, we like to take up to another example of nonlinear equation, that has a connection with the UNS equation. That is the one with the UNS type, whose sign of nonlinear term is negative. As one of the example for its physical picture, in §5.3, consider the case where the wave number equals to the critical value k_c. Then the nonlinear term in (5.51) has a negative sign (this means that $s = +1$ in (5.51)), and the UNS equation with negative nonlinear term emerges. A plane wave solution is stable for (5.51). In this case, we have the dark soliton solution given by

$$q(X,\tau) = \frac{(\lambda + i\nu)^2 + \exp 2\nu(\tau - \tau_0 - 2\lambda X)}{1 + \exp 2\nu(\tau - \tau_0 - 2\lambda X)} e^{-2iX},$$
$$\nu = \sqrt{1-\lambda^2}, \tag{6.2}$$

Initial value problem of (5.51) under the boundary condition $|q|^2 \to 1 \quad (x \to \pm\infty)$ is physically and mathematically very interesting. It was solved very recently (Iizuka, Wadati and Yajima 1991). Including their result, we can say that the outcomes of this thesis are commonly acceptable for the equations that describe time developments of unstable nonlinear systems.

In conclusions, it should be emphasized that in this thesis, we have thoroughly investigated the UNS model, which is one of the canonical equations describing the nonlinear evolutions in unstable media, and have clarified the roles of the localized wave modes.

Appendix A

In this appendix, we present the asymptotic expansion of the matrix Jost function, and get some information for the 'generator' of the conserved densities Γ.

First, let us consider the eq.(3.4). In the limit $x \to -\infty$, we have

$$\psi_{jk} e^{2i\zeta^2\sigma_j x} \to \sigma_j \delta_{jk}.$$

From (3.1a), the x-derivative of the left hand side is

$$\frac{\partial}{\partial x}(\psi_{jk} e^{2i\zeta^2\sigma_j x}) = e^{2i\zeta^2\sigma_j x} \sum_{\ell} \hat{L}_{j\ell} \psi_{\ell k}. \tag{A.1a}$$

The element of the matrix $\hat{L}$ is defined by

$$\hat{L}_{jk} = L_{jk} + 2i\zeta^2 \sigma_j \delta_{jk}. \tag{A.1b}$$

Integrating both sides of (A.1a) taking care of the boundary condition, we get

$$\psi_{jk} = e^{-2i\zeta^2\sigma_j x}\left\{\delta_{jk} + \int_{-\infty}^{x} dy\, e^{2i\zeta^2\sigma_j y} \sum_{\ell} \hat{L}_{j\ell}\psi_{\ell k}(y)\right\}. \tag{A.2}$$

We split the matrix $\hat{L}$ according to the order in ζ:

$$\hat{L} = i|q|^2 A + B + \zeta C,$$
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & iq_t \\ iq_t^* & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 2q \\ -2q^* & 0 \end{pmatrix}, \tag{A.3}$$

The matrix $\hat{L}$ is $\mathcal{O}(\zeta)$, then we have from integrating by parts, the RHS of (A.2) has $\mathcal{O}(\zeta)$. By definition, the diagonal element of Γ is the unity, then

$$\Gamma_{jk} = \mathcal{O}(\zeta^{-1}).$$

Next, we derive the recursion relation (3.33). The matrix $\tilde{L}$ can be written as $\tilde{L} = B + \zeta C$, where B and C are defined in the (A.3), and Γ_{jj} is unity, so the second and the third terms in the right hand side of (3.31) are

$$\sum_{\ell} \tilde{L}_{k\ell}\Gamma_{\ell j} = B_{kj} + \zeta C_{kj},$$
$$\sum_{\ell} \tilde{L}_{j\ell}\Gamma_{kj}\Gamma_{\ell j} = \sum_{\substack{p=1 \\ \neq j}}^{2} B_{jp}\Gamma_{pj}\Gamma_{kj} + \zeta \sum_{\substack{p=1 \\ \neq j}}^{2} C_{jp}\Gamma_{pj}\Gamma_{kj}. \tag{A.4}$$

where we have used the fact that the diagonal elements of $\tilde{L}$ (so are those of B and C) are zero. As the elements Γ_{jj}'s are unity, the non-trivial expressions for $\Gamma_{jk}^{(n)}$'s are gotten only for $j \neq k$. Then for $k \neq j$, the factor $\sigma_k - \sigma_j$ is $2(-1)^j$. Substituting (3.32) and (A.4) into (3.31) and comparing the same order in ζ, we have (3.33).

Appendix B

Here we treat the initial conditions (b) and (c) in (4.14), and get Jost functions. Both conditions have common properties: $q_t(x,0)$ is identically zero and $q(x,0)$ is bounded. As we have mentioned in the text, we first think real valueξ for ζ. We introduce a set of quantities:

$$
\begin{aligned}
F &= |Q|^2 - 2\xi^2, \\
Q &= 2\xi q(x,0), \qquad R = -2\xi q(x,0)^*.
\end{aligned} \tag{B.1}
$$

Then, the equation (3.1a) becomes

$$
\chi_x = \begin{pmatrix} iF & Q \\ R & -iF \end{pmatrix} \chi, \tag{B.2}
$$

We can get a set of differential equations for the elements of the function χ by eliminating the other element. The equations are:

$$
\begin{aligned}
\chi_{1xx} - \frac{Q_x}{Q}\chi_{1x} + (F^2 - iF_x - QR + i\frac{FQ_x}{Q})\chi_1 = 0, \\
\chi_{2xx} - \frac{R_x}{R}\chi_{2x} + (F^2 + iF_x - QR - i\frac{FR_x}{R})\chi_1 = 0.
\end{aligned} \tag{B.3}
$$

We shall discuss each initial condition separately.

1) initial condition (b)

We substitute the explicit form of the initial condition, and set

$$
\chi_1 = e^{iA^2 \tanh x} \cdot e^{ikx/2} \cdot w_1, \tag{B.4a}
$$

$$
\chi_2 = e^{-iA^2 \tanh x} \cdot e^{-ikx/2} \cdot w_2. \tag{B.4b}
$$

Then as in the case (1), we get a differential equations. For w_1, we have

$$
\begin{aligned}
z(1-z)\frac{d^2 w_1}{dz^2} + \left(\frac{1}{2} - z\right)\frac{dw_1}{dz} \\
+ \left\{ 4\xi^2 A^2 + \frac{4\xi^4 + 2k\xi^2 + k^2/4 + i(2\xi^2 + k/2)(1-2z)}{4z(1-z)} \right\} v_1 = 0.
\end{aligned} \tag{B.5}
$$

We introduce a variable $\lambda \equiv \xi^2 + k/4$, and have two independent solutions of (B.5):

$$
\begin{aligned}
w_1^{(1)} &= z^{i\lambda}(1-z)^{-i\lambda}\, F(2\xi A, -2\xi A; \frac{1}{2} + 2i\lambda; z), \\
w_1^{(2)} &= z^{1/2 - i\lambda}(1-z)^{-i\lambda} \\
&\quad \times F(\frac{1}{2} - 2i\lambda + 2\xi A, \frac{1}{2} - 2i\lambda - 2\xi A; \frac{3}{2} - 2i\lambda; z).
\end{aligned} \tag{B.6}
$$

Therefore, the two independent solutions for the first component in (3.1a) are:

$$
\begin{aligned}
\chi_1^{(1)} &= e^{iA^2(1-2z)} e^{ikx/2} z^{i\lambda}(1-z)^{-i\lambda}\, F(2\xi A, -2\xi A; \frac{1}{2} + 2i\lambda; z), \\
\chi_1^{(2)} &= e^{iA^2(1-2z)} e^{ikx/2} z^{1/2 - i\lambda}(1-z)^{-i\lambda} \\
&\quad \times F(\frac{1}{2} - 2i\lambda + 2\xi A, \frac{1}{2} - 2i\lambda - 2\xi A; \frac{3}{2} - 2i\lambda; z).
\end{aligned} \tag{B.7a}
$$

Similarly, we have for the function χ_2:

$$
\begin{aligned}
\chi_2^{(1)} &= e^{iA^2(1-2z)} e^{-ikx/2} z^{-i\lambda}(1-z)^{i\lambda}\, F(2\xi A, -2\xi A; \frac{1}{2} - 2i\lambda; z), \\
\chi_2^{(2)} &= e^{iA^2(1-2z)} e^{-ikx/2} z^{1/2 + i\lambda}(1-z)^{i\lambda} \\
&\quad \times F(\frac{1}{2} + 2i\lambda + 2\xi A, \frac{1}{2} + 2i\lambda - 2\xi A; \frac{3}{2} + 2i\lambda; z).
\end{aligned} \tag{B.7b}
$$

The asymptotic forms of these functions are in Table B.1. These forms give us the explicit forms of the Jost functions:

$$
\psi = \begin{pmatrix} \dfrac{2\xi A\, v_1^{(2)}}{2\lambda + i/2} \\ e^{-iA^2} v_2^{(1)} \end{pmatrix}, \qquad \bar{\psi} = \begin{pmatrix} e^{iA^2} v_2^{(1)*} \\ -\dfrac{2\xi A\, v_1^{(2)*}}{2\lambda - i/2} \end{pmatrix}, \tag{B.8}
$$

Table B.1

Of course, these satisfy the relation (3.11). The asynptotic form of ϕ decides the scattering amplitudes and ϕ itself. From the Table B.1, the Jost function ψ is in the limit of $x \to -\infty$:

$$
\psi = \begin{pmatrix} \dfrac{i|\Gamma(\frac{1}{2} - 2i\lambda)|^2}{\Gamma(2\xi A)\Gamma(1 - 2\xi A)} e^{-iA^2} e^{-2i\xi^2 x} \\ \dfrac{\Gamma(\frac{1}{2} - 2i\lambda)^2}{\Gamma(\frac{1}{2} - 2i\lambda + 2\xi A)\Gamma(\frac{1}{2} - 2i\lambda - 2\xi A)} e^{-iA^2} e^{-2i\xi^2 x} \end{pmatrix}. \tag{B.9}
$$

So we have

$$
\phi = \begin{pmatrix} a e^{iA^2} v_2^{(1)*} + b\dfrac{2\xi A\, v_1^{(2)}}{2\lambda + i/2} \\ -a\dfrac{2\xi A\, v_1^{(2)*}}{2\lambda - i/2} + b e^{-iA^2} v_2^{(1)} \end{pmatrix},
$$

$$a = \frac{\Gamma(\frac{1}{2} - 2i\lambda)^2}{\Gamma(\frac{1}{2} - 2i\lambda + 2\xi A)\Gamma(\frac{1}{2} - 2i\lambda - 2\xi A)} e^{-2iA^2},$$

$$b = \frac{i|\Gamma(\frac{1}{2} - 2i\lambda)|^2}{\Gamma(2\xi A)\Gamma(1 - 2\xi A)} e^{-iA^2}. \quad \text{(B.10)}$$

The function $\bar{\phi}$ can be gotten from (3.11). Then, the Jost functions for the initial condition (b) have been derived. It should be noted that when we make analytic continuation of the coefficients a and b, we can get the complex variable version of the scattering data (4.22).

3) initial condition (c)

Because the potential has discontinuities at $x = \pm L$, we think in each continuous region and make continuation at the point of discontinuity. When $|x| > L$, the equation (3.1a) is very simple. The matrix of the equation has only constant diagonal elements, so the general solution in this regime is

$$\chi = \begin{pmatrix} Ae^{-2i\xi^2 x} \\ Be^{2i\xi^2 x} \end{pmatrix}, \qquad |x| > L, \quad \text{(B.11)}$$

where A and B are the integral constants. In the other region $|x| < L$, we use (B.3). These equations become

$$\chi_{jxx} + (-1)^j ik\chi_{jx} + (V^4 + 4\xi^4 - kV^2 + 2k\xi^2)\chi_j = 0,$$

then the characteristic equation is

$$\mu^2 \pm ik\mu + (V^4 + 4\xi^4 - kV^2 + 2k\xi^2) = 0. \quad \text{(B.12)}$$

For the first component χ_1, the sign of the second term in RHS is minus and for the second component χ_2, it is plus. Now we have four fundamental solutions and four coupling constants to get general solutions, but among these coefficients, the independent one are only two. From the equation (3.1a) we have relations for the components of χ. Then the solution is

$$\chi = \begin{pmatrix} i\lambda_2\alpha_1 e^{iK_1 x} + i\lambda_1\alpha_2 e^{iK_2 x} \\ \alpha_1 e^{-iK_2 x} + \alpha_2 e^{-iK_1 x} \end{pmatrix}, \quad \text{(B.13)}$$

where

$$\omega = \sqrt{(V^2 - 2\xi^2 - k/2)^2 + 4\xi^2 V^2},$$
$$K_j = \frac{1}{2} - (-1)^j\omega, \qquad \lambda_j = \frac{1}{2\xi V}\left(\frac{k}{2} + 2\xi^2 - V^2 - (-1)^j\omega\right), \qquad j = 1, 2, \quad \text{(B.14)}$$

and α_j's $(j = 1, 2)$ are the coupling constants. We should note a relation $\lambda_1\lambda_2 = -1$.

Now from (B.11) and (B.13) we get the explicit forms of the Jost functions. Let us first consider ψ, whose asymptotic form is (3.4a). In the region of $x > L$,

$$\psi = \begin{pmatrix} 0 \\ e^{2i\xi^2 x} \end{pmatrix}, \quad \text{(B.15)}$$

because of (3.4a) and (B.11). The continuity of the each component of ψ at $x = L$ determines the coefficients in (B.13):

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{\xi V}{\omega} e^{i(k/2+\xi^2)L} \begin{pmatrix} \lambda_1 e^{-i\omega L} \\ -\lambda_2 e^{i\omega L} \end{pmatrix}. \quad \text{(B.16)}$$

This gives a result

$$\psi = e^{i(k/2+\xi^2)L} \begin{pmatrix} \frac{2\xi V}{\omega} e^{ikx/2} \sin(\omega(x - L)) \\ e^{-ikx/2}\left[\cos(\omega(x - L)) + i\frac{k/2 + 2\xi^2 - V^2}{\omega}\sin(\omega(x - L))\right] \end{pmatrix}. \quad \text{(B.17)}$$

Similarly, we impose the continuity conditions on ψ at $x = -L$, and we can have the form of ψ in the region $x < -L$. The result is

$$\psi = \begin{pmatrix} \frac{-2\xi V}{\omega} \sin 2\omega L e^{-2i\xi^2 x} \\ e^{i(k/2+4\xi^2)L}(\cos 2\omega L - i\frac{k/2 + 2\xi^2 - V^2}{\omega}\sin 2\omega L)e^{2i\xi^2 x} \end{pmatrix} \quad \text{(B.18)}$$

This is summarized in Table 4.1. As for the other Jost function ϕ, we can get its expressions in a similar way.

Appendix C

We seek the periodic solutions for (4.29). An expected function for the solution is elliptic functions because of the third order nonlinearity. We can set the general forms of the solutions as in the equation (4.30). We put them into the (4.29) to get the solution. We easily have a set of condition for the cnoidal case by substitution:

$$\begin{aligned}\phi &= A\,\mathrm{cn}(\alpha(y-y_0),k),\\ 2A^2 &= \alpha^2 + K + \Omega^2,\\ A^2 &= \alpha^2 k^2.\end{aligned} \tag{C.1}$$

We see at a glance that a function $-f(y)$ is also a solution if $f(y)$ is the solution of (4.29). Then we can confine ourselves to the case $A = \alpha k$ and get

$$\begin{aligned}\phi &= A\,\mathrm{cn}(\frac{A}{k}(y-y_0),k),\\ (2-\frac{1}{k^2})A^2 &= K + \Omega^2.\end{aligned} \tag{C.2}$$

The quantity A is real, so the following condition emerges:

(1) $0 < k^2 \leq 1/2, \quad K + \Omega^2 \leq 0$, or,

(2) $1/2 < k^2 \leq 1, \quad K + \Omega^2 > 0.$ (C.3)

Now we adopt a constant a defined in the equation (4.30) for A. We have from (C.2) that

$$\begin{aligned}\phi(y) &= a\mathrm{cn}(\frac{a}{k}(y-y_0),k),\\ k^2 &= a^2/\sqrt{(K+\Omega^2)^2+C}, \qquad 0<k<1.\end{aligned} \tag{C.4}$$

The explicit form of k is

$$\begin{aligned}k^2 =& a^2/(2a^2-(K+\Omega^2)),\\ =&\frac{1}{2}\left[1-\frac{K+\Omega^2}{\sqrt{(K+\Omega^2)^2+C}}\right].\end{aligned} \tag{C.5}$$

The condition $0 < k < 1$ corresponds to $C > 0$, and the form of (C.5) automatically guarantees the condition (C.3) as far as this condition is valid. This is the solution (4.32b).

Similarly, we can treat the solution of dn-type. The conditions for A and α, which correspond to (C.1), are

$$\begin{aligned}\phi &= A\,\mathrm{dn}(\alpha(y-y_0),k),\\ 2A^2 &= \alpha^2 k^2 + K + \Omega^2,\\ A^2 &= \alpha^2.\end{aligned} \tag{C.6}$$

Then the solution is

$$\begin{aligned}\phi(y) &= a\mathrm{dn}(a(y-y_0),k),\\ k &= \frac{1}{a^2}\sqrt{(K+\Omega^2)^2+C}, \qquad 0<k<1.\end{aligned} \tag{C.7}$$

We have from (C.6) that

$$k^2 = \frac{2\sqrt{(K+\Omega^2)^2+C}}{[(K+\Omega^2)+\sqrt{(K+\Omega^2)^2+C}]}. \tag{C.5}$$

This time, the condition for $0 < k < 1$ is $C < 0$. In addition, the reality of the quantities A and k requires $-(K+\Omega^2)^2 < C$. Then we have confirmed the solution (4.32a).

Another possibility, whose form is $A\,\mathrm{sn}(\alpha(y-y_0),k)$, brings a result $A = 0$, and this is trivial.

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Jost functions	$x \to +\infty$	$x \to -\infty$	$\|\zeta\| \to \infty$
$\phi(x,\zeta)$	$\begin{pmatrix} a(\zeta)e^{-2i\zeta^2 x} \\ b(\zeta)e^{2i\zeta^2 x} \end{pmatrix}$	$\begin{pmatrix} e^{-2i\zeta^2 x} \\ 0 \end{pmatrix}$	$\begin{pmatrix} e^{-2i\zeta^2 x} \\ 0 \end{pmatrix}$
$\psi(x,\zeta)$	$\begin{pmatrix} 0 \\ e^{2i\zeta^2 x} \end{pmatrix}$	$\begin{pmatrix} \bar{b}(\zeta)e^{-2i\zeta^2 x} \\ a(\zeta)e^{2i\zeta^2 x} \end{pmatrix}$	$\begin{pmatrix} 0 \\ e^{2i\zeta^2 x} \end{pmatrix}$
$\bar{\phi}(x,\zeta)$	$\begin{pmatrix} \bar{b}(\zeta)e^{-2i\zeta^2 x} \\ -\bar{a}(\zeta)e^{2i\zeta^2 x} \end{pmatrix}$	$\begin{pmatrix} 0 \\ -e^{2i\zeta^2 x} \end{pmatrix}$	$\begin{pmatrix} 0 \\ -e^{2i\zeta^2 x} \end{pmatrix}$
$\bar{\psi}(x,\zeta)$	$\begin{pmatrix} e^{-2i\zeta^2 x} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \bar{a}(\zeta)e^{-2i\zeta^2 x} \\ -b(\zeta)e^{2i\zeta^2 x} \end{pmatrix}$	$\begin{pmatrix} e^{-2i\zeta^2 x} \\ 0 \end{pmatrix}$

Table 3.1 Relations among the Jost functions and the scattering data.

Jost functions	$x < -L$	$\|x\| < L$	$x > L$
$\phi(x,\xi)$	$\begin{pmatrix} e^{-2i\xi^2 x} \\ 0 \end{pmatrix}$	$\begin{pmatrix} Ae^{ikx/2}(\cos\theta_1 - i\frac{K}{\omega}\sin\theta_1) \\ -\frac{2\xi VA}{\omega}e^{-ikx/2}\sin\theta_1 \end{pmatrix}$	$\begin{pmatrix} \alpha e^{-2i\xi^2 x} \\ \beta e^{2i\xi^2 x} \end{pmatrix}$
$\psi(x,\zeta)$	$\begin{pmatrix} \beta e^{-2i\xi^2 x} \\ \alpha e^{2i\xi^2 x} \end{pmatrix}$	$\begin{pmatrix} \frac{2\xi VA}{\omega}e^{ikx/2}\sin\theta_2 \\ Ae^{-ikx/2}(\cos\theta_2 + i\frac{K}{\omega}\sin\theta_2) \end{pmatrix}$	$\begin{pmatrix} 0 \\ e^{2i\xi^2 x} \end{pmatrix}$

$$A = e^{i(k/2+2\xi^2 x)}, \qquad K = k/2 + 2\xi^2 - V^2, \qquad \omega = \sqrt{K^2 + 4\xi^2 V^2},$$
$$\theta_1 = \omega(x+L), \qquad \theta_2 = \omega(x-L),$$
$$\alpha = A(\cos 2\omega L - i\frac{K}{\omega}\sin 2\omega L), \qquad \beta = -\frac{2\xi V}{\omega}\sin 2\omega L$$

Table 4.1 The Jost functions for the initial condition (4.14c).

Function	$x \to +\infty$ $(z \to 0)$	$x \to -\infty$ $(z \to 1)$
$\chi_1^{(1)}$	$e^{iA^2}e^{-2i\zeta^2 x}$	$\dfrac{\Gamma(\frac{1}{2}+2i\lambda)\Gamma(\frac{1}{2}+2i\lambda)}{\Gamma(\frac{1}{2}+2i\lambda+2\zeta A)\Gamma(\frac{1}{2}+2i\lambda-2\zeta A)} e^{-iA^2}e^{-2i\zeta^2 x}$
$\chi_1^{(2)}$	0	$\dfrac{\Gamma(\frac{3}{2}-2i\lambda)\Gamma(\frac{1}{2}+2i\lambda)}{\Gamma(1-2\zeta A)\Gamma(1+2\zeta A)} e^{-iA^2}e^{-2i\zeta^2 x}$
$\chi_2^{(1)}$	$e^{iA^2}e^{2i\zeta^2 x}$	$\dfrac{\Gamma(\frac{1}{2}-2i\lambda)\Gamma(\frac{1}{2}-2i\lambda)}{\Gamma(\frac{1}{2}-2i\lambda+2\zeta A)\Gamma(\frac{1}{2}-2i\lambda-2\zeta A)} e^{-iA^2}e^{2i\zeta^2 x}$
$\chi_2^{(2)}$	0	$\dfrac{\Gamma(\frac{3}{2}+2i\lambda)\Gamma(\frac{1}{2}-2i\lambda)}{\Gamma(1-2\zeta A)\Gamma(1+2\zeta A)} e^{-iA^2}e^{2i\zeta^2 x}$

Table B.1 Asymptotic forms of $\chi_1^{(1)}$, $\chi_1^{(2)}$, $\chi_2^{(1)}$ and $\chi_2^{(2)}$.

Figures

Figure Captions

Fig.3.1 The integral contour in calculating (3.14). The direction is anti-clockwise.

Fig.4.1 Zeros of transmission amplitude for the initial condition (4.14a). There appear even number of zeros.

Fig.5.1 A mecanical model of the USG equation. Pendulii are connected with rubber band.

Fig.5.2 Dispersion relation (5.24) near the critical wave number k_c when $V^2 > 2(1+\kappa)$. Solid line indecates the real part of ω, and dashed line the absolute value of the imaginary part (linear growth rate).

Fig.5.3 A configuration of the two dimensional Rayleigh-Taylor instability problem.

Fig.5.4 Dispersion relation $\omega^2 = (Tk^3/\rho - gk)\tanh kh$. The critical wave number is denoted by k_c.

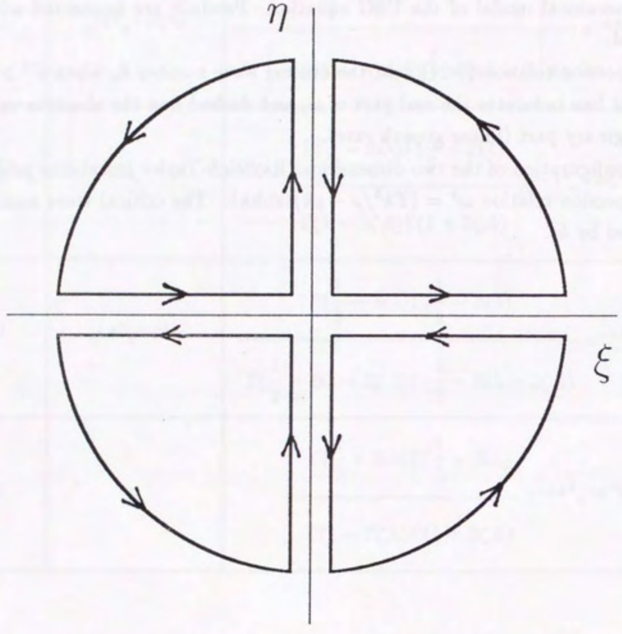

Fig.3.1

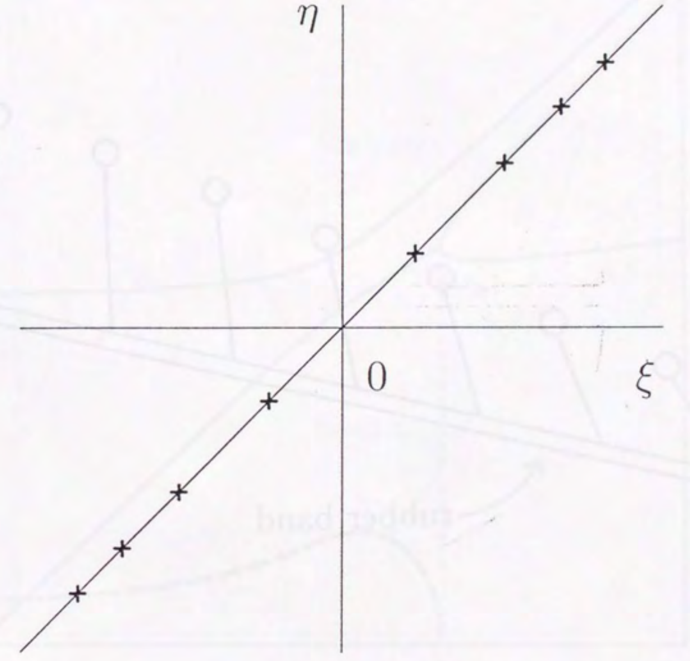

Fig.4.1

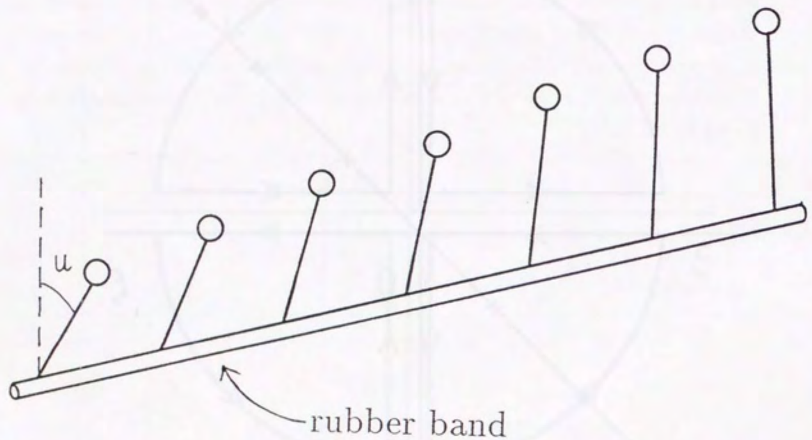

Fig.5.1

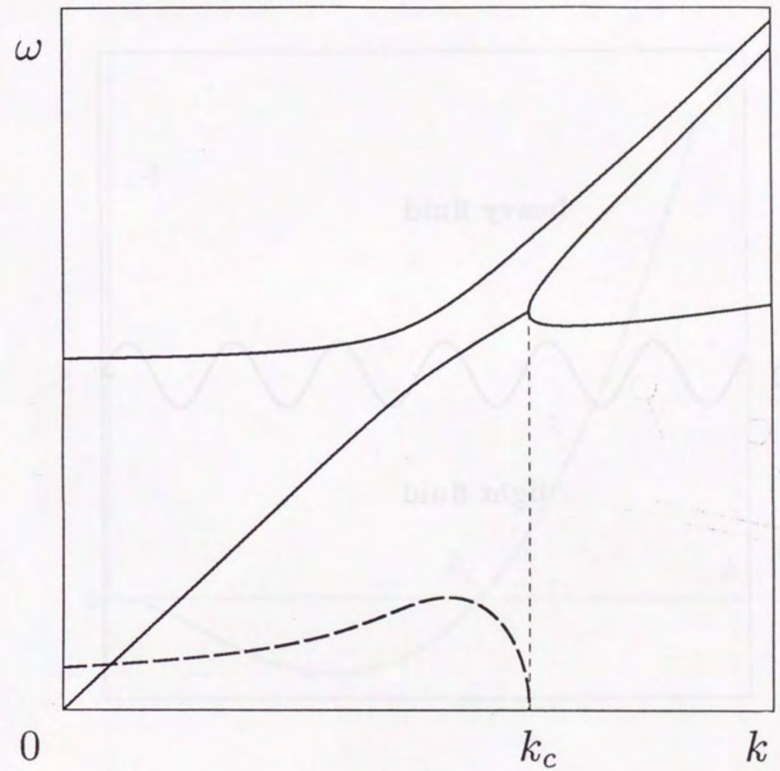

Fig.5.2

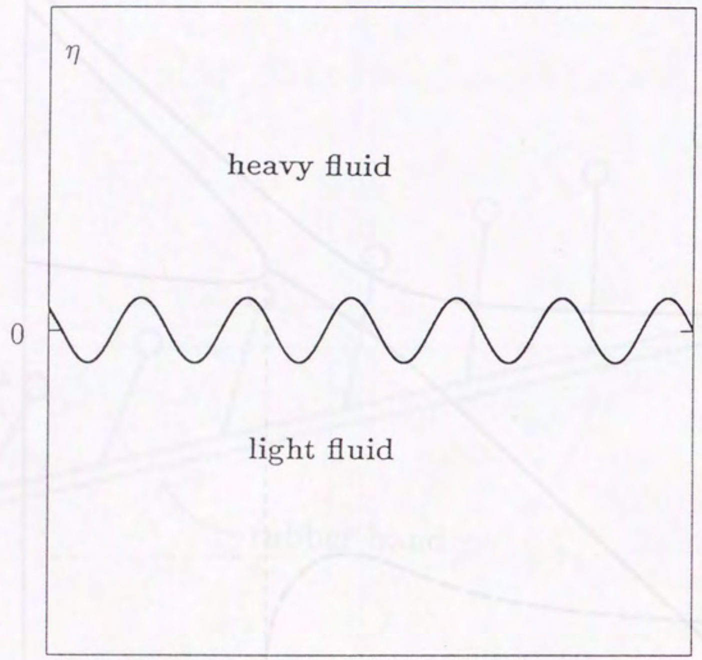

Fig.5.3

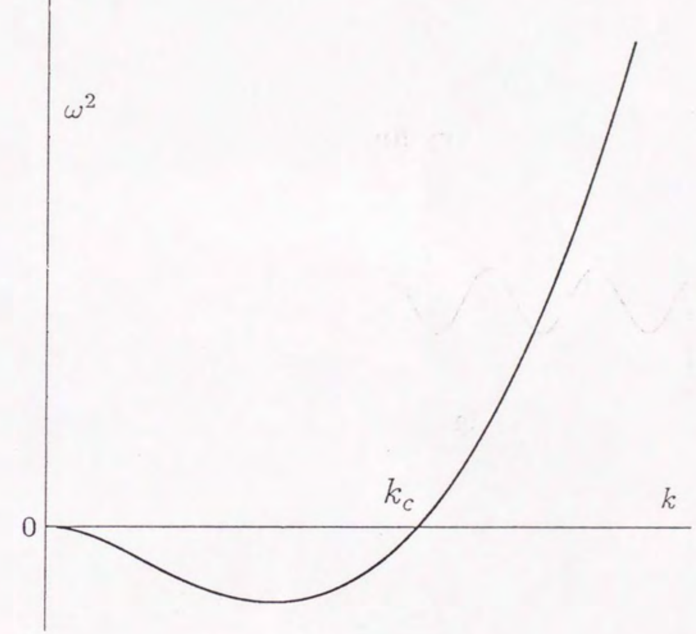

Fig.5.4