# CALCULATION OF THE JOSEPHSON CURRENT IN P-WAVE AND S-WAVE SUPERCONDUCTING HETERO-JUNCTIONS BY BOGOLIUBOV TRANSFORMATION

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## Abstract

Superconducting hetero-junction made of the p-wave superconductor and the normal s-wave superconductor has been considered. Bogoliubov transformation in the p-wave superconductor has been obtained by diagonalizing the superconducting Hamiltonian. Moreover, the value of the Josephson current that flows in the hetero-junctions of p-wave/s-wave superconductors has been calculated by this Bogoliubov transformation, and proved to be zero in the first order perturbation.

## Introduction

Discovery of high-temperature oxide superconductors has enhanced the field of superconducting electronics. Though the mechanism of the superconductivity of high-temperature oxides is not clear yet, devices with high-temperature oxide superconductors are very important. Moreover, it is thought that "heavy-fermion" compounds such as $CeCu_2Si_2$, $UBe_{13}$, $UPt_3$, and $UPt_2C$, are likely not to be normal "singlet" superconductors, but to be "triplet" superconductors. Superconducting hetero-junctions such as metal-superconductor / oxide-superconductor junction and "triplet" superconductor/"singlet" superconductor junction are so useful for electrical application and also interesting from physical point. Maximum dc Josephson current[1] of s-wave/s-wave homo-junction was first calculated by Ambegaokar and Baratoff,[2] and was expressed in the frequency domain by Werthamer.[3] In this paper, as a example that is typical of the hetero-junction of "triplet" superconductor / "singlet" superconductor, p-wave-supercomdutor/s-wave-superconductor hetero-junction has been considered. Josephson current between p-wave and s-wave superconducting electrodes has been calculated with a generalized Bogoliubov transformation,[4,5] which has been obtained from diagonalizing the superconducting Hamiltonian. As the p-wave state we have considered Balian-Werthamer (BW) state,[5] which also appears in $He^3$ liquid.

## Josephson Current in Tunnel Junctions

Following Ambegaokar and Baratoff,[2] calculation of the tunneling current begins with the assumption of the tunneling interaction Hamiltonian $H_T$. The junction is described by the total Hamiltonian $H = H_L+H_R+H_T$, as shown in Fig. 1, where $H_L$ and $H_R$ describe the Hamiltonian of each left and right superconducting electrode, and $H_T$ transfers electrons from one superconductor to the other:

$$H_T = \sum_{kq\sigma} T_{kq} a^+_{k\sigma} b_{q\sigma} + h.c., \tag{1}$$

where $a_{k\sigma}$ ($b_{q\sigma}$) denotes the annihilation operator of a electron of wave vector $k$ ($q$) and spin state $\sigma$ in the left (right) electrode, and $T_{kq}$ is a tunneling matrix element connecting a $k$-state on the left with a $q$-state on the right. In the tunneling process conservation of the direction of spin is assumed, and this assumption will later play an important role in the calculation of the Josephson current. We calculate the tunneling current $I$ from the expectation value of the rate of change of the number operator

$$N_L = \sum_{k\sigma} a^+_{k\sigma} a_{k\sigma}, \tag{2}$$

for electrons on the left side. By the first order perturbation method, we obtain[6,7]

$$I = \frac{\mathrm{Tr}\{ e^{-H/k_BT} (e\dot{N}_L) \}}{\mathrm{Tr}\{ e^{-H/k_BT} \}} = -\frac{2e}{\hbar^2} \mathrm{Re} \sum_{kq\sigma} \int_{-\infty}^{t} dt' e^{\eta t'} < [T_{kq} \hat{a}^+_{k\sigma}(t)\, b_{q\sigma}(t), H_T(t')] >, \tag{3}$$

where $e = |e|$ is the absolute value of electron charge, Tr{ } is the trace of the operator inside the brackets, [A,B] = AB - BA, and symbol < > denotes an expectation value referred to the unperturbed Hamiltonian $H_0 = H_L+H_R$:

$$<O> = \frac{\mathrm{Tr}\{ e^{-H_0/k_BT} (O) \}}{\mathrm{Tr}\{ e^{-H_0/k_BT} \}}, \tag{4}$$

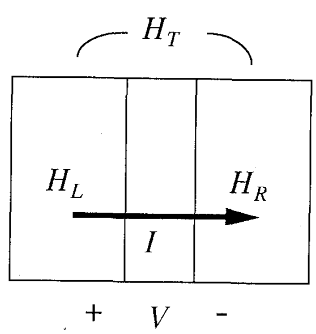


Figure 1. The tunnel junction structure. The left and right superconductors are coupled by the tunneling Hamiltonian $H_t$.

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and $a(t)$, $b(t)$ and $H(t)$ are the operators in the interaction representation, such as

$$\frac{d}{dt} a^+_{k\sigma}(t) = \frac{i}{h}[H_0, a^+_{k\sigma}(t)], \qquad (5)$$

and

$$\hat{a}^+_{k\sigma}(t) = e^{ieVt/h} a^+_{k\sigma}(t), \qquad (6)$$

where V is the voltage difference between the left and right electrodes. After the integration, a positive small value $\eta$ tends to zero.

In particular, dc Josephson current is obtained as:

$$J_s = -\frac{2e}{h^2}\mathrm{Re}\sum_{kq\sigma}\sum_{k'q'\sigma'}\int_{-\infty}^{t} dt' e^{\eta t'}\{-T_{kq}T_{k'q'}\langle a^+_{k\sigma}(t)\, a^+_{k'\sigma'}(t')\rangle\langle b_{q\sigma}(t)\, b_{q'\sigma'}(t')\rangle + T_{kq}T_{k'q'}\langle a^+_{k'\sigma'}(t')\, a^+_{k\sigma}(t)\rangle\langle b_{q'\sigma'}(t')\, b_{q\sigma}(t)\rangle\}. \qquad (7)$$

In this section, the expression of Josephson current of eq. (7) has been obtained.

## Diagonalizing Superconducting Hamiltonians

### S-wave Superconductors

In this section we discuss the Diagonalization of each Hamiltonian $H_L$ and $H_R$ for the left and right electrodes. We assume that the left electrode has a s-wave superconducting state and that the right electrode has a p-wave superconducting state.

For diagonalizing it is convenient to use a four component matrix notation:

$$a^k = \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \\ a^+_{-k\uparrow} \\ a^+_{-k\downarrow} \end{pmatrix}, \qquad \alpha^k = \begin{pmatrix} \alpha_{k+} \\ \alpha_{k-} \\ \alpha^+_{-k+} \\ \alpha^+_{-k-} \end{pmatrix}, \qquad (8)$$

where $\alpha$ and $\alpha^+$ are a quasi-particles annihilation and creation operator, respectively, and quantities with superscript $k$ are matrices in either 2- or 4-dimensional spin space, following Balian and Werthamer.[5] In the Hartree-Fock approximation, the Hamiltonian of the left superconducting electrode is as follows:

$$H_L = \frac{1}{2}\sum_k a^{k+}\begin{pmatrix} e_k\mathrm{I} & \Delta^k \\ \Delta^{k+} & -e_k\mathrm{I} \end{pmatrix} a^k, \qquad (9)$$

where I is a unit matrix, $e_k$ is the kinetic energy of electron measured from the Fermi level, and $\Delta^k$ is the 2 x 2 gap matrix. We can diagonalize this superconducting Hamiltonian by the Bogoliubov transformation[4,5] in the form:

$$a^k = U^k \alpha^k, \qquad (10)$$

where $U^k$ is a 4 x 4 unitary matrix.

The left superconductor is assumed to have normal s-wave state, then the gap matrix is exhibited:

$$\Delta^k = \Delta_1 e^{i\theta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (11)$$

where $e^{i\theta}$ is a phase factor and $\Delta_1$ is a positive value which means the magnitude of the gap:

$$|\Delta^k|^2 = \Delta^{k+}\Delta^k = \Delta_1^2\mathbf{I}, \qquad (12)$$

Hence, the Hamiltonian $H_L$ can be diagonalized by the usual Bogoliubov transformation:

$$a^k = \begin{pmatrix} u_k & 0 & 0 & -v_k e^{i\theta} \\ 0 & u_k & v_k e^{i\theta} & 0 \\ 0 & -v_k e^{-i\theta} & u_k & 0 \\ v_k e^{-i\theta} & 0 & 0 & u_k \end{pmatrix} \alpha^k, \qquad (13)$$

where $u_k$ and $v_k$ are positive value:

$$u_k = \frac{E_k + e_k}{\sqrt{2E_k(E_k+e_k)}}, \quad v_k = \frac{\Delta_1}{\sqrt{2E_k(E_k+e_k)}}, \qquad (14a,b)$$

and

$$E_k = \sqrt{e_k^2 + \Delta_1^2}, \qquad (15)$$

in which $E_k$ is the energy of the quasi-particle of the wave vector $k$ in the s-wave superconductor.

### P-wave Superconductors

We assume that the right p-wave superconductor has a Balian-Werthamer (BW) state.[5] In this right electrode, we denote the wave vector by $q$ and the annihilation operators of an electron and a quasi-particle by b and $\beta$, respectively. The gap matrix $\Delta_q$ can be written as

$$\Delta^q = \Delta_{q2}\, e^{i\theta}\begin{pmatrix} -\sqrt{2}Y_{1,-1}(\hat{q}) & Y_{1,0}(\hat{q}) \\ Y_{1,0}(\hat{q}) & \sqrt{2}Y_{1,1}(\hat{q}) \end{pmatrix}$$
$$= \Delta_{q2}\, e^{i\theta}\begin{pmatrix} e^{-i\psi}\sin\chi & \cos\chi \\ \cos\chi & -e^{i\psi}\sin\chi \end{pmatrix}, \qquad (16)$$

where $e^{i\theta'}$ is a phase factor, $\Delta_{q2}$ is a isotropic positive value that means the magnitude of the gap, $\chi$ and $\psi$ are the usual latitude and longitude angles defining the direction of the wave vector $q$ as shown in Fig. 2, and the functions $Y_{l,m}$ are spherical harmonics, and $\hat{q}$ denotes $\hat{q} = q/|q|$.

The Hamiltonian $H_R$ of the BW-state of the right electrode is

$$H_R = \frac{1}{2}\sum_q b^{q+}\begin{pmatrix} e_q\mathrm{I} & \Delta^q \\ \Delta^{q+} & -e_q\mathrm{I} \end{pmatrix} b^q, \qquad (17)$$

and can be diagonalized by the Bogoliubov transformation as:

$$b^{q} = \begin{pmatrix} u_q & 0 & -v_q e^{i(\theta'-\psi)}\sin\chi & -v_q e^{i\theta'}\cos\chi \\ 0 & u_q & -v_q e^{i\theta'}\cos\chi & v_q e^{i(\theta'+\psi)}\sin\chi \\ v_q e^{-i(\theta'-\psi)}\sin\chi & v_q e^{-i\theta'}\cos\chi & u_q & 0 \\ v_q e^{-i\theta'}\cos\chi & -v_q e^{-i(\theta'+\psi)}\sin\chi & 0 & u_q \end{pmatrix} \beta^{q}, \quad (18)$$

where

$$u_q = \frac{E_q+e_q}{\sqrt{2E_q(E_q+e_q)}}, \quad v_q = \frac{\Delta_{q2}}{\sqrt{2E_q(E_q+e_q)}}, \qquad (19)$$

and

$$E_q = \sqrt{e_q^{2}+\Delta_{q2}^{2}}\ , \qquad (20)$$

in which $E_q$ is the energy of the quasi-particle of the wave vector $q$ in the BW-state superconductor. Ambiguity factors in coefficients of unitary matrix of the Bogoliubov transformation are removed so that $\beta_{q\sigma}$ can satisfy the relation $\beta_{q\sigma} \to b_{q\sigma}$ when the absolute value of the wave vector $q$ tends to infinite.

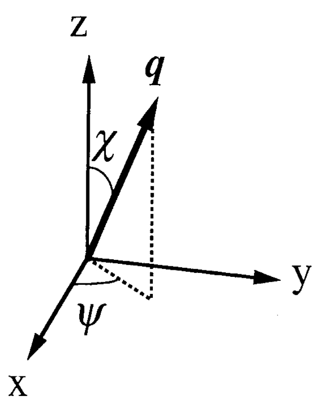


Figure 2. Representation of the direction of the wave vector $q$ in the spherical coordinates. $\chi$ and $\psi$ are the usual latitude and longitude angles defining the direction of the wave vector $q$.

Especially at 0 K, the ground state of the BW-state superconductor satisfies the relations:

$$(^{\forall}q,\sigma) \qquad \beta_{q\sigma}\,\Phi_{BW} = 0\ , \qquad (21)$$

because no quasi-particle exists in the ground state at 0 K. From these relations we obtain the BW ground state at 0 K as follows:

$$\Phi_{\rm BW} = \prod_q \{(u_q+v_q e^{i(\theta'-\psi)}\sin\chi\, b^+_{-q\uparrow}b^+_{q\uparrow}+v_q\cos\chi e^{i\theta'}b^+_{-q\downarrow}b^+_{q\uparrow})$$
$$(u_q-v_q e^{i(\theta'+\psi)}\sin\chi\, b^+_{-q\downarrow}b^+_{q\downarrow}+v_q\cos\chi e^{i\theta'}b^+_{-q\uparrow}b^+_{q\downarrow})\} \quad (22)$$

In this section Bogoliubov transformation in the s-wave and p-wave superconductors has been obtained by the diagonalization of the superconducting Hamiltonian. The ground state of the BW-state superconductor at 0 K has also been obtained in the second-quantization formalism.

## Calculation of the Josephson Current

We define $F(k,t-t')$ and $F'(q,t-t')$ as follows:

$$F(t-t') \equiv \langle a_{k\uparrow}(t)a_{-k\downarrow}(t')\rangle \qquad (23)$$
$$F'(t-t') \equiv \langle b_{q\uparrow}(t)b_{-q\downarrow}(t')\rangle\ . \qquad (24)$$

With the following relations:

$$\alpha_{k\sigma}(t) = \alpha_{k\sigma}e^{-iE_k t/h}, \qquad (25)$$
$$\langle\alpha^+_{k\sigma}\alpha_{k'\sigma'}\rangle = \delta_{kk'}\delta_{\sigma\sigma'}f(E_k), \qquad (26)$$
$$\beta_{q\sigma}(t) = \beta_{q\sigma}e^{-iE_q t/h}, \qquad (27)$$
$$\langle\beta^+_{q\sigma}\beta_{q'\sigma'}\rangle = \delta_{qq'}\delta_{\sigma\sigma'}f(E_q), \qquad (28)$$
$$f(X) = \{1+\exp(X/k_B T)\}^{-1}, \qquad (29)$$

we can calculate $F$, $PF$, $TF$, $PTF$, $F'$, $PF'$, $TF'$ and $PTF'$. Then we obtain

$$PF(k,t-t') \equiv \langle a_{-k\uparrow}(t)a_{k\downarrow}(t')\rangle = F(k,t-t'),$$
$$TF(k,t-t') \equiv \langle a_{-k\downarrow}(t)a_{k\uparrow}(t')\rangle = -F(k,t-t'), \quad (30a,b,c)$$
$$PTF(k,t-t') \equiv \langle a_{k\downarrow}(t)a_{-k\uparrow}(t')\rangle = -F(k,t-t'),$$

and

$$PF'(q,t-t') \equiv \langle b_{-q\uparrow}(t)b_{q\downarrow}(t')\rangle = -F'(q,t-t'),$$
$$TF'(q,t-t') \equiv \langle b_{-q\downarrow}(t)b_{q\uparrow}(t')\rangle = -F'(q,t-t'), \quad (31a,b,c)$$
$$PTF'(q,t-t') \equiv \langle b_{q\downarrow}(t)b_{-q\uparrow}(t')\rangle = F'(q,t-t'),$$

where P and T are the parity and time reversal opertor,[8] respectively, such as

$$Pk\uparrow = -k\uparrow,\ Tk\uparrow = -k\downarrow,\ PTk\uparrow = k\downarrow. \qquad (32a,b,c)$$

The dc Josephson current $J_S$ of eq. (7) is calculated as follows:

$$J_s = -\frac{e}{h^2}\mathrm{Re}\sum_{kq}\int_{-\infty}^{t} dt' e^{\eta t'} T_{kq} T_{-k-q}\Big\{ -F(k,t-t')F'(q,t-t') - \mathrm{P}F(k,t-t')\mathrm{P}F'(q,t-t') - \mathrm{T}F(k,t-t')\mathrm{T}F'(q,t-t') - \mathrm{PT}F(k,t-t')\mathrm{PT}F'(q,t-t')$$
$$+ F(k,t'-t)F'(q,t'-t) + \mathrm{P}F(k,t'-t)\mathrm{P}F'(q,t'-t) + \mathrm{T}F(k,t'-t)\mathrm{T}F'(q,t'-t) + \mathrm{PT}F(k,t'-t)\mathrm{PT}F'(q,t'-t)\Big\} \tag{33}$$

With the relations (30) and (31), the magnitude of the Josephson current is zero in the first order perturbation.

The BW state is, in general, degenerated; the state constructed by replacing $Y_{1,m}(\hat{q})$ in eq. (16) with $Y_{1,m}(R\hat{q})$ has the same energy, where R is a rotation in the momentum space only. The gap matrix of this degenerated state is as follows:

$$\Delta^q = \Delta_{q2}\, e^{i\theta}\begin{pmatrix} -\sqrt{2}Y_{1,-1}(R\hat{q}) & Y_{1,0}(R\hat{q}) \\ Y_{1,0}(R\hat{q}) & \sqrt{2}Y_{1,1}(R\hat{q}) \end{pmatrix}$$
$$= \Delta_{q2}\, e^{i\theta}\begin{pmatrix} e^{-i\psi'}\sin\chi' & \cos\chi' \\ \cos\chi' & -e^{i\psi'}\sin\chi' \end{pmatrix}, \tag{34}$$

where $\chi$ 'and $\psi$ ' are the latitude and longitude angles defining the direction of the wave vector $\boldsymbol{q}' = \mathrm{R}\boldsymbol{q}$. In this case, the Hamiltonian $H_{\mathrm{R}}$ can be diagonalized by the following Bogoliubov transformation:

$$b^q = \begin{pmatrix} u_q & 0 & -v_q e^{i(\theta'-\psi')}\sin\chi' & -v_q e^{i\theta'}\cos\chi' \\ 0 & u_q & -v_q e^{i\theta'}\cos\chi' & v_q e^{i(\theta'+\psi')}\sin\chi' \\ v_q e^{-i(\theta'-\psi')}\sin\chi' & v_q e^{-i\theta'}\cos\chi' & u_q & 0 \\ v_q e^{-i\theta'}\cos\chi' & -v_q e^{-i(\theta'+\psi')}\sin\chi' & 0 & u_q \end{pmatrix}\beta^q, \tag{35}$$

then the similar relations with eqs. (30)-(31) are obtained. Hence, also in this case $J_s$ is zero in the first order perturbation.

The degenerated ground state of the BW-state superconductor at 0 K can be obtained by replacing $\chi$ and $\psi$ in eq. (22) with the latitude angle $\chi$ 'and the longitude angle $\psi$ 'of the direction of the wave vector $\boldsymbol{q}' = \mathrm{R}\boldsymbol{q}$, respectively.

## Conclusions

In conclusions, Josephson currents have been calculated with Bogoliubov transformations, which have been obtained by the diagonalization of the Hamiltonians of the p-wave and the s-wave superconducting electrodes, and proved to be zero in the first order perturbation in the p-wave/s-wave superconducting hetero-junctions at 0 K and at the finite temperature. Moreover, the ground state of the BW-state superconductor at 0 K has been obtained in the second-quantization formalism.

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