Multivariable Invariants of Colored Links and Related Solvable Models in Statistical Mechanics

色付き絡み目の多変数不変量および関連する統計力学の可解模型

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Thesis:

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March 1992

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学位論文

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１９９２年３月

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Acknowledgement

The author would like to thank Prof. M. Wadati for critical reading of this manuscript and continuous encouragement. He is grateful to Prof. Y. Akutsu, Dr. A. Fujii and Dr. T. Ohtsuki for helpful discussions and fruitful collaborations. He is thankful to graduate students of Prof. M. Wadati for discussions on their studies of nonlinear physics and statistical physics. He would also like to thank Prof. K. Murasugi for continuous encouragement.

Abstract

We formulate two families of exactly solvable models, colored vertex models and $gl(M|N)$ vertex models. In association with these solvable models, we discuss construction of two families of topological invariants, link polynomials related to representations of the the Lie superalgebra $gl(M|N)$, and multivariable invariants of colored oriented links and trivalent colored graphs generalizing the multivariable Alexander polynomial. We derive representations of the braid group and state models for the (one-variable) Alexander polynomial from the $gl(M|N)$ vertex and IRF (Interaction Round a Face) models. We discuss fusion models of $gl(M|N)$ vertex and IRF models, and give link invariants related to the fusion models. From the colored vertex models we construct the colored link invariants through tangle diagrams. We derive the braid matrices for the colored link invariants from roots of unity representations of $U_q(sl(2))$. Using the Clebsch-Gordan coefficients of the roots of unity representation of $U_q(sl(2))$, we formulate exactly solvable vertex models and IRF models related to the colored link invariants. We construct invariants of framed graphs (ribbon graphs) associated with the colored link polynomials.

Contents

Part I

$gl(m|n)$ Solvable Models and Link Polynomials

1 Introduction

It is known that classification of knots and links is a difficult and unsolved problem in topology. There has been given no explicit and systematic method to determine whether any given two knots are equivalent or not. Topological invariants such as link polynomials are important in classification of knots and links. We call a Laurent polynomial link polynomial if it is a topological invariant of knots and links.

In many physical systems, we encounter knotted configurations of string-like objects. For example, we meet with those of polymers, vortex lines, and dislocations, etc.. The effects of entanglements are important in the thermodynamic properties of the systems, because they can severely restrict the available degrees of freedom in the configuration spaces. In order to study knotted configurations, we can use topological invariants of knots and links, in particular, link polynomials. In statistical physics the Alexander polynomial has been applied to analyses of entanglement problems of polymers. For example, knotting probability of self-avoiding random walks has been estimated through computer simulation using the Alexander polynomial. [98]

It has been shown by many researchers that topological invariants of knots and links and representations of the braid group are closely related to various interesting branches of mathematics and theoretical physics such as quantum groups, C^* algebra, conformal field theories, topological quantum field theory, quantum gravity and exactly solvable models. [100,94,60,41,79,69,99,86,6] From the viewpoint of exactly solvable lattice models in statistical mechanics we can show that the partition functions of the models correspond to link polynomials. We call solution of the Yang-Baxter relation exactly solvable model.

After the discovery of the Jones polynomial, [51] link polynomials have been studied in many different fields. From the solutions of the Yang-Baxter relation link polynomials and representations of the braid group were derived. [3,4] From the vertex models related to simple Lie algebras the representations of the braid group corresponding to the HOMFLY polynomial [39,78] and the Kauffman polynomial [56] were obtained. [95] The representations of the braid group were systematically discussed from the quantum groups. [80,59] From topological quantum field theory invariants of three dimensional manifolds were constructed. [99]

The Alexander polynomial vanishes for disconnected links. We note that the Jones polynomial does not have this property. The representations of the braid group related to the finite dimensional representations of the simple Lie algebras do not lead to link polynomials that have the vanishing property. Thus we can not derive the Alexander polynomial from the representations of the braid group related to the finite dimensional representations of the simple Lie algebras.

In this paper we discuss two families of link invariants, multivariable polynomial invariants of colored oriented links and (one-variable) link polynomials related to representations of the the Lie superalgebra $gl(m|n)$. The former family gives generalization of the multivariable Alexander polynomial. The latter gives state models for HOMFLY polynomial including the case of the Alexander polynomial, and extensions of them into those related to higher representations of $gl(M|N)$. Both the two families of link invariants are generalizations of the (one-variable) Alexander polynomial. The braid matrices for the two families of invariants can be considered as generalizations of the R matrix associated with the algebra $gl(1|1)$. The multivariable invariants and the link polynomials associated with representations of $gl(m|m)$ also have the vanishing property. We generalize the invariants of colored links into multivariable invariants of colored ribbon graphs.

This paper consists of three parts. (1) In part I we show the following. We derive representations of the braid group and state models for the Alexander polynomial from the $gl(m|n)$ vertex and IRF models. We discuss fusion models of $gl(m|n)$ vertex and IRF models associated with representations of the Lie superalgebra $gl(m|n)$, and formulate link invariants related to the fusion models. (2) In part II we discuss construction of multivariable invariants of colored links which give generalizations of the multivariable Alexander polynomial. The colored link invariants are associated with the colored braid matrices and the colored vertex models. We introduce the colored vertex models. (3) In part III we show that the colored braid matrices are derived from the universal R matrices of $U_q(sl(2))$ at q roots of unity. From roots of unity representations of $U_q(sl(2))$ we construct the colored vertex models and IRF models corresponding to the colored link invariants. Through the Clebsch-Gordan coefficients of the roots of unity representations we construct ribbon graph

invariants related to the colored link polynomials.

2 Exactly solvable models and the braid group

2.1 Vertex and IRF models in statistical mechanics

We introduce two types of solvable models, vertex models and IRF models, in two-dimensional statistical mechanics. [12] Models whose Boltzmann weights satisfy the Yang-Baxter relation are called to be solvable. The Yang-Baxter relation gives a sufficient condition for the fact that the transfer matrices of the model mutually commute. In this sense it gives the solvability of the model. There are various methods to calculate physical quantities (free energy, one-point function, etc.) for the solvable models, such as Bethe ansatz method, corner transfer method, inversion method, etc.. [12,6]

Let us introduce vertex models. The Boltzmann weight (statistical weight) $X^{ab}_{cd}(u)$ of a vertex model defined is for a configuration $\{a,b,c,d\}$.

Fig.2.1.1

Here the parameter u is called spectral parameter which controls the anisotropy (and strength) of the interactions for the model. The Yang-Baxter relation for vertex models is given by

$$\begin{aligned} &\sum_{c_1c_2c_3} X^{a_1a_2}_{c_2c_1}(u)X^{c_1a_3}_{c_3b_1}(u+v)X^{c_2c_3}_{b_3b_2}(v) \\ &= \sum_{c_1c_2c_3} X^{a_2a_3}_{c_3c_2}(v)X^{a_1c_3}_{b_3c_1}(u+v)X^{c_1c_2}_{b_2b_1}(u). \end{aligned} \tag{2.1.1}$$

Let us define an operator $X(u) \in End(V \otimes V)$ for vertex models, which is a construction unit ('building block') of the diagonal-to-diagonal transfer matrix. [12]

$$X(u) = \sum_{abcd} X^{ab}_{cd}(u) e_{ac} \otimes e_{bd}, \tag{2.1.2}$$

where

$$(e_{ab})_{jk} = \delta_{ja}\delta_{kb}. \tag{2.1.3}$$

Let the symbol I denote the identity matrix of the size N. We define $X_i(u) \in End(\otimes^n V)$ by

$$X_i(u) = I^{(1)} \otimes \cdots \otimes^{(i-1)} X(u) \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \tag{2.1.4}$$

Equivalently we have

$$X_i(u) = \sum_{abcd} X^{ab}_{cd}(u) I^{(1)} \otimes \cdots \otimes e^{(i)}_{ac} \otimes e^{(i+1)}_{bd}$$
$$\otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \qquad (2.1.5)$$

We recall that the operators $\{X_i(u)\}$ act on the tensor product space $V^{(1)} \otimes V^{(2)} \otimes \cdots \otimes V^{(n)}$. In terms of the operators $X_i(u)$ the Yang-Baxter relation is written as follows.

$$\begin{aligned} X_i(u)X_{i+1}(u+v)X_i(v) &= X_{i+1}(v)X_i(u+v)X_{i+1}(u), \\ X_i(u)X_j(v) &= X_j(v)X_i(u), \qquad |i-j| \geq 2. \end{aligned} \qquad (2.1.6)$$

It is remarked that the operator $X(u)$ is equivalent to the R matrix $\check{R}(u)$ in the contexts of the quantum groups and the quantum inverse scattering method. Usually we have $X(u) = \check{R}(u) = \pi R(u)$, where π is the permutation operator such that $\pi(e_1 \otimes e_2) = e_2 \otimes e1$.

Almost all known solvable vertex models satisfy the following relations, which we call basic relations.

1) standard initial condition

$$X^{ab}_{cd}(u=0) = \rho(0) \times \delta_{ac}\delta_{bd}. \qquad (2.1.7)$$

Here $\rho(0)$ is a constant.

2) first inversion relation (unitarity condition)

$$\sum_{mp} X^{ab}_{mp}(u) X^{mp}_{cd}(-u) = C(u)\delta_{ac}\delta_{bd}, \qquad (2.1.8)$$

where the function $C(u)$ is related to the normalization of the Boltzmann weights and often can be written as $C(u) = \rho(u)\rho(-u)$.

3) Second inversion relation

$$\sum_{mp} X^{am}_{cp}(\lambda+u) X^{dp}_{bm}(\lambda-u) h(m)/h(c) = C(u)\delta_{ac}\delta_{bd}. \qquad (2.1.9)$$

We shall see the basic relations and the Yang-Baxter relation are related to the local moves on link diagrams, the Reidemeister moves in knot theory. We can construct invariants of links in terms of oriented tangles.

Let us introduce IRF (Interaction Round a Face) models. The Boltzmann weight of an IRF model $w(a,b,c,d;u)$ is defined on a configuration $\{a,b,c,d\}$ round a face.

Fig.2.1.2

IRF models have constraints on the configurations. The symbol $b \sim a$ means that the "spin" b is admissible to the "spin" a under the constraint of the model. If the conditions $b \sim a, a \sim d, b \sim c$ and $c \sim d$ are all satisfied, then the configuration $\{a,b,c,d\}$ in Fig. 2.1.2 is called to be allowed. The Boltzmann weights for not-allowed configurations are set to be 0. For IRF models the Yang-Baxter relation is written as

$$\begin{aligned} &\sum_c w(b,d,c,a;u)w(d,e,f,c;u+v)w(c,f,g,a;v) \\ = &\sum_c w(d,e,c,b;v)w(b,c,g,a;u+v)w(c,e,f,g;u) \end{aligned} \qquad (2.1.10)$$

The Yang-Baxter relation for vertex models and IRF models is written in the same form as (2.1.6) in terms of the Yang-Baxter operators for IRF models. The transfer matrix of IRF model is defined on the Hilbert space [5] which consists of admissible sequences of local states: $\{\ell_i; \ell_{i+1} \sim \ell_i, (i = 0, \cdots n-1)\}$. [12,6] We call the admissible sequence and the Hilbert space as path and path space, respectively. We introduce Yang-Baxter operator [12,6] acting on the i-th site of the path space in the following:

$$\{X_i\}^{p_0,\cdots p_n}_{k_0\cdots k_n}(u) = \prod_{j=0}^{i-1} \delta^{p_j}_{k_j} \cdot w(k_i, p_{i+1}, p_i, p_{i-1}; u) \cdot \prod_{j=i+1}^{n} \delta^{p_j}_{k_j}. \qquad (2.1.11)$$

We recall that the Yang-Baxter operator $X_i(u)$ is a constuction unit (building block) of transfer matrix of the IRF model.

2.2 Braids and closed braids

We introduce braids and the braid group. [15] The braid group B_n is defined by a set of generators, $b_1, \cdots, b_{n-1}$ which satisfy

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \\ b_i b_j &= b_j b_i, \qquad |i-j| \geq 2. \end{aligned} \qquad (2.2.1)$$

It is known that any oriented link can be expressed by a closed braid. The equivalent braids expressing the same link are mutually transformed by a finite sequence of two types of operations, Markov moves I and II. The Markov trace $\phi(\cdot)$ is a linear functional on the representation of the braid group which have the following properties (the Markov properties):

$$I. \qquad \phi(AB) = \phi(BA), \qquad A, B \epsilon B_n, \qquad (2.2.2)$$

$$
\begin{aligned}
II. \quad \phi(Ab_n) &= \tau\phi(A), \\
\phi(Ab_n^{-1}) &= \bar{\tau}\phi(A), \\
& \quad A\epsilon B_n, \quad b_n \epsilon B_{n+1},
\end{aligned}
\tag{2.2.3}
$$

where

$$
\tau = \phi(b_i), \quad \bar{\tau} = \phi(b_i^{-1}), \text{ for all } i. \tag{2.2.4}
$$

From the Markov trace we obtain a link polynomial $\alpha(\cdot)$ as [6]

$$
\alpha(A) = (\tau\bar{\tau})^{-\frac{n-1}{2}} (\frac{\bar{\tau}}{\tau})^{\frac{1}{2}e(A)} \phi(A), A\epsilon B_n \tag{2.2.5}
$$

Here $e(A)$ is the exponent sum of b_i's in the braid A, which is equivalent to the writhe of the link diagram. For instance, if $A = b_1^4 b_2^{-2} b_3 b_1^{-1}$, then $e(A) = 4 - 2 + 1 - 1 = 2$.

2.3 Derivation of braid matrix

The braid operator $G_i(+)$, the inverse operator $G_i(-)$ and the identity operator I are given by

$$
G_i(\pm) = \lim_{u \to \infty} X_i(\pm u)/\rho(\pm u), \tag{2.3.1}
$$

$$
I = X_i(0). \tag{2.3.2}
$$

Here the function $\rho(u)$ is related to the normalization of the Boltzmann weights of the model. The function is appeared in the inversion relations discussed in the previous subsection. The limit $u \to \infty$ (more precisely, an infinite limit in a certain direction in the complex u-plane) requires that the Boltzmann weights be parametrized by hyperbolic (trigonometric) functions.

We define the matrix elements of the braid operator by the following.

$$
G_{cd}^{ab}(\pm) = \lim_{u \to \infty} X_{cd}^{ab}(\pm u)/\rho(\pm u). \tag{2.3.3}
$$

The braid operator constructed from the operator $X_i(u)$ is given by

$$
G_i(\pm) = \sum_{abcd} G_{cd}^{ab}(\pm) I^{(1)} \otimes \cdots \otimes e_{ac}^{(i)} \otimes e_{bd}^{(i+1)} \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \tag{2.3.4}
$$

In the operator form we have

$$
G_i(\pm) = I \otimes \cdots \otimes G(\pm) \otimes \cdots \otimes I, i = 1, \cdots, n-1. \tag{2.3.5}
$$

The operators $\{G_i(\pm)\}$ give representation of the braid group : $B_n \to End(\otimes V^n)$, where the space V is given by $V = \mathbf{C}^d$ and d is the dimension.

2.4 Construction of the Markov trace

We discuss construction of the Markov trace on the representations of the braid group derived from the solvable models. The Markov trace takes the following form [6]

$$
\begin{aligned}
\phi(A) &= \frac{\mathrm{Tr}(H(n)A)}{\mathrm{Tr}(H(n))}, \; A\epsilon B_n, \\
& \quad H(n) = h^{(1)} \otimes \cdots \otimes h^{(n)}.
\end{aligned}
\tag{2.4.1}
$$

The quantity $h(p)$ is nothing but the crossing multiplier of the model in the second inversion relation. We present sufficient conditions for the Markov properties explicitly. We can show that the trace $\phi(\cdot)$ defined in is the Markov trace by proving for the Markov property I the "charge conservation" property and for the Markov property II the following conditions:

$$
\Sigma_b G_{ab}^{ab}(\pm) h(b) = \chi(\pm) \quad \text{(independent of } a). \tag{2.4.2}
$$

The τ-factors are related to $\chi(\pm)$ as $\bar{\tau}/\tau = \chi(-)/\chi(+)$.

Most of the solvable models from which we can derive braid matrices and the Markov trace have the extended Markov trace property, which is an extension of the Markov trace property with finite spectral parameter.

$$
\sum_b X_{ab}^{ab}(u) h(b) = h(u;\eta)\rho(u) \quad \text{(independent of } a), \tag{2.4.3}
$$

where we call the function $h(u;\eta)$ characteristic function.

We note that the form of the Markov trace (2.4.1) is discussed in references [3,95], and is consistent with the representations of the quantum groups. [80,82] Our approach to the Markov trace is based on the basic relations of solvable models, in particular the crossing multiplier in the second inversion relation. We can apply the approach to the elliptic vertex models such as the 8 vertex model and the Z_n Baxter model (Belavin model).

3 $gl(M|N)$ vertex model

3.1 Lie superalgebra $gl(M|N)$

We briefly introduce the notion of the Lie superalgebras. [54] A Lie superalgebra is a Z_2-graded algebra over $\mathbf{C}$. We define parity $p(A)$ of an element A by

$$p(A) = \begin{cases} 0, & \text{if } A \text{ is bosonic}, \\ 1, & \text{if } A \text{ is fermionic}. \end{cases} \tag{3.1.1}$$

The bracket given by

$$[A, B] = AB - (-1)^{p(A)p(B)} BA \tag{3.1.2}$$

satisfies the super-Jacobi identity:

$$\begin{aligned} (-1)^{p(A)p(C)}[A,[B,C]] &+ (-1)^{p(B)p(A)}[B,[C,A]] \\ &+ (-1)^{p(C)p(B)}[C,[A,B]] = 0. \end{aligned} \tag{3.1.3}$$

We define tensor product $A \otimes B$ with the induced grading:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{p(b_1)p(a_2)} a_1 a_2 \otimes b_1 b_2, \; a_i \in A, b_i \in B. \tag{3.1.4}$$

Let us introduce the Lie superalgebra $gl(M|N)$. The generators $\{E^a_b\}$ of $gl(M|N)$ satisfy the defining relations

$$[E^a_b, E^c_d] = \delta_{cb} E^a_d - (-1)^{(p(a)+p(b))(p(c)+p(d))} \delta_{ad} E^c_b. \tag{3.1.5}$$

The parity of the generator E^a_b is given by $p(E^a_b) = p(a) + p(b)$. The Cartan subalgebra of the $gl(M|N)$ is given by $gl(M) \oplus gl(N)$. We choose the generators $E^a_a, (1 \le a \le M+N)$ as the basis of the Cartan algebra. Every finite dimensional irreducible $gl(M|N)$ modules are uniquely characterized by their highest weights Λ. [54] We can write the highest weight as

$$\Lambda = \sum_{j=1}^{M+N} \lambda_j \epsilon_j, \tag{3.1.6}$$

where $(\epsilon_i, \epsilon_j) = \sigma_j \delta_{ij}$.

The Lie superalgebra has even roots and odd roots. [54] Hereafter in this subsection we assume for simplicity that $p(a) = 0$ if $1 \le a \le M$ and $p(a) = 1$ if

$M + 1 \le a \le M + N$. The set of even positive roots Φ^+_0 and the set of odd positive roots Φ^+_0 are given by the following.

$$\begin{aligned} \Phi^+_0 &= \{\epsilon_i - \epsilon_j | 1 \le i < j \le M \quad \text{or} \quad M+1 \le i < j \le M+N\}, \\ \Phi^-_0 &= \{\epsilon_i - \epsilon_j | 1 \le i \le M, M+1 \le j \le M+N\}. \end{aligned} \tag{3.1.7}$$

The symbols ρ_0 and ρ_1 denote half the sum of even positive roots and odd positive roots, respectively.

$$\begin{aligned} \rho_0 &= \frac{1}{2}\sum_{j=1}^{m}(m+1-2j)\epsilon_j + \frac{1}{2}\sum_{j=m+1}^{M+N}(2M+N+1-2j)\epsilon_j, \\ \rho_1 &= \frac{n}{2}\sum_{j=1}^{m}\epsilon_j - \frac{m}{2}\sum_{j=m+1}^{M+N}\epsilon_j. \end{aligned} \tag{3.1.8}$$

We define "half the sum of positive roots" ρ by

$$\rho = \rho_0 - \rho_1. \tag{3.1.9}$$

3.2 Vertex models associated with $gl(M|N)$

Let us introduce a family of solvable vertex models associated with $gl(M|N)$. [88, 24,27] The models are given in the case IB in Ref.[88]. We introduce a set of signs $\{\sigma_i\}$

$$\sigma_a = (-1)^{p(a)}, \quad \text{for } a = 1, \cdots, M+N. \tag{3.2.1}$$

The number of positive (resp. negative) signs is given by M (resp. N). Here we do not assume that $p(a) = 0$ if $1 \le a \le M$ and $p(a) = 1$ if $M+1 \le a \le M+N$.

For any set of signs $\{\sigma_i\}$ we have a solution of the Yang-Baxter relation. The Boltzmann weights are given as follows:

$$\begin{aligned} X^{aa}_{aa}(u) &= \sinh(\eta - \sigma_a u)/\sinh\eta, \\ X^{ab}_{ba}(u) &= \sinh u/\sinh\eta, \qquad (a \ne b) \\ X^{ab}_{ab}(u) &= \exp(sign(a-b)u) \qquad (a \ne b). \end{aligned} \tag{3.2.2}$$

Here η is a parameter and the edge variables a and b take values $1, 2, \cdots, M+N$. The models have the charge conservation property: $w(a,b,c,d;u) = 0$, unless $\vec{a}+\vec{b} = \vec{c}+\vec{d}$. Here $\vec{a}$ represents 'charge' of the state a, which is vector-valued in general. The Boltzmann weights satisfy the reflection symmetry $w(a,b,c,d;u) = w(c,d,b,a;u)$. They also satisfy the standard initial condition and the inversion relation.

1) standard initial condition

$$X_{cd}^{ab}(u=0) = \delta_{ac}\delta_{bd}. \tag{3.2.3}$$

2) first inversion relation (unitarity condition)

$$\sum_{mp} X_{mp}^{ab}(u)X_{cd}^{mp}(-u) = \rho(u)\rho(-u)\delta_{ac}\delta_{bd}, \tag{3.2.4}$$

where $\rho(u) = \sinh(\eta - u)/\sinh\eta$.

3) Second inversion relation

$$\sum_{mp} X_{cp}^{am}(\lambda+u)X_{bm}^{dp}(\lambda-u)h(m)/h(c) = C(u)\delta_{ac}\delta_{bd}, \tag{3.2.5}$$

where $h(a)$ is given by

$$h(j) = \sigma_j \exp\{\eta(\sum_{k=1}^{j-1} 2\sigma_k + \sigma_j - m + n)\}, \text{ for } j = 1\cdots M+N. \tag{3.2.6}$$

The model for the case $(\sigma_1, \sigma_2) = (1, -1)$ (and (-1,1)) corresponds to the free fermion 6-vertex model. [91] We note that by changing the sign of the spectral parameter as $u \to -u$ and changing signs in the Boltzmann weights as $w(a,b,b,a;u) \to -w(a,b,b,a;u)$ $(a \neq b)$, the models for (-1,-1) and (-1, 1) are transformed into those for (1,1) and (1,-1), respectively. For $M + N > 2$, the two cases: $\sigma_i = 1$ (for all i) and $\sigma_i = -1$ (for all i) are the M-state vertex models associated with sl(M). [88,17]

We introduce graded permutation operator π_i as

$$\begin{aligned}\pi_i(u) = \sum_{abcd} \pi_{cd}^{ab}(u) I^{(1)} \otimes \cdots \otimes e_{ac}^{(i)} \otimes e_{bd}^{(i+1)} \\ \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)},\end{aligned} \tag{3.2.7}$$

where

$$\pi_{cd}^{ab} = (-1)^{p(a)p(b)}\delta_{ad}\delta_{bc}. \tag{3.2.8}$$

The R-matrix $R(u)$ in the context of the quantum inverse scattering method is related to the Yang-Baxter operator $X(u)$ as

$$R_{cd}^{ab}(u) = X_{dc}^{ab}(u)\pi_{cd}^{dc}. \tag{3.2.9}$$

The matrix elements $R_{cd}^{ab}(u)$ satisfy the graded Yang-Baxter relation

$$\begin{aligned} & R_{cb}^{rq}(u)R_{ka}^{cp}(u+v)R_{ji}^{ba}(v)(-1)^{p(b)p(c)+p(a)p(k)+p(i)p(j)} \\ = \; & R_{ba}^{qp}(v)R_{ci}^{ra}(u+v)R_{kj}^{cb}(u)(-1)^{p(a)p(b)+p(i)p(c)+p(j)p(k)}. \end{aligned} \tag{3.2.10}$$

By rescaling the variables as $u \to \epsilon u$, $\eta \to \epsilon\eta$ and taking the limit: $\epsilon \to 0$, we derive a rational solution of the Yang-Baxter relation from the vertex model The Yang-Baxter operator $\tilde{X}_i(u)$ for the rational solution has the form

$$\tilde{X}_i(u) = I - \frac{u}{\eta}\pi_i \tag{3.2.11}$$

Here π_i is the graded permutation operator The rational solution for the R-matrix satisfies the graded Yang-Baxter relation (3.2.10).

3.3 $gl(M|N)$ Representations of the Hecke algebra

The A_n type Hecke algebra $H_n(q)$ is generated by the generators $\{1, g_1, \cdots, g_{n-1}\}$ with the defining relations given in the following. [51]

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \\ g_i g_i &= (1-q^2)g_i + q^2 I, \\ g_i g_j &= g_j g_i \text{ for } |i-j| \geq 2. \end{aligned} \tag{3.3.1}$$

If we define operators A_i $(i = 1, \cdots n-1)$ by $g_i = I - qA_i$, then the operators satisfy the relations:

$$\begin{aligned} A_i A_{i+1} A_i - A_i &= A_{i+1} A_i A_{i+1} - A_{i+1}, \\ A_i A_i &= (q + q^{-1})A_i, \\ A_i A_j &= A_j A_i \text{ for } |i-j| \geq 2. \end{aligned} \tag{3.3.2}$$

Here we have introduced a parameter t by $t = \exp 2\eta$. We note that, for the case gl(2|0), $\{A_i\}$ satisfy the Temperley-Lieb algebra. [93]

We construct representations of the Hecke algebra from the vertex model related to the Lie super algebra $gl(M|N)$. [27] We apply the formula (2.3.1) to the vertex model (3.2.2), then we have the following braid matrices: [24]

$$\begin{aligned} G_{aa}^{aa}(+) &= \begin{cases} 1 & \text{for } \sigma_a = 1, \\ -t & \text{for } \sigma_a = -1, \end{cases} \\ G_{ab}^{ab}(+) &= \begin{cases} 0 & \text{for } a < b, \\ 1-t & \text{for } a > b, \end{cases} \\ G_{ba}^{ab}(+) &= -t^{1/2}, \text{ for } a \neq b. \end{aligned} \tag{3.3.3}$$

Here a variable t is defined by $t = q^2 = \exp(2\eta)$. We call the representation $gl(M|N)$ representation of the braid group.

The braid matrices of the $gl(M|N)$ representations have only two eigenvalues 1 and $-t$, and therefore they satisfy the defining relations of the Hecke algebra. Thus we have seen that the Hecke algebra also appears in the braid matrices associated with the Lie superalgebra $gl(M|N)$.

By taking the limit $\eta \to 0$ we derive the graded permutation operator from the representation of the braid group Thus the braid operator is a q-analog of the graded permutation operator.

Recently the $gl(M|N)$ representations of the Hecke algebra have been studied from the viewpoints of integrable spin chains [68] and representation theory of specht modules [34]. In these contexts the following proposition is useful for the study of the $gl(M|N)$ representation.

Proposition 3.1 *[68] The $gl(M|N)$ representation contains every irreducible representation of $H_n(q)$ with multiplicity at least one, except those associated to partition shapes containing as a subdiagram the rectangular diagram of height $M+1$ and width $N+1$.*

3.4 Fusion of the vertex model associated with $gl(M|N)$

We discuss construction of fusion models of the $gl(M|N)$ vertex model, which are vertex models associated with higher dimensional representations of $gl(M|N)$. By constructing the "affinization" of the finite dimensional representations of $U_q(gl(M|N))$ we can construct the fusion models of the $gl(M|N)$ vertex model. In this approach we follow the cases of the simple Lie algebra in ref. [46]. As far as the finite dimensional representations are concerned, the "fusion method" through the generalized Young operators (primitive idempotents) gives another equivalent approach to the fusion models. [27,34] For both the "affinization" and the fusion approaches, construction of fusion model is essentially related to the properties of the generalized Young operators of the Hecke algebra. For $gl(M|N)$ fusion models we can show the following proposition.

Proposition 3.2 *[34] The $gl(M|N)$ fusion model is trivial if the shape of the Young diagram for the projection operators of the model contains as a subdiagram the rectangular diagram of height $M+1$ and width $N+1$.*

Let us discuss construction of composite models by the fusion method in terms of the exactly solvable models and the Hecke algebra. [32,35] These models are special cases of generalized inhomogeneous models consistent with the Z-invariance.

[13,64,62] In the following discussion we use only one property that the Yang-Baxter operator $X_i(u)$ for the vertex model has two eigenvalues, more precisely, has quadratic minimal polynomial. From this property the Yang-Baxter operator becomes projection operator when the spectral parameter $u = \pm\eta$ (see also [35]).

$$A_i = X_i(u=\eta). \tag{3.4.1}$$

Using the operator A_i the Yang-Baxter operator $X_i(u)$ for the vertex model is written as

$$X_i(u) = \rho(u)\,(I + f(u)A_i)\,, \tag{3.4.2}$$

where

$$f(u) = \frac{\sinh u}{\sinh(\eta - u)}, \tag{3.4.3}$$

$$\rho(u) = \frac{\sinh(\eta - u)}{\sinh \eta}. \tag{3.4.4}$$

We consider the generators of the Hecke algebra as generalized Young operators. [44,32,34] If we define

$$P_i^{[S]} = P_i^{[sym,2]} = X_i(u=-\eta)/(t^{1/2}+t^{-1/2}), \tag{3.4.5}$$

$$P_i^{[A]} = P_i^{[anti,2]} = X_i(u=\eta)/(t^{1/2}+t^{-1/2}), \tag{3.4.6}$$

then $P_i^{[sym,2]}$ and $P_i^{[anti,2]}$ are the projectors with one row and one column, respectively. Hereafter we write $P_i^{[sym,k]}$ as $P^{[k]}$, where $\mu = [sym,k]$ corresponds to the weight with the Dynkin coefficients $\mu = (k, 0, \cdots, 0)$. The projectors $P^{[k]}$ corresponding to the Young diagram with one row (k boxes) are recursively given by

$$P_i^{[k]} = P_i^{[k-1]} X_{i+k-2}(-(k-1)\eta) P_i^{[k-1]}. \tag{3.4.7}$$

Note that k is the number of strings in a composite string. The identity $I^{(k)}$ is given by

$$I^{[k]} = P_1^{[k]} P_{k+1}^{[k]} \cdots P_{(n-1)k+1}^{[k]}. \tag{3.4.8}$$

Using the projectors we construct composite Yang-Baxter operators $\{Y_i^{[k]}(u); \text{for } i = 1, \cdots, n\}$ as [32,35]

$$Y_i^{[k]}(u) = P_{(i-1)k+1}^{[k]} P_{ik+1}^{[k]} \left(\prod_{j=1}^{k} \hat{X}_i^{(j)}(u) \right) P_{(i-1)k+1}^{[k]} P_{ik+1}^{[k]}, \tag{3.4.9}$$

where

$$\hat{X}_i^{(j)}(u) = \prod_{m=1}^{k} X_{ik+m-j}(u - (k-j-m+1)\eta). \tag{3.4.10}$$

For example in the case of 2 strings ($k = 2$) it is given by

$$\begin{aligned} Y_i^{[2]}(u) &= P^{[2]}_{2i-1} P^{[2]}_{2i+1} X_{2i}(u-\eta) X_{2i-1}(u) X_{2i+1}(u) \\ &\quad \times X_{2i}(u+\eta) P^{[2]}_{2i-1} P^{[2]}_{2i+1} \\ &= X_{2i-1}(-\eta) X_{2i+1}(-\eta) X_{2i}(u-\eta) X_{2i-1}(u) \\ &\quad \times X_{2i+1}(u) X_{2i}(u+\eta). \end{aligned} \tag{3.4.11}$$

The composite operators act on the composite space which is constructed by applying the identity operator $I^{[k]}$ to the space $V^{(1)} \otimes V^{(2)} \otimes \cdots \otimes V^{(kn)}$.

The composite model of the free fermion model is equivalent to the free fermion model itself. We note that for the free fermion model the dimensions of the eigenspaces for the projection operators $P^{[S]}$ and $P^{[A]}$ are both equal to 2. Therefore by the method of composition the dimensions of the edge variables do not increase.

4 $gl(M|N)$ IRF models

4.1 The Boltzmann weights and basic relations

We shall define unrestricted IRF models associated with the fundamental representations of $gl(M|N)$ and $sl(M|N)$. For the IRF models, the weight lattices for local states are different. The difference is related to the supertraceless condition and important in the case $m = n$. The $sl(M|N)$ IRF models are independently constructed by Deguchi and Fujii, and by Okado. [33,74]

We assign a local state variable to each site of the lattice. The Boltzmann weights of IRF models are defined for the configurations of the local state variables round a face. For the four state variables a, b, c, d assigned counterclockwise round a face (a plaquette in a square lattice), we write the Boltzmann weight as

$$w(b, c, d, a|u) = W\left(\begin{array}{ccc} & d & \\ a & & c \\ & b & \end{array}\middle| u\right). \tag{4.1.1}$$

Here the variable u is the spectral parameter. The local state a takes values in the $(M+N)$ dimensional vector space as $(a_1, a_2, \ldots a_{M+N})$. We introduce constraints on the local state variables a, b, c, d, counterclockwise round a face. For $gl(M|N)$ and $sl(M|N)$, the symbol σ_μ denotes the parity of the suffix μ, which takes $+1$ or -1. Let a set S be the set of vectors related to the fundamental

representations of $sl(M|N)$ ($gl(M|N)$). [20] For $gl(M|N)$ the set S is given by $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{M+N}\}$, where $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{M+N}$ are linearly independent. For $sl(M|N)$, the vectors $\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{M+N}\}$ have a condition: $\sum \sigma_\mu \hat{e}_\mu = 0$. Hereafter we shall sometimes use notation $\hat{e}_\mu = \hat{\mu}$. The Boltzmann weight of the IRF model is set to be zero unless $b-a, d-a, c-d$ and $c-b$ are the weight vectors in the set S.

We introduce a short-handed symbol as follows,

$$[x] = \theta_1(\pi x/L). \tag{4.1.2}$$

Here L is a fixed constant, $\theta_1(u)$ is the Jacobi's elliptic function and p is its nome,

$$\theta_1(x, p) = 2p^{\frac{1}{4}} \sin x \prod_{k=1}^{\infty} (1 - 2p^k \cos 2x + p^{2k})(1 - p^k). \tag{4.1.3}$$

The Boltzmann weights of the unrestricted models associated with the vector representations of $gl(M|N)$ and $sl(M|N)$ are given in the following:

$$W\left(\begin{array}{ccc} & a+\hat{\mu} & \\ a & & a+2\hat{\mu} \\ & a+\hat{\mu} & \end{array}\middle| u\right) = \frac{[1+\sigma_\mu u]}{[1]}, \tag{4.1.4}$$

$$W\left(\begin{array}{ccc} & a+\hat{\mu} & \\ a & & a+\hat{\mu}+\hat{\nu} \\ & a+\hat{\mu} & \end{array}\middle| u\right) = \frac{[a_{\mu\nu} - u]}{[a_{\mu\nu}]}, \tag{4.1.5}$$

$$W\left(\begin{array}{ccc} & a+\hat{\nu} & \\ a & & a+\hat{\mu}+\hat{\nu} \\ & a+\hat{\mu} & \end{array}\middle| u\right) = \frac{[u][a_{\mu\nu} - 1]}{[1][a_{\mu\nu}]}. \tag{4.1.6}$$

We assume that the quantities $\{a_{\mu\nu}\}$ in the Boltzmann weights satisfy the following relations:

$$a_{\mu\nu} = -a_{\nu\mu}, \text{and} \tag{4.1.7}$$

$$(a+\hat{\kappa})_{\mu\nu} = a_{\mu\nu} + \sigma_\mu \delta_{\kappa\mu} - \sigma_\nu \delta_{\kappa\nu}. \tag{4.1.8}$$

We may write $a_{\mu\nu}$ in the following way.

$$a_{\mu\nu} = \sigma_\mu m_\mu - \sigma_\nu m_\nu + (\omega_0)_{\mu\nu}. \tag{4.1.9}$$

Here ω_0 is an arbitrary antisymmetric $N \times N$ matrix.

It is not difficult to see that if $a_{\mu\nu}$ satisfy the relations (4.1.8), then the Boltzmann weights given in [?] satisfy the Yang-Baxter relation.

$$\sum_g W\left(\begin{matrix} & f & \\ a & & e \\ & g & \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} & e & \\ g & & d \\ & c & \end{matrix}\,\middle|\, u+v\right) W\left(\begin{matrix} & g & \\ a & & c \\ & b & \end{matrix}\,\middle|\, v\right)$$
$$= \sum_g W\left(\begin{matrix} & e & \\ f & & d \\ & g & \end{matrix}\,\middle|\, v\right) W\left(\begin{matrix} & f & \\ a & & g \\ & b & \end{matrix}\,\middle|\, u+v\right) W\left(\begin{matrix} & g & \\ b & & d \\ & c & \end{matrix}\,\middle|\, u\right). \tag{4.1.10}$$

It is remarked that the IRF model for sl(2) is nothing but the 8VSOS model [11], and that the IRF model for $sl(M)$ ($= sl(M,0)$) is the unrestricted $A^{(1)}_{M-1}$ IRF [50] model.

These IRF models satisfy the following fundamental relations.

(1) The standard initial condition

$$w(a,b,c,d|u=0) = W\left(\begin{matrix} & c & \\ d & & b \\ & a & \end{matrix}\,\middle|\, u=0\right) = \delta_{ac}. \tag{4.1.11}$$

Here δ_{ab} is the Kronecker delta.

(2) The inversion relation

$$\sum_g W\left(\begin{matrix} & d & \\ a & & c \\ & g & \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} & g & \\ a & & c \\ & b & \end{matrix}\,\middle|\, -u\right) = \rho_1(u)\delta_{bd}, \tag{4.1.12}$$

Here we have set

$$\rho_1(u) = \frac{[1+u][1-u]}{[1]^2}. \tag{4.1.13}$$

(3) The second inversion relations

$$\sum_g W\left(\begin{matrix} & b & \\ a & & g \\ & d & \end{matrix}\,\middle|\, u\right) W\left(\begin{matrix} & d & \\ c & & g \\ & b & \end{matrix}\,\middle|\, -2\lambda-u\right) \frac{\psi(g)}{\psi(b)} \Big/ \frac{\psi(d)}{\psi(c)} = \rho_2(u)\delta_{ac}, \tag{4.1.14}$$

Here we have set

$$\psi(a) = \phi(a)\prod_{\kappa<\rho}[a_{\kappa\rho} - \omega_{\kappa\rho}]^{\sigma_\kappa\sigma_\rho}, \tag{4.1.15}$$

$$\omega_{\kappa\rho} = \frac{1}{2}(\sigma_\kappa - \sigma_\rho) \quad , \quad \phi(a+\hat{\mu})/\phi(a) = \sigma_\mu, \tag{4.1.16}$$

$$\rho_2(u) = \frac{[u][-2\lambda-u]}{[1]^2}. \tag{4.1.17}$$

The constant λ is given by

$$\lambda = (M-N)/2 \tag{4.1.18}$$

The IRF models have "gauge" symmetry. The Yang-Baxter relation is invariant under the following transformation.

$$W\left(\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}\,\middle|\, u\right) \to \left(\frac{s(a,b)s(b,d)}{s(a,c)s(c,d)}\right)^{\frac{1}{2}} W\left(\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}\,\middle|\, u\right), \tag{4.1.19}$$

We call (4.1.19) gauge transformation or symmetry breaking transformation. If we set

$$s(a, a+\hat{\mu}) = \prod_{\lambda\neq\mu}([a_{\mu\lambda}][a_{\mu\lambda}+\sigma_\lambda])^{\frac{1}{2}\sigma_\lambda}, \tag{4.1.20}$$

then only changes the form as

$$W\left(\begin{matrix} & a+\hat{\nu} & \\ a & & a+\hat{\mu}+\hat{\nu} \\ & a+\hat{\mu} & \end{matrix}\,\middle|\, u\right) = \frac{[u]}{[1]}\frac{\sqrt{[a_{\mu\nu}+1][a_{\mu\nu}-1]}}{[a_{\mu\nu}]}, \tag{4.1.21}$$

and other two types of the Boltzmann weights are invariant.

The Boltzmann weights have a larger symmetry than the transformation (4.1.19). We can show that the Boltzmann weights transformed by the following transformation also satisfy the Yang-Baxter relation,

$$W\left(\begin{matrix} & d & \\ a & & c \\ & b & \end{matrix}\,\middle|\, u\right) \to f\left(\begin{matrix} & d & \\ a & & c \\ & b & \end{matrix}\right) W\left(\begin{matrix} & d & \\ a & & c \\ & b & \end{matrix}\,\middle|\, u\right) \tag{4.1.22}$$

where

$$f\left(\begin{matrix} & d & \\ a & & c \\ & b & \end{matrix}\right) = f\left(\begin{matrix} & b & \\ a & & c \\ & d & \end{matrix}\right)^{-1}. \tag{4.1.23}$$

We note that the transformation (4.1.19) is defined on the edges whereas the transformation (4.1.23) is defined on the face. We call the transformations (4.1.19) and (4.1.23) edgewise and facewise gauge transformations, respectively. For an illustration we give a facewise gauge transformation,

$$f\left(\begin{matrix} & d & \\ a & & c \\ & b & \end{matrix}\right) = \exp(a_{\mu\nu} + a_{\kappa\lambda}). \tag{4.1.24}$$

Here $\mu = d-a, \nu = c-d, \kappa = b-a$, and $\lambda = c-b$.

4.2 Connection of the IRF models to graded vertex models

By taking some limiting processes the IRF models reduce to vertex models, which are related to $gl(M|N)$ ($sl(M|N)$).

Let us derive the vertex model from the IRF model. In the Boltzmann weights of the IRF model we take the trigonometric limit

$$p \to 0. \tag{4.2.1}$$

We substitute $a, b, \ldots$ by $a' = a + \Omega, b' = b + \Omega, \ldots$. Based on the Wu-Kadanoff-Wegner transformation [90] we introduce the following limit (the base point infinity limit) [5]:

$$\lim_{\Omega\to\infty} W \left(\begin{array}{ccc} & d' & \\ a' & & c' \\ & b' & \end{array} \middle| u \right) = X_{\kappa\lambda}^{\mu\nu}(u). \tag{4.2.2}$$

Here $\mu = d' - a', \nu = c' - d', \kappa = b' - a'$, and $\lambda = c' - b'$. By putting

$$1/L \to \lambda \ , \ u/L \to iu, \tag{4.2.3}$$

we have the Boltzmann weights of the $gl(M|N)$ vertex model. We note that the following equality is useful in the calculation.

$$\lim_{\Omega\to\infty} \frac{[a'_{\mu\nu} - u]}{[a'_{\mu\nu}]} = \exp \theta(\Omega_{\mu\nu}) u, \tag{4.2.4}$$

where $\theta(\Omega_{\mu\nu})$ is the sign of $\Omega_{\mu\nu}$. Thus we have shown that the IRF models are generalizations of vertex models.

4.3 Fusion IRF models

We can construct IRF models related to higher representations of the Lie superalgebras $gl(M|N)$ and $sl(M|N)$. [33] More precisely, we can construct IRF models for Young operators constructed from the graded permutation operators associated with the Lie superalgebras. We apply the fusion method to the Boltzmann weights of IRF models associated with vector representations of $gl(M|N)$ and $sl(M|N)$. Construction of fusion models has been discussed in various contexts. [18,62,64,49]. Our discussion consists of the following two points:

(1) Z-invariance of the model. [13]

(2) Projecting into irreducible components by multiplying projection operators acting path space of the model.

The transfer matrix $T_n(u)$ of IRF model is defined on the Hilbert space [5] which consists of admissible sequences of local states: $\{\ell_i; \ell_{i+1} \sim \ell_i, (i = 0, \cdots n-1)\}$. We recall that the admissible sequence and the Hilbert space are called path and path space, respectively. We use the following notation [12,6] for local operators acting on the i-th site of the path space: We use the

$$\{R_i\}_{k_0 \cdots k_n}^{p_0, \cdots p_n} = \prod_{j=0}^{i-1} \delta_{k_j}^{p_j} \cdot R(k_i, p_{i+1}, p_i, p_{i-1}) \cdot \prod_{j=i+1}^{n} \delta_{k_j}^{p_j}. \tag{4.3.1}$$

The symbol $R(k_i, p_{i+1}, p_i, p_{i-1})$ corresponds to the notation for the Boltzmann weight.

Let us discuss construction of projection operator R^μ from the Yang-Baxter operators of the IRF models associated with $gl(M|N)$ and $sl(M|N)$. Here the symbol μ represents the Young diagram for the projection operator. We assume that we can apply the construction of the projection operators R^μ for the $gl(N)$ vertex models given in the reference to the IRF models associated with $gl(M|N)$ and $sl(M|N)$. For Young diagrams with one row or one column we can check the construction of the projection operator R^μ by an elementary method.

The projection operator R^μ is a generalization of the Young operator corresponding to the Young diagram μ. Taking the limit $L \to \infty$ in the Boltzmann weights, we see that the projection operator R^μ reduces to the Young operator of the symmetric group.

For an illustration, we consider the symmetric and antisymmetric projection operators $R^S = R^\mu$, $\mu = (2,0)$, and $R^A = R^\mu$, $\mu =(1,1)$. Here R^S and R^A are given by $R^S = W(u = 1)$ and $R^A = W(u = -1)$, respectively. Taking the limit $L \to \infty$, we find that $R^S \to P^S$ and $R^A \to P^A$. Here we have introduced the Young operators P^S and P^A acting on the path space.

$$\begin{aligned} P^S(a+\mu, a+2\mu, a+\mu, a) &= 1, \text{ for } \quad \sigma_\mu = 1, \\ &= 0, \text{ for } \quad \sigma_\mu = -1, \\ P^S(a+\mu, a+\mu+\nu, a+\mu, a) &= 1/2, \text{ for } \mu \neq \nu, \\ P^S(a+\mu, a+\mu+\nu, a+\nu, a) &= 1/2, \text{ for } \mu \neq \nu, \end{aligned} \tag{4.3.2}$$

$$\begin{aligned} P^A(a+\mu, a+2\mu, a+\mu, a) &= 0, \text{ for } \sigma_\mu = 1, \\ &= 1, \text{ for } \sigma_\mu = -1, \end{aligned}$$

$$P^A(a+\mu,a+\mu+\nu,a+\mu,a) = 1/2, \text{ for } \mu \neq \nu,$$
$$P^A(a+\mu,a+\mu+\nu,a+\nu,a) = -1/2, \text{ for } \mu \neq \nu. \tag{4.3.3}$$

The Young operators P^S and P^A are written in terms of the graded permutation operator π_i associated with the Lie superalgebra $gl(M|N)$ ($sl(M|N)$) acting on the path space. The relation between $P^S(P^A)$ and π is $P_i^S = (1+\pi_i)/2$ ($P_i^A = (1-\pi_i)/2$). The elements of the graded permutation operator π are given in the following.

$$\begin{aligned} \pi(a+\mu,a+2\mu,a+\mu,a) &= 1, \text{ for } \sigma_\mu = 1, \\ &= -1, \text{ for } \sigma_\mu = -1, \\ \pi(a+\mu,a+\mu+\nu,a+\mu,a) &= 0, \text{ for } \mu \neq \nu, \\ \pi(a+\mu,a+\mu+\nu,a+\nu,a) &= 1, \text{ for } \mu \neq \nu. \end{aligned} \tag{4.3.4}$$

We construct composite Yang-Baxter operators by using the projection operators R^μ. For an illustration we define the Yang-Baxter operator for the fusion model of (k_1,k_2)-strings with symmetric projectors $R^{[sym,k_1]}$ and $R^{[sym,k_2]}$. We use the notation $R^{[sym,n]} = R^\mu$ for $\mu = (n,0,\cdots,0)$.

$$Y_I^{[sym,k_1],[sym,k_2]}(u) = R_{I-k_2}^{[sym,k_1]} R_{I+k_1}^{[sym,k_2]} \left(\prod_{j=1}^{k_2} \hat{W}_{I-j+1}^{(j)}(u) \right) R_{I-k_2}^{[sym,k_2]} R_{I+k_1}^{[sym,k_1]}, \tag{4.3.5}$$

where

$$\hat{W}_I^{(j)}(u) = \prod_{m=1}^{k_1} W_{I+m-j}(u + \epsilon(j - k_1 + m - 1))). \tag{4.3.6}$$

Here ϵ =1 for symmetric projectors. For an illustration we give the case of (2,2) strings ($k=2$) in the following:

$$\begin{aligned} Y_I^{[sym,2]}(u) &= R_{I-1}^{[sym,2]} R_{I+1}^{[sym,2]} W_I(u+1) W_{I-1}(u) W_{I+1}(u) \\ &\quad \times W_I(u-1) R_{I-1}^{[sym,2]} R_{I+1}^{[sym,2]}. \end{aligned} \tag{4.3.7}$$

The spectral parameter dependence of the Yang-Baxter operators in the product (4.3.6) should be chosen in accordance with the Z-invariance. [18,62,64] For antisymmetric projectors R^μ ($\mu = (1,1,\cdots,1)$), we set ϵ=-1 in (4.3.6).

We can check existence of the fusion models for symmetric projectors $R^{[sym,n]}$ by an elementary way. We use the fact $W(u=-1) \sim P^A$ and the factorizable property of the projection operator R^μ. In the λ, κ sector (charge $\lambda + \kappa$ subspace, $\lambda \neq \kappa$) $W(u=-1) \sim P^A$. Here we have chosen the weight as $[u][a_{\lambda\kappa}+1]/[1][a_{\lambda\kappa}]$ by using the gauge transformation (4.1.19).

The projection operators R^μ 's decompose the tensor product space (or the path space) into subspaces. The decomposed subspaces are invariant under the gauge transformation (4.1.19) (or more generally the similarity transformation (4.1.23)), although explicit forms of the Boltzmann weights of the fusion models depend on the choice of the transformation (4.1.19) (or (4.1.23)) on the weight

For an illustration we give the Boltzmann weights of fusion models of gl(1,1) IRF model. We consider the case (1): (2,1)-strings with the projector P^S, the case (2): (2,2)-strings with the projectors P^S, and the case (3): (3,1)-strings for the mixed symmetry.

(1) (2,1)-strings with the projector R^μ, $\mu = (2,0)$.

$$\begin{aligned} w(a+e_1,a+2e_1+e_2,a+e_1+e_2,a;u) &= \frac{[u+1][a_{12}+2]}{[1][a_{12}+1]}, \\ w(a+e_1,a+3e_1,a+2e_1,a;u) &= \frac{[u+2]}{[1]}, \\ w(a+e_2,a+2e_1+e_2,a+e_1+e_2,a;u) &= \frac{[u+1+a_{12}][2]}{[1][a_{12}+1]}, \\ w(a+e_1,a+2e_1+e_2,a+2e_1,a;u) &= \frac{[a_{12}-u]}{[a_{12}+1]}, \\ w(a+e_2,a+2e_1+e_2,a+2e_1,a;u) &= \frac{[u][a_{12}-1]}{[1][a_{12}+1]}, \\ w(a+e_2,a+2e_1+e_2,a+e_1+e_2,a;u) &= \frac{[1-u][a_{12}]}{[1][a_{12}+1]}. \end{aligned} \tag{4.3.8}$$

(2) (2,2)-strings with the projectors R^μ, μ=(2,0).

Here we use the notation $e_1+e_2 = f_1$ and $2e_1 = b_1$.

$$\begin{aligned} w(a+b_1,a+b_1+f_1,a+f_1,a;u) &= \frac{[u][a_{12}+3]}{[1][a_{12}+1]}, \\ w(a+b_1,a+2b_1,a+b_1,a;u) &= \frac{[u+2]}{[1]}, \\ w(a+f_1,a+f_1+b_2,a+b_1,a;u) &= \frac{[a_{12}-1][u]}{[1][a_{12}+1]}, \\ w(a+f_1,a+f_1+b_1,a+f_1,a;u) &= \frac{[2][a_{21}-1-u]}{[1][a_{21}-1]}, \\ w(a+b_1,a+b_1+f_1,a+b_1,a;u) &= \frac{[2][a_{12}+1-u]}{[1][a_{12}+1]}, \\ w(a+f_1,a+2f_1,a+f_1,a;u) &= \frac{[2-u]}{[1]}. \end{aligned} \tag{4.3.9}$$

(3) (3,1)-strings for the mixed symmetry R^μ, $\mu = (2,1)$.

$$w(a,a+e_1,a+2e_1+e_2,a+3e_1+e_2) = \frac{[u+2][a_{12}+1]}{[1][a_{12}]},$$

$$
\begin{aligned}
w(a, a+e_2, a+e_1+2e_2, a+3e_1+e_2) &= \frac{[a_{12}][u-2]}{[1][a_{12}+1]}, \\
w(a, a+e_2, a+2e_1+e_2, a+2e_1+2e_2) &= \frac{[1-u][a_{12}+1][a_{12}-1]}{[1][a_{12}][a_{12}+2]}, \\
w(a, a+e_1, a+e_1+2e_2, a+2e_1+2e_2) &= \frac{[u+1][a_{12}+3]}{[1][a_{12}+1]}, \\
w(a, a+e_1, a+2e_1+e_2, a+2e_1+2e_2) &= -\frac{[3][u-a_{12}-1]}{[1][a_{12}]}, \\
w(a, a+e_2, a+e_1+2e_2, a+2e_1+2e_2) &= -\frac{[a_{12}+1+u]}{[a_{12}+2]}. \qquad (4.3.10)
\end{aligned}
$$

It is emphasized that in the case of the $gl(1|1)$ IRF model, the fusion model of the case (2) is equivalent to the $gl(1|l)$ IRF model for the vector representation. This fact is explained in the following way. [27] If we multiply two vector representations, then we have symmetric and antisymmetric representations. For $gl(1|1)$ case, the dimensions of the symmetric and antisymmetric representations are both equal to 2, which is equal to the dimension of the vector representation. Therefore we do not have multi-state model by the fusion method of the case (2) for the $gl(1|1)$ model.

4.4 Restricted IRF models

We construct restricted IRF models related to representations of the Lie superalgebras $sl(m|n)$ and $gl(m|n)$.

Let us discuss three typical restriction mechanisms. By letting L take some discrete values and choosing $\omega(\mu,\nu)$ properly, we can construct restricted IRF models. We consider the following three types of restriction mechanisms.

I. Restriction with restriction lines

$$
\begin{aligned}
\omega(\mu,\nu)/r &= 0 \pmod{Z + Zi\tau}, \\
\frac{1}{L} &= \frac{s_1}{r} + \frac{s_2}{r} i\tau, \quad (s_1, r) = 1 \text{ or } (s_2, r) = 1, \qquad (4.4.1)
\end{aligned}
$$

where $p = \exp(-\tau)$, and r, s_1, s_2 are positive integers. The symbol (a,b) expresses that the integers a and b are coprime, i.e., having no nontrivial common divisors. We note that $\mathrm{Z} + \mathrm{Z}i\tau$ is the set of the (quasi-) periods of elliptic theta functions.

II. Cyclic restriction

$$
\begin{aligned}
\omega(\mu,\nu)/r &\neq 0 \pmod{Z + Zi\tau}, \text{ for any } \mu \neq \nu, \\
\frac{1}{L} &= \frac{s_1}{r} + \frac{s_2}{r} i\tau, \text{ for any } s_1, s_2 \in Z. \qquad (4.4.2)
\end{aligned}
$$

III. Mixed type restriction

For higher dimensional weight lattices, we can construct weight lattice restricted in some directions by the mechanism I and the other directions by II.

$$
\begin{aligned}
\omega(\mu,\nu)/r &= 0 \pmod{Z + Zi\tau}, \text{ for some } \mu \neq \nu \\
\omega(\mu,\nu)/r &\neq 0 \pmod{Z + Zi\tau}, \text{ for other } \mu \neq \nu, \\
\frac{1}{L} &= \frac{s_1}{r} + \frac{s_2}{r} i\tau, \text{ for any } s_1, s_2 \in Z. \qquad (4.4.3)
\end{aligned}
$$

Fig.4.4.1

The restricted 8VSOS (8 vertex solid-on-solid) model (or Andrew-Baxter-Forrester model) is a typical case of the restricted IRF model of the mechanism I. Usually the restricted models of this kind are studied. However, there can be various kinds of restricted IRF models which are worth studying in the context of exactly solvable lattice models in statistical physics.

It is interesting to note that in the case of vertex models, the Boltzmann weights for vector representations of $gl(m|n)$ and $sl(m|n)$ are equivalent, while in the case of (restricted) IRF models they are different.

5 Link polynomials associated with representations of $gl(M|N)$

5.1 The Markov trace

Let us construct the Markov trace on the representations derived in the section 3. Let us consider the sufficient condition for the Markov trace property. We introduce a diagonal matrix h is

$$
\begin{aligned}
(h)_{ij} &= h(j)\delta_{ij} \\
&= \sigma_j \delta_{ij} \exp\{\eta(\sum_{k=1}^{j-1} 2\sigma_k + \sigma_j - M + N)\}, \\
&\quad \text{for } j = 1 \cdots M+N. \qquad (5.1.1)
\end{aligned}
$$

It is easy to see that the matrix h given by ((5.1.1)) satisfies the sufficient condition (2.4.2).

Thus for the case $M \neq N$ we have the Markov trace of the following form.

$$
\begin{aligned}
\phi(A) &= \frac{\mathrm{Tr}(H(n)A)}{\mathrm{Tr}(H(n))}, \ A \in B_n, \\
&[H(n)]_{b_1 b_2 \cdots b_n}^{a_1 a_2 \cdots a_n} = \prod_{j=1}^{n} h(a_j)\delta_{b_j}^{a_j}. \qquad (5.1.2)
\end{aligned}
$$

From the explicit form of the Markov trace we find that

$$\chi(\pm) = \exp\{\pm(M - N - 1)\eta\}. \tag{5.1.3}$$

If we define $q^{1/2} = \sum_j h(j)$, then

$$q^{1/2} = \frac{\sinh((M-N)\eta)}{\sinh\eta}, \tag{5.1.4}$$

and

$$Tr(H(n)) = q^{n/2} = \left(\frac{\sinh((M-N)\eta)}{\sinh\eta}\right)^n. \tag{5.1.5}$$

We remark that in the limit $\eta \to 0$, the Markov trace reduces to the supertrace [54] $strA = \sum_j \sigma_j A_{jj}$. [27] Hence the Markov trace is an extension (q-analog) of the supertrace.

We can prove the extended Markov property, which is an extension of the Markov property with finite spectral parameter.

$$\sum_b X_{ab}^{ab}(u)h(b) = h(u;\eta)\rho(u) \text{ (independent of } a). \tag{5.1.6}$$

where the function $h(u;\eta)$ is given by

$$h(u;\eta) = \frac{\sinh((M-N)\eta - u)}{\sinh(\eta - u)}. \tag{5.1.7}$$

The construction of the Markov trace is consistent with the quantum trace of the quantum group $U_q(gl(M|N))$. We can show that $h(j)$ is related to "the half the sum of positive roots" ρ by

$$h(j) = -2(\rho, \epsilon_j). \tag{5.1.8}$$

For an illustration we consider the case $p(a) = 0$ if $1 \le a \le m$, $p(a) = 1$ if $1 \le a \le m$. Then we can show (5.1.8) through the following

$$\rho = \frac{1}{2}\sum_{j=1}^{M}(M - N + 1 - 2j)\epsilon_j + \frac{1}{2}\sum_{j=M+1}^{M+N}(3M + N + 1 - 2j)\epsilon_j. \tag{5.1.9}$$

For the case $M = N$, the diagonal matrix h gives vanishing trace: $\sum_j h_{jj} = 0$. This property is related to the vanishing property of the Alexander polynomial. In this case, the definition of the trace (5.1.2) has an ambiguity. We define the Markov trace by a proper modification of the trace. [27]

$$\phi^*(A) = \mathrm{Tr}(H^*(n)A)/\Sigma_j k(j), \quad A\sigma B_n. \tag{5.1.10}$$

Here $k(a)$ is arbitrary but $\sum_j k(j) \neq 0$.

We can show the following.

Proposition 5.1 *For the $gl(M|N)$ representation of the braid group the value of the trace $\phi^*(\cdot)$ is independent of the choice of $\{k(j)\}$.*

We can prove the proposition through invariants of oriented (1,1) tangles (see also discussion in Part II).

Let us discuss the IRF model. We can derive the representations of the braid group from the Boltzmann weights of the $gl(M|N)$ IRF model in the critical case: $p = 1$. We apply the formula (2.3.1) to the Boltzmann weights of the IRF model. We can construct the Markov trace of the IRF type [5] from the IRF models related to representations of the Lie superalgebras $gl(M|N)$ and $sl(M|N)$. [33,25] For IRF models we introduce a "constrained trace" $\tilde{Tr}(A)$ [5]

$$\tilde{Tr}(A) = \tilde{\sum_{\ell_1\ell_2\cdots\ell_n}} A_{\ell_0\ell_1\cdots\ell_n}^{\ell_0\ell_1\cdots\ell_n}\frac{\psi(\ell_n)}{\psi(\ell_0)}, \qquad (\ell_0 : fixed) \tag{5.1.11}$$

where the symbol $\tilde{\Sigma}$ represents the summation over admissible multi-indices $\ell_i : \ell_{i+1} \sim \ell_i$ for $i = 0, \cdots, n-1$ with ℓ_0 being fixed. For simplicity we assume $M \neq N$. We can define the Markov trace also for the IRF models related to representations of $gl(M|N)$ and $sl(M|N)$.

$$\phi(A) = \frac{\tilde{Tr}(A)}{\tilde{Tr}(I(n))}, \qquad A\epsilon B_n, \tag{5.1.12}$$

where $I(n)$ is the "identity" operator for n strings. We can prove the extended Markov property also for IRF models.

For the case $M = N$, we can define the Markov trace of the IRF type in the same way as the Markov trace (5.1.10) of the $gl(M|M)$ vertex model.

5.2 Link polynomials

The link polynomial obtained from the vertex model associated with $gl(M|N)$ has the skein relation: [24]

$$\alpha(L_+) = t^{\ell/2}(1-t)\alpha(L_0) + t^{\ell+1}\alpha(L_-). \tag{5.2.1}$$

Here we have defined a number ℓ as

$$\ell = M - N - 1. \tag{5.2.2}$$

Since the skein relation is of second degree, the link polynomial is completely determined by the relation. Thus we obtain a hierarchy of link polynomials which depends on the number $\ell = M - N - 1$. The most characteristic point of this hierarchy is that from the Markov traces and the braid matrices with different sizes the same link polynomial for an integer ℓ is constructed. [24] From different models related to gl(M|N) with $\ell = M - N - 1$ we obtain the same link polynomial. Note that the hierarchy includes the case $\ell = 0$ where $\bar{\tau}/\tau = 1$. For any integer ℓ we have a link polynomial with the skein relation. We remark that the link polynomial for an integer ℓ corresponds to that for $-2-\ell$ under the replacement of t by $1/t$.

The HOMFLY polynomial[39,78] is characterized by the second degree skein relation:

$$\alpha(L_+) = \Omega^{1/2}(1-t)\alpha(L_0) + \Omega t\alpha(L_-). \tag{5.2.3}$$

Here t and Ω are independent (continuous) variables. We see that the link polynomials constructed from the gl(M|N) type vertex models correspond to the cases $\Omega = t^\ell$, $\ell \in Z$ of the HOMFLY polynomial. Based on the Markov traces we thus obtain a hierarchy of state models for the HOMFLY polynomial. [24]

Let us discuss link polynomials derived from the composite models (fusion models). [24,27] We can show the existence of the Markov trace for the composite string representations by two different methods. First one is to prove the extended Markov property for the composite model. From the operator form of the composite Yang-Baxter operator $Y_i^{[k]}(u)$ (for k strings) the characteristic function $H^{[k]}(u;\eta)$ is recursively calculated as

$$H^{[k]}(u;\eta) = \prod_{r=1}^{k} \frac{\sinh((M-N-1+r)\eta - u)}{\sinh(r\eta - u)}. \tag{5.2.4}$$

The other method is to consider the composite string representation. [32] A sufficient condition for the Markov property is the following eigenvalue equation: [32]

$$P_i^{[\lambda]}\Delta_i^2 = \alpha_\lambda P_i^{[\lambda]}, \tag{5.2.5}$$

where Δ_i is half twist [15,32] and λ represents the symmetry of the projector. Note that Δ^2 is in the center of $H_k(q)$. We construct link polynomials by solving this equation (5.2.5).

There are various known methods for construction of projection operators of the Hecke algebra. We may classify the methods into the two types, the "additive" types [44] and the "multiplicative" types. [18,32,67] For the multiplicative case, there is a simple algorithm for calculation of projection operators for any given Young diagrams. [18] We can calculate the projection operators also by explicitly

diagonalizing Δ^2 in the regular representations of the Hecke algebra. Since the eigenvalue $\alpha(\mu)$ can be easily obtained from the knowledge of the q-analogs we can calculate the projectors (idempotents) through diagonalizing the eigenvector space. For an illustration we show the case $k = 3$. [32,6]

$$\begin{aligned}
\Delta &= g_i g_{i+1} g_i, \\
P^{(3)} &= \frac{1}{(1+q^2)(1+q^2+q^4)}(q^6 + q^4(g_i + g_{i+1}) + q^2(g_i g_{i+1} + g_{i+1} g_i) + g_i g_{i+1} g_i), \\
P^{(2,1)} &= \frac{1}{2q(1+q^2+q^4)}(2q^3 + q(1-q-q^2)(g_i + g_{i+1}) \\
&\quad +(-1-q+q^2)(g_i g_{i+1} + g_{i+1} g_i) + 2g_i g_{i+1} g_i), \\
P^{(2,1)'} &= \frac{1}{2q(1+q^2+q^4)}(2q^3 + q(1+q-q^2)(g_i + g_{i+1}) \\
&\quad +(1-q-q^2)(g_i g_{i+1} + g_{i+1} g_i) - 2g_i g_{i+1} g_i), \\
P^{(1^3)} &= \frac{1}{(1+q^2)(1+q^2+q^4)}(1 - (g_i + g_{i+1}) + (g_i g_{i+1} + g_{i+1} g_i) - g_i g_{i+1} g_i)
\end{aligned} \tag{5.2.6}$$

For the $gl(M|N)$ representations the propositions 3.1 and 3.2 are helpful for construction of the projection operators and calculation of the link polynomial. For example, the projection operator P^μ for $\mu = (3^2)$ vanishes for the case of $gl(2|1)$, while it does not for the case of $gl(3|2)$.

Through the projection operators the braid operators for the composite string are given by

$$G_i^{[\lambda]} = P_{(i-1)k+1}^{[\lambda]} P_{ik+1}^{[\lambda]} \left(\prod_{j=1}^{k} \hat{G}_i^{(j)} \right) P_{(i-1)k+1}^{[\lambda]} P_{ik+1}^{[\lambda]}, \tag{5.2.7}$$

where

$$\hat{G}_i^{(j)} = \prod_{m=1}^{k} G_{ik+m-j}. \tag{5.2.8}$$

The Markov trace $\psi^{[\lambda]}(\cdot)$ is given by (for the cases gl(M|N) with $M \neq N$) [32]

$$\psi^{[\lambda]}(A) = \phi(A)/[\phi(P_i^{[\lambda]})]^n, \ A \in B_n^{[\lambda]}. \tag{5.2.9}$$

Here $\phi(\cdot)$ is defined in the last subsection. For the cases $gl(M|M)$, we introduce the Markov trace by

$$\psi^{[\lambda]}(A) = \phi^*(A), \quad A \in B_n^{[\lambda]}, \tag{5.2.10}$$

where $\phi^*(\cdot)$ is given by (5.1.10). We remark that we can use the definition (5.2.10) also for the cases gl(M|N) with $M \neq N$.

Link polynomials are given by

$$\alpha^{[\lambda]}(L) = (Z_\lambda \bar{Z}_\lambda)^{-(n-1)/2} \left(\frac{\bar{Z}_\lambda}{Z_\lambda} \right)^{e(A)/2} \psi^{[\lambda]}(A), \ A \in B_n^{[\lambda]}, \tag{5.2.11}$$

where A is a braid whose closed braid is equivalent to the link L, $\epsilon(A)$ is the exponent sum of the braid A and

$$\begin{aligned} Z_\lambda &= \psi^{[\lambda]}(G_j), \ \ G_j \in B_n^{[\lambda]}, & (5.2.12) \\ \bar{Z}_\lambda &= \psi^{[\lambda]}(G_j^{-1}), \ \ G_j^{-1} \in B_n^{[\lambda]}. & (5.2.13) \end{aligned}$$

We shall sometimes write the invariant as $\alpha^{[\lambda]}_{gl(M|N)}(L)$.

The skein relations for the link polynomials constructed from the composite models for the Young diagram of one row are given as follows.

$$\begin{aligned} \alpha^{[(2)]}(L_{3+}) &= t^{\ell}(1-t^2+t^3)\alpha^{[(2)]}(L_{2+}) + t^{2\ell}(t^2-t^3+t^5)\alpha^{[(2)]}(L_+) \\ &- t^{3\ell+5}\alpha^{[(2)]}(L_0), & (5.2.14) \\ \alpha^{[(3)]}(L_4+) &= t^{3\ell/2}(1-t^3+t^5-t^6)\alpha^{[(3)]}(L_{3+}) \\ &+ t^{3\ell}(t^3-t^5+t^6+t^8-t^9+t^11)\alpha^{[(3)]}(L_{2+}) \\ &+ t^{9\ell/2}(-t^8+t^9-t^11+t^14)\alpha^{[(3)]}(L_+) \\ &- t^{6\ell}t^{14}\alpha^{[(3)]}(L_0). & (5.2.15) \end{aligned}$$

We see that the skein relations of the link polynomials correspond to those for the two-variable link invariants with $\Omega = t^{\ell}$. [32,6]

It is remarked that the factor $\Omega = t^{\ell}$ is different from the $sl(M)$ cases for $\ell = 0, -1$ and therefore the link polynomials for $\ell = 0, -1$ are new in this sense. It is interesting that the Alexander polynomial is related to the free fermion model. Note that generalizations of the Alexander polynomial are obtained from the composite models related to $gl(M|M)$ $(M > 1)$. These link polynomials have higher degree skein relations.

The link invariants $\alpha^{[\lambda]}(L)$ for the $gl(M|N)$ representations are not derived by simply replacing the variable Ω by $\Omega = t^{\ell}$ in HOMFLY polynomial. We recall that the projection operator P^{μ} for $\mu = (3^2)$ vanishes for the case of $gl(2|1)$ while it does not for the case of $gl(3|2)$. Note that both the cases $gl(2|1)$ and $gl(3|2)$ have the same $\ell = M - N - 1 = 0$. Thus we see that $\alpha^{[\mu]}_{gl(2|1)}(L)$ $(\Omega = 1)$ vanishes while $\alpha^{[\mu]}_{gl(3|2)}(L)$ $(\Omega = 1)$ does not.

In general, link polynomials associated with fusion models or higher representations of quantum groups are related to the invariants for parallel links. The parallel version of link invariants has been studied in Ref. [70]. The problem to establish precise connections between the link polynomials derived from the $gl(M|N)$ fusion models (higher representations of quantum groups $U_q(gl(M|N))$) and the HOMFLY

polynomial for parallel links is worth studying. However, the problem is beyond the scope of this paper and we leave it untouched. We recall that there are various approaches to explicit construction of projection operators (primitive idempotents) of the Hecke algebra. In this section, we have concentrated on the approach of fusion models to the link polynomials, which are related to higher representations of $gl(m|n)$ (and also those of $sl(M|N)$) and the Specht module theory of the Hecke algebra.

Part II

Colored Vertex Models and Colored Link Polynomials

6 Colored Vertex Models

6.1 Introduction to Parts II and III

Recently a hierarchy of representations of the colored braid group has been introduced. [1,28] We regard colored braid as a braid on strings with colors. We call the matrix elements of the representations colored braid matrices. We can derive the colored braid matrices from solvable vertex models, which we call colored vertex models. [29]

In part II we discuss construction of multivariable invariants of colored links which give generalizations of the multivariable Alexander polynomial. In part III we show that the braid matrices are derived from the universal R matrices of $U_q(sl(2))$ at q roots of unity.

The hierarchy of the colored vertex model is related to the free fermion model. [37] The 2-state case of the colored vertex model corresponds to the trigonometric limit of Felderhof's solution of the free fermion model. [38] The colored vertex models can be considered as extensions of the free fermion model [37,38] into N-state vertex models related to "Z_N analog" of the graded symmetry.

A state model for the multivariable Alexander polynomial has been constructed from the trigonometric limit of the Felderhof's solution. [71] The $N=2$ case of the hierarchy of the colored link invariants corresponds to the multivariable Alexander polynomial. In this sense the hierarchy of the colored link invariants gives generalizations of the multivariable Alexander polynomial.

The invariants of colored links vanish for disconnected links. Due to this property the standard state sum vanishes, if we simply take summation over all possible configurations. Both the (one-variable) Alexander polynomial and the multivariable Alexander polynomial have this property. It is remarked that this property is characteristic of the Markov trace on the representation of the braid group derived from the $gl(m|m)$ vertex model. [27] Several examples of braid matrices with the

vanishing property were also given by Lee, Couture and Schmeing, by directly solving the braid relation up to the 6 state case. [22] The braid matrices are equivalent to the non-colored limit of the N-state colored braid matrices ($N=2,\cdots,6$).

There are various viewpoints associated with the colored vertex models. We recall that the $N=2$ case of the colored vertex model corresponds to the trigonometric limit of the Felderhof's solution [38] of the free fermion model. [37] The Felderhof parametrization of the colored Boltzmann weights of the free fermion model was introduced from the condition that the transfer matrix of the free fermion model commutes with the XY Hamiltonian. [38] The non-colored case of the trigonometric limit is related to the Lie superalgebra $gl(1|1)$ ($sl(1|1)$). [27] [88,77,63] Recently it has been shown [87] that the trigonometric limit of the Felderhof's solution is related to $U_q(gl(1|1))$. The 2-state colored braid matrix has been derived from the universal R matrix of $U_q(gl(1|1))$. From the different viewpoint it has also been pointed out [72] that the trigonometric limit of the Felderhof's solution is related to $U_q(sl(2))$.

It has been explicitly shown that the colored braid matrices are given by the matrix elements of the universal R matrix of $U_q(sl(2))$ for color representations. [30] Here we call by color representation a root of unity N-dimensional representation with $q^{2N}=1$ which has a continuous Dynkin label and both highest and lowest weight vectors. Using the limit $q^2 \to \omega$, we can derive the N-state colored braid matrices from infinite dimensional representations of $U_q(sl(2))$. [30] Here ω is a primitive N-th root of unity. We call infinite dimensional representation with a free parameter infinite dimensional color representation.

Finally we give comments on roots of unity representations of quantum groups. When a parameter q is not a root of unity, finite dimensional irreducible representations of quantum groups [36,45] are (one-parameter) deformations of those for classical Lie algebras. However, the theory becomes not so simple when the q-parameter is a root of unity. Representations theories of roots of unity representations are studied. [66,19,20] Cyclic representations of $U_q(sl(n))$ with q a root of unity have been discussed. [10,23,9] It is remarked that the cyclic representations do not have neither highest weight vectors nor lowest weight vectors. We also note that infinite dimensional representations of the Sklyanin algebra [89] are related to the root of unity representations of $U_q(sl(2))$ (see also [83]).

6.2 Colored Yang-Baxter relation

Let us introduce some symbols for the Boltzmann weights of the N-state colored vertex model.

Fig. 6.2.1

The symbol $X_{\gamma_i,\gamma_j}(u_i - u_j)_{b_j b_i}^{a_i a_j}$ represents the Boltzmann weight for the vertex configuration $\{a_i, a_j, b_j, b_i\}$ at the intersection of the i-th and j-th strings, where the state variables a_i, a_j, b_i, b_j take N values $0, 1, 2, \cdots, N-1$.

Let us introduce the colored Yang-Baxter relation.

$$\begin{aligned} & \sum_{c_1,c_2,c_3} X_{\gamma_1,\gamma_2}(u_1 - u_2)_{c_2 c_1}^{a_1 a_2} X_{\gamma_1,\gamma_3}(u_1 - u_3)_{c_3 b_1}^{c_1 a_3} X_{\gamma_2,\gamma_3}(u_2 - u_3)_{b_3 b_2}^{c_2 c_3} \\ = & \sum_{c_1,c_2,c_3} X_{\gamma_2,\gamma_3}(u_2 - u_3)_{c_3 c_2}^{a_2 a_3} X_{\gamma_1,\gamma_3}(u_1 - u_3)_{b_3 c1}^{a_1 c_3} X_{\gamma_1,\gamma_2}(u_1 - u_2)_{b_2 b1}^{c_1 c_2}. \end{aligned} \tag{6.1.1}$$

Here u_i is spectral parameter and γ_i is color variable. Both u_i and γ_i are attached to the i-th string. The variables a_i, b_i and c_i are state variables defined on the i-th string. Setting $u_1 = u + v, u_2 = v, u_3 = 0$, we have the usual form of the spectral parameter dependence. If we assume $\gamma_1 = \gamma_2 = \gamma_3$, then we recover the standard form of the Yang-Baxter relation.

The color variable γ_i plays a similar role with the spectral parameter u_i in the colored Yang-Baxter relation. Both the color variable and the spectral parameter are assigned on the strings. The Boltzmann weights, however, depend on the color variables in a different way from the spectral parameters. The spectral parameters $\{u_i\}$ appear in the Boltzmann weights $X_{\gamma_i,\gamma_j}(u_i - u_j)$ through the difference $u_i - u_j$, while the color variables $\{\gamma_i\}$ appear in the Boltzmann weights nontrivially and not in the difference form.

6.3 Symmetries of colored vertex model

We discuss basic properties and symmetries of the colored vertex models. We shall study construction of the colored vertex model from $U_q(sl(2))$ in §11 (see Appendix E for the Boltzmann weights of the colored vertex model).

In this subsection we use the following notation for the spectral parameter u.

$$x = \exp u. \tag{6.1.1}$$

The symbol ω denotes a primitive N-th root of unity

$$\omega = \exp(2\pi i s/N), \quad (N, s) = 1. \tag{6.1.2}$$

Here the symbol $(N, s) = 1$ means that s is an integer coprime to N.

The colored vertex models have charge conservation condition, first and second inversion relations.

1) charge conservation condition

$$X_{\alpha,\beta}(u)_{cd}^{ab} = 0, \text{ unless } a + b = c + d. \tag{6.1.3}$$

2) first inversion relation

$$\sum_{ef} X_{\alpha,\beta}(u)_{ef}^{ab} X_{\beta,\alpha}(-u)_{cd}^{ef} = C_1(u)\delta_{ac}\delta_{bd},$$

where

$$C_1(u) = \prod_{n=1}^{N-1}(1 - xw^{n-1}\alpha\beta)(1 - x^{-1}w^{n-1}\alpha\beta). \tag{6.1.4}$$

3) second inversion relation

$$\sum_{ef} X_{\alpha,\beta}(u)_{cf}^{ae} X_{\beta,\alpha}(u')_{be}^{df} w^{-e} = C_2(u)\delta_{ab}\delta_{cd} w^{-c},$$

where

$$\begin{aligned} & \exp(u') = x^{-1}w^{2-N}, \\ & C_2(u) = \prod_{n=1}^{N-1}(\alpha - xw^{n-1}\beta)(\alpha - x^{-1}w^{1-n}\beta). \end{aligned} \tag{6.1.5}$$

The Boltzmann weights of colored vertex models have the following symmetries.

4) reflection symmetry

$$X_{\alpha,\beta}(u)_{cd}^{ab} = X_{\beta,\alpha}(u)_{ab}^{cd}. \tag{6.1.6}$$

5) parity symmetry

$$X_{\alpha,\beta}(u)_{cd}^{ab} = X_{\beta,\alpha}(u)_{dc}^{ba} \cdot x^{(b+d-a-c)/2}. \tag{6.1.7}$$

If we set $\alpha = \beta$, then the colored vertex model recovers the standard initial condition.

6) standard initial condition

$$X_{\alpha,\beta=\alpha}(u = 0)_{cd}^{ab} = \delta_{ac}\delta_{bd}. \tag{6.1.8}$$

We note that when $\alpha \neq \beta$, the colored vertex model does not satisfy the standard initial condition. The Boltzmann weights $X_{\alpha,\beta}^{(N)}(u)_{cd}^{ab}$ of the colored vertex model at

$u = 0$ give a permutation operator $\pi^{(N)}_{\alpha,\beta}$ due to the first inversion relation (6.1.4). The permutation operator $\pi^{(N)}_{\alpha,\beta}$ acts on the colored strings that have the color variables α and β. We call the operator colored permutation operator.

The colored vertex models have invariant transformations on the Boltzmann weights. The transformed Boltzmann weights also satisfy the colored Yang-Baxter relation (6.1.1). Let us formulate the transformation.

7) invariant transformation (symmetry breaking transformation [90])

$$X_{\alpha\beta}(u)^{ab}_{cd} \to \exp(\mu(a+c-b-d)u)\frac{F(\alpha;a)F(\beta;b)}{F(\beta;c)F(\alpha;d)}X_{\alpha\beta}(u)^{ab}_{cd}. \tag{6.1.9}$$

Here $F(\gamma;a)$ is an arbitrary function of the color variables γ and the state variable a. The variable μ is a free parameter. The reflection symmetry (6.1.6) and the parity symmetry (6.1.7) can be modified by the transformation (6.1.9).

7 Colored link polynomials

7.1 Colored tangles

As in the reference [96] we introduce a category of colored oriented tangles and a category of colored oriented tangle diagrams.

A *(k, l)-tangle* T is a finite set of disjoint oriented arcs and circles properly embedded (up to isotopy) in $\mathbf{R}^2 \times [0,1]$ such that

$$\partial T = \{(i,0,0); i = 1,2,\dots,k\} \cup \{(j,0,1); j = 1,2,\dots,l\}, \tag{7.1.1}$$

and such that T is perpendicular to $\mathbf{R}^2 \times 0$ and $\mathbf{R}^2 \times 1$. A *colored (k, l)-tangle* (T, α) is a (k, l)-tangle $T = T_1 \cup \dots \cup T_n$ with color $\alpha = (\alpha_1, \dots, \alpha_n)$; each component T_j has a color α_j which is a complex parameter.

To each colored (k, l)-tangle (T, α), we define two sequences of numbers $\partial_+(T,\alpha) = ((\nu_1,\dots,\nu_l),(\gamma_1,\dots,\gamma_l))$ and $\partial_-(T,\alpha) = ((\varepsilon_1,\dots,\varepsilon_k),(\beta_1,\dots,\beta_k))$. For $\partial_+(T,\alpha)$, $\nu_j = 1$ if the tangent vector of T at $(j,0,1)$ is outward with respect to $\mathbf{R}^2 \times [0,1]$, and $\nu_j = -1$ if it is inward. γ_j is the color (a complex number) of a component which bounds $(j,0,1)$. For $\partial_-(T,\alpha)$, $\varepsilon_i = 1$ if the tangent vector at $(i,0,0)$ is inward and $\varepsilon_i = -1$ if it is outward, and β_i is the color at $(i,0,0)$.

We introduce a *composition* $\circ$ and a *tensor product* $\otimes$ on the set of tangles as in Fig 7.1.1.

Fig 7.1.1

The composition $T_1 \circ T_2$ is defined only when $\partial_- T_1 = \partial_+ T_2$.

We define the category $\mathcal{COT}$ of oriented tangles. The objects of $\mathcal{COT}$ are the sequences $((\varepsilon_1,\dots,\varepsilon_k),(\alpha_1,\dots,\alpha_k))$ with $\varepsilon_i = \pm 1$ and complex parameters α_i, including the empty sequence. A morphism of (ε,β) to (ν,γ) is a colored (k, l)-tangle (T,α) such that $\partial_-(T,\alpha) = (\varepsilon,\beta)$ and $\partial_+(T,\alpha) = (\nu,\gamma)$. The tensor product of objects (ε,β) and (ν,γ) is $(\varepsilon\nu,\beta\gamma)$, and the tensor product of morphisms is given by the tensor product of tangles.

We now introduce oriented tangle diagrams. A *(k, l)-diagram* D is a finite set of disjoint oriented arcs and circles immersed in $\mathbf{R} \times [0,1]$ such that

$$\partial D = \{(i,0); i = 1,\dots,k\} \cup \{(j,1); j = 1,\dots,l\}, \tag{7.1.2}$$

and D is perpendicular to $\mathbf{R} \times 0$ and $\mathbf{R} \times 1$; at each point of self-intersection just two branches meet transversally with upward orientations, one of which is located on top and the other below. In other words a diagram is obtained from generators in Fig 7.1.2 by using operations "composition" and "tensor product". A colored (k, l)-diagram (D,α), the operations composition and tensor product on diagrams, and notations $\partial_+ D$ and $\partial_- D$ are defined in the same way as those of tangles.

Fig 7.1.2

We define the category $\mathcal{COD}$ of colored oriented tangle diagrams. The objects of $\mathcal{COD}$ are equal to the objects of $\mathcal{COT}$. A morphism of (ε,β) to (ν,γ) is an equivalence class of colored (k, l)-diagrams such that $\partial_- D = (\varepsilon,\beta)$ and $\partial_+ D = (\nu,\gamma)$, with an equivalence relation generated by relations D1,...,D6 in Fig 7.1.3. The operations composition and tensor product on $\mathcal{COT}$ carry over to the case of $\mathcal{COD}$.

Fig 7.1.3

There is a natural covariant functor $\mathcal{F} : \mathcal{COD} \to \mathcal{COT}$ defined by passing from a diagram to the tangle represented by it.

Theorem 7.1 ([96]) *The functor $\mathcal{F}$ is isomorphic, i.e. $\mathcal{F}$ induces a bijection of the morphisms of $\mathcal{COD}$ to the morphisms of $\mathcal{COT}$.*

Remark 7.2 *The tangle T in Fig 7.1.4 corresponds to the diagram D (note that the tangent vectors must be upward at each crossing in $\mathcal{COD}$).*

Fig 7.1.4

7.2 Colored braid matrices

In this subsection we give colored braid matrices which are solutions of the colored braid relation. We will use colored braid matrices to construct invariants of colored tangles in the next section. The colored braid matrices $G^{ab}_{cd}(\alpha,\beta;\pm)$ defined below correspond to two vertex configurations shown in Fig 7.2.1 (a) and (b), respectively.

Fig 7.2.1

In Fig. 7.2.1 (a), the variables (charges) a and d are assigned on the α-string, and b and c are assigned on the β-string. Here the "α-string" means that the string has a color α.

We introduce some symbols for q-analogs. For a non-negative integer n we define q-analog of factorial by

$$\begin{aligned}(z;n)_q &= (1-z)(1-zq)\cdots(1-zq^{n-1}) \\ &= \prod_{k=0}^{n-1}(1-zq^k), \quad \text{for } n>0, \\ &= 1, \quad \text{for } n=0. \end{aligned} \tag{7.2.1}$$

We introduce q-analog of combinatorial for two non-negative integers m and n:

$$\begin{aligned}\begin{bmatrix} m \\ n \end{bmatrix}_q &= \frac{(q;m)_q}{(q;m-n)_q(q;n)_q} \quad \text{for } m-n\geq 0, \\ &= 0, \quad \text{for } m-n<0. \end{aligned} \tag{7.2.2}$$

It is sometimes convenient to use the following symbol.

$$\begin{pmatrix} m \\ n \end{pmatrix}_{z,q} = \frac{(z;m)_q}{(z;n)_q}, \quad \text{for } m,n\geq 0. \tag{7.2.3}$$

Hereafter the symbol ω represents a primitive N-th root of unity.

We now define the colored braid matrix $G^{ab}_{cd}(\alpha,\beta;\pm)$. Let N be an integer greater than or equal to two. In the colored braid matrices, the variables (charges) a,b,c,d take N values $0,1,\ldots,N-1$, and the colors α,β are complex parameters. The colored braid matrix has the *charge conservation* condition

$$G^{ab}_{cd}(\alpha,\beta;\pm) = 0, \text{ unless } a+b=c+d. \tag{7.2.4}$$

The matrix elements for $a+b=c+d$ are given in the following. [28]

$$\begin{aligned} & G^{ab}_{cd}(\alpha,\beta;+) \\ &= \begin{bmatrix} a \\ d \end{bmatrix}_\omega \begin{pmatrix} c \\ b \end{pmatrix}_{\beta,\omega} \beta^d\omega^{bd}\cdot\alpha^{\mu b+\nu c}\beta^{-\mu d-\nu a} \\ &\quad \times f(\alpha,\beta,\omega)\frac{F(\alpha,a)F(\beta,b)}{F(\alpha,d)F(\beta,c)}\cdot e^{\eta(a+d-b-c)+\kappa(ab-cd)}. \end{aligned} \tag{7.2.5}$$

$$\begin{aligned} & G^{ab}_{cd}(\alpha,\beta;-) \\ &= \begin{bmatrix} b \\ c \end{bmatrix}_{1/\omega} \begin{pmatrix} d \\ a \end{pmatrix}_{1/\beta,1/\omega} \beta^{-c}\omega^{-ac}\cdot\beta^{\mu b+\nu c}\alpha^{-\mu d-\nu a} \\ &\quad \times f(\alpha,\beta,\omega)^{-1}\frac{F(\alpha,b)F(\beta,a)}{F(\alpha,c)F(\beta,d)}\cdot e^{\eta(a+d-b-c)+\kappa(ab-cd)}. \end{aligned} \tag{7.2.6}$$

Let V be an N-dimensional vector space over $\mathbf{C}$ with basis $e_0,e_1,\ldots,e_{N-1}$. We define $G(\alpha,\beta;\pm)\in \mathrm{Hom}(V\otimes V,V\otimes V)$ by

$$G(\alpha,\beta;\pm) : e_c\otimes e_d \mapsto \sum_{a,b} G^{ab}_{cd}(\alpha,\beta;\pm)e_a\otimes e_b. \tag{7.2.7}$$

Remark 7.3 *By using charge conservation, we can show that values of the invariants defined in §7.4 do not depend on* F,μ,ν,η,κ.

For the colored braid matrices we can show the following propositions.

Proposition 7.4 (colored braid relation)

$$\begin{aligned} & (G(\alpha,\beta;+)\otimes id_V)(id_V\otimes G(\alpha,\gamma;+))(G(\beta,\gamma;+)\otimes id_V) \\ &= (id_V\otimes G(\beta,\gamma;+))(G(\alpha,\gamma;+)\otimes id_V)(id_V\otimes G(\alpha,\beta;+)) \end{aligned} \tag{7.2.8}$$

We call this relation colored braid relation. This proposition is proved in Appendix A.

Proposition 7.5 ([28])

(i) *First inversion relation:*

$$G(\alpha,\beta;+)G(\alpha,\beta;-) = G(\alpha,\beta;-)G(\alpha,\beta;+) = id_{V\otimes V}. \tag{7.2.9}$$

(ii) *Second inversion relation:*

$$\begin{aligned} \sum_{e,f} G^{ae}_{cf}(\alpha,\beta;+)G^{df}_{be}(\alpha,\beta;-)\omega^{d-e} &= \delta^a_b\delta^c_d, \\ \sum_{e,f} G^{ae}_{cf}(\beta,\alpha;-)G^{df}_{be}(\beta,\alpha;+)\omega^{d-e} &= \delta^a_b\delta^c_d. \end{aligned} \tag{7.2.10}$$

(iii) *Markov trace property:*

$$\sum_b G^{ab}_{ab}(\alpha, \alpha; +)\omega^{-b} = \alpha^{-(N-1)/2}, \tag{7.2.11}$$

$$\sum_b G^{ab}_{ab}(\alpha, \alpha; -)\omega^{-b} = \alpha^{-(N-1)/2}. \tag{7.2.12}$$

In the equations (7.2.11) and (7.2.12) we have assumed that the normalization of the colored braid matrices (7.2.5) satisfies

$$f(\alpha, \alpha; \omega) = \alpha^{-(N-1)/2}. \tag{7.2.13}$$

The first and second inversion relations (7.2.9) and (7.2.10), and the Markov trace property (7.2.11) were proved in the reference [28]. We show the proof of this proposition (the relations (7.2.9) ~ (7.2.11)) in Appendix B.

7.3 Invariants of colored oriented tangles

In this subsection we define invariants of colored oriented tangles by using the colored braid matrices $G(\alpha, \beta; \pm)$ given in the previous section. In §7.3 and §7.4 we assume the normalization of the braid matrices as $f(\alpha, \beta; \omega) = \alpha^{-(N-1)/4}\beta^{-(N-1)/4}$.

Let N and V be as in the previous section. The symbol V^* denotes the dual vector space of V, and $\{e_i^*\}$ the dual basis of $\{e_i\}$.

We define the category $\mathcal{L}$ of linear maps. The objects of $\mathcal{L}$ are the vector spaces $V^{\varepsilon_1} \otimes \ldots \otimes V^{\varepsilon_k}$ with $\varepsilon_i = \pm 1$ and $k \geq 0$, where $V^1 = V$ and $V^{-1} = V^*$. The morphisms of $\mathcal{L}$ are the linear maps between two objects. The operation composition is the composition of linear maps, and the operation tensor product is the tensor product of vector spaces and linear maps.

We define a functor ϕ of $\mathcal{COD}$ to $\mathcal{L}$. An object $((\varepsilon_1, \ldots, \varepsilon_k), (\alpha_1, \ldots, \alpha_k))$ of $\mathcal{COD}$ is mapped by ϕ to an object $V^{\varepsilon_1} \otimes \ldots \otimes V^{\varepsilon_k}$ of $\mathcal{L}$. The generators of morphisms of $\mathcal{COD}$ in Fig 7.3.1 are mapped as follows.

Fig 7.3.1

$$\begin{aligned}
\phi(I) &= \mathrm{id}_V \in \mathrm{Hom}(V, V), \\
\phi(I^*) &= \mathrm{id}_{V^*} \in \mathrm{Hom}(V^*, V^*), \\
\phi(U_r) &= \alpha^{-(N-1)/4} \sum_{a=0}^{N-1} \omega^{a/2} e^*_{N-1-a} \otimes e_a \in \mathrm{Hom}(\mathbf{C}, V^* \otimes V) = V^* \otimes V, \\
\phi(U_l) &= \alpha^{(N-1)/4} \sum_{a=0}^{N-1} \omega^{-a/2} e_a \otimes e^*_{N-1-a} \in \mathrm{Hom}(\mathbf{C}, V \otimes V^*) = V \otimes V^*,
\end{aligned}$$

$$\begin{aligned}
\phi(\bar{U}_r) &= \alpha^{(N-1)/4} \sum_{a=0}^{N-1} \omega^{-a/2} e^*_a \otimes e_{N-1-a} \in \mathrm{Hom}(V \otimes V^*, \mathbf{C}) = V^* \otimes V, \\
\phi(\bar{U}_l) &= \alpha^{-(N-1)/4} \sum_{a=0}^{N-1} \omega^{a/2} e_{N-1-a} \otimes e^*_a \in \mathrm{Hom}(V^* \otimes V, \mathbf{C}) = V \otimes V^*, \\
\phi(X_+) &= G(\alpha, \beta; +) \in \mathrm{Hom}(V \otimes V, V \otimes V) \\
\phi(X_-) &= G(\alpha, \beta; -) \in \mathrm{Hom}(V \otimes V, V \otimes V).
\end{aligned} \tag{7.3.1}$$

We request ϕ to be equivariant with respect to the operations composition and tensor product. So we can calculate the value of ϕ for any tangle diagram in $\mathcal{COD}$.

Theorem 7.6 *The functor $\phi : \mathcal{COD} \to \mathcal{L}$ is well-defined.*

Proof. It is sufficient to check that ϕ is invariant under the moves D1,...,D6. The invariance under D1 immediately follows by the definition of ϕ. The invariance under D2 is obtained by using charge conservation. We obtain D3,D4,D5 by Proposition 3.3 (i)(ii)(iii) respectively. D6 follows by Proposition 3.2. □

Definition 7.7 *We also write $\phi(D, \alpha)$ as $\phi(T, \alpha)$ if (T, α) is a colored tangle represented by a colored diagram (D, α). Then $\phi(T, \alpha)$ is an isotopy invariant of a colored tangle (T, α).*

We obtain invariants of colored tangles. However these invariants vanish for $(0, 0)$-tangles, that is, links.

Proposition 7.8 *For any colored $(0, 0)$-tangle (T, α),*

$$\phi(T, \alpha) = 0 \in \mathit{Hom}(\mathbf{C}, \mathbf{C}) = \mathbf{C}. \tag{7.3.2}$$

Proof. For any $(0, 0)$-tangle T, there exists some $(1, 1)$-tangle T_1 such that T is obtained by closing open string of T_1 as in Fig 7.3.2.

Fig 7.3.2

By Lemma 7.9 below, we have $\phi(T, \alpha) = \lambda \cdot \mathrm{id}_V$ with some λ. So we can calculate the invariant of T as follows.

$$\phi(T, \alpha) = \sum_{a=0}^{N-1} \alpha^{(N-1)/2} \omega^{-a} \lambda = 0 \tag{7.3.3}$$

(recall that ω is an N-th root of unity.) □

Lemma 7.9 *For any colored* $(1,1)$*-tangle* (T, α),

$$\phi(T, \alpha) = \lambda \cdot \ id_V \in \ Hom(V, V) \tag{7.3.4}$$

with some scalar λ.

Proof. By charge conservation we can put

$$\phi(T, \alpha) = \sum \lambda_i e_i^* \otimes e_i. \tag{7.3.5}$$

Fig 7.3.3

Since two $(2,2)$-tangles in Fig 7.3.3 give the same invariant, we have

$$\lambda_a G_{cd}^{ab}(\alpha, \beta; +) = G_{cd}^{ab}(\alpha, \beta; +)\lambda_d \quad \text{for any } a, b, c, d. \tag{7.3.6}$$

Hence we have $\lambda_i = \lambda_j$ for any i, j. This completes the proof. □

7.4 Invariants of colored links

We define another invariant of a link through $(1,1)$-tangle.

Let T be a $(1,1)$-tangle. We assume that $\hat{T}$ is the link (in S^3) obtained by closing the open string of T.

Proposition 7.10 *Let* T_1 *and* T_2 *are two* $(1,1)$*-tangles. If* $\hat{T}_1$ *is isotopic to* $\hat{T}_2$ *as a link in* S^3 *by an isotopy which carries the closing component of* $\hat{T}_1$ *to that of* $\hat{T}_2$*. Then* T_1 *is isotopic to* T_2 *as a* $(1,1)$*-tangle.*

Proof. Put $L_1 = \hat{T}_1$. We can obtain T_1 from L_1 as $T_1 = L_1 \cap (S^3 - B_1)$ where B_1 is a small 3-ball in S^3 on a closing component of $\hat{T}_1 = L_1$. In the same way we can put $T_2 = L_2 \cap (S^3 - B_2)$. Since B_1 and B_2 are on the corresponding component of L_1 and L_2, we may assume that the isotopy between L_1 and L_2 carries B_1 to B_2. It follows that T_1 is isotopic to T_2 with the isotopy. □

Let T be a $(1,1)$-tangle. We put $L = \hat{T} = L_1 \cup \ldots \cup L_n$ and L_s is the closing component of $\hat{T}$.

Definition 7.11 *For a colored link* (L, α) *and a component* L_s*, we define* Φ *by* $\Phi(L, s, \alpha) = \lambda$ *where* L, T, s *are as above and* $\phi(T, \alpha) = \lambda \cdot \ id_V$ *by Lemma 4.4.*

By Proposition 7.10, Φ is well-defined, i.e. $\Phi(L, s, \alpha)$ does not depend on a choice of T.

Further we have the next proposition, to obtain invariants which do not depend on s.

Proposition 7.12 *For a link* $L = L_1 \cup \ldots \cup L_n$ *and its color* $\alpha = (\alpha_1, \ldots, \alpha_n)$*, we have the next formula.*

$$\Phi(L, s, \alpha)(\alpha_s; N-1)_\omega^{-1} \alpha_s^{(N-1)/2} = \Phi(L, s', \alpha)(\alpha_{s'}; N-1)_\omega^{-1} \alpha_{s'}^{(N-1)/2} \tag{7.4.1}$$

This proposition is proved in Appendix C. By this proposition we obtain the next definition.

Definition 7.13 *For a colored link* (L, α)*, we define an isotopy invariant* $\hat{\Phi}$ *of* (L, α) *by*

$$\hat{\Phi}(L, \alpha) = \Phi(L, s, \alpha)(\alpha_s; N-1)_\omega^{-1} \alpha_s^{(N-1)/2}. \tag{7.4.2}$$

Remark 7.14 *For a link* L *which has at least two components,* $\hat{\Phi}(L, \alpha)$ *is a polynomial in* $\alpha_i^{1/2}$ *and* $\alpha_i^{-1/2}$*. However for a knot* K*,* $\hat{\Phi}(K, \alpha)$ *is not necessarily a polynomial.*

Remark 7.15 *If we renormalize the colored braid matrix by modifying the factor* $f(\alpha, \beta; \omega)$ *such that* $f(\alpha, \alpha; \omega) = \alpha^{-(N-1)/2}$ *holds, then the corresponding change in the colored link invariant* $\phi(L, \alpha)$ *is determined by the linking matrix of the link* L.

We discuss symmetries of the colored braid matrices such as the crossing symmetry in appendix D.

Let us discuss connection of the new colored link invariants to the multivariable Alexander polynomial. It was shown by J. Murakami [71] that a colored link invariant which corresponds to $\hat{\Phi}(L, \alpha)$ for the N=2 case is a version of the multivariable Alexander polynomial (the Conway potential function [21]). Therefore the new colored link invariants $\hat{\Phi}(L, \alpha)$ for $N = 3, 4, \cdots$, are generalizations of the multivariable Alexander polynomial.

Part III

From Quantum Groups to Colored Graph Invariants

8 Colored Braid Matrices from $U_q(sl(2))$

8.1 Infinite dimensional representations of $U_q(sl(2))$

Let us introduce the algebra $U_q(sl(2))$.[36,45] We follow the notation in the reference [48]. The generators of the algebra are $\{X^+, X^-, K^{\pm 1}\}$ with the defining relations

$$KX^{\pm}K^{-1} = q^{\pm 2}X^{\pm}, \quad [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{8.1.1}$$

The comultiplication is given by

$$\begin{aligned} \Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(X^+) = X^+ \otimes I + K \otimes X^+, \\ \Delta(X^-) &= X^- \otimes K^{-1} + I \otimes X^-. \end{aligned} \tag{8.1.2}$$

We use the following symbols for q-analogs.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \quad (z;q)_n = \prod_{k=0}^{n-1} (1 - zq^k). \tag{8.1.3}$$

We assume $[0]_q! = (z;q)_0 = 1$. The universal R matrix of $U_q(sl(2))$ is given by

$$\mathcal{R} = q^{-H \otimes H/2} \exp_q\left(-(q - q^{-1})K^{-1}X^+ \otimes X^- K\right), \tag{8.1.4}$$

where

$$\exp_q(x) = \sum_{n=0}^{\infty} q^{-n(n-1)/2} \frac{1}{[n]!} x^n. \tag{8.1.5}$$

The operator H is related to the generator K by $K = q^H$. We define permutation operator τ by $\tau(t_1 \otimes t_2) = t_2 \otimes t_1$, for $t_1, t_2 \in U_q(sl(2))$. The universal R matrix satisfies the following.

$$\mathcal{R}\Delta(a) = \tau \circ \Delta(a)\mathcal{R}, \quad a \in U_q(sl(2)). \tag{8.1.6}$$

The R matrix for representations (π_i, V_i) for $i = 1, 2$ is defined by

$$R_{V_1,V_2} = P_{V_1,V_2} \circ (\pi_1 \otimes \pi_2(\mathcal{R})), \tag{8.1.7}$$

where P_{V_1,V_2} is given by $P_{V_1,V_2}(e_1 \otimes e_2) = e_2 \otimes e_1$ for $e_1 \in V_1$ and $e_2 \in V_2$.

Let us define an infinite dimensional representation $(\pi^{\alpha}, V^{(\infty)})$ of $U_q(sl(2))$. We assume α is a complex parameter. Let $V^{(\infty)}$ be an infinite dimensional vector space over $\mathbf{C}$ with basis $e_0, e_1, \cdots$, where e_a is a basis vector in $V^{(\infty)}$ with the property $(e_a)_\mu = \delta_{\mu a}$. We define matrix elements of the representations of the generators X^+, X^- for $(\pi^{\alpha}, V^{(\infty)})$ as follows.

$$\begin{aligned} (\pi^{\alpha}(X^+)_q)^a_b &= [p - a]_q \cdot \delta_{a+1,b} \cdot \frac{F(\alpha, a)}{F(\alpha, b)}, \\ (\pi^{\alpha}(X^-)_q)^a_b &= [a]_q \cdot \delta_{a-1,b} \frac{F(\alpha, a)}{F(\alpha, b)}, \\ (\pi^{\alpha}(K)_q)^a_b &= q^{(p-2a)} \cdot \delta_{a,b}. \end{aligned} \tag{8.1.8}$$

Here a, b are nonnegative integers $(a, b = 0, 1, \cdots)$, p is a complex parameter $(p \in \mathbf{C})$ and α is given by $\alpha = q^p$. We remark that the factor $F(\alpha, a)$ corresponds to the gauge factor in the case of the Boltzmann weights of exactly solvable models. [28] It is easy to see that the operators defined in (8.1.8) satisfy the defining relations of the algebra $U_q(sl(2))$ with q generic. We call the representation in (8.1.8) infinite dimensional color representation.

Let $R(\alpha_1, \alpha_2; q)$ denote the R matrix of the infinite dimensional representations $(\pi^{\alpha_1}, V_1^{(\infty)})$ and $(\pi^{\alpha_2}, V_2^{(\infty)})$. The action of $R(\alpha, \beta; q) \in \mathrm{Hom}(V_1^{(\infty)} \otimes V_2^{(\infty)}, V_2^{(\infty)} \otimes V_1^{(\infty)})$ is given by the following.

$$R(\alpha, \beta; q) : e_c^{(1)} \otimes e_d^{(2)} \mapsto \sum_{a,b} R^{ab}_{cd}(\alpha_1, \alpha_2; q) e_b^{(2)} \otimes e_a^{(1)}, \tag{8.1.9}$$

where $e_a^{(1)}, e_c^{(1)} \in V_1^{(\infty)}$ and $e_b^{(2)}, e_d^{(2)} \in V_2^{(\infty)}$. Here the matrix elements $R^{a_1,a_2}_{b_1,b_2}(\alpha_1, \alpha_2; q)$ are given by

$$R^{a_1,a_2}_{b_1,b_2}(\alpha_1, \alpha_2; q) = (\pi^{\alpha_1} \otimes \pi^{\alpha_2}(\mathcal{R}))^{a_1 a_2}_{b_1 b_2}. \tag{8.1.10}$$

It is easy to show the following

$$\begin{aligned} (\pi^{\alpha}(X^+)^n)^a_b &= \prod_{k=0}^{n-1} [p - a - k]_q \cdot \delta_{a+n+1,b} \cdot \frac{F(\alpha, a)}{F(\alpha, b)}, \\ (\pi^{\alpha}(X^-)^n)^a_b &= \prod_{k=0}^{n-1} [a - k]_q \cdot \delta_{a-n-1,b} \cdot \frac{F(\alpha, a)}{F(\alpha, b)}. \end{aligned} \tag{8.1.11}$$

Using the relations (8.1.11) we can calculate the matrix elements $R^{a_1,a_2}_{b_1,b_2}(\alpha_1, \alpha_2; q)$.

Proposition 8.1 *The matrix elements $R^{a_1 a_2}_{b_1 b_2}(\alpha_1, \alpha_2; q)$ are given by*

$$R^{a_1,a_2}_{b_1,b_2}(\alpha_1, \alpha_2; q) = \sum_{n=0}^{\infty} q^{-(p_1 - 2a_1)(p_2 - 2a_2)/2 - n(p_1 - 2a_1) + n(p_2 - 2b_2)}$$

$$
\begin{aligned}
& \times \quad (-1)^n (q-q^{-1})^n q^{-n(n-1)/2} \\
& \times \quad \frac{[a_2]_q!}{[n]_q![a_2-n]_q!} \prod_{k=0}^{n-1} [p_1 - a_1 - k]_q \cdot \delta_{a_1+n,b_1} \delta_{a_2-n,b_2} \\
& \times \quad \frac{F(\alpha_1, a_1) F(\alpha_2, a_2)}{F(\alpha_2, b_2) F(\alpha_1, b_1)}. \qquad (8.1.12)
\end{aligned}
$$

We remark that setting $p_i = j_i$ where j_i is a half integer, replacing $V_i^{(\infty)}$ by $V_i^{(N_i)}$ ($N_i = 2j_i + 1$), and choosing the gauge factor, we have from (8.1.12) the R matrix [59] for the spin representations j_1 and j_2 (see [28]).

8.2 Colored braid matrix as an infinite dimensional R matrix

Let us introduce an infinite-state colored braid matrix $\hat{G}^{ab}_{cd}(\alpha, \beta; t)$. The variables (charges) a, b, c, d take infinite values $a = 0, 1, \cdots$, and the colors α, β and the variable t are complex parameters. [1,28,2] The infinite state colored braid matrix has the *charge conservation* condition: $\hat{G}^{ab}_{cd}(\alpha, \beta; t) = 0$, unless $a+b = c+d$. We also assume $\hat{G}^{ab}_{cd}(\alpha, \beta; t) = 0$ for $a < d$. The non-zero matrix elements for $a + b = c + d$ are given in the following.

$$
\begin{aligned}
\hat{G}^{ab}_{cd}(\alpha, \beta; t) &= \frac{(t;t)_a}{(t;t)_{a-d}(t;t)_d} \frac{(\beta;t)_c}{(\beta;t)_b} \alpha^{b/2} \beta^{d/2} t^{bd} \cdot \alpha^{\mu b + \nu c} \beta^{-\mu d - \nu a} \\
&\quad \times f(\alpha, \beta, t) \frac{E(\alpha, a) E(\beta, b)}{E(\alpha, d) E(\beta, c)} \cdot e^{\eta(a+d-b-c) + \kappa(ab-cd)}. \qquad (8.2.1)
\end{aligned}
$$

Here t is indefinite. The expression (8.2.1) is equivalent to the formula for the weights $W(+)$ given in the reference [28].

Let $V^{(\infty)}$ be the infinite dimensional vector space over $\mathbf{C}$, with basis $e_0, e_1, \cdots$. We define $\hat{G}(\alpha, \beta; t) \in \mathrm{Hom}(V^{(\infty)} \otimes V^{(\infty)}, V^{(\infty)} \otimes V^{(\infty)})$ by

$$
\hat{G}(\alpha, \beta; t) : e_c \otimes e_d \mapsto \sum_{a,b} \hat{G}^{ab}_{cd}(\alpha, \beta; t) e_a \otimes e_b. \qquad (8.2.2)
$$

It has been proved that the infinite-state colored braid matrix satisfies the colored braid relation. [2]

Proposition 8.2 (colored braid relation for the infinite-state colored braid matrix)

$$
\begin{aligned}
& (\hat{G}(\alpha, \beta; t) \otimes \mathit{id}_V)(\mathit{id}_V \otimes \hat{G}(\alpha, \gamma; t))(\hat{G}(\beta, \gamma; t) \otimes \mathit{id}_V) \\
= \; & (\mathit{id}_V \otimes \hat{G}(\beta, \gamma; t))(\hat{G}(\alpha, \gamma; t) \otimes \mathit{id}_V)(\mathit{id}_V \otimes \hat{G}(\alpha, \beta; t)). \qquad (8.2.3)
\end{aligned}
$$

Let us introduce a matrix $\check{R}$ by $\check{R}^{ab}_{cd}(\alpha_1, \alpha_2; q) = R^{ba}_{cd}(\alpha_1, \alpha_2; q)$. By comparing the explicit matrix elements, we have the following theorem.

Theorem 8.3 *The matrix elements $\check{R}^{ab}_{cd}(\alpha_1, \alpha_2; q)$ for the infinite dimensional color representation of $U_q(sl(2))$ are equivalent to the matrix elements $\hat{G}^{ab}_{cd}(\alpha, \beta; t)$ of the infinite-sate colored braid matrix by setting $t = q^{-2}$, $\alpha = q^{2p_2}$, $\beta = q^{2p_1}$, $\mu = -1/2$, $\nu = 1/2$, $\eta = 0$, $\kappa = \log(q)$, $f = q^{-p_1 p_2/2}$ and $F(\alpha; a) = E(\alpha; a)$.*

Thus we have derived the infinite-state colored braid matrix from the quantum group $U_q(sl(2))$.

8.3 Finite dimensional color representations

Let us discuss restriction of the infinite dimensional representation into finite dimensional ones. Let ω be a primitive N-th root of unity:

$$
\omega = \exp(\frac{2\pi i s}{N}), \qquad (N, s) = 1. \qquad (8.3.1)
$$

Here we recall that the symbol $(a, b) = 1$ means that the integers a and b have no common divisor except 1. Let ϵ denote a square root of ω: $\epsilon = \exp(\pi i s/N)$, $\quad (N, s) = 1$. Hereafter we assume that the gauge factor is given by $F(\alpha, a) = ([p-a]_q![a]_q!)^{-1/2}$.

We take the limit $q \to \epsilon$ in the infinite dimensional representation (8.1.8). Then we have

$$
\lim_{q \to \epsilon} (\pi^\alpha(X^-)_q)^N_{N-1} = 0. \qquad (8.3.2)
$$

From (8.3.2) we can restrict the infinite dimensions into N dimensions: $V^{(\infty)} \to V^{(N)}$, where $V^{(N)}$ is an N-dimensional vector space with basis $e_0, \cdots, e_{N-1}$. For $a, b = 0, 1, \cdots N-1$ we have the following matrices

$$
\begin{aligned}
(\pi^\alpha(X^+)_{q=\epsilon})^a_b &= ([p-a]_\epsilon [a+1]_\epsilon)^{1/2} \cdot \delta_{a+1,b}, \\
(\pi^\alpha(X^-)_{q=\epsilon})^a_b &= ([p-a+1]_\epsilon [a]_\epsilon)^{1/2} \cdot \delta_{a-1,b}, \\
(\pi^\alpha(K)_{q=\epsilon})_{ab} &= \epsilon^{(p-2a)} \cdot \delta_{a,b}. \qquad (8.3.3)
\end{aligned}
$$

From the fact that there is no singularity in the limiting process $q \to \epsilon$ that yields (8.3.3), we have the following.

Proposition 8.4 *The matrix representations (8.3.3) of the generators $X^\pm, K^{\pm 1}$ satisfy the defining relations of the $U_q(sl(2))$ with $q = \epsilon$.*

Thus we have a finite dimensional representation of $U_q(sl(2))$ with $q = \epsilon$. We call the representation (finite dimensional) color representation. We assume that the notation $(\pi^{(\alpha,\epsilon)}, V^{(N)})$ represents the color representation.

Let us discuss R matrix of the color representation. We first consider restriction of the infinite dimensional R matrix (8.1.12) into a finite dimensional one. We use the following (see [2]).

$$\frac{[m]_\epsilon!}{[m-n]_\epsilon![n]_\epsilon!} = 0, \quad \text{for} \quad m \geq N > \quad max\{n, m-n\}. \tag{8.3.4}$$

By taking the limit $q \to \epsilon$ and using (8.3.4), the infinite dimensional representation (8.1.12) of the universal $\mathcal{R}$ matrix essentially reduces to a finite dimensional matrix (see [2] for the colored braid matrix): $R^{a_1,a_2}_{b_1,b_2}(\alpha_1, \alpha_2; q \to \epsilon) = 0$, unless $a_i, b_i \leq N$. We define matrix elements $R^{ab}_{cd}(\alpha_1, \alpha_2)^{(N)}$ by $R^{ab}_{cd}(\alpha_1 \alpha_2)^{(N)} = R^{ab}_{cd}(\alpha_1, \alpha_2; q \to \epsilon)$ where $a, b, c, d = 0, 1, \cdots, N-1$. We define the action of $R(\alpha_1, \alpha_2)^{(N)}$ in the same way as (2.9). Since both the matrix $R(\alpha_1, \alpha_2)^{(N)}$ and the color representations $(\pi^{(\alpha_i,\epsilon)}, V_i^{(N)})$ $(i = 1, 2)$ are obtained from the infinite dimensional representation $(\pi^{\alpha_i}, V_i^{(\infty)})$ by taking the limit $q \to \epsilon$, we can show the following relation.

$$R(\alpha_1, \alpha_2)^{(N)}(\pi_1 \otimes \pi_2(\Delta(a))) = (\pi_2 \otimes \pi_1(\Delta(a)))R(\alpha_1, \alpha_2)^{(N)} \quad \text{for } a = X^{\pm}, K^{\pm 1}, \tag{8.3.5}$$

where $\pi_i = \pi^{(\alpha_i,\epsilon)}$ for $i = 1, 2$. Thus we have the following.

Theorem 8.5 *The matrix $R(\alpha_1, \alpha_2)^{(N)}$ gives the R matrix of the color representations $(\pi^{(\alpha_i,\epsilon)}, V_i^{(N)})$ $(i = 1, 2)$ of $U_q(sl(2, \mathbf{C}))$ with $q = \epsilon$.*

Thus we have shown that the N-state colored braid matrix corresponds to the R matrix for the color representation.

9 CG coefficients of color representations of $U_q(sl(2))$

9.1 CGC for infinite dim. representations of $U(sl(2))$

Let us discuss representations of the universal enveloping algebra $U(sl(2))$. The defining relations of the algebra $U(sl(2))$ are given by the following.

$$\begin{aligned} [H, X^{\pm}] &= \pm 2X^{\pm}, \\ [X^+, X^-] &= H. \end{aligned} \tag{9.1.1}$$

The comultiplication is given by

$$\begin{aligned} \Delta(X^{\pm}) &= X^{\pm} \otimes I + I \otimes X^{\pm}, \\ \Delta(H) &= H \otimes I + I \otimes H. \end{aligned} \tag{9.1.2}$$

Let us define infinite dimensional representation $(\pi^p, V^{(\infty)})$ of $sl(2)$. Let $V^{(\infty)}$ be an infinite dimensional vector space over $\mathbf{C}$ with basis $e_0, e_1, \cdots$, where e_a is a basis vector in $V^{(\infty)}$ with the property $(e_a)_b = \delta_{ba}$. We define the matrix elements of the generators as follows.

$$\begin{aligned} (\pi^p(X^+))^a_b &= \sqrt{(2p-a)(a+1)} \cdot \delta_{a+1,b}, \\ (\pi^p(X^-))^a_b &= \sqrt{a(2p-a+1)} \cdot \delta_{a-1,b}, \\ (\pi^p(H))^a_b &= (2p-2a) \cdot \delta_{a,b}. \end{aligned} \tag{9.1.3}$$

The variable a takes a nonnegative integer; $a = 0, 1, \cdots$. The basis vectors for the infinite dimensional representation are labeled by (p, z), where $p \in \mathbf{C}$ and $z \in \mathbf{Z}_{\geq 0}$. Here the symbol $\mathbf{Z}_{\geq 0}$ represents the set of nonnegative integers. We write the representation $(\pi^p, V^{(\infty)})$ as $V^{(\infty)}(p)$, and the basis vector e_z as $|p, z>$. In terms of the basis vector the representation (9.1.3) is expressed as follows.

$$\begin{aligned} X^+|p, z> &= \sqrt{(2p-z+1)z}|p, z-1>, \\ X^-|p, z> &= \sqrt{(2p-z)(z+1)}|p, z+1>, \\ H|p, z> &= (2p-2z)|p, z>. \end{aligned} \tag{9.1.4}$$

Let us discuss decomposition of the tensor product $V^{(\infty)}(p_1) \otimes V^{(\infty)}(p_2)$. We assume the following

$$|p_1, p_2; p_3, z_3> = \sum_{z_1, z_2} C(p_1, p_2, p_3; z_1, z_2, z_3)|p_1, z_1> \otimes |p_2, z_2>. \tag{9.1.5}$$

From the comultiplication rule $\Delta(H) = H \otimes I + I \otimes H$ $(H_{tot} = H_1 + H_2)$ we can show

$$C(p_1, p_2, p_3; z_1, z_2, z_3) = 0, \text{ unless } \quad p_1 - z_1 + p_2 - z_2 = p_3 - z_3. \tag{9.1.6}$$

The relation (9.1.6) gives the charge conservation law. Using (9.1.6) and the condition $z_i \in \mathbf{Z}_{\geq 0}$, we can show

$$C(p_1, p_2, p_3; z_1, z_2, z_3) = 0, \text{ unless } \quad p_3 = p_1 + p_2 - n, n \in \mathbf{Z}_{\geq 0}. \tag{9.1.7}$$

Thus we have for $n \in \mathbf{Z}_{\geq 0}$

$$\begin{aligned} & |p_1, p_2; p_1 + p_2 - n, z_3 > \\ = & \sum_{z_1=0}^{z_3+n} C(p_1, p_2, p_1 + p_2 - n; z_1, z_3 + n - z_1, z_3)|p_1, z_1 > \otimes |p_2, z_3 + n - z_1 > . \end{aligned} \tag{9.1.8}$$

The fusion rules for the tensor product $V^{(\infty)}(p_1) \otimes V^{(\infty)}(p_2)$ are summarized as follows.

$$V^{(\infty)}(p_1) \otimes V^{(\infty)}(p_2) = \sum_{p_3} N_{p_1 p_2}^{p_3} V^{(\infty)}(p_3) \tag{9.1.9}$$

where

$$\begin{aligned} N_{p_1 p_2}^{p_3} &= 1 \text{ for } p_3 = p_1 + p_2 - n, n \in \mathbf{Z}_{\geq 0}, \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{9.1.10}$$

We assume the following convention of the phase factors of the CG coefficients.

$$C(p_1, p_2, p_1 + p_2 - n; 0, n, 0) = 1. \tag{9.1.11}$$

Then the Clebsch-Gordan coefficients for infinite dimensional representations are given by the following.

$$\begin{aligned} & C(p_1, p_2, p_1 + p_2 - n; z_1, z_2, z_3) = \delta(z_3, z_1 + z_2 - n) \\ & \times \sqrt{(2p_1 + 2p_2 - 2n + 1)} \left(\frac{(2p_1 - n)!(2p_2 - n)!(n)!}{(2p_1 + 2p_2 - n + 1)!} \right)^{1/2} \\ & \times \sqrt{(2p_1 - z_1)!(2p_2 - z_2)!(2p_1 + 2p_2 - 2n - z_3)!(z_1)!(z_2)!(z_3)!} \\ & \times \sum_{\nu} \frac{(-1)^{\nu}}{(\nu)!(n - \nu)!} \frac{1}{(z_1 - \nu)!(z_2 - n + \nu)!} \\ & \times \frac{1}{(2p_1 - n - z_1 + \nu)!(2p_2 - z_2 - \nu)!}. \end{aligned} \tag{9.1.12}$$

Here we have defined the symbol $(p)!$ through the Gamma function

$$(p)! = \Gamma(p + 1). \tag{9.1.13}$$

The sum over the integer ν in (9.1.12) is taken under the following condition

$$\max\{0, n - z_3\} \leq \nu \leq \min\{n, z_1\}. \tag{9.1.14}$$

9.2 Infinite dimensional case of $U_q(sl(2))$

In this section we introduce the following forms of the defining relations of $U_q(sl(2))$. [36,45]

$$\begin{aligned} [H, X^{\pm}] &= \pm 2X^{\pm}, \\ [X^+, X^-] &= \frac{q^H - q^{-H}}{q - q^{-1}}. \end{aligned} \tag{9.2.1}$$

The comultiplication is given by

$$\begin{aligned} \Delta(H) &= H \otimes I + I \otimes H, \\ \Delta(X^{\pm}) &= X^{\pm} \otimes q^{H/2} + q^{-H/2} \otimes X^{\pm}. \end{aligned} \tag{9.2.2}$$

It is noted that the defining relations are slightly different from those in the section 8. But the difference is not important. We use the following symbols for q-analogs.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \prod_{k=1}^{n} [k]_q, \quad [p; n]_q! = \prod_{k=0}^{n-1} [p - k]_q. \tag{9.2.3}$$

Here n is a positive integer. For the case $n = 0$ we assume

$$[0]_q! = [p; 0]_q! = 1. \tag{9.2.4}$$

Let us define an infinite dimensional representation $(\pi_q^p, V^{(\infty)})$ of $U_q(sl(2))$. We assume p is a complex parameter. Let $V^{(\infty)}$ be an infinite dimensional vector space over $\mathbf{C}$ with basis $e_0, e_1, \cdots$, where e_a is a basis vector in $V^{(\infty)}$ with the property $(e_a)_b = \delta_{ba}$. We define matrix elements of the representations of the generators X^+, X^- for $(\pi_q^p, V^{(\infty)})$ as follows.

$$\begin{aligned} (\pi^p(X^+)_q)_b^a &= \sqrt{[2p - a]_q[a + 1]_q} \cdot \delta_{a+1,b}, \\ (\pi^p(X^-)_q)_b^a &= \sqrt{[2p - a + 1]_q[a]_q} \cdot \delta_{a-1,b}, \\ (\pi^p(H)_q)_b^a &= (2p - 2a) \cdot \delta_{a,b}. \end{aligned} \tag{9.2.5}$$

Here a,b are nonnegative integers $(a, b = 0, 1, \cdots)$, and $p \in \mathbf{C}$. It is easy to see that the operators defined in (9.2.5) satisfy the defining relations of the algebra $U_q(sl(2))$ with q generic. We call the representation in (9.2.5) infinite dimensional color representation. The symbols $e_a(p)$ and $V_q^{(\infty)}(p)$ represent the basis vector $|p, z>_q$, and the representation $(\pi_q^p, V^{(\infty)})$, respectively.

Let us discuss decomposition of the tensor product $V_q^{(\infty)}(p_1) \otimes V_q^{(\infty)}(p_2)$. We define the Clebsch-Gordan coefficients by the following

$$|p_1, p_2; p_3, z_3>_q = \sum_{z_1, z_2} C(p_1, p_2, p_3; z_1, z_2, z_3)_q |p_1, z_1>_q \otimes |p_2, z_2>_q . \tag{9.2.6}$$

From the comultiplication rule we can show

$$C(p_1,p_2,p_3;z_1,z_2,z_3)_q = 0, \ \text{unless} \quad p_1 - z_1 + p_2 - z_2 = p_3 - z_3. \qquad (9.2.7)$$

The relation (9.2.7) leads to the charge conservation law. Using (9.2.7) and the condition $z_i \in \mathbf{Z}_{\geq 0}$, we can show

$$C(p_1,p_2,p_3;z_1,z_2,z_3)_q = 0, \ \text{unless} \quad p_3 = p_1 + p_2 - n, n \in \mathbf{Z}_{\geq 0}. \qquad (9.2.8)$$

Thus we have following.

$$\begin{aligned} & |p_1,p_2;p_1+p_2-n,z_3>_q \\ = & \sum_{z_1=0}^{z_3+n} C(p_1,p_2,p_1+p_2-n;z_1,z_3+n-z_1,z_3)_q |p_1,z_1>_q \otimes |p_2,z_3+n-z_1>_q . \end{aligned} \qquad (9.2.9)$$

Here $n \in \mathbf{Z}_{\geq 0}$. The Clebsch-Gordan coefficients for infinite dimensional representations are given by the following.

$$\begin{aligned} & C(p_1,p_2,p_1+p_2-n;z_1,z_2,z_3)_q = \delta(z_3,z_1+z_2-n) \\ & \times\sqrt{[2p_1+2p_2-2n+1]_q}\sqrt{[n]_q![z_1]_q![z_2]_q![z_3]_q!} \\ & \times q^{(n-n^2)/2+(n-z_2)p_1+(n+z_1)p_2} \\ & \times \sum_{\nu} \frac{(-1)^{\nu} q^{-\nu(2p_1+2p_2-n+1)}}{[\nu]_q![n-\nu]_q![z_1-\nu]_q![z_2-n+\nu]_q!} \\ & \times\sqrt{\frac{[2p_1-n;z_1-\nu]_q![2p_1-z_1;n-\nu]_q![2p_2-n;z_2+\nu-n]_q![2p_2-z_2;\nu]_q!}{[2p_1+2p_2-n+1;z_1+z_2+1]_q!}}. \end{aligned} \qquad (9.2.10)$$

Here $p_3 = p_1 + p_2 - n, n \in \mathbf{Z}_{\geq 0}$, and the sum over the integer ν in (9.2.10) is taken under the following condition

$$\max\{0, n - z_2\} \leq \nu \leq \min\{n, z_1\}. \qquad (9.2.11)$$

Let $V_q^{(j)}$ be the spin j representation of $U_q(sl(2))$. Let us discuss decomposition of the tensor product $V_q^{(\infty)}(p_1) \otimes V_q^{(j_2)}$. It is easy to see the following rules.

$$\begin{aligned} C(p_1,j_2,p_3;z_1,z_2,z_3)_q &= 0, \ \text{unless} \quad p_1 - z_1 + j_2 - z_2 = p_3 - z_3, \\ C(p_1,j_2,p_3;z_1,z_2,z_3)_q &= 0, \ \text{unless} \quad p_3 = p_1 + j_2 - n \\ & \text{and} \quad 0 \leq n \leq 2j_2, n \in \mathbf{Z}_{\geq 0}. \end{aligned} \qquad (9.2.12)$$

The Clebsch-Gordan coefficients are given by

$$\begin{aligned} & C(p_1,j_2,p_1+j_2-n;z_1,z_2,z_3)_q = \delta(z_3,z_1+z_2-n) \\ & \times\sqrt{[2p_1+2j_2-2n+1]_q}\sqrt{[n]_q![z_1]_q![z_2]_q![z_3]_q!} \\ & \times q^{(n-n^2)/2+(n-z_2)p_1+(n+z_1)j_2} \\ & \times \sum_{\nu} \frac{(-1)^{\nu} q^{-\nu(2p_1+2j_2-n+1)}}{[\nu]_q![n-\nu]_q![z_1-\nu]_q![z_2-n+\nu]_q!} \\ & \times \left(\frac{[2p_1-n;z_1-\nu]_q![2p_1-z_1;n-\nu]_q![2j_2-n;z_2+\nu-n]_q![2j_2-z_2;\nu]_q!}{[2p_1+2j_2-n+1;z_1+z_2+1]_q!}\right)^{1/2}. \end{aligned} \qquad (9.2.13)$$

Here the sum over the integer ν in (9.2.13) is taken under the following condition

$$\max\{0, n - z_2\} \leq \nu \leq \min\{n, z_1, 2j_2 - z_2\}. \qquad (9.2.14)$$

Let us discuss decomposition of the tensor product $V_q^{(j_1)} \otimes V_q^{(\infty)}(p_2)$. We can show the following.

$$\begin{aligned} C(j_1,p_2,p_3;z_1,z_2,z_3)_q &= 0, \ \text{unless} \quad j_1 - z_1 + p_2 - z_2 = p_3 - z_3, \\ C(j_1,p_2,p_3;z_1,z_2,z_3)_q &= 0, \ \text{unless} \quad p_3 = j_1 + p_2 - n, \\ & \text{and} \quad 0 \leq n \leq 2j_1, n \in \mathbf{Z}_{\geq 0}. \end{aligned} \qquad (9.2.15)$$

The Clebsch-Gordan coefficients are given by the following.

$$\begin{aligned} & C(j_1,p_2,j_1+p_2-n;z_1,z_2,z_3)_q = \delta(z_3,z_1+z_2-n) \\ & \times\sqrt{[2j_1+2p_2-2n+1]_q}\sqrt{[n]_q![z_1]_q![z_2]_q![z_3]_q!} \\ & \times q^{(n-n^2)/2+(n-z_2)j_1+(n+z_1)p_2} \\ & \times \sum_{\nu} \frac{(-1)^{\nu} q^{-\nu(2j_1+2p_2-n+1)}}{[\nu]_q![n-\nu]_q![z_1-\nu]_q![z_2-n+\nu]_q!} \\ & \times\sqrt{\frac{[2j_1-n;z_1-\nu]_q![2j_1-z_1;n-\nu]_q![2p_2-n;z_2+\nu-n]_q![2p_2-z_2;\nu]_q!}{[2j_1+2p_2-n+1;z_1+z_2+1]_q!}}. \end{aligned} \qquad (9.2.16)$$

The sum over the integer ν in (9.2.16) is taken under the following condition

$$\max\{0, n - z_2, n + z_1 - 2j_1\} \leq \nu \leq \min\{n, z_1\}. \qquad (9.2.17)$$

For convenience we give the Clebsch-Gordan coefficients for finite dimensional representations $V^{(j_i)}, i = 1,2,3$ in the following [59] (see also [73]).

$$C(j_1,j_2,j_3;z_1,z_2,z_3)_q = \delta(z_3,z_1+z_2-n)$$

$$\times([2j_3+1]_q)^{1/2}\Delta_q(j_1j_2j_3)$$
$$\times q^{j_1(j_1+1)+j_2(j_2+1)-j_3(j_3+1)+2(j_1j_2+j_1(j_2-z_2)-j_2(j_1-z_1))/2}$$
$$\times([2j_1-z_1]_q![z_1]_q![2j_2-z_2]_q![z_2]_q![2j_3-z_3]_q![z_3]_q!)^{1/2}$$
$$\times\sum_{\nu}\frac{(-1)^{\nu}}{[\nu]_q!}\frac{q^{-\nu(j_1+j_2+j_3+1)}}{[j_1+j_2-j_3-\nu]_q![z_1-\nu]_q!}$$
$$\times\frac{1}{[2j_2-z_2-\nu]_q![j_3+j_1-j_2-z_1+\nu]_q![j_3-j_1-j_2+z_2+\nu]_q!}. \qquad (9.2.18)$$

Here the sum over the integer ν is such that

$$\max\{0,j_2-j_3-j_1+z_1,j_1+j_2-j_3-z_2\}\le\nu\le\min\{z_1,j_1+j_2-j_3,2j_2-z_2\}. \qquad (9.2.19)$$

The symbol $\Delta_q(abc)$ has been defined by

$$\Delta_q(abc)=\left(\frac{[a+b-c]_q![c+a-b]_q![b+c-a]_q!}{[a+b+c+1]_q!}\right)^{1/2}. \qquad (9.2.20)$$

If we replace p_i by j_i $(i=1,2,3)$, then the formal expression of the Clebsch-Gordan coefficients for infinite dimensional representations is consistent with the finite dimensional one (9.2.18).

The Clebsch-Gordan coefficients satisfy the orthogonality relation.

$$\delta_{nn'} = \sum_{z_1}C(p_1,p_2,p_1+p_2-n;z_1,\tau-z_1,\tau-n)_q \times C(p_1,p_2,p_1+p_2-n';z_1,\tau-z_1,\tau-n')_q, \qquad (9.2.21)$$

$$\delta_{nn'} = \sum_{z_1}C(p_1,j_2,p_1+p_2-n;z_1,\tau-z_1,\tau-n)_q \times C(p_1,j_2,p_1+j_2-n';z_1,\tau-z_1,\tau-n')_q, \qquad (9.2.22)$$

$$\delta_{nn'} = \sum_{z_1}C(j_1,p_2,j_1+p_2-n;z_1,\tau-z_1,\tau-n)_q \times C(j_1,p_2,j_1+p_2-n';z_1,\tau-z_1,\tau-n')_q. \qquad (9.2.23)$$

We can calculate the Racah coefficients using the Clebsch-Gordan coefficients and the orthogonality relations (9.2.21),(9.2.22),(9.2.23) as in §2. The Racah coefficients are given by

$$R_{\mu''\mu'}(q) = \sqrt{[2\mu''+1]_q[2\mu'+1]_q}W(\mu_1,\mu_2,\mu,\mu_3;\mu',\mu'')_q$$
$$= \sum_{w_1}\sum_{w_2}C(\mu_1,\mu_2,\mu';w_1,w_2,w_1+w_2-n')_q$$

$$\times C(\mu',\mu_3,\mu;w_1+w_2-n',z''-w_2,w_1+z''-n)_q$$
$$\times C(\mu_2,\mu_3,\mu_2+\mu_3-n'';w_2,z''-w_2,z''-n'')_q$$
$$\times C(\mu_1,\mu'',\mu;w_1,z'',w_1+z''+n''-n)_q. \qquad (9.2.24)$$

Here we have assumed that $2\mu_i(i=1,2,3)$ is either a nonnegative integer ($2\mu_i\in \mathbf{Z}_{\ge0}$) or a complex parameter ($\mu_i\in\mathbf{C}$). We have defined μ,μ',μ'' by

$$\mu=\mu_1+\mu_2+\mu_3-n,\ \mu'=\mu_1+\mu_2-n',\ \mu''=\mu_2+\mu_3-n''. \qquad (9.2.25)$$

where $n,n',n''\in\mathbf{Z}_{\ge0}$.

9.3 Finite dimensional case of $U_q(sl(2))$

Let ω be a primitive N-th root of unity:

$$\omega=\exp(\frac{2\pi is}{N}),\qquad (N,s)=1. \qquad (9.3.1)$$

Here we recall that by the symbol $(a,b)=1$ we express that the integers a and b have no common divisor except 1. Let ϵ represent a square root of ω: $\epsilon=\omega^{1/2}=\exp(\pi is/N)$, $(N,s)=1$.

We take the limit $q\to\epsilon$ in the infinite dimensional representation (9.2.5). Then we have

$$\lim_{q\to\epsilon}(\pi^p(X^-)_q)^N_{N-1}=0. \qquad (9.3.2)$$

From the property (9.3.2) we can restrict the infinite dimensions into N dimensions: $V^{(\infty)}\to V^{(N)}$, where $V^{(N)}$ is an N-dimensional vector space with basis $e_0,\cdots,e_{N-1}$. For $a,b=0,1,\cdots N-1$ we have the following matrix representations

$$(\pi^p(X^+)_{q=\epsilon})^a_b = ([2p-a]_\epsilon[a+1]_\epsilon)^{1/2}\cdot\delta_{a+1,b},$$
$$(\pi^p(X^-)_{q=\epsilon})^a_b = ([2p-a+1]_\epsilon[a]_\epsilon)^{1/2}\cdot\delta_{a-1,b},$$
$$(\pi^p(K)_{q=\epsilon})_{ab} = \epsilon^{(2p-2a)}\cdot\delta_{a,b}. \qquad (9.3.3)$$

The symbol $V^{(N)}_\epsilon(p)$ represents the finite dimensional color representation $\{\pi^p_\epsilon,V^{(N)}\}$. We write the basis vector e_a as $|p,z>_\epsilon$. The basis vectors for $V^{(N)}_\epsilon(p)$ are $\{|p,z>_\epsilon, z=0,\cdots,N-1\}$.

Let us discuss decomposition of the tensor product $V^{(N)}_\epsilon(p_1)\otimes V^{(N)}_\epsilon(p_2)$. We take the limit $q\to\epsilon$ in the expression of the Clebsch-Gordan coefficients (9.2.10). Since there is no singularity in the limiting process in the expression of the Clebsch-Gordan coefficients, we have the fusion rule in the following.

$$V^{(N)}_\epsilon(p_1)\otimes V^{(N)}_\epsilon(p_2)=\sum_{p_3}N^{p_3}_{p_1,p_2}V^{(N)}_\epsilon(p_3), \qquad (9.3.4)$$

where

$$\begin{aligned} N_{p_1 p_2}^{p_3} &= 1, \text{ for } p_3 = p_1 + p_2, p_1 + p_2 - 1, \cdots, p_1 + p_2 - N + 1. \\ &= 0, \text{ otherwise.} \end{aligned} \tag{9.3.5}$$

We note that the condition $0 \le n \le N-1$ is derived from the factor $\sqrt{[n]_q!}$ in (9.2.10). In terms of the Clebsch-Gordan coefficients we have

$$\begin{aligned} C(p_1, p_2, p_3; z_1, z_2, z_3)_\epsilon &= 0, \text{ unless } \quad p_1 - z_1 + p_2 - z_2 = p_3 - z_3 \\ & \quad \text{and } 0 \le z_3 \le N-1, \\ C(p_1, p_2, p_3; z_1, z_2, z_3)_\epsilon &= 0, \text{ unless } \quad p_3 = p_1 + p_2 - n, \\ & \quad \text{and } \quad 0 \le n \le N-1, n \in \mathbf{Z}_{\ge 0}. \end{aligned} \tag{9.3.6}$$

We again note that the condition $0 \le z_3 \le N-1$ comes from the factor $\sqrt{[z_3]_q!}$ in (9.2.10).

By taking the limit $q \to \epsilon$ in (9.2.10) we obtain the Clebsch-Gordan coefficients for finite dimensional color representations. Let us set $p_3 = p_1 + p_2 - n$, where n is an integer with $0 \le n \le N-1$, and $0 \le z_i \le N-1$, for i=1,2,3. The Clebsch-Gordan coefficients for $V_\epsilon^{(N)}(p_i)$ for $i = 1, 2, 3$ are given as follows.

$$\begin{aligned} & C(p_1, p_2, p_1 + p_2 - n; z_1, z_2, z_3)_\epsilon = \delta(z_3, z_1 + z_2 - n) \\ & \times \sqrt{[2p_1 + 2p_2 - 2n + 1]_\epsilon} \sqrt{[n]_\epsilon! [z_1]_\epsilon! [z_2]_\epsilon! [z_3]_\epsilon!} \\ & \times \epsilon^{(n-n^2)/2 + (n - z_2) p_1 + (n + z_1) p_2} \\ & \times \sum_\nu \frac{(-1)^\nu \epsilon^{-\nu(p_1 + p_2 + p_3 + 1)}}{[\nu]_\epsilon! [n-\nu]_\epsilon! [z_1 - \nu]_\epsilon! [z_2 - n + \nu]_\epsilon!} \\ & \times \sqrt{\frac{[2p_1 - n; z_1 - \nu]_\epsilon! [2p_1 - z_1; n - \nu]_\epsilon! [2p_2 - n; z_2 + \nu - n]_\epsilon! [2p_2 - z_2; \nu]_\epsilon!}{[2p_1 + 2p_2 - n + 1; z_1 + z_2 + 1]_\epsilon!}}. \end{aligned} \tag{9.3.7}$$

Here the sum over the integer ν in (9.3.7) is taken under the following condition

$$\max\{0, n - z_2\} \le \nu \le \min\{n, z_1\}. \tag{9.3.8}$$

Let $V_\epsilon^{(j)}$ be the spin j representation of $U_\epsilon(sl(2))$. When q is a root of unity ($q^{2N} = 1$), the spin j representations of $U_q(sl(2))$ has the following constraint. [82,83]

$$0 \le 2j \le N - 2, \quad 2j \in \mathbf{Z}_{\ge 0}. \tag{9.3.9}$$

Let us discuss decomposition of the tensor product $V_\epsilon^{(N)}(p_1) \otimes V_\epsilon^{(j_2)}$. By taking the limit $q \to \epsilon$ in (9.2.13) we have the following fusion rule.

$$V_\epsilon^{(N)}(p_1) \otimes V_\epsilon^{(j_2)} = \sum_{p_3} N_{p_1 j_2}^{p_3} V_\epsilon^{(N)}(p_3) \tag{9.3.10}$$

where

$$\begin{aligned} N_{p_1 j_2}^{p_3} &= 1 \text{ for } p_3 = p_1 + j_2 - n, 0 \le n \le 2j_2, n \in \mathbf{Z}, \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{9.3.11}$$

We recall that $2j_2 < N - 1$. In terms of the Clebsch-Gordan coefficients we have

$$\begin{aligned} C(p_1, j_2, p_3; z_1, z_2, z_3)_\epsilon &= 0, \text{ unless } \quad p_1 - z_1 + j_2 - z_2 = p_3 - z_3 \\ & \quad \text{and } \quad 0 \le z_3 \le N-1, \\ C(p_1, j_2, p_3; z_1, z_2, z_3)_\epsilon &= 0, \text{ unless } \quad p_3 = p_1 + j_2 - n, \\ & \quad \text{and } \quad 0 \le n \le 2j_2, n \in \mathbf{Z}_{\ge 0}. \end{aligned} \tag{9.3.12}$$

Let us set $p_3 = p_1 + p_2 - n$, where n is an integer with $0 \le n \le N-1$, and $0 \le z_i \le N-1$, for i=1,3, and $0 \le z_2 \le N-2$. Then, by taking the limit: $q \to \epsilon$ in (9.2.13), we have the Clebsch-Gordan coefficients. We note there is no singularity in the limiting procedure.

$$\begin{aligned} & C(p_1, j_2, p_1 + j_2 - n; z_1, z_2, z_3)_\epsilon = \delta(z_3, z_1 + z_2 - n) \\ & \times \sqrt{[2p_1 + 2j_2 - 2n + 1]_\epsilon} \sqrt{[n]_\epsilon! [z_1]_\epsilon! [z_2]_\epsilon! [z_3]_\epsilon!} \\ & \times \epsilon^{(n-n^2)/2 + (n - z_2) p_1 + (n + z_1) j_2} \\ & \times \sum_\nu \frac{(-1)^\nu \epsilon^{-\nu(2p_1 + 2j_2 - n + 1)}}{[\nu]_\epsilon! [n-\nu]_\epsilon! [z_1 - \nu]_\epsilon! [z_2 - n + \nu]_\epsilon!} \\ & \times \left(\frac{[2p_1 - n; z_1 - \nu]_\epsilon! [2p_1 - z_1; n - \nu]_\epsilon! [2j_2 - n; z_2 + \nu - n]_\epsilon! [2j_2 - z_2; \nu]_\epsilon!}{[2p_1 + 2j_2 - n + 1; z_1 + z_2 + 1]_\epsilon!} \right)^{1/2}. \end{aligned} \tag{9.3.13}$$

Here the sum over the integer ν in (9.3.13) is taken under the following condition

$$\max\{0, n - z_2\} \le \nu \le \min\{n, z_1, 2j_2 - z_2\}. \tag{9.3.14}$$

In the same way we have the following.

$$V_\epsilon^{(j_1)} \otimes V_\epsilon^{(N)}(p_2) = \sum_{p_3} N_{j_1 p_2}^{p_3} V_\epsilon^{(N)}(p_3) \tag{9.3.15}$$

where

$$
\begin{aligned}
N_{j_1 p_2}^{p_3} &= 1 \text{ for } p_3 = j_1 + p_2 - n, 0 \le n \le 2j_1, n \in \mathbf{Z}, \\
&= 0, \quad \text{otherwise.}
\end{aligned} \tag{9.3.16}
$$

We recall that $2j_1 < N - 1$. In terms of the Clebsch-Gordan coefficients we have

$$
\begin{aligned}
C(j_1, p_2, p_3; z_1, z_2, z_3)_\epsilon &= 0, \text{ unless } \quad j_1 - z_1 + p_2 - z_2 = p_3 - z_3 \\
&\quad \text{and} \quad 0 \le z_3 \le N - 1 \\
C(j_1, p_2, p_3; z_1, z_2, z_3)_\epsilon &= 0, \text{ unless } \quad p_3 = j_1 + p_2 - n, \\
&\quad \text{and} \quad 0 \le n \le 2j_1, \quad n \in \mathbf{Z}_{\ge 0}.
\end{aligned} \tag{9.3.17}
$$

By taking the limit: $q \to \epsilon$ in (9.2.16), we have the Clebsch-Gordan coefficients.

$$
\begin{aligned}
& C(j_1, p_2, j_1 + p_2 - n; z_1, z_2, z_3)_\epsilon = \delta(z_3, z_1 + z_2 - n) \\
& \times \sqrt{[2j_1 + 2p_2 - 2n + 1]_\epsilon} \sqrt{[n]_\epsilon! [z_1]_\epsilon! [z_2]_\epsilon! [z_3]_\epsilon!} \\
& \times \epsilon^{(n - n^2)/2 + (n - z_2) j_1 + (n + z_1) p_2} \\
& \times \sum_\nu \frac{(-1)^\nu \epsilon^{-\nu(2j_1 + 2p_2 - n + 1)}}{[\nu]_\epsilon! [n - \nu]_\epsilon! [z_1 - \nu]_\epsilon! [z_2 - n + \nu]_\epsilon!} \\
& \times \left(\frac{[2j_1 - n; z_1 - \nu]_\epsilon! [2j_1 - z_1; n - \nu]_\epsilon! [2p_2 - n; z_2 + \nu - n]_\epsilon! [2p_2 - z_2; \nu]_\epsilon!}{[2j_1 + 2p_2 - n + 1; z_1 + z_2 + 1]_\epsilon!} \right)^{1/2}
\end{aligned} \tag{9.3.18}
$$

Here the sum over the integer ν in (9.3.18) is taken under the following condition

$$
\max\{0, n - z_2, n + z_1 - 2j_1\} \le \nu \le \min\{n, z_1\}. \tag{9.3.19}
$$

The Clebsch-Gordan coefficients have the following orthogonality relations.

$$
\begin{aligned}
\delta_{nn'} &= \sum_{z_1} C(p_1, p_2, p_1 + p_2 - n; z_1, \tau - z_1, \tau - n)_\epsilon \\
& \quad \times C(p_1, p_2, p_1 + p_2 - n'; z_1, \tau - z_1, \tau - n')_\epsilon,
\end{aligned} \tag{9.3.20}
$$

$$
\begin{aligned}
\delta_{z_1 z_1'} &= \sum_{n} C(p_1, p_2, p_1 + p_2 - n; z_1, z + n - z_1, z)_\epsilon \\
& \quad \times C(p_1, p_2, p_1 + p_2 - n; z_1', z + n - z_1', z)_\epsilon,
\end{aligned} \tag{9.3.21}
$$

$$
\begin{aligned}
\delta_{nn'} &= \sum_{z_1} C(p_1, j_2, p_1 + j_2 - n; z_1, \tau - z_1, \tau - n)_\epsilon \\
& \quad \times C(p_1, j_2, p_1 + j_2 - n'; z_1, \tau - z_1, \tau - n')_\epsilon,
\end{aligned} \tag{9.3.22}
$$

$$
\begin{aligned}
\delta_{z_1 z_1'} &= \sum_{n} C(p_1, j_2, p_1 + j_2 - n; z_1, z + n - z_1, z)_\epsilon \\
& \quad \times C(p_1, j_2, p_1 + j_2 - n; z_1', z + n - z_1', z)_\epsilon,
\end{aligned} \tag{9.3.23}
$$

$$
\begin{aligned}
\delta_{nn'} &= \sum_{z_1} C(j_1, p_2, j_1 + p_2 - n; z_1, \tau - z_1, \tau - n)_\epsilon \\
& \quad \times C(j_1, p_2, j_1 + p_2 - n'; z_1, \tau - z_1, \tau - n')_\epsilon,
\end{aligned} \tag{9.3.24}
$$

$$
\begin{aligned}
\delta_{z_1 z_1'} &= \sum_{n} C(j_1, p_2, j_1 + p_2 - n; z_1, z + n - z_1, z)_\epsilon \\
& \quad \times C(j_1, p_2, j_1 + p_2 - n; z_1', z + n - z_1', z)_\epsilon.
\end{aligned} \tag{9.3.25}
$$

Let $2\mu_i$ $(i = 1, 2, 3)$ be either a nonnegative integer $(2\mu_i \in \mathbf{Z}_{\ge 0})$ or a complex parameter $(\mu_i \in \mathbf{C})$. We introduce μ, μ', μ'' by

$$
\mu = \mu_1 + \mu_2 + \mu_3 - n, \mu' = \mu_1 + \mu_2 - n', \mu'' = \mu_2 + \mu_3 - n''. \tag{9.3.26}
$$

where $n, n', n'' \in \mathbf{Z}_{\ge 0}$. In terms of the Clebsch-Gordan coefficients the Racah coefficients $W(\mu_1, \mu_2, \mu, \mu_3; \mu', \mu'')_\epsilon$ are given by the following.

$$
\begin{aligned}
R_{\mu'' \mu'}(\epsilon) &= \sqrt{[2\mu' + 1]_\epsilon [2\mu'' + 1]_\epsilon} W(\mu_1, \mu_2, \mu, \mu_3; \mu', \mu'')_\epsilon \\
&= \sum_{w_1} \sum_{w_2} C(\mu_1, \mu_2, \mu'; w_1, w_2, w_1 + w_2 - n')_\epsilon \\
& \quad \times C(\mu', \mu_3, \mu; w_1 + w_2 - n', z'' - w_2, w_1 + z'' - n)_\epsilon \\
& \quad \times C(\mu_2, \mu_3, \mu_2 + \mu_3 - n''; w_2, z'' - w_2, z'' - n'')_\epsilon \\
& \quad \times C(\mu_1, \mu'', \mu; w_1, z'', w_1 + z'' + n'' - n)_\epsilon.
\end{aligned} \tag{9.3.27}
$$

Finally we give comments. (1) The finite dimensional color representation in this section is equivalent to the N dimensional representation of $U_q(sl(2))$ with $q^{2N} = 1$ in the reference [83]. We have discussed the representation from the view point of the infinite dimensional representation of $U_q(sl(2))$ and the limit $q \to \epsilon$ introduced in the reference [30], and then through the limit we have obtained the formulas for the Clebsch-Gordan coefficients and the Racah coefficients. (2) Fusion rules similar to (9.3.5) and (9.3.11) have been given in the reference [8]. The fusion rules are for the m dimensional representations (semi-periodic representations) of $U_q(sl(2))$ with $q^m = 1$, which are different from the finite dimensional color representations (the N dimensional representation with $q^{2N} = 1$).

10 Invariants of Colored Framed Graphs

10.1 Invariants of colored oriented framed graph tangles

We can show the following relation for the Clebsch-gordan coefficients and the colored braid matrix

$$
\begin{aligned}
& \sum_{c_1 c_2} G^{a_1 c_3}_{b_3 c_1}(p_1, p_3; +) G^{a_2 a_3}_{c_3 c_2}(p_2, p_3; +) C(p_1, p_2, p; c_1, c_2, b)_e \\
= & \sum_{c} C(p_1, p_2, p; a_1, a_2, c)_e G^{c a_3}_{b_3 b}(p, p_3; +), \\
& \sum_{c_1 c_2} G^{a_1 c_3}_{b_3 c_1}(p_3, p_1; -) G^{a_2 a_3}_{c_3 c_2}(p_3, p_2; -) C(p_1, p_2, p; c_1, c_2, b)_e \\
= & \sum_{c} C(p_1, p_2, p; a_1, a_2, c)_e G^{c a_3}_{b_3 b}(p_3, p; -). \qquad (10.1.1)
\end{aligned}
$$

We consider trivalent graphs, which have vertices with three edges. Framed graphs have framing vector fields. We assume that all three edges have common tangent vector in the trivalent vertex, and the framing vector field is always normal to the tangent vector to the vertex.

Fig. 10.1.1

We introduce "graph tangle". We define (k, l)-oriented graph tangle T by a finite set of disjoint oriented arcs, oriented trivalent vertices, and oriented circles properly embedded in $\mathbf{R}^2 \times [0, 1]$ such that

$$\partial T = \{(i, 0, 0); i = 1, 2, \cdots, k\} \cup \{(j, 0, 1); j = 1, \cdots, l\}, \qquad (10.1.2)$$

where the symbol ∂T denotes the upper and lower boundaries of the graph tangle. We define colored oriented graph tangles (T, α) by assigning colors on edges, arcs and circles of graph tangles.

We can express trivalent framed graph tangle by plane diagram. [81,56] We can choose the diagram such that the framing vector field is normal to the plane of projection and directed "up".

Proposition 10.1 *[81] The isotopy invariants of framed graph tangles are functions on their plane diagrams, invariant under the following local moves $E_1 \sim E_7$.*

Fig. 10.1.2

We introduce weights for diagrams. We assign the colored braid matrices, the Clebsch-Gordan coefficients, and the weights U_i to the braiding diagrams, trivalent vertex diagrams, and the creation-annihilation diagrams, respectively.

Fig. 10.1.3

We define $\phi(T, \alpha)$ for a graph tangle T by the summation over all possible configurations of variables z_i on the edges (or segments) of the graph. In the summation we fix the colors p_i. The sum corresponds to partition function in statistical mechanics. Then from the braid relation and the basic relations corresponding to the Reidemeister moves the sum $\phi(T, \alpha)$ for the graph tangle T is invariant under the moves $E_1 \sim E_7$. Thus we have isotopy invariants of trivalent colored oriented framed graph tangles.

10.2 Invariants of trivalent framed graphs

We construct invariants of trivalent colored oriented framed graphs (colored oriented ribbon graphs) by an approach parallel to that for the colored link invariants. [2,31] Through $(1, 1)$-graph tangle we introduce another invariant of a framed graph. [2]

Let T be a $(1, 1)$- graph tangle. We assume that $\hat{T}$ represents the graph obtained by closing the open strings of T. It is easy to show the following proposition. [2]

Proposition 10.2 *Let T_1 and T_2 are two $(1, 1)$-graph tangles. If $\hat{T}_1$ is isotopic to $\hat{T}_2$ as a graph in S^3 by an isotopy which carries the closing component of $\hat{T}_1$ to that of $\hat{T}_2$. Then T_1 is isotopic to T_2 as a $(1, 1)$-graph tangle.*

Let T be a $(1, 1)$-graph tangle. We put $F = \hat{T}$ and s is the color of the closing component (or edge) of $\hat{T}$. We denote by $\phi(T, \alpha)^a_b$ the value ϕ for the graph tangle with variables a and b on the closing component (or edge).

Fig. 10.2.1

Then from the discussion given in the reference [2] we can show that

$$\phi(T, \alpha)^a_b = \lambda \delta_{ab}. \qquad (10.2.1)$$

The value of $\phi(T, \alpha)^a_b$ do not depend on a or b.

For a colored graph (F, α) and a color s of closing component (or edge), we define Φ by $\Phi(F, s, \alpha) = \lambda$ where F, T, s are above and $\phi(T, \alpha)^a_b = \lambda \delta_{ab}$. By the basic relations given in §7.2 and §10.1, Φ is well-defined, i.e. $\Phi(F, s, \alpha)$ does not depend on a choice of T.

Further we have the following proposition, to obtain invariants which do not depend on s.

Proposition 10.3 *For a graph F and its color $\alpha = (p_1, \cdots, p_n)$, we have the following formula.*

$$\Phi(F, s, \alpha)([p_s; N-1]_e!)^{-1} = \Phi(F, s', \alpha)([p_{s'}; N-1]_e!)^{-1}. \tag{10.2.2}$$

The proof of this proposition is equivalent to that given in Appendix C. [2,31] By this proposition we obtain the next definition.

Definition 10.4 *For a trivalent colored oriented framed graph (F, α), we define an isotopy invariant $\hat{\Phi}$ of (F, α) by*

$$\hat{\Phi}(F, \alpha) = \Phi(F, s, \alpha)([p_s; N-1]_e!)^{-1}. \tag{10.2.3}$$

Thus we obtain new invariants $\hat{\Phi}(F, \alpha)$ of trivalent colored oriented framed graphs (F, α).

11 Colored IRF models and vertex models

11.1 Quantum affine algebra $U_q(\hat{sl(2)})$

We express the Boltzmann weights of the colored vertex models in terms of the Clebsch-Gordan coefficients of color representations. Let us consider the q-analog of the universal enveloping algebra $U_q(\hat{sl}(2, C))$ of the affine Kac-Moody algebra $\hat{sl}(2)$. [45] The generators $\{X_i^{\pm}, H_i; i = 0, 1\}$ satisfy the following defining relations $(i, j = 0, 1)$.

$$\begin{aligned} [H_i, H_j] &= 0, \\ [H_i, X_i^{\pm}] &= \pm 2X_i^{\pm}, \quad [H_i, X_j^{\pm}] = \mp 2X_j^{\pm} \quad (i \neq j), \\ [X_i^+, X_j^-] &= \delta_{ij}\frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \\ \sum_{\nu=0}^{3} (-1)^{\nu} & \frac{[3]_q!}{[3-\nu]_q![\nu]_q!}(X^{\pm})^{3-\nu}X_j^{\pm}(X_i^{\pm})^{\nu} = 0 \quad (i \neq j). \end{aligned} \tag{11.1.1}$$

The comultiplication is given by

$$\begin{aligned} \Delta(H_i) &= H_i \otimes I + I \otimes H_i, \\ \Delta(X_i^{\pm}) &= X_i^{\pm} \otimes q^{H_i/2} + q^{-H_i/2} \otimes X_i^{\pm}. \end{aligned} \tag{11.1.2}$$

Let $\mathcal{R}$ represent the universal R matrix of $U_q(\hat{sl}(2, C))$. The universal R matrix satisfies the following

$$\mathcal{R}\Delta(a) = \tau \circ \Delta(a)\mathcal{R}, \quad a \in U_q(\hat{sl}(2, C)), \tag{11.1.3}$$

where τ is the permutation operator $\tau(t_1 \otimes t_2) = t_2 \otimes t_1$, for $t_1, t_2 \in U_q(\hat{sl}(2, C))$. For simplicity we sometimes write $U_q(\hat{sl}(2, \mathbf{C}))$ and $U_q(sl(2, \mathbf{C}))$ as $\hat{U}_q$ and U_q, respectively.

Let us discuss solvable models from the viewpoint of $\hat{U}_q$. We define a homomorphism ϕ

$$\phi : \hat{U}_q \to U_q, \tag{11.1.4}$$

by

$$\begin{aligned} \phi(X_0^{\pm}) &= X^{\mp}, \quad \phi(X_1^{\pm}) = X^{\pm}, \\ \phi(H_0) &= -H, \quad \phi(H_1) = H. \end{aligned} \tag{11.1.5}$$

We introduce an automorphism T_x

$$T_x : \hat{U}_q \to \hat{U}_q \tag{11.1.6}$$

by

$$\begin{aligned} T_x(X_0^{\pm}) &= x^{\pm 1}X_0^{\pm}, \\ T_x(X) &= X, \quad \text{for} \quad X = X_1^{\pm}, H_0, H_1. \end{aligned} \tag{11.1.7}$$

We define an operator $R(x)$ by

$$R(x) = \phi((T_x \otimes I(\mathcal{R})). \tag{11.1.8}$$

Then the operator $R(x)$ satisfies the following for $i = 0, 1$.

$$R(x)q^{\hat{H}_i/2} \otimes q^{\hat{H}_i/2} = q^{\hat{H}_i/2} \otimes q^{\hat{H}_i/2}R(x), \tag{11.1.9}$$

$$\begin{aligned} & R(x)(x^{\delta(i,0)}\hat{X}_i^+ \otimes q^{\hat{H}_i/2} + q^{-\hat{H}_i/2} \otimes \hat{X}_i^+) \\ = \; & (x^{\delta(i,0)}\hat{X}_i^+ \otimes q^{-\hat{H}_i/2} + q^{\hat{H}_i/2} \otimes \hat{X}_i^+)R(x), \end{aligned} \tag{11.1.10}$$

$$\begin{aligned} & R(x)(x^{-\delta(i,0)}\hat{X}_i^- \otimes q^{\hat{H}_i/2} + q^{-\hat{H}_i/2} \otimes \hat{X}_i^-) \\ = \; & (x^{-\delta(i,0)}\hat{X}_i^- \otimes q^{-\hat{H}_i/2} + q^{\hat{H}_i/2} \otimes \hat{X}_i^-)R(x). \end{aligned} \tag{11.1.11}$$

Here $\hat{X}_i^{\pm} = \phi(X_i^{\pm})$, $\hat{H}_i = \phi(H_i)$ for $i = 0, 1$.

Let us take arbitrary two representations μ_1, μ_2 of U_q. We can take as μ_1 and μ_2 the spin j representations of $U_q(sl(2))$, or the (infinite dim. or finite dim.) color representations of $U_q(sl(2))$. We define R matrix and its matrix elements for the representations by

$$\begin{aligned} R_{\mu_1\mu_2}(x) &= \pi^{\mu_1}\otimes\pi^{\mu_2}(R(x)), \\ R_{\mu_1\mu_2}(x)^{a_1a_2}_{b_1b_2} &= (\pi^{\mu_1}\otimes\pi^{\mu_2}(R(x)))^{a_1a_2}_{b_1b_2}. \end{aligned} \tag{11.1.12}$$

Then the relations (11.1.9), (11.1.10) and (11.1.11) give linear equations for the R matrix elements. It is easy to see that if the matrix elements satisfy the linear equations, then the matrix elements satisfy the Yang-Baxter equations. [46]

$$\begin{aligned} & \sum_{c_1c_2c_3} R_{p_1p_2}(u)^{c_1c_2}_{b_1b_2} R_{p_1p_3}(u+v)^{a_1c_3}_{c_1b_3} R_{p_2p_3}(v)^{a_2a_3}_{c_2c_3} \\ = & \sum_{c_1c_2c_3} R_{p_1p_2}(u)^{a_1a_2}_{c_1c_2} R_{p_1p_3}(u+v)^{c_1a_3}_{b_1c_3} R_{p_2p_3}(v)^{c_2c_3}_{b_2b_3}. \end{aligned} \tag{11.1.13}$$

Here we have defined the spectral parameter u by $x=\exp u$. In operator formalism, solutions of the linear equations (R matrix) can be written as

$$R(q;u)=\sum_{\mu}|\mu_1\mu_2;\mu> g(\mu_1,\mu_2,\mu;u) < \mu_1\mu_2;\mu| \tag{11.1.14}$$

Here the sum is taken over all μ appearing in the decomposition of the tensor product $\mu_1\otimes\mu_2$ ($\mu\subset\mu_1\otimes\mu_2$) , and $g(\mu_1,\mu_2,\mu;u)$ is some function of the spectral parameter $x=\exp u$.

11.2 Infinite dimensional vertex model

In this subsection we construct the colored vertex models by calculating the matrix elements of the R matrix (11.1.14) on the color representations. Let us consider color representations $(\pi_q^{(p_i)}, V^{(\infty)})$ $(i=1,2)$ of $U_q(sl(2))$. We define $R_{p_1p_2}(q;u)$ and its matrix elements by

$$\begin{aligned} R_{p_1p_2}(q;u) &= \pi_q^{(p_1)}\otimes\pi_q^{(p_2)}(R(x)), \\ R_{p_1p_2}(q;u)^{a_1a_2}_{b_1b_2} &= \left(\pi_q^{(p_1)}\otimes\pi_q^{(p_2)}(R(x))\right)^{a_1a_2}_{b_1b_2}. \end{aligned} \tag{11.2.1}$$

Here $a_1,a_2,b_1,b_2=0,1,\cdots,\infty$.

We can show that the following gives a solution of the linear equations (11.1.9), (11.1.10) and (11.1.11).

$$\begin{aligned} R_{p_1p_2}(q;u)^{a_1a_2}_{b_1b_2} = & \sum_{n=0}^{\infty} g(p_1,p_2,n;u)_q \\ & C(p_1,p_2,p_1+p_2-n;b_1,b_2,b_1+b_2-n)_q \\ & C(p_2,p_1,p_1+p_2-n;a_2,a_1,a_1+a_2-n)_q. \end{aligned} \tag{11.2.2}$$

where

$$g(p_1,p_2,n;u)_q=(-1)^n\prod_{k=0}^{n-1}\frac{[u-p_1-p_2+k]_q}{[u+p_1+p_2-k]_q}, \text{ for } n\in \mathbf{Z}_{\geq 0}. \tag{11.2.3}$$

We assume $g(p_1,p_2,n=0;u)_q=1$. For the fixed values of $a_i,b_i(i=1,2)$, the sum in (11.2.2) reduces to a finite sum. Therefore the sum in (11.2.2) is well defined. Thus we have an explicit formula for the R matrix elements for the infinite dimensional color representation. It is remarked that the case $p_1=p_2$, the infinite dimensional representation of the R matrix with a spectral parameter were discussed by Jimbo by using projection operators. [45]

Through the R matrix (11.2.2) the Boltzmann weights of the colored vertex model are written as follows

$$X^{(\infty)}_{\alpha\beta}(u)^{ab}_{cd}=R_{p_1p_2}(q;u)^{ba}_{cd}, \tag{11.2.4}$$

where $\alpha=q^{p_2}$ and $\beta=q^{p_1}$. Thus we obtain the colored vertex model of the infinite state case introduced in the reference [30].

Using the Clebsch-Gordan coefficients we can calculate R matrix for the representations V^j and $V^{(\infty)}$. We define $R_{p_1j_2}(x)$ and its matrix elements by

$$\begin{aligned} R_{p_1j_2}(x) &= \pi_q^{(p_1)}\otimes\pi_q^{(j_2)}(R(x)), \\ R_{p_1j_2}(x)^{a_1a_2}_{b_1b_2} &= \left(\pi_q^{(p_1)}\otimes\pi_q^{(j_2)}(R(x))\right)^{ab}_{cd}. \end{aligned} \tag{11.2.5}$$

Here $a_1,b_1=0,1,\cdots,\infty$, and $a_2,b_2=0,1,\cdots 2j_2$. The matrix elements are given by the following.

$$\begin{aligned} R_{p_1j_2}(x)^{a_1a_2}_{b_1b_2} = & \sum_{n=0}^{2j_2} g(p_1,j_2,n;u)_q \\ & C(p_1,j_2,p_1+p_2-n;b_1,b_2,b_1+b_2-n)_q \\ & C(j_2,p_1,p_1+p_2-n;a_2,a_1,a_1+a_2-n)_q. \end{aligned} \tag{11.2.6}$$

Here $g(p_1,j_2,n;u)_q$ is given by (11.2.3) with $p_2=j_2$.

11.3 Finite dimensional vertex model

We discuss finite dimensional colored vertex models. In Appendix E we give the Boltzmann weights of the colored vertex models for the cases $N=2,3,4$.

Let us consider color representations $(\pi_\epsilon^{(p_i)}, V^{(N)})$ $(i = 1, 2)$ of $U_q(sl(2))$ with $q = \epsilon$. We define $R_{p_1 p_2}(\epsilon; x)$ and its matrix elements by

$$\begin{aligned} R_{p_1 p_2}(\epsilon; x) &= \pi_\epsilon^{(p_1)} \otimes \pi_\epsilon^{(p_2)}(R(x)), \\ R_{p_1 p_2}(\epsilon; x)_{b_1 b_2}^{a_1 a_2} &= \left(\pi_\epsilon^{(p_1)} \otimes \pi_\epsilon^{(p_2)}(R(x))\right)_{b_1 b_2}^{a_1 a_2}. \end{aligned} \tag{11.3.1}$$

Then we have the following.

$$\begin{aligned} R_{p_1 p_2}(\epsilon; u)_{b_1 b_2}^{a_1 a_2} &= \sum_{n=0}^{N-1} g(p_1, p_2, n; u)_\epsilon \\ &\times C(p_1, p_2, p_1 + p_2 - n; b_1, b_2, b_1 + b_2 - n)_\epsilon \\ &\times C(p_2, p_1, p_1 + p_2 - n; a_2, a_1, a_1 + a_2 - n)_\epsilon, \end{aligned} \tag{11.3.2}$$

where

$$g(p_1, p_2, n; u)_\epsilon = (-1)^n \prod_{k=0}^{n-1} \frac{[u - p_1 - p_2 + k]_\epsilon}{[u + p_1 + p_2 - k]_\epsilon}, \text{ for } n \in \mathbf{Z}_{\geq 0}. \tag{11.3.3}$$

We assume $g(p_1, p_2, n = 0; u)_\epsilon = 1$.

Through the R matrix (11.2.2) the Boltzmann weights of the colored vertex model are written as follows

$$X_{\alpha\beta}^{(N)}(u)_{cd}^{ab} = R_{p_1 p_2}(\epsilon; u)_{cd}^{ba}, \tag{11.3.4}$$

where $\alpha = \epsilon^{p_2}$ and $\beta = \epsilon^{p_1}$. Thus we obtain the colored vertex model of the finite state case given in the reference [29].

Using the Clebsch-Gordan coefficients we can calculate R matrix for the representations V^j and $V^{(N)}$. We define $R_{p_1 j_2}(x)$ and its matrix elements by

$$\begin{aligned} R_{p_1 j_2}(\epsilon; x) &= \pi_\epsilon^{(p_1)} \otimes \pi_\epsilon^{(j_2)}(R(x)), \\ R_{p_1 j_2}(\epsilon; x)_{b_1 b_2}^{a_1 a_2} &= \left(\pi_q^{(p_1)} \otimes \pi_\epsilon^{(j_2)}(R(x))\right)_{cd}^{ab}. \end{aligned} \tag{11.3.5}$$

Here $a_1, b_1 = 0, 1, N - 1$, and $a_2, b_2 = 0, 1, \cdots 2j_2$. The matrix elements are given by the following.

$$\begin{aligned} R_{p_1 j_2}(\epsilon; x)_{b_1 b_2}^{a_1 a_2} &= \sum_{n=0}^{2j_2} g(p_1, j_2, n; u)_\epsilon \\ & C(p_1, j_2, p_1 + p_2 - n; b_1, b_2, b_1 + b_2 - n)_\epsilon \\ & C(j_2, p_1, p_1 + p_2 - n; a_2, a_1, a_1 + a_2 - n)_\epsilon. \end{aligned} \tag{11.3.6}$$

Here the function $g(p_1, j_2, n; u)_\epsilon$ is given by (11.3.3) with $p_2 = j_2$. It is easy to show that the R matrix (11.3.6) gives a solution of the linear equation (11.1.10).

11.4 Infinite dimensional colored IRF models

Using the Racah coefficients for color representations we define colored IRF (Interaction Round a Face) models. Let the symbol $w(a, b, c, d; p_1, p_2; u)_q$ denote the Boltzmann weight of colored IRF model for the configuration $\{a, b, c, d; p_1, p_2\}$. The admissible condition is determined by the fusion rules. In the configuration $\{a, b, c, d; p_1, p_2\}$, a is admissible to d if $a \subset p_1 \otimes d$, i.e., $N_{p_1, d}^{a} \neq 0$. We express We assume that the symbol $a \sim d$ means this condition. The Boltzmann weight $w(a, b, c, d; p_1, p_2; u)_q$ is defined to be zero, unless $a \sim d$, $b \sim a$, $c \sim d$ and $b \sim c$.

The Yang-Baxter relation for the colored IRF model is given by the following.

$$\begin{aligned} & \sum_g w(g_1, g_2, g, g_0; p_1, p_2; u) w(g_2, g_3, g_2', g; p_1, p_3; u + v) \\ & w(g, g_2', g_1', g_0; p_2, p_3; v) \\ = & \sum_g w(g_2, g_3, g, g_1; p_2, p_3; v) w(g_1, g, g_1', g_0; p_1, p_3; u + v) \\ & w(g, g_3, g_2', g_1'; p_1, p_2; u). \end{aligned} \tag{11.4.1}$$

The Boltzmann weights of the colored IRF models are given as follows.

$$\begin{aligned} & w(g_{12}, g_1, g_{21}, g_0; p_1, p_2; u)_q \\ = & \sum_{n=0}^{\infty} g(p_1, p_2, n; u)_q [2p + 1]_q \sqrt{[2g_{12} + 1]_q [2g_{21} + 1]_q} \\ & W(p_2, p_1, g_1, g_0; p, g_{12})_q W(p_1, p_2, g_1, g_0; p, g_{21})_q, \end{aligned} \tag{11.4.2}$$

where $p = p_1 + p_2 - n, n \in \mathbf{Z}_{\geq 0}$, and the function $g(p_1, p_2, n; u)_q$ is given by (11.2.3) . We recall that the symbols W and w represent the Racah coefficient and the Boltzmann weight of the IRF model, respectively.

The expression (11.4.2) can be derived from (11.2.2) by using the definition of the Racah coefficients. For the cases of the spin j representations, connection of the IRF model to the quantum 6 j symbol was discussed in Ref. [75] (see also [73]). Therefore the Boltzmann weights given in (11.4.2) satisfy the Yang-Baxter relation for the IRF models (11.4.1).

We can discuss hybrid type colored IRF models, i.e., $2\mu_1 \in \mathbf{Z}_{\geq 0}$ or $2\mu_2 \in \mathbf{Z}_{\geq 0}$. For simplicity we consider only the case $p_1 \in \mathbf{C}$ and $2\mu_2 = 2j_2 \in \mathbf{Z}_{\geq 0}$. The Boltzmann weights of the colored IRF models are given as follows.

$$\begin{aligned} & w(g_{12}, g_1, g_{21}, g_0; p_1, j_2; u)_q \\ = & \sum_{n=0}^{2j_2} g(p_1, j_2, n; u)_q [2p + 1]_q \sqrt{[2g_{12} + 1]_q [2g_{21} + 1]_q} \\ & W(j_2, p_1, g_1, g_0; p, g_{12})_q W(p_1, j_2, g_1, g_0; p, g_{21})_q, \end{aligned} \tag{11.4.3}$$

where $p = p_1 + j_2 - n, 0 \le n \le 2j_2, n \in Z_{\ge 0}$, and the function $g(p_1, p_2, n; u)_q$ is given by (11.2.3).

11.5 Finite dimensional colored IRF models

Using the Racah coefficients for color representations we define colored IRF (Interaction Round a Face) models. Let the symbol $w(a, b, c, d; p_1, p_2; u)_\epsilon$ represent the Boltzmann weight of colored IRF model associated with finite dimensional color representations of $U_q(sl(2))$.

The Boltzmann weights of the colored IRF model are given by the following.

$$\begin{aligned} & w(g_{12}, g_1, g_{21}, g_0; p_1, p_2; u)_\epsilon \\ = & \sum_{n=0}^{N-1} g(p_1, p_2, n; u)_\epsilon [2p+1]_q \sqrt{[2g_{12}+1]_\epsilon [2g_{21}+1]_\epsilon} \\ & \times W(p_2, p_1, g_1, g_0; p, g_{12})_\epsilon W(p_1, p_2, g_1, g_0; p, g_{21})_\epsilon. \end{aligned} \quad (11.5.1)$$

Here the function $g(p_1, p_2, n; u)_\epsilon$ is given by (11.3.3).

We can discuss hybrid type colored IRF models, i.e., $2\mu_1 \in Z_{\ge 0}$ or $2\mu_2 \in Z_{\ge 0}$. For simplicity we consider only the case $p_1 \in \mathbf{C}$ and $2\mu_2 = 2j_2 \in Z_{\ge 0}$. The Boltzmann weights of the colored IRF models are given as follows.

$$\begin{aligned} & w(g_{12}, g_1, g_{21}, g_0; p_1, j_2; u)_\epsilon \\ = & \sum_{n=0}^{2j_2} g(p_1, j_2, n; u)_\epsilon [2p+1]_\epsilon \sqrt{[2g_{12}+1]_\epsilon [2g_{21}+1]_\epsilon} \\ & W(j_2, p_1, g_1, g_0; p, g_{12})_\epsilon W(p_1, j_2, g_1, g_0; p, g_{21})_\epsilon, \end{aligned} \quad (11.5.2)$$

where $p = p_1 + j_2 - n$, $0 \le n \le 2j_2, n \in Z_{\ge 0}$, and the function $g(p_1, j_2, n)_\epsilon$ is given by (11.3.3).

12 Discussion

12.1 Color representations of $U_q(g)$

Let us consider $U_q(g)$, where g is an arbitrary Lie algebra (see Appendix B). We can construct color representations of $U_q(g)$ through the following procedures: (1) We construct infinite dimensional representations with color variables by taking the Dynkin coefficients (or the labels for the highest weight vectors of finite dimensional representations) complex parameters, which give color variables. (2) Taking the

limit $q \to \epsilon$, where ϵ is a square root of a primitive root of unity ω, we restrict the infinite dimensional representation into a finite dimensional one.

Through the q-analog of the Gelfand-Zetlin basis of $sl(n)$ [47,97], we construct color representations of $U_q(sl(n, \mathbf{C}))$. We introduce the Gelfand pattern. [42,97] Each finite dimensional irreducible representation space of $sl(n)$ is specified by a set of n ordered integers

$$[[m]]_n = (m_{1n}, m_{2n}, \cdots, m_{nn}), \quad m_{1n} \ge m_{2n} \ge \cdots \ge m_{nn}. \quad (12.1.1)$$

We write this vector space as $V([[m]]_n)$. The vector space is spanned by a set of vectors which are labeled by the integers $m_{ij}(i \le j = 1, 2, \cdots n-1)$. The set of $n(n+1)/2$ integers is arranged in a *Gelfand pattern*, a triangular array , denoted by $(m)_n$:

$$(m)_n = \begin{pmatrix} m_{1n}, & m_{2n} & \cdots & m_{nn} \\ & m_{1n-1} & \cdots & m_{n-1n-1} \\ & \cdot & & \cdot \\ & & m_{11} & \end{pmatrix} \quad (12.1.2)$$

For a specified set of integers $[[m]]_n$ the remaining integers m_{ij} $(i \le j = 1, \cdots n-1)$ assume any values consistent with the "betweenness" conditions

$$m_{i,j+1} \ge m_{i,j} \ge m_{i+1,j+1}. \quad (12.1.3)$$

Finite dimensional representations of $U_q(sl(n))$ are labeled by the Gelfand patterns. [85,48] We can write down matrix representations of the generators $X_i^{\pm}, K_i$ $(i = 1, \cdots, n)$ of $U_q(sl(n))$ (see [48]).

Let us discuss infinite dimensional color representations of $U_q(sl(n))$. We replace the integer m_{1n} in $[[m]]_n$ by a complex parameter p, and we let m_{1n-1} take any integers but a condition $m_{1n-1} \ge m_{2n}$. Then we have infinite dimensional color representations, where p is the color parameter. We can construct matrix representations of the generators $X_i^{\pm}, K_i$ $(i = 1, \cdots, n)$ in the infinite dimensional representation using the q-analog of the Gelfand-Zetlin basis.

Let us construct finite dimensional color representations. We take the limit $q \to \epsilon$ in the infinite dimensional representation. We recall that ϵ^2 is a primitive root of unity. We consider the case $m_{2n} = \cdots = m_{nn} = 0$, i.e. $[[m]]_n = (p, m_{2n} = 0, \cdots, m_{nn} = 0)$. It is easy to see the following from the fact that $[m_{11}+1]_\epsilon = 0$ for $m_{11} = N-1$

$$X_1^- | [[m]]_n, m_{ij} >= 0, \quad \text{for} \quad m_{11} = N-1, \quad (12.1.4)$$

where $|[[m]]_n, m_{ij}>$ is a basis vector corresponding to the Gelfand pattern for the integers $\{m_{ij}\}$. Therefore we can restrict the infinite dimensional representation space into a finite dimensional one. Since the limit $q \to \epsilon$ is well-defined, we have a finite dimensional representation of $U_q(sl(n))$. In the Appendix F, a 9-dimensional color representation of $U_q(sl(3))$ is presented for an illustration.

It is interesting to note that the color representation is closely related to the cyclic representation [10,23,9] of $U_q(g)$ at roots of unity. From the cyclic representations we can derive the color representations by taking the continuous parameters some special values. However, the R matrices of the color and the cyclic representations are completely different. We can not derive the R matrices of the color representations from the R matrices of the cyclic representations. We note that the formers are derived from the universal R matrix while the latters are not.

Summary: The color representations have the following different properties than the cyclic representations. (1) The color representations have both the highest weight and the lowest weight vectors. (2) The standard universal R matrix gives the R matrix of the color representation.

A Proof of colored braid relation for $G(\alpha, \beta; +)$

Recall the next useful formulas (see [28]).

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{bmatrix} m-1 \\ n \end{bmatrix}_q + q^{m-n} \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}_q, \tag{A.1}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{bmatrix} m-1 \\ n \end{bmatrix}_q q^n + \begin{bmatrix} m-1 \\ n-1 \end{bmatrix}_q, \tag{A.2}$$

$$(z;m)_q = \sum_{n=0}^{m} (-z)^n \begin{bmatrix} m \\ n \end{bmatrix}_q q^{n(n-1)/2}. \tag{A.3}$$

Moreover we use the following formulas.

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \begin{bmatrix} m \\ m-n \end{bmatrix}_q, \tag{A.4}$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} m-k \\ m-n \end{bmatrix}_q. \tag{A.5}$$

In this appendix we will prove that the colored braid matrix given in section 3 satisfies colored braid relation:

$$\sum_{j,k,l} G^{ab}_{jk}(\alpha,\beta;+) G^{ke}_{lf}(\alpha,\gamma;+) G^{jl}_{cd}(\beta,\gamma;+) \tag{A.6}$$

$$= \sum_{j,k,l} G^{be}_{lj}(\beta,\gamma;+) G^{al}_{ck}(\alpha,\gamma;+) G^{kj}_{df}(\alpha,\beta;+), \tag{A.7}$$

where

$$G^{ab}_{cd}(\alpha,\beta;+) = \begin{cases} \left[{c \atop b}\right]_\omega \left({a \atop d}\right)_{\alpha,\omega} \alpha^b \omega^{bd}, & \text{if } a+b = c+d, \\ 0, & \text{if } a+b \neq c+d, \end{cases} \tag{A.8}$$

(note that it is sufficient to prove (A.7) in the case that $f, F, \mu, \nu, \eta, \kappa$ have special values, so we may put

$$F(\alpha, a) = \frac{(\alpha; a)_\omega}{(\omega; a)_\omega}, \quad f = \mu = 1, \quad \nu = \eta = \kappa = 0, \tag{A.9}$$

see [28]).

The colored braid relation for the N-state colored braid matrix is derived from that for the infinite-state colored braid matrix using the following.

$$\begin{bmatrix} m \\ n \end{bmatrix}_\omega = 0, \quad \text{for} \quad m \geq N > \quad max\{n, m-n\}, \tag{A.10}$$

where ω is a primitive N-th root of unity. Therefore we show the colored braid relation for the colored braid matrix of the infinite-state case.

$$\begin{aligned}&\sum_{j,k,l} \hat{G}^{ab}_{jk}(\alpha,\beta;+)\hat{G}^{ke}_{lj}(\alpha,\gamma;+)\hat{G}^{jl}_{cd}(\beta,\gamma;+)\\ &= \sum_{j,k,l} \hat{G}^{be}_{lj}(\beta,\gamma;+)\hat{G}^{al}_{ck}(\alpha,\gamma;+)\hat{G}^{kj}_{df}(\alpha,\beta;+).\end{aligned} \tag{A.11}$$

Here the infinite-state colored braid matrix is given by

$$\hat{G}^{ab}_{cd}(\alpha,\beta;+) = \begin{cases} \begin{bmatrix} c \\ b \end{bmatrix}_q \binom{a}{d}_{\alpha,q} \alpha^b q^{bd}, & \text{if } a+b=c+d \\ 0, & \text{if } a+b \neq c+d. \end{cases} \tag{A.12}$$

Replacing a,c,k,l by $a+f, c+e, k+f, l+e$ respectively, each side of (A.11) is as follows.

$$\begin{aligned}\text{LHS} &= \textstyle\sum_{j,k,l} \begin{bmatrix} j \\ b \end{bmatrix}_q \binom{a+f}{k+f}_{\alpha,q} \alpha^b q^{b(k+f)} \begin{bmatrix} l+e \\ e \end{bmatrix}_q \binom{k+f}{f}_{\alpha,q} \alpha^e q^{ef} \begin{bmatrix} c+e \\ l+e \end{bmatrix}_q \binom{j}{d}_{\beta,q} \beta^{l+e} q^{d(l+e)} \\ &= \binom{a+f}{f}_{\alpha,q} \begin{bmatrix} c+e \\ e \end{bmatrix}_q \alpha^{b+e}\beta^e q^{bf+ef+ed} \textstyle\sum_{j,k,l} \begin{bmatrix} j \\ b \end{bmatrix}_q \begin{bmatrix} c \\ l \end{bmatrix}_q \binom{j}{d}_{\beta,q} \beta^l q^{bk+dl} \\ &\qquad (l=k,\ j=a+b-k \quad \text{by charge conservation})\end{aligned} \tag{A.13}$$

$$\begin{aligned}\text{RHS} &= \binom{a+f}{f}_{\alpha,q} \begin{bmatrix} c+e \\ e \end{bmatrix}_q \alpha^{b+e}\beta^e q^{bf+ef+ed} \textstyle\sum_{j,k,l} \begin{bmatrix} d \\ j \end{bmatrix}_q \begin{bmatrix} c \\ l \end{bmatrix}_q \binom{b}{j}_{\beta,q} q^{kl} \\ &\qquad (l=c+k-a,\ j=d-k)\end{aligned} \tag{A.14}$$

We introduce the following symbols.

$$L^{ab}_{cd} = \sum_k \begin{bmatrix} a+b-k \\ b \end{bmatrix}_q \begin{bmatrix} c \\ k \end{bmatrix}_q \binom{a+b-k}{d}_{\beta,q} \beta^k q^{(b+d)k} \tag{A.15}$$

$$R^{ab}_{cd} = \sum_k \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} c \\ a-k \end{bmatrix}_q \binom{b}{d-k}_{\beta,q} q^{(c+k-a)k} \tag{A.16}$$

$$M^{ab}_{cd} = \sum_k \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} c \\ a-k \end{bmatrix}_q \binom{b+k}{d}_{\beta,q} q^{(a-k)(d-k)} \tag{A.17}$$

The proof of (A.11) is reduced to Lemmas A.1 and A.3 below.

Lemma A.1 $L^{ab}_{cd}(\beta) = M^{ab}_{cd}(\beta)$.

Proof. By (A.2) we have

$$\begin{bmatrix} a+b-k \\ b \end{bmatrix}_q = \begin{bmatrix} a-1+b-k \\ b \end{bmatrix}_q + q^{a-k} \begin{bmatrix} a+b-1-k \\ b-1 \end{bmatrix}_q \tag{A.18}$$

Further we have

$$\binom{a+b-k}{d}_{\beta,q} = \binom{a+b-1-k}{d-1}_{\beta,q} \tag{A.19}$$

We use (A.18) and (A.19), to obtain

$$L^{ab}_{cd}(\beta) = L^{a-1\,b}_{c\,d-1}(\beta q) + q^a L^{a\,b-1}_{c\,d-1}(\beta q) \tag{A.20}$$

In a similar way, by using

$$\begin{bmatrix} d \\ k \end{bmatrix}_q = \begin{bmatrix} d-1 \\ k \end{bmatrix}_q q^k + \begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q \tag{A.21}$$

$$\binom{b+k}{d}_{\beta,q} = \binom{b-1+k}{d-1}_{\beta,q} \tag{A.22}$$

we have

$$M^{ab}_{cd}(\beta) = q^a M^{a\,b-1}_{c\,d-1}(\beta q) + M^{a-1\,b}_{c\,d-1}(\beta q). \tag{A.23}$$

Hence $L^{ab}_{cd}(\beta)$ and $M^{ab}_{cd}(\beta)$ have the same recursive formula. Since we can check that they have the same initial value, we obtain the required formula. □

Remark A.2 *The initial condition $L^{a,b}_{a+b,0}(\beta) = M^{ab}_{a+b,0}(\beta)$ is reduced to the following relation:*

$$\sum_{k=0}^{a} \begin{bmatrix} a \\ k \end{bmatrix}_q (\beta; a+b-k)_q \beta^k q^{bk} = (\beta; b)_q. \tag{A.24}$$

We can show the equation (A.24) by using (A.1) and by induction on a.

Lemma A.3 $M^{ab}_{cd}(\beta) = R^{ab}_{cd}(\beta)$.

Proof. Since we have

$$\binom{d}{b}_{\beta,q} \binom{b+k}{d}_{\beta,q} = (\beta q^b; k)_q = \sum_{n=0}^{k} (-\beta q^b)^n \begin{bmatrix} k \\ n \end{bmatrix}_q q^{n(n-1)/2}, \tag{A.25}$$

the coefficient of $(-\beta)^n q^{n(n-1)/2}$ in $\binom{d}{b}_{\beta,q} M^{ab}_{cd}(\beta)$ equals to

$$\begin{aligned}&\sum_k \begin{bmatrix} d \\ k \end{bmatrix}_q \begin{bmatrix} c \\ a-k \end{bmatrix}_q \begin{bmatrix} k \\ n \end{bmatrix}_q q^{bn+(a-k)(d-k)} \\ &= \begin{bmatrix} d \\ n \end{bmatrix}_q q^{bn} \sum_k \begin{bmatrix} c \\ a-k \end{bmatrix}_q \begin{bmatrix} d-n \\ k-n \end{bmatrix}_q q^{(a-k)(d-k)} \quad (\text{use (A.4) and (A.5)}) \\ &= \begin{bmatrix} d \\ n \end{bmatrix}_q q^{bn} \sum_{k'} \begin{bmatrix} c \\ a'-k' \end{bmatrix}_q \begin{bmatrix} d' \\ k' \end{bmatrix}_q q^{(a'-k')(d'-k')}. \quad (a'=a-n,\ d'=d-n,\ k'=k-n)\end{aligned} \tag{A.26}$$

In a similar way the coefficient of $(-\beta)^n q^{n(n-1)/2}$ in $\binom{d}{b}_{\beta,q} R^{ab}_{cd}(\beta)$ equals to

$$\begin{bmatrix} d \\ n \end{bmatrix}_q q^{(d+c-a)n} \sum_{k'} \begin{bmatrix} c \\ a'-k' \end{bmatrix}_q \begin{bmatrix} d' \\ k' \end{bmatrix}_q q^{(c+k'-a')k'} \tag{A.27}$$

(note that $d + c - a = b$ by charge conservation).

Now we introduce the following notations.

$$M_{a,d} = \sum_k \begin{bmatrix} c \\ a-k \end{bmatrix}_q \begin{bmatrix} d \\ k \end{bmatrix}_q q^{(a-k)(d-k)} \tag{A.28}$$

$$R_{a,d} = \sum_k \begin{bmatrix} c \\ a-k \end{bmatrix}_q \begin{bmatrix} d \\ k \end{bmatrix}_q q^{(c+k-a)k} \tag{A.29}$$

We can reduce this proof to the next lemma. □

Lemma A.4 $M_{a,d} = R_{a,d}$.

Proof. By (A.2) we have

$$\begin{bmatrix} d \\ k \end{bmatrix}_q = \begin{bmatrix} d-1 \\ k \end{bmatrix}_q q^k + \begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q. \tag{A.30}$$

By using (A.30) we can obtain

$$M_{a,d} = q^a M_{a,d-1} + M_{a-1,d-1}. \tag{A.31}$$

With this recursive formula we can prove

$$(1-q^a)M_{a,d-1} = (1-q^{c-a+d})M_{a-1,d-1} \tag{A.32}$$

by induction of d. By (A.31) and (A.32) we have another recursive formula

$$M_{a,d} = M_{a,d-1} + q^{c-a+d} M_{a-1,d-1}. \tag{A.33}$$

On the other hand we use

$$\begin{bmatrix} d \\ k \end{bmatrix}_q = \begin{bmatrix} d-1 \\ k \end{bmatrix}_q + q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q, \tag{A.34}$$

to obtain

$$R_{a,d} = R_{a,d-1} + q^{c-a+d} R_{a-1,d-1}. \tag{A.35}$$

Hence $M_{a,d}$ and $R_{a,d}$ have the same recursive formula. Since we can check that they have the same initial value, we obtain $M_{a,d} = R_{a,d}$. □

B Proof of Inversion relations and the Markov trace property

In this appendix we show the first and second inversion relations and the Markov trace property. For the first and second inversion relations we prove the relations

for the infinite state braid matrix $\hat{G}(\pm)$. By substituting $q = \omega$ in the first and second inversion relations for the infinite state colored braid matrix, we obtain these relations for the N-state colored braid matrices.

We use the following expression for the inverse of the infinite state colored braid matrix $\hat{G}(-)$.

$$\hat{G}^{ab}_{cd}(\alpha,\beta;-) = \begin{cases} \begin{bmatrix} d \\ a \end{bmatrix}_{1/q} \begin{pmatrix} b \\ c \end{pmatrix}_{1/\alpha,1/q} \alpha^{-a} q^{-ac}, & \text{if } a+b=c+d, \\ 0, & \text{if } a+b \neq c+d. \end{cases} \tag{B.1}$$

(1) First inversion relation:

$$\hat{G}(\alpha,\beta;+)\hat{G}(\alpha,\beta;-) = \hat{G}(\alpha,\beta;-)\hat{G}(\alpha,\beta;+) = \mathrm{id}_{V\otimes V}. \tag{B.2}$$

Proof.

$$\begin{aligned} &\sum_{e,f} \hat{G}^{ab}_{ef}(\alpha,\beta,+)\hat{G}^{ab}_{ef}(\alpha,\beta,-) = \\ &\sum_{e,f} \begin{bmatrix} b \\ e \end{bmatrix}_{1/q} \begin{bmatrix} e \\ d \end{bmatrix}_q \begin{pmatrix} f \\ a \end{pmatrix}_{1/\alpha,1/q} \begin{pmatrix} c \\ f \end{pmatrix}_{\alpha,q} \alpha^{f-a} q^{fd-ae} \\ =\ & \begin{bmatrix} d \\ d \end{bmatrix}_q \begin{pmatrix} c \\ a \end{pmatrix}_{\alpha,q} q^{a(d-b)-(b-d)(b-d-1)/2} \sum_m (-1)^m \begin{bmatrix} b-d \\ m \end{bmatrix}_q q^{m(m-1)/2} \\ =\ & \begin{bmatrix} b \\ d \end{bmatrix}_q \begin{pmatrix} c \\ a \end{pmatrix}_{\alpha,q} q^{a(b-d)-(b-d)(b-d-1)/2}(1;b-d)_q \\ =\ & \delta_{bd}\delta_{ac}. \end{aligned} \tag{B.3}$$

□

(2) Second inversion relation:

$$\begin{aligned} \sum_{e,f} G^{ae}_{cf}(\alpha,\beta;+)G^{df}_{be}(\alpha,\beta;-)\omega^{d-e} &= \delta^a_b\delta^c_d, \\ \sum_{e,f} G^{ae}_{cf}(\beta,\alpha;-)G^{df}_{be}(\beta,\alpha;+)\omega^{d-e} &= \delta^a_b\delta^c_d. \end{aligned} \tag{B.4}$$

Proof. By setting $x = 1/q$, we can show the second inversion relations (B.4) for the infinite-state colored braid matrix from the following lemma.

Lemma B.1

$$\sum_{e,f} \hat{G}^{ae}_{cf}(\alpha,\beta;+)\hat{G}^{df}_{be}(\alpha,\beta;-)x^{e-d} = \begin{bmatrix} c \\ d \end{bmatrix}_q \begin{pmatrix} a \\ d \end{pmatrix}_{\alpha,q} (qx;c-d)_q, \tag{B.5}$$

$$\sum_{e,f} G^{ae}_{cf}(\beta,\alpha;-)G^{df}_{be}(\beta,\alpha;+)x^{e-d} = \begin{bmatrix} d \\ c \end{bmatrix}_q \begin{pmatrix} b \\ a \end{pmatrix}_{\beta,q} (qx;d-c)_q. \tag{B.6}$$

Proof.

$$
\begin{aligned}
&\sum_{e,f} \dot{G}^{ae}_{cf}(\alpha,\beta;+)\dot{G}^{df}_{be}(\alpha,\beta;-)x^{e-d} \\
&= \sum_{ef} \begin{bmatrix} c \\ e \end{bmatrix}_q \begin{bmatrix} e \\ d \end{bmatrix}_{1/q} \begin{pmatrix} a \\ f \end{pmatrix}_{\alpha,q} \begin{pmatrix} f \\ b \end{pmatrix}_{1/\alpha,1/q} \alpha^{e-d} q^{ef-bd} x^{e-d} \\
&= \begin{bmatrix} c \\ d \end{bmatrix}_q \begin{pmatrix} a \\ b \end{pmatrix}_{\alpha,q} \sum_{ef} \begin{bmatrix} c-d \\ e-d \end{bmatrix}_q (-x)^{e-d} q^{(e-d)(e-d+1)/2} \\
&= \begin{bmatrix} c \\ d \end{bmatrix}_q \begin{pmatrix} a \\ b \end{pmatrix}_{\alpha,q} (xq;c-d)_q.
\end{aligned} \tag{B.7}
$$

We can show the equation (B.6) in the same way as (B.5). □

(3) Markov trace property:

$$\sum_b G^{ab}_{ab}(\alpha,\alpha;+)\omega^{-b} = 1, \tag{B.8}$$

$$\sum_b G^{ab}_{ab}(\alpha,\alpha;-)\omega^{-b} = \alpha^{-(N-1)}. \tag{B.9}$$

In the eqs. (B.8) and (B.9) we have assumed the normalization of the colored braid matrices (7.2.5) as

$$f(\alpha,\beta;\omega) = 1. \tag{B.10}$$

(a) Equation (B.8)

Proof. Let us introduce the following notation

$$Z_m(z,q) = \sum_k^m Q_{mk}(z,q), \tag{B.11}$$

where

$$Q_{mn}(z,q) = \begin{bmatrix} m \\ n \end{bmatrix}_q \begin{pmatrix} m \\ n \end{pmatrix}_{zq,q} z^n q^{n^2}. \tag{B.12}$$

Then we have

$$
\begin{aligned}
\sum_{k=0}^{N-1} G^{ak}_{ak}(\alpha,\alpha;+)\omega^{-k} &= \sum_{k=0}^{N-1} \begin{bmatrix} a \\ k \end{bmatrix}_\omega \begin{pmatrix} a \\ k \end{pmatrix}_{\alpha,q} \alpha^k \omega^{k^2-k} \\
&= \sum_{k=0}^{a} Q_{ak}(\alpha\omega^{-1},\omega) = Z_a(\alpha\omega^{-1},\omega).
\end{aligned} \tag{B.13}
$$

We can show the following recursion relation (see [28]).

$$Q_{mn}(z,q) = (1-zq^{m+n})Q_{m-1n}(z,q) + zq^{m+n-1}Q_{m-1n-1}(z,q). \tag{B.14}$$

Using the equation (B.14) we have the following:

$$
\begin{aligned}
Z_m(z,q) &= \sum_{k=0}^{m-1}(1-zq^{m+k})Q_{m-1k}(z,q) + \sum_{k=1}^{m} zq^{m+k-1}Q_{m-1k-1}(z,q) \\
&= \sum_{k=0}^{m-1}(1-zq^{m+k})Q_{m-1k}(z,q) + \sum_{k=0}^{m-1} zq^{m+k}Q_{m-1k}(z,q) \\
&= \sum_{k=0}^{m-1} Q_{m-1k}(z,q) \\
&= Z_{m-1}(z,q).
\end{aligned} \tag{B.15}
$$

It is easy to see that $Z_0 = 1$. Therefore we obtain $Z_m(z,q) = 1$ by induction on m. Thus we have the Markov trace property (B.8). □

(b) Equation (B.9)

Proof. Let us calculate the left hand side of the equation (B.9) in the case $a = 0$.

$$
\begin{aligned}
\sum_{k=0}^{N-1} G^{0k}_{0k}(\alpha,\alpha;-)\omega^{-k} &= \sum_{k=0}^{N-1} (\alpha^{-1};k)_{1/\omega}\omega^{-k} \\
&= \alpha \sum_{k=0}^{N-1} \left\{(\alpha^{-1};k)_{1/\omega} - (\alpha^{-1};k+1)_{1/\omega}\right\} \\
&= \alpha(1-(\alpha^{-1};N)_{1/\omega}) \\
&= \alpha(1-(1-\alpha^{-N})) = \alpha^{-(N-1)}.
\end{aligned} \tag{B.16}
$$

We now define the following quantity.

$$S^a_{\alpha,q} = \sum_{k=a}^{N-1} \begin{bmatrix} k \\ a \end{bmatrix}_q \begin{pmatrix} k \\ a \end{pmatrix}_{\alpha,q} \alpha^a q^{a^2+k}. \tag{B.17}$$

Then we have

$$
\begin{aligned}
&\sum_{k=0}^{N-1} G^{ak}_{ak}(\alpha,\alpha;-)\omega^{-k} \\
&= \sum_{k=a}^{N-1} \begin{bmatrix} k \\ a \end{bmatrix}_{1/\omega} \begin{pmatrix} k \\ a \end{pmatrix}_{1/\alpha,1/q} \alpha^{-a}\omega^{-a^2-k} \\
&= S^a_{1/\alpha,1/\omega}.
\end{aligned} \tag{B.18}
$$

We can show the following lemma.

Lemma B.2

$$S^a_{\alpha,q} = (1-\alpha^{-1})\frac{\alpha q^{a+1}}{1-q^a} S^{a-1}_{\alpha q,q} + (1-\alpha q^a)\frac{q^2}{1-q^a} S^{a-1}_{\alpha q^2,q} + \begin{bmatrix} N \\ a \end{bmatrix}_q \begin{pmatrix} N \\ a \end{pmatrix}_{\alpha,q} q^N. \tag{B.19}$$

By substituting in (B.19) α and q by $1/\alpha$ and $1/\omega$, respectively, and by discussing induction on a, we obtain the Markov trace property (B.9). □

C Proof of Proposition 7.12

In this appendix we use the colored braid matrix used in Appendix A.

Let T be any (2,2)-tangle whose open strings have colors α, β and let T_1, T_2 be tangles closing one component of T as in Fig C.1.

Fig C.1

Let W^{ab}_{cd} denote the invariant of T when each end has a bond charge a, b, c, d. That is, W^{ab}_{cd} is defined by

$$\phi(T) = \sum W^{ab}_{cd} e_a \otimes e_b \otimes e^*_c \otimes e^*_d, \tag{C.1}$$

(note that $W^{ab}_{cd} = 0$ unless $a + b = c + d$ by charge conservation). The invariants ϕ of T_1 and T_2 are given as follows by using Lemma 4.4.

$$\phi(T_1) = \alpha^{-(N-1)/2} \left(\sum_{k=0}^{N-1} W^{k0}_{k0} \omega^k \right) \mathrm{id}_V \tag{C.2}$$

$$\phi(T_2) = \beta^{(N-1)/2} \left(\sum_{k=0}^{N-1} W^{0k}_{0k} \omega^{-k} \right) \mathrm{id}_V \tag{C.3}$$

Since the formula of Proposition 7.12 becomes

$$\phi(T_1)(\beta; N-1)^{-1}_\omega \beta^{(N-1)/2} = \phi(T_2)(\alpha; N-1)^{-1}_\omega \alpha^{(N-1)/2}, \tag{C.4}$$

we can reduce the proof of Proposition 5.3 to the next lemma.

Lemma C.1 *We have the following two formulas.*

$$\sum_{k=0}^{N-1} W^{k0}_{k0} \omega^k = W^{0\,N-1}_{N-1\,0} \epsilon_N, \tag{C.5}$$

$$\sum_{k=0}^{N-1} W^{0k}_{0k} \omega^{-k} = W^{0\,N-1}_{N-1\,0} (\beta; N-1)^{-1}_\omega (\alpha; N-1)_\omega \alpha^{-(N-1)} \epsilon_N, \tag{C.6}$$

where

$$\epsilon_N = \begin{cases} 1, & \text{for} \quad \omega^{N/2} = -1, \\ (-1)^{N-1}, & \text{for} \quad \omega^{N/2} = 1. \end{cases} \tag{C.7}$$

Proof.

Fig C.2

Since two tangles in Fig C.2 give the same invariant, we have

$$W^{ab}_{c0}(1 - \omega^a) = W^{a-1\,b}_{c-1\,0}(1 - \omega^c) - W^{a-1\,b+1}_{c\,0}(1 - \omega^{b+1})\omega^{a-1}. \tag{C.8}$$

With (C.8) we can prove the next formula by induction of a from $a = N - 1$ down to $a = 0$.

$$\sum_{k=a}^{N-1} W^{k0}_{k0} \omega^k = \sum_{b=0}^{N-1-a} W^{a\,b}_{a+b\,0} \begin{bmatrix} a+b \\ a \end{bmatrix}^{-1}_\omega \begin{bmatrix} N-a \\ b+1 \end{bmatrix}_\omega (-1)^b \omega^{(2a+b)(b+1)/2}. \tag{C.9}$$

Put $a = 0$, then we obtain (C.5).

Fig C.3

In a similar way with Fig C.3, we obtain the next formulas and (C.6).

$$W^{0b}_{cd}(1 - \beta\omega^{d-1})\alpha\omega^c = W^{0\,b-1}_{c\,d-1}(1 - \beta\omega^{b-1})\alpha - W^{0\,b}_{c+1\,d-1}(1 - \alpha\omega^c), \tag{C.10}$$

$$\sum_{k=d}^{N-1} W^{0k}_{0k} \omega^{-k} = \sum_{c=0}^{N-1-d} W^{0\,c+d}_{c\,d} \begin{bmatrix} N-d \\ c+1 \end{bmatrix}_\omega \begin{pmatrix} d \\ c+d \end{pmatrix}_{\beta,\omega} (\alpha; c)_\omega (-\alpha)^{-c} \omega^{c(c+2d+3)/2+1}. \tag{C.11}$$

□

D Symmetries of the colored braid matrices

We discuss symmetries of the colored braid matrices in this section. These symmetries will be useful for calculation of the colored link invariants. We introduce the following notation for the matrix elements of the tangle diagrams.

$$\begin{aligned} U^r(\alpha,\omega)_{ab} = (\phi(U_r))_{ab} &= \alpha^{-(N-1)/4} \omega^{b/2} \delta_{a+b,N-1}, \\ U^l(\alpha,\omega)_{ab} = (\phi(U_l))_{ab} &= \alpha^{(N-1)/4} \omega^{-a/2} \delta_{a+b,N-1}, \\ \bar{U}^r(\alpha,\omega)_{ab} = (\phi(\bar{U}_r))_{ab} &= \alpha^{(N-1)/4} \omega^{-a/2} \delta_{a+b,N-1}, \\ \bar{U}^l(\alpha,\omega)_{ab} = (\phi(\bar{U}_l))_{ab} &= \alpha^{-(N-1)/4} \omega^{b/2} \delta_{a+b,N-1}. \end{aligned} \tag{D.1}$$

We have the following proposition.

Proposition D.1 *Mirror symmetry:*

$$\begin{aligned} G^{ab}_{cd}(\alpha,\beta;\omega;+) &= G^{ba}_{dc}(1/\alpha, 1/\beta; 1/\omega; -), \\ U^r(\alpha,\omega)_{ab} &= U^l(1/\alpha, 1/\omega)_{ba}, \\ \bar{U}^r(\alpha,\omega)_{ab} &= \bar{U}^l(1/\alpha, 1/\omega)_{ba}. \end{aligned} \tag{D.2}$$

The N-state colored braid matrices have crossing symmetry. For simplicity we use the symmetric expression for the colored braid matrices.

$$
\begin{aligned}
G^{ab}_{cd}(\alpha,\beta;+) &= \left(\begin{bmatrix} a \\ d \end{bmatrix}_{\omega} \begin{bmatrix} c \\ b \end{bmatrix}_{\omega} \begin{pmatrix} a \\ d \end{pmatrix}_{\alpha,\omega} \begin{pmatrix} c \\ b \end{pmatrix}_{\beta,\omega} \right)^{1/2} \left(\alpha^{b}\beta^{d}\right)^{1/2} \omega^{bd}, && \text{(D.3)} \\
G^{ab}_{cd}(\alpha,\beta;-) &= \left(\begin{bmatrix} d \\ a \end{bmatrix}_{1/\omega} \begin{bmatrix} b \\ c \end{bmatrix}_{1/\omega} \begin{pmatrix} d \\ a \end{pmatrix}_{1/\beta,1/\omega} \begin{pmatrix} b \\ c \end{pmatrix}_{1/\alpha,1/\omega} \right)^{1/2} \\
&\quad \left(\alpha^{-a}\beta^{-c}\right)^{1/2} \omega^{-ac}. && \text{(D.4)}
\end{aligned}
$$

Proposition D.2 *Crossing symmetry*

$$
G^{ab}_{cd}(\alpha,\beta;+) = \epsilon_{b-c}\omega^{(d-a)/2}\beta^{-(N-1)/2}G^{b,N-1-d}_{N-1-a,c}(\omega^2/\alpha,\beta,-), \qquad \text{(D.5)}
$$

where

$$
\epsilon_{b-c} = \begin{cases} 1, & \text{for} \quad \omega^{N/2} = -1, \\ (-1)^{b-c}, & \text{for} \quad \omega^{N/2} = 1. \end{cases} \qquad \text{(D.6)}
$$

E Colored vertex model for $N = 2, 3$ and 4

We present the Boltzmann weights of a hierarchy of the N-state colored vertex model for $N = 2, 3, \cdots$. [29] We recall that we use the notation $x = \exp u$.

(1) 2-state colored vertex model

$$
\begin{aligned}
X_{\alpha,\beta}(u)^{1,1}_{1,1} &= (1-\alpha\beta x), \\
X_{\alpha\beta}(u)^{1,2}_{1,2} &= x\sqrt{(1-\alpha^2)(1-\beta^2)}, \\
X_{\alpha\beta}(u)^{1,2}_{2,1} &= (\alpha-\beta x), \\
X_{\alpha\beta}(u)^{2,1}_{1,2} &= (\beta-\alpha x), \\
X_{\alpha\beta}(u)^{2,1}_{2,1} &= \sqrt{(1-\alpha^2)(1-\beta^2)}, \\
X_{\alpha,\beta}(u)^{2,2}_{2,2} &= (x-\alpha\beta). && \text{(E.1)}
\end{aligned}
$$

The Boltzmann weights (E.1) of the 2-state colored vertex models are equivalent to the trigonometric limit of the Felderhof parametrization [38] of the free fermion model. We can apply the transformation (6.1.9) to the Boltzmann weights so that we have $X_{\alpha\beta}(u)^{1,2}_{1,2} = X_{\alpha\beta}(u)^{2,1}_{2,1}$. The color variables α and β are related to the parameters of the Felderhof parametrization by the following relations.

$$
\begin{aligned}
\alpha &= -\frac{\sinh(\gamma+\delta)-i}{\cosh(\gamma+\delta)}, \\
\beta &= \frac{\sinh(\gamma-\delta)+i}{\cosh(\gamma-\delta)}. && \text{(E.2)}
\end{aligned}
$$

Here γ and δ are parameters defined in eq.(2.7) in the Reference [38].

(2) 3-state colored vertex model

$$
\begin{aligned}
X_{\alpha,\beta}(u)^{1,1}_{1,1} &= (1-\alpha\beta x)(1-\alpha\beta\omega x), \\
X_{\alpha\beta}(u)^{1,2}_{1,2} &= x\sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\omega x), \\
X_{\alpha\beta}(u)^{1,2}_{2,1} &= (\alpha-\beta x)(1-\alpha\beta\omega x), \\
X_{\alpha\beta}(u)^{1,3}_{1,3} &= x^2\sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\beta^2)(1-\beta^2\omega)}, \\
X_{\alpha\beta}(u)^{1,3}_{2,2} &= \sqrt{(1-\alpha^2)(1-\beta^2\omega)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}} \times \\
&\quad \times x(\alpha-\beta x), \\
X_{\alpha\beta}(u)^{1,3}_{3,1} &= (\alpha-\beta x)(\alpha-\beta\omega x), \\
\\
X_{\alpha\beta}(u)^{2,1}_{1,2} &= (\beta-\alpha x)(1-\alpha\beta\omega x), \\
X_{\alpha\beta}(u)^{2,1}_{2,1} &= \sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\omega x), \\
X_{\alpha\beta}(u)^{2,2}_{1,3} &= \sqrt{(1-\alpha^2\omega)(1-\beta^2)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}} \times \\
&\quad \times x(\beta-\alpha x) \\
X_{\alpha\beta}(u)^{2,2}_{2,2} &= (\left(1-\alpha^2\right)\left(1-\beta^2\omega\right)x - \\
&\quad -(\beta-\alpha x)(-\alpha\omega+\beta x)), \\
X_{\alpha\beta}(u)^{2,2}_{3,1} &= \sqrt{(1-\alpha^2)(1-\beta^2\omega)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}}(\alpha-\beta x), \\
X_{\alpha\beta}(u)^{2,3}_{2,3} &= x(x-\alpha\beta)\sqrt{(1-\alpha^2\omega)(1-\beta^2\omega)}, \\
X_{\alpha\beta}(u)^{2,3}_{3,2} &= (1+\omega)(\alpha-\beta x)(x-\alpha\beta), \\
\\
X_{\alpha\beta}(u)^{3,1}_{1,3} &= (\beta-\alpha x)(\beta-\alpha\omega x), \\
X_{\alpha\beta}(u)^{3,1}_{2,2} &= \sqrt{(1-\beta^2)(1-\alpha^2\omega)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}}(\beta-\alpha x), \\
X_{\alpha\beta}(u)^{3,1}_{3,1} &= \sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\beta^2)(1-\beta^2\omega)}, \\
X_{\alpha\beta}(u)^{3,2}_{2,3} &= (1+\omega)(\beta-x\alpha)(x-\alpha\beta), \\
X_{\alpha\beta}(u)^{3,2}_{3,2} &= \sqrt{(1-\alpha^2\omega)(1-\beta^2\omega)}(x-\alpha\beta), \\
X_{\alpha\beta}(u)^{3,3}_{3,3} &= (x-\alpha\beta)(x-\alpha\beta\omega). && \text{(E.3)}
\end{aligned}
$$

We recall that ω is a 3-rd root of unity but 1 for N=3 case.

The explicit forms of the Boltzmann weights of the 3-state colored vertex model are introduced in Reference [29].

(3) 4-state colored vertex model

$$\begin{aligned}
X_{\alpha,\beta}(u)_{1,1}^{1,1} &= (1-\alpha\beta x)(1-\alpha\beta\omega x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{1,2}^{1,2} &= x\sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\omega x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{2,1}^{1,2} &= (\alpha-\beta x)(1-\alpha\beta\omega x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{1,3}^{1,3} &= x^2\sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\beta^2)(1-\beta^2\omega)}\times \\
&\quad \times\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{2,2}^{1,3} &= \sqrt{(1-\alpha^2)(1-\beta^2\omega)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}}\times \\
&\quad \times x(\alpha-\beta x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{3,1}^{1,3} &= (\alpha-\beta x)(\alpha-\beta\omega x)\left(1-\alpha\beta\omega^2 x\right),
\end{aligned}$$

$$\begin{aligned}
X_{\alpha\beta}(u)_{1,4}^{1,4} &= x^3\sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\alpha^2\omega^2)} \\
&\quad \times\sqrt{(1-\beta^2)(1-\beta^2\omega)(1-\beta^2\omega^2)}, \\
X_{\alpha\beta}(u)_{2,3}^{1,4} &= \sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\beta^2\omega)(1-\beta^2\omega^2)} \\
&\quad \frac{\sqrt{1-\omega^3}}{\sqrt{1-\omega}}x^2(\alpha-\beta x), \\
X_{\alpha\beta}(u)_{3,2}^{1,4} &= \sqrt{(1-\alpha^2)(1-\beta^2\omega^2)}\frac{\sqrt{1-\omega^3}}{\sqrt{1-\omega}}\times \\
&\quad \times x(\alpha-\beta x)(\alpha-\beta\omega x), \\
X_{\alpha\beta}(u)_{4,1}^{1,4} &= (\alpha-\beta x)(\alpha-\beta\omega x)\left(\alpha-\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{1,2}^{2,1} &= (\beta-\alpha x)(1-\alpha\beta\omega x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{2,1}^{2,1} &= \sqrt{(1-\alpha^2)(1-\beta^2)}(1-\alpha\beta\omega x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{1,3}^{2,2} &= \sqrt{(1-\alpha^2\omega)(1-\beta^2)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}}\times \\
&\quad \times x(\beta-\alpha x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{2,2}^{2,2} &= (1-\alpha\beta\omega^2 x)(\left(1-\alpha^2\right)\left(1-\beta^2\omega\right)x- \\
&\quad -(\beta-\alpha x)(-\alpha\omega+\beta x)), \\
X_{\alpha\beta}(u)_{3,1}^{2,2} &= \sqrt{(1-\alpha^2)(1-\beta^2\omega)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}}\times \\
&\quad \times(\alpha-\beta x)\left(1-\alpha\beta\omega^2 x\right), \\
X_{\alpha\beta}(u)_{1,4}^{2,3} &= \sqrt{(1-\alpha^2\omega)(1-\alpha^2\omega^2)(1-\beta^2)(1-\beta^2\omega)}\times \\
&\quad \times\frac{\sqrt{1-\omega^3}}{\sqrt{1-\omega}}x^2(\beta-\alpha x),
\end{aligned}$$

$$\begin{aligned}
X_{\alpha\beta}(u)_{2,3}^{2,3} &= \sqrt{(1-\alpha^2\omega)(1-\beta^2\omega)}(\left(1-\beta^2\right)\left(1-\alpha^2\omega^2\right)x^2- \\
&\quad -(1+\omega)x(-\beta\omega+\alpha x)(\alpha-\beta x)), \\
X_{\alpha\beta}(u)_{3,2}^{2,3} &= (\alpha-\beta x)(\left(1-\alpha^2\beta^2\right)\left(1-\omega^3\right)x- \\
&\quad -\omega(-\beta\omega+\alpha x)(\alpha-\beta x)), \\
X_{\alpha\beta}(u)_{4,1}^{2,3} &= \frac{\sqrt{1-\omega^3}}{\sqrt{1-\omega}}\sqrt{(1-\alpha^2)(1-\beta^2\omega^2)}(\alpha-\beta x)(\alpha-\beta\omega x), \\
X_{\alpha\beta}(u)_{2,4}^{2,4} &= x^2\sqrt{1-\alpha^2\omega}\sqrt{(1-\alpha^2\omega^2)(1-\beta^2\omega)(1-\beta^2\omega^2)}\times \\
&\quad \times(-(\alpha\beta)+x), \\
X_{\alpha\beta}(u)_{3,3}^{2,4} &= x\frac{\sqrt{(1-\omega^2)(1-\omega^3)}}{1-\omega}\sqrt{(1-\alpha^2\omega)(1-\beta^2\omega^2)}\times \\
&\quad \times(-(\alpha\beta)+x)(\alpha-\beta x), \\
X_{\alpha\beta}(u)_{4,2}^{2,4} &= \frac{(1-\omega^3)(-(\alpha\beta)+x)(\alpha-\beta x)(\alpha-\beta\omega x)}{1-\omega},
\end{aligned}$$

$$\begin{aligned}
X_{\alpha\beta}(u)_{1,3}^{3,1} &= (\beta-\alpha x)(\beta-\alpha\omega x)(1-\alpha\beta\omega^2 x), \\
X_{\alpha\beta}(u)_{2,2}^{3,1} &= \sqrt{(1-\beta^2)(1-\alpha^2\omega)}\frac{\sqrt{1-\omega^2}}{\sqrt{1-\omega}}(\beta-\alpha x)(1-\alpha\beta\omega^2 x), \\
X_{\alpha\beta}(u)_{3,1}^{3,1} &= \sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\beta^2)(1-\beta^2\omega)}\left(1-\alpha\beta\omega^2 x\right),
\end{aligned}$$

$$\begin{aligned}
X_{\alpha\beta}(u)_{1,4}^{3,2} &= \sqrt{(1-\alpha^2\omega^2)(1-\beta^2)}\frac{\sqrt{1-\omega^3}}{\sqrt{1-\omega}}x(\beta-\alpha x)(\beta-\alpha\omega x), \\
X_{\alpha\beta}(u)_{2,3}^{3,2} &= (\beta-\alpha x)(\left(1-\alpha^2\beta^2\right)\left(1-\omega^3\right)x- \\
&\quad -\omega(\beta-\alpha x)(-(\alpha\omega)+\beta x)), \\
X_{\alpha\beta}(u)_{3,2}^{3,2} &= \sqrt{(1-\alpha^2\omega)(1-\beta^2\omega)}(\left(1-\alpha^2\right)\left(1-\beta^2\omega^2\right)x- \\
&\quad -(1+\omega)(\beta-\alpha x)(-\alpha\omega+\beta x)), \\
X_{\alpha\beta}(u)_{4,1}^{3,2} &= \sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\beta^2\omega)(1-\beta^2\omega^2)}\times \\
&\quad \times\frac{\sqrt{1-\omega^3}}{\sqrt{1-\omega}}(\alpha-\beta x), \\
X_{\alpha\beta}(u)_{2,4}^{3,3} &= x\frac{\sqrt{(1-\omega^2)(1-\omega^3)}}{1-\omega}\sqrt{(1-\alpha^2\omega^2)(1-\beta^2\omega)}\times \\
&\quad \times(-\alpha\beta+x)(\beta-\alpha x), \\
X_{\alpha\beta}(u)_{3,3}^{3,3} &= (\left(1-\alpha^2\omega\right)\left(1-\beta^2\omega^2\right)x- \\
&\quad -\left(1+\omega+\omega^2\right)(\beta-\alpha x)(-\alpha\omega+\beta x))\times \\
&\quad \times(-\alpha\beta+x), \\
X_{\alpha\beta}(u)_{4,2}^{3,3} &= \sqrt{(1-\alpha^2\omega)(1-\beta^2\omega^2)}\frac{\sqrt{(1-\omega^2)(1-\omega^3)}}{1-\omega}\times
\end{aligned}$$

$$
\begin{aligned}
& \times(-\alpha\beta+x)(\alpha-\beta x), \\
X_{\alpha\beta}(u)^{3,4}_{3,4} &= x\sqrt{1-\alpha^2\omega^2}\sqrt{1-\beta^2\omega^2}(-\alpha\beta+x)(-\alpha\beta\omega+x), \\
X_{\alpha\beta}(u)^{3,4}_{4,3} &= \frac{1-\omega^3}{1-\omega}(-(\alpha\beta)+x)(-(\alpha\beta\omega)+x)(\alpha-\beta x), \\
X_{\alpha\beta}(u)^{4,1}_{1,4} &= (\beta-\alpha x)(\beta-\alpha\omega x)\left(\beta-\alpha\omega^2 x\right), \\
X_{\alpha\beta}(u)^{4,1}_{2,3} &= \frac{\sqrt{(1-\alpha^2\omega^2)(1-\beta^2)}\sqrt{1-\omega^3}(\beta-\alpha x)(\beta-\alpha\omega x)}{\sqrt{1-\omega}}, \\
X_{\alpha\beta}(u)^{4,1}_{3,2} &= \sqrt{(1-\alpha^2\omega)(1-\alpha^2\omega^2)(1-\beta^2)(1-\beta^2\omega)}\times \\
& \times\frac{\sqrt{1-\omega^3}}{\sqrt{1-\omega}}(\beta-\alpha x), \\
X_{\alpha\beta}(u)^{4,1}_{4,1} &= \sqrt{(1-\alpha^2)(1-\alpha^2\omega)(1-\alpha^2\omega^2)}\times \\
& \times\sqrt{(1-\beta^2)(1-\beta^2\omega)(1-\beta^2\omega^2)}, \\
X_{\alpha\beta}(u)^{4,2}_{2,4} &= \frac{(1-\omega^3)(-(\alpha\beta)+x)(\beta-\alpha x)(\beta-\alpha\omega x)}{1-\omega}, \\
X_{\alpha\beta}(u)^{4,2}_{3,3} &= \sqrt{(1-\alpha^2\omega^2)(1-\beta^2\omega)}\frac{\sqrt{(1-\omega^2)(1-\omega^3)}}{1-\omega}\times \\
& \times(-(\alpha\beta)+x)(\beta-ax), \\
X_{\alpha\beta}(u)^{4,2}_{4,2} &= \sqrt{(1-\alpha^2\omega)(1-\alpha^2\omega^2)(1-\beta^2\omega)(1-\beta^2\omega^2)}\times \\
& \times(-(\alpha\beta)+x), \\
X_{\alpha\beta}(u)^{4,3}_{3,4} &= \frac{(1-\omega^3)(-(\alpha\beta)+x)(-(\alpha\beta\omega)+x)(\beta-\alpha x)}{1-\omega}, \\
X_{\alpha\beta}(u)^{4,3}_{4,3} &= \sqrt{(1-\alpha^2\omega^2)(1-\beta^2\omega^2)}(-\alpha\beta+x)(-\alpha\beta\omega+x), \\
X_{\alpha\beta}(u)^{4,4}_{4,4} &= (-\alpha\beta+x)(-\alpha\beta\omega+x)\left(-\alpha\beta\omega^2+x\right). \qquad \text{(E.4)}
\end{aligned}
$$

We recall that $\omega=\pm\sqrt{-1}$ for $N=4$.

F Color representation of $U_q(sl(n))$

We present matrix representations of the generators of $X_i^{\pm}$, K_i for $i=1,2$ of $U_q(sl(3))$ for color representation with $[[m]]_3=(p,0,0)$ $(p\in\mathbf{C})$ and $q=\epsilon=\exp(\pi i/3)$ $(N=3)$. All the possible Gelfand patterns $|m_{12},m_{11},m_{22}>$ are given by the following: $|0>=(0,0,0)$, $|1>=(1,0,0)$, $|2>=(1,1,0)$, $|3>=(2,0,0)$, $|4>=(2,1,0)$, $|5>=(2,2,0)$, $|6>=(3,1,0)$, $|7>=(3,2,0)$, $|8>=(4,2,0)$. We define matrix elements on the Gelfand-Zetlin basis by $X|m>=\sum_{n=0}^{8}(X)^m_n|n>$ for $m=0,1,\cdots,8$.

The explicit matrix elements of the generators are given as follows.

$$
\begin{aligned}
(X_1^+)^1_2 &= 1, \quad (X_1^+)^3_4=\sqrt{[2]_\epsilon}, \quad (X_1^+)^4_5=\sqrt{[2]_\epsilon}, \quad (X_1^+)^6_7=[2]_\epsilon,
\end{aligned}
$$

$$
\begin{aligned}
(X_1^-)^2_1 &= 1, \quad (X_1^-)^4_3=\sqrt{[2]_\epsilon}, \quad (X_1^-)^5_4=\sqrt{[2]_\epsilon}, \quad (X_1^+)^7_6=[2]_\epsilon, \\
(X_2^+)^0_1 &= \sqrt{[p]_\epsilon}, \quad (X_2^+)^1_3=\sqrt{[2]_\epsilon[p-1]_\epsilon}, \quad (X_2^+)^2_4=\sqrt{[p-1]_\epsilon}, \\
(X_2^+)^4_6 &= \sqrt{[p-2]_\epsilon[2]_\epsilon}, \quad (X_2^+)^5_7=\sqrt{[p-2]_\epsilon}, \quad (X_2^+)^7_8=\sqrt{[p-3]_\epsilon[2]_\epsilon}, \\
(X_2^-)^1_0 &= \sqrt{[p]_\epsilon}, \quad (X_2^-)^3_1=\sqrt{[2]_\epsilon[p-1]_\epsilon}, \quad (X_2^-)^4_2=\sqrt{[p-1]_\epsilon}, \\
(X_2^-)^6_4 &= \sqrt{[p-2]_\epsilon[2]_\epsilon}, \quad (X_2^-)^7_5=\sqrt{[p-2]_\epsilon}, \quad (X_2^-)^8_7=\sqrt{[p-3]_\epsilon[2]_\epsilon}, \\
(K_1)^m_n &= \epsilon^{m_{12}-2m_{11}}\delta^m_n, \\
(K_2)^m_n &= \epsilon^{p+m_{11}-2m_{12}}\delta^m_n \quad \text{for } m,n=0,1,\cdots,8. \qquad \text{(F.1)}
\end{aligned}
$$

Here we recall p is a complex parameter.

Let us construct color representations of $U_q(sl(n))$. We assume $m_{2n}=m_{3n}=\cdots=m_{nn}=0$, i.e. $[[m]]_n=(p,m_{2n}=0,\cdots,m_{nn}=0)$. Let p be a complex parameter. We replace the integer m_{1n} in $[[m]]_n$ by p. Let m_{1n-1} take any integers but conditions $m_{1n-1}\geq m_{2n}$ and $m_{1n-1}\geq m_{1n-2}$. Then we have infinite dimensional color representations, where p is the color variable. The matrix representations of the generators $X_i^{\pm}, K_i$ $(i=1,\cdots,n)$ in the infinite dimensional representation are given in the following.

$$
\begin{aligned}
X_j^-|m> &= \sqrt{[m_{1j+1}-m_{1j}]_q[m_{1j}-m_{1j-1}+1]_q}|m_{1j}+1>, \\
X_j^+|m> &= \sqrt{[m_{1j+1}-m_{1j}+1]_q[m_{1j}-m_{1j-1}]_q}|m_{1j}-1>, \\
K_j|m> &= q^{m_{1j+1}+m_{1j-1}-2m_{1j}}|m>, \\
& \qquad \text{for} \quad j=1,\cdots,n-2, \\
X_{n-1}^-|m> &= \sqrt{[p-m_{1n-1}]_q[m_{1n-1}-m_{1n-2}+1]_q}|m_{1n-1}+1>, \\
X_{n-1}^+|m> &= \sqrt{[p-m_{1n-1}+1]_q[m_{1n-1}-m_{1n-2}]_q}|m_{1n-1}-1>, \\
K_{n-1}|m> &= q^{p+m_{1n-2}-2m_{1n-1}}|m>. \qquad \text{(F.2)}
\end{aligned}
$$

Here we have assumed that $m_{10}=0$.

Let us restrict the infinite dimensional representation into a finite dimensional one. Let ω be a primitive N-th root of unity:

$$
\omega=\exp(\frac{2\pi is}{N}), \qquad (N,s)=1. \qquad \text{(F.3)}
$$

Here the symbol $(a,b)=1$ means that the integers a and b have no common divisor except 1. Let ϵ denote a square root of ω: $\epsilon=\exp(\pi is/N)$,

$(N,s)=1$. We take the limit $q\to\epsilon$ in the infinite dimensional color representation (F.2). Note that $[m_{11}+1]_\epsilon=0$ for $m_{11}=N-1$. We have

$$
X_1^-|m>=0, \quad \text{for} \quad |m> \text{with } m_{11}=N-1. \qquad \text{(F.4)}
$$

Therefore we can restrict the infinite dimensional representation space into a finite dimensional one. Since the limit $q \to \epsilon$ is well defined, we have a finite dimensional representation of $U_q(sl(n))$. The dimension of the color representation is given by N^{n-1}. We have explicit matrix representations for the generators.

$$
\begin{aligned}
X_j^-|m> &= \sqrt{[m_{1j+1} - m_{1j}]_\epsilon [m_{1j} - m_{1j-1} + 1]_\epsilon} |m_{1j} + 1>, \\
X_j^+|m> &= \sqrt{[m_{1j+1} - m_{1j} + 1]_\epsilon [m_{1j} - m_{1j-1}]_\epsilon} |m_{1j} - 1>, \\
K_j|m> &= \epsilon^{m_{1j+1} + m_{1j-1} - 2m_{1j}} |m>, \\
& \quad \text{for} \quad j = 1, \cdots, n-2, \\
X_{n-1}^-|m> &= \sqrt{[p - m_{1n-1}]_\epsilon [m_{1n-1} - m_{1n-2} + 1]_\epsilon} |m_{1n-1} + 1>, \\
X_{n-1}^+|m> &= \sqrt{[p - m_{1n-1} + 1]_\epsilon [m_{1n-1} - m_{1n-2}]_\epsilon} |m_{1n-1} - 1>, \\
K_{n-1}|m> &= \epsilon^{p + m_{1n-2} - 2m_{1n-1}} |m>.
\end{aligned} \tag{F.5}
$$

We note again that p is a complex parameter. Thus we have obtained finite dimensional color representations of $U_q(sl(n))$ with q roots of unity.

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Figeure Captions

Fig. 2.1.1 Vertex Configuration $\{a, b, c, d\}$.

Fig. 2.1.2 IRF Configuration $\{a, b, c, d\}$.

Fig. 4.4.1 Restricted weight lattice and restriction lines for $sl(2|1)$ IRF model of the mechanism I ($r = 5$).

Fig. 4.4.2 Restricted weight lattice for $gl(1|1)$ IRF model with $r = 3$.

Fig. 6.2.1 $X_{\gamma_i,\gamma_j}(u_i - u_j)_{b_j b_i}^{a_i a_j}$ denotes the Boltzmann weight for the vertex configuration $\{a_i, a_j, b_j, b_i\}$ at the intersection of the i-th and j-th strings.

Fig. 10.1.1 Trivalent vertex The edges have colors p_1, p_2, p and variables z_1, z_2, z.

Fig. 10.1.2 Relations E1 $\sim$ E7.

Fig. 10.1.3 We assign the following weights to the elements of the graph tangle diagrams.

(a) Identity diagram. $I_b^a = \delta_{ab}$, $(I^*)_b^a = \delta_{ab}$

(b) Creation-annihilation diagrams.

$(U_r)_{ab} = q^{-p(N-1)}\epsilon^{-b}\delta_{a+b,N-1}$

$(U_l)_{ab} = q^{p(N-1)}\epsilon^{a}\delta_{a+b,N-1}$

$(\bar{U}_r)_{ab} = q^{p(N-1)}\epsilon^{a}\delta_{a+b,N-1}$

$(\bar{U}_l)_{ab} = q^{-p(N-1)}\epsilon^{-b}\delta_{a+b,N-1}$

(c) Braiding diagrams

$G(p_1, p_2; +)_{cd}^{ab}$ and $G(p_1, p_2; -)_{cd}^{ab}$

(d) Vertex diagrams.

We assign the Clebsch-Gordan coefficient $C(p_1, p_2, p; z_1, z_2, z)$ both to the two vertex diagrams V and V'.

Fig. 10.2.1. The color of the closing edge (or component) is p_3.

Fig. 2.1.1

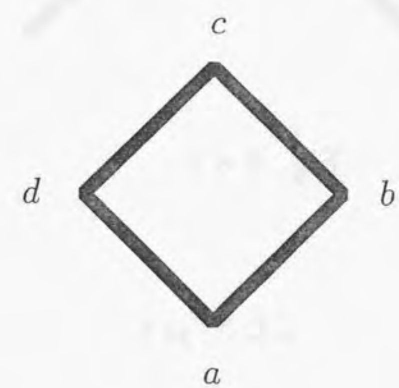
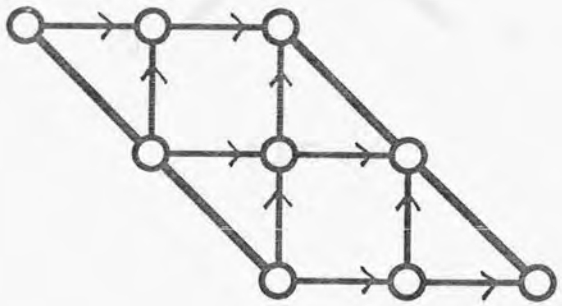


Fig. 2.1.2



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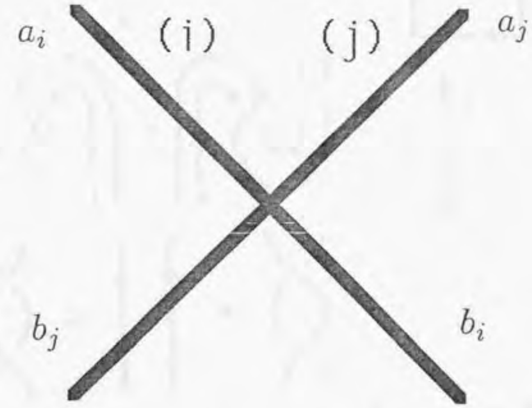

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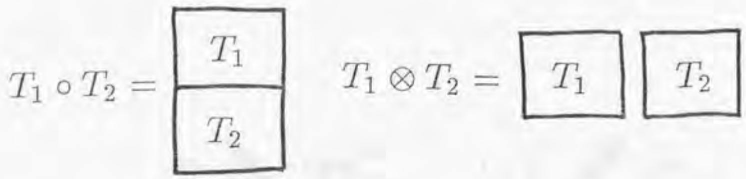
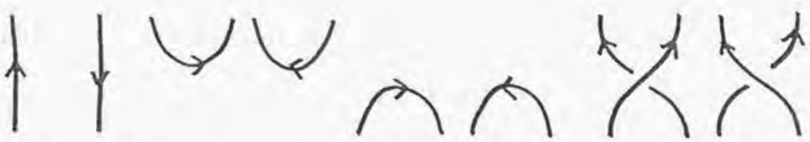


Fig. 7.1.1



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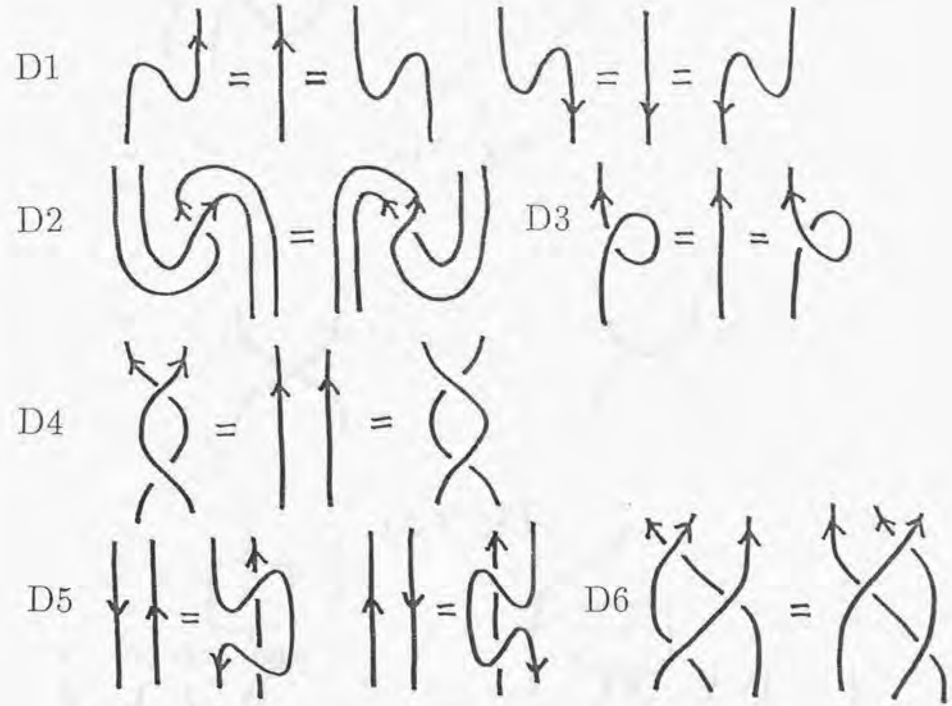

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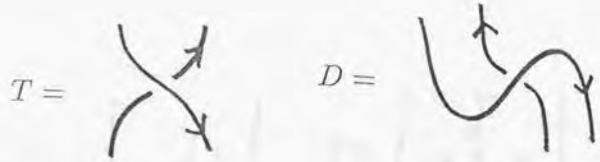

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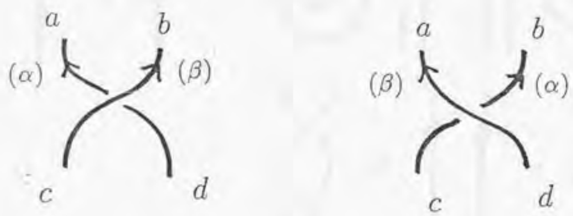

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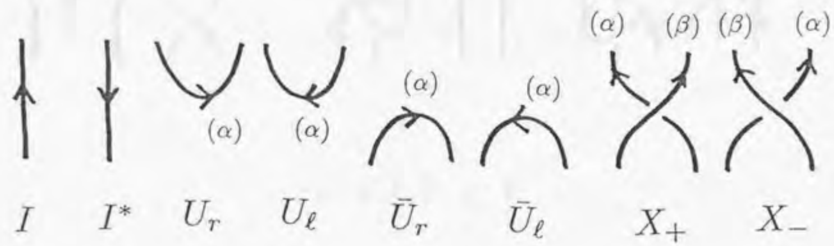

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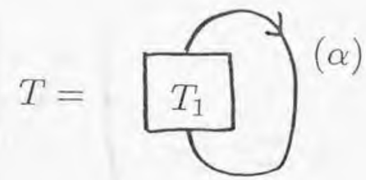

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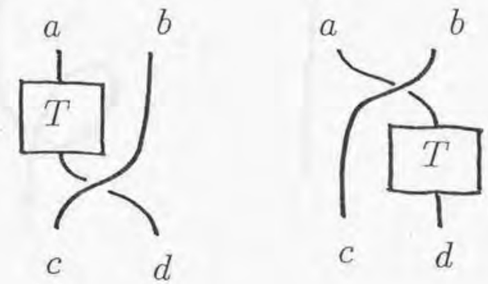
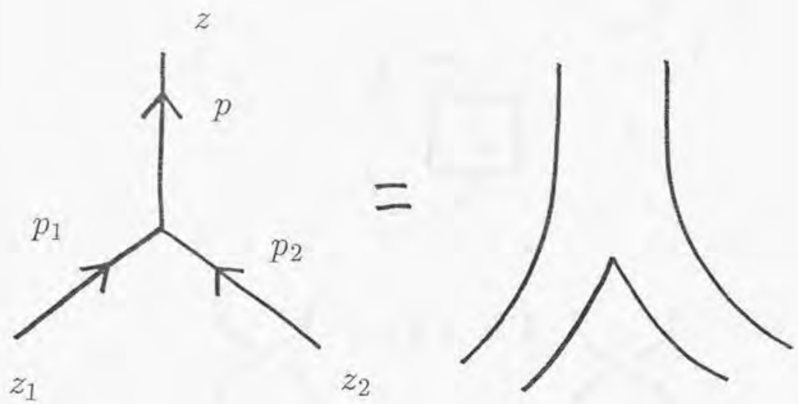


Fig. 7.3.3



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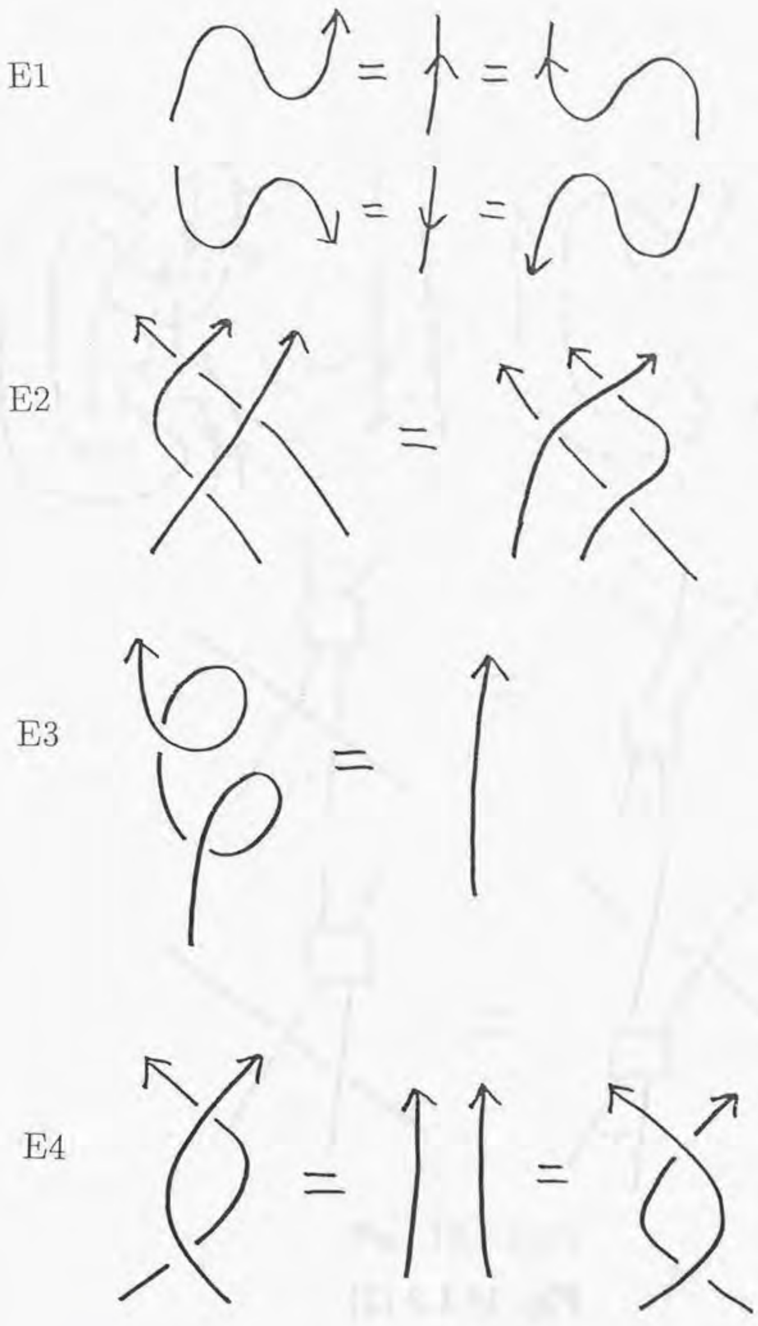
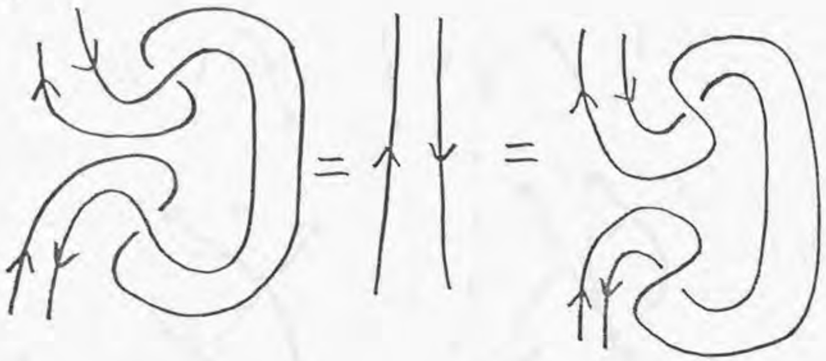


Fig. 10.1.2 (1)

E5



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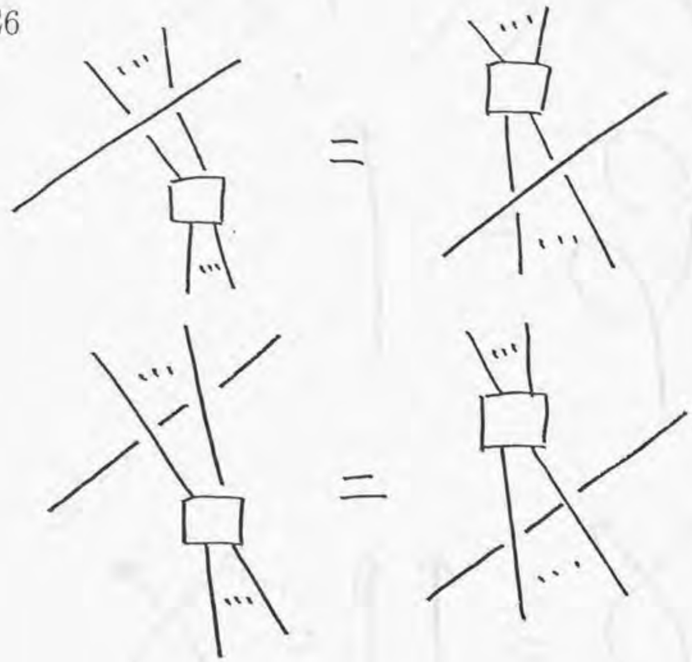
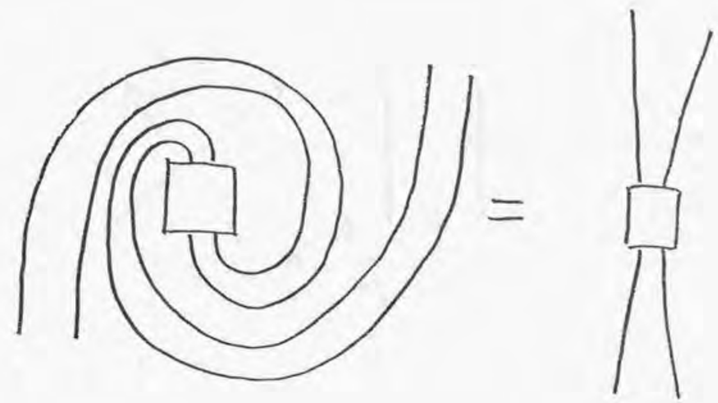


Fig. 10.1.2 (2)

E7



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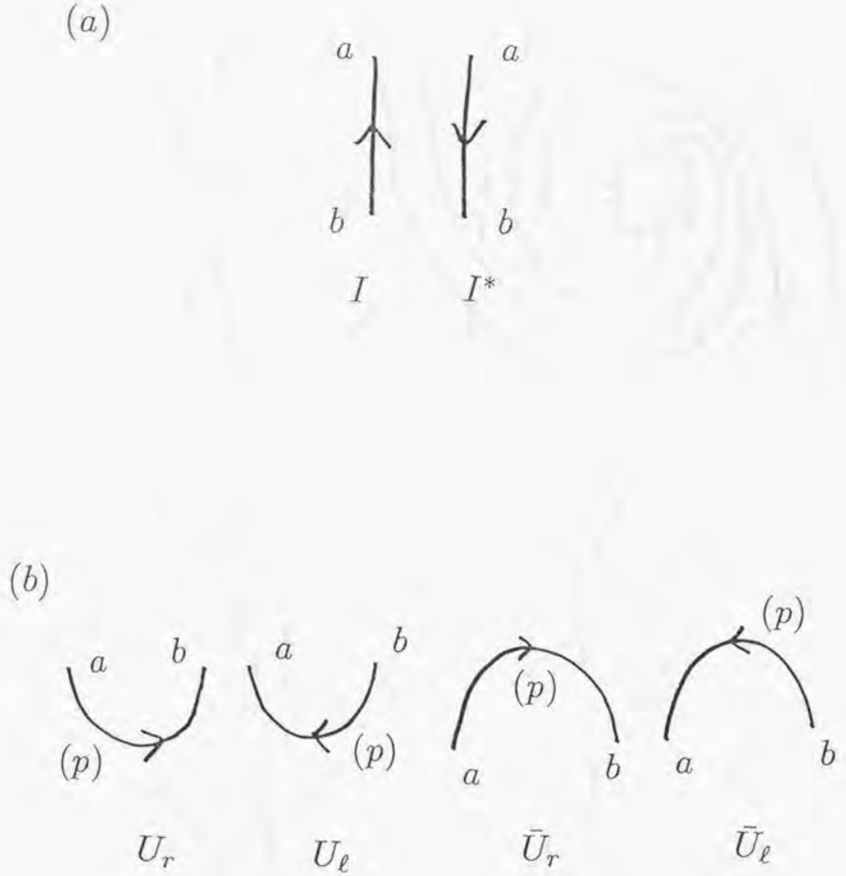

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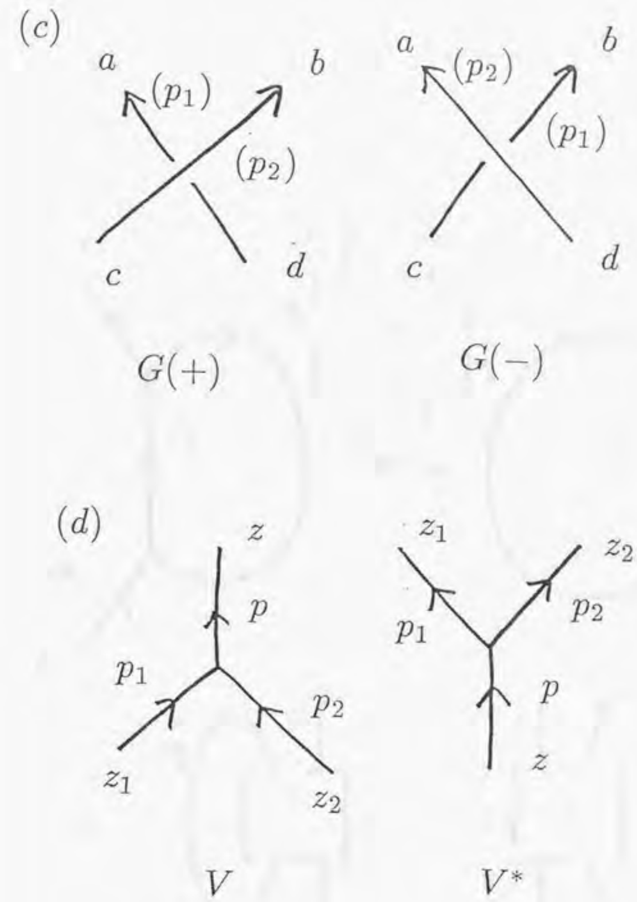

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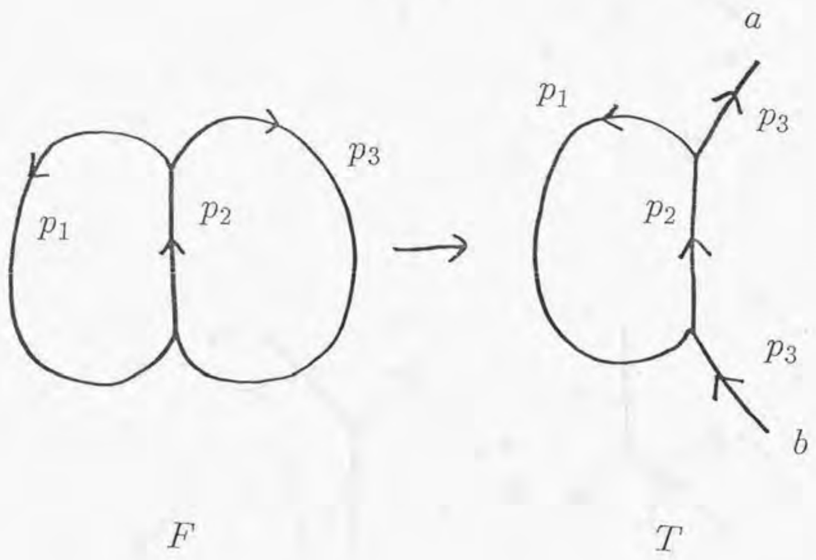

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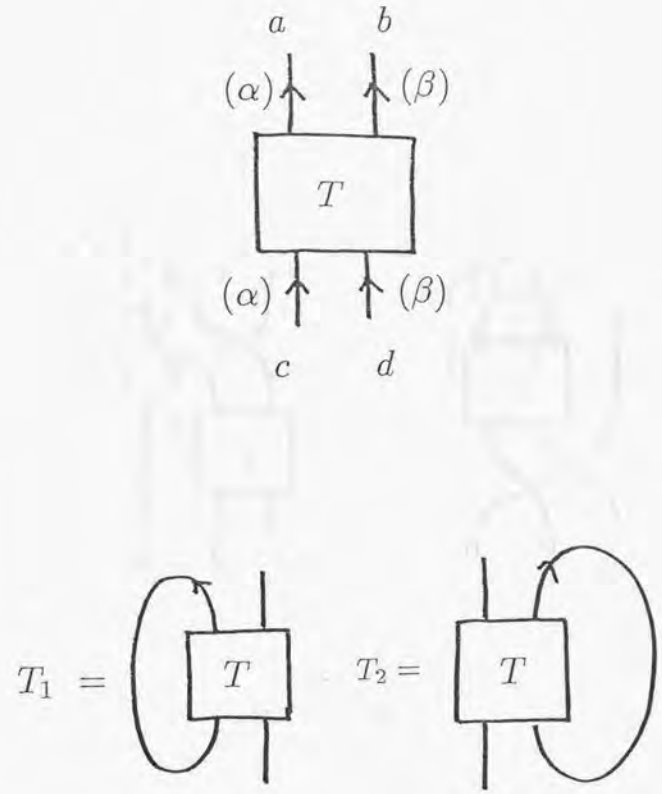

Fig. C.1

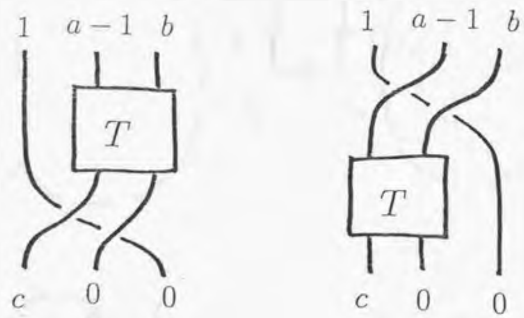

Fig. C.2

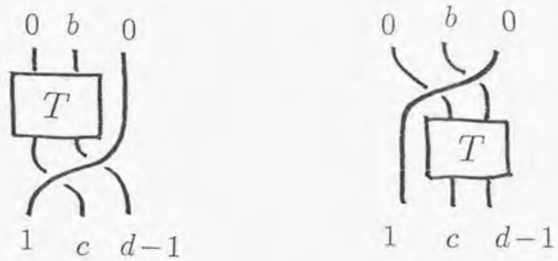

Fig. C.3