12. *The Long Wave in a Bay of Variable Section.* (1)

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1. After the tunami of March 3, 1933, on the Sanriku Coast, Nippon, two papers dealing with the problem of long wave in a bay have been published in connection with the study of the tunamis. Arakawa[1] generalized Green's law on wave motion in a canal, while Homma[2] solved the wave equation, taking into account water friction, and gave several numerical examples.

We have also made a theoretical study of the long wave in a bay of variable section, the method and the results of which differ from those of the above two papers, and which we shall describe in this paper.

Using Stokes' method, which has recently been adopted[3] for the study of elastic waves, we solved the long wave equation and obtained a general expression for the surface elevation of water due to changes in the elevation at the mouth of a bay expressed by $f(t)$, where t is the time, during which the water is still in the initial state $t=0$, and of which the breadth $b(x)$ and the depth $h(x)$ are any functions of the distance x from its close end, where there is friction between the water and the wall enclosing it. Using the general expression of the water elevation thus obtained, we studied the case of the bay of rectangular section with horizontal bed.[4]

In the present study we ignore the decay of wave motion in the bay that occurs from the fact that a part of the energy of the reflected waves in the bay is propagated in the open sea as circular wave of diverging type. As in the case of sound wave in a tube, one end of which is open, what has just been said must be taken into consideration when studying wave motion in a bay. The theory of wave motion as thus corrected will give us such a result as may be suf-

1) H. ARAKAWA, *Geophy. Mag.*, Tokyo, **7**, 1933.
2) M. HOMMA, *Jour. Civil Eng. Soc.*, Tokyo, **19**, 1933.
3) G. NISHIMURA, *Disin* **5**, 1933.
4) Other case will be discussed on another occasion.

ficient to remove the position of the mouth of the bay from the position which is assumed in the uncorrected theory of the bay of the present study as in the case of the corrected theory of sound wave in an open pipe.[5)]

In this paper therefore we disregard the decay of energy from the mouth of the bay. We intend to deal with the corrected theory above mentioned on an other occasion.

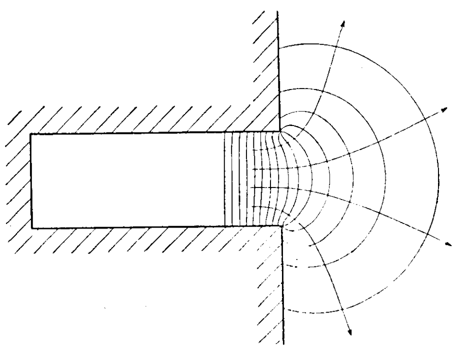

Fig. 1.

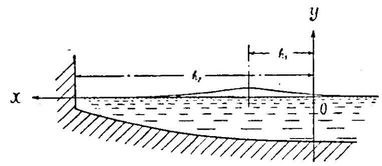

Fig. 2.

2. Let the axis of x be parallel to the length of the bay, that of y vertical and upwards, and let us suppose that the motion takes place in these two dimensions x, y. Let the ordinate for the free surface, corresponding to the abscissa x, at time t, be denoted by $\eta+y_0$, where y_0 is the ordinate in the undisturbed state. We shall assume that the vertical acceleration is the same for all particles in a plane perpendicular to x, and that all the particles which lie in such a plane do so always; in other worde, that the horizontal velocity u is a function of x and t only. Now let $\xi=\int u dt$; neglecting $u\frac{\partial u}{\partial x}$, which is of the second order in the case of infinitely small motions, the equation of horizontal motion may then be written

$$\frac{\partial^2 \xi}{\partial t^2}=-g\frac{\partial \eta}{\partial x} \qquad \text{(1)}$$

where g is the acceleration of gravity. When the section (S, say) of the bay is not uniform, but varies gradually from point to point, the equation of continuity of flow is

5) The corrected theory is treated in Lamb's Sound Dynamics.

$$\left\{b(x)h(x)+\frac{d(b(x)h(x))}{dx}\xi+b(x)\eta\right\}\left\{1+\frac{\partial\xi}{\partial x}\right\}=b(x)h(x).$$

Neglecting the higher order of infinitesimals, we obtain

$$\frac{\partial}{\partial x}\{b(x)h(x)\xi\}+b(x)\eta=0, \qquad \text{.................(2)}$$

where $b(x)$ denotes the breadth at the surface, and $h(x)$ denotes the mean depth over the width $b(x)$.

Eliminating ξ between (1) and (2), the equation in η is

$$\frac{\partial^2\eta}{\partial t^2}=\frac{g}{b(x)}\frac{\partial}{\partial x}\left\{b(x)h(x)\frac{\partial\eta}{\partial x}\right\}. \qquad \text{.................(3)}$$

In the present study the conditions are as follows. The initial and the boundary conditions which we have already discussed in section 1 are expressed by the following mathematical expressions.

When $\qquad t=0, \quad \eta=0, \qquad$ ....(4)

and $\qquad \dfrac{\partial\eta}{\partial t}=0, \qquad$ ....(5)

in the bay. And at the open end $x=h_1$ which communicates with the open sea,

$$\eta=f(t), \qquad \text{.........................(6)}$$

and at the closed end $x=h_2$ of the bay

$$\frac{\partial\eta}{\partial x}=k\eta, \qquad \text{.........................(7)}$$

where k is a constant related to the friction between the water and the wall of the bay at $x=h_2$.

The boundary condition at $x=h_2$ expressed by (7) is accordant with the assumption that when the water particle along the plane $x=h_2$ moves there is a resistance proportional to the shear strain of water, and it is found by means of the ordinary form of the equation of continuity.

Now the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \qquad \text{.........................(8)}$$

Whence $\qquad v=-y\dfrac{\partial u}{\partial x}. \qquad$ ....(9)

If the origin (for the time being) is taken to be in the bottom of the bay,

$$\int v\,dt = -y\frac{\partial\xi}{\partial x}. \quad\text{.......................(10)}$$

Using expression (2), and differentiating (10) with respect to t we obtain

$$v = y\frac{b}{S}\frac{\partial\eta}{\partial t} + \frac{y}{S}\frac{dS}{dx}\frac{\partial\xi}{\partial t}. \quad\text{.................(11)}$$

Now generally in two dimemsional problems the shear strain is expressed by

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \quad\text{.........................(12)}$$

but, as already discussed in this section, $\frac{\partial u}{\partial y}=0$. Therefore, if there is friction at $x=h_2$, it may be expressed by an expression such that

$$\frac{\partial v}{\partial x} = kv, \quad\text{.......................(13)}$$

where k is a certain constant when there is slidable friction at $x=h_2$.

From the two expressions (11) and (13) we obtain

$$\frac{\partial}{\partial x}\left\{\frac{b}{S}\frac{\partial\eta}{\partial t}\right\} + y\frac{\partial}{\partial x}\left\{\frac{1}{S}\frac{dS}{dx}\frac{\partial\xi}{\partial t}\right\} = k\left\{\frac{b}{S}\frac{\partial\eta}{\partial t} + \frac{1}{S}\frac{dS}{dx}\frac{\partial\xi}{\partial t}\right\}. \quad\text{....(14)}$$

Moreover, if we assume that at $x=h_2$, the space derivatives of b and h are very small, so that by neglecting the higher order of the derivatives of these quantities, we obtain expression (7).

Applying now Stokes' method, we shall solve equation (3) by the initial and the boundary conditions (4), (5), (6), (7) as follows.

We assume that the function of x, such as $\chi_s(x)$, is the solution of the differential equation of

$$\frac{1}{b(x)}\frac{d}{dx}\left\{b(x)\,h(x)\frac{d\chi_s(x)}{dx}\right\} + \frac{\lambda_s^2}{h_1}\chi_s(x) = 0, \quad\text{..........(15)}$$

with the conditions such that

$$\chi_s(h_1) = 0, \quad\text{.......................(16)}$$

and $$\frac{d\chi_s(h_2)}{dh_2} = k\chi_s(h_2). \quad\text{.....................(17)}$$

Then we can prove that $\chi_s(x)$, which is the solution of (15) and which satisfies conditions (16) and (17), is a normal function as follows.

Generally we assume $s \neq p$, so that from equation (15) we have

$$\frac{1}{b(x)}\frac{d}{dx}\left\{b(x)\,h(x)\frac{d\chi_s(x)}{dx}\right\}+\frac{\lambda_s^2}{h_1}\chi_s(x)=0,\qquad\text{..........(18)}$$

and

$$\frac{1}{b(x)}\frac{d}{dx}\left\{b(x)\,h(x)\frac{d\chi_p(x)}{dx}\right\}+\frac{\lambda_p^2}{h_1}\chi_p(x)=0.\qquad\text{........(19)}$$

Then

$$\frac{1}{h_1}(\lambda_s^2-\lambda_p^2)\int_{h_1}^{h_2}b(x)\chi_s(x)\chi_p(x)\,dx$$

$$=b(h_2)\,h(h_2)\left\{\chi_s(h_2)\frac{d\chi_p(h_2)}{dh_2}-\chi_p(h_2)\frac{d\chi_s(h_2)}{dh_2}\right\}$$

$$-b(h_1)\,h(h_1)\left\{\chi_s(h_1)\frac{d\chi_p(h_1)}{dh_1}-\chi_p(h_1)\frac{d\chi_s(h_1)}{dh_1}\right\}.\qquad\text{....(20)}$$

From condtions (16) and (17) and

$$\chi_p(h_1)=0,\qquad\text{........................(16)}'$$

$$\frac{d\chi_p(h_2)}{dh_2}=k\chi_p(h_2),\qquad\text{....................(17)}'$$

we obtain

$$\frac{1}{h_1}(\lambda_s^2-\lambda_p^2)\int_{h_1}^{h_2}b(x)\,\chi_s(x)\,\chi_p(x)\,dx=0.\qquad\text{..........(21)}$$

If then we can obtain

$$\int_{h_1}^{h_2}b(x)\,\chi_s^2(x)\,dx,\qquad\text{.....................(22)}$$

we can easily expand η into series of $\chi_s(x)$, thus:

$$\eta(x,t)=\sum_{s=1}^{\infty}A_s\chi_s(x),\qquad\text{.................(23)}$$

where

$$A_s=\frac{\displaystyle\int_{h_1}^{h_2}b(x)\eta(x,t)\chi_s(x)dx}{\displaystyle\int_{h_1}^{h_2}b(x)\,\chi_s^2(x)\,dx}.\qquad\text{..................(24)}$$

If we can normalize

$$A_s=\int_{h_1}^{h_2}b(x)\,\eta(x,t)\,\chi_s(x)\,dx.\qquad\text{..................(25)}$$

we can then also expand $\dfrac{1}{b(x)}\dfrac{\partial}{\partial x}\left\{b(x)\,h(x)\dfrac{\partial\eta}{\partial x}\right\}$ into series of $\chi_s(x)$ as follows.

$$\frac{1}{b(x)}\frac{\partial}{\partial x}\left\{b(x)h(x)\frac{\partial\eta}{\partial x}\right\}=\sum_{s=1}^{\infty}\chi_s(x)\frac{\int_{h_1}^{h_2}\frac{\partial}{\partial\xi}\left\{b(\xi)h(\xi)\frac{\partial\eta}{\partial\xi}\right\}\chi_s(\xi)d\xi}{\int_{h_1}^{h_2}b(\xi)\chi_s^2(\xi)\,d\xi}. \quad \ldots.(26)$$

Now

$$\int_{h_1}^{h_2}\frac{\partial}{\partial\xi}\left[b(\xi)h(\xi)\frac{\partial\eta}{\partial\xi}\right]\chi_s(\xi)\,d\xi$$

$$=\left[\left\{\chi_s(h_2)b(h_2)h(h_2)\frac{d\chi_s(h_2)}{dh_2}\right\}+\left\{\eta(h_1)b(h_1)h(h_1)\frac{d\chi_s(h_1)}{dh_1}\right\}\right]$$

$$-\left[\left\{\eta(h_2)b(h_2)h(h_2)\frac{d\chi_s(h_2)}{dh_2}\right\}-\left\{\chi_s(h_1)b(h_1)h(h_1)\frac{\partial\eta(h_1)}{\partial h_1}\right\}\right]$$

$$+\int_{h_1}^{h_2}\eta(\xi,t)\frac{d}{d\xi}\left[b(\xi)h(\xi)\frac{d\chi_s(\xi)}{d\xi}\right]d\xi. \quad \ldots\ldots\ldots\ldots\ldots\ldots(27)$$

By using boundary conditions (6), (7) and also relations (15), (16), (17), we can reduce (27) to the following:

$$\int_{h_1}^{h_2}\frac{\partial}{\partial\xi}\left[b(\xi)h(\xi)\frac{\partial\eta}{\partial\xi}\right]\chi_s(\xi)\,d\xi$$

$$=\chi_s(h_2)b(h_2)h(h_2)k\eta(h_2,t)-\eta(h_2,t)b(h_2)h(h_2)\frac{d\chi_s(h_2)}{dh_2}$$

$$+b(h_1)h(h_1)\frac{d\chi_s(h_1)}{dh_1}f(t)-\frac{\lambda_s^2}{h_1}A_s\int_{h_1}^{h_2}b(x)\chi_s^2(x)\,dx$$

$$=b(h_1)h(h_1)\frac{d\chi_s(h_1)}{dh_1}f(t)-\frac{\lambda_s^2}{h_1}A_s\int_{h_1}^{h_2}b(\xi)\chi_s^2(\xi)\,d\xi, \quad \ldots\ldots\ldots(28)$$

so that (26) reduces to

$$\frac{1}{b(x)}\frac{\partial}{\partial x}\left\{b(x)h(x)\frac{\partial\eta}{\partial x}\right\}=\sum_{s=1}^{\infty}\chi_s(x)\frac{b(h_1)h(h_1)\frac{d\chi_s(h_1)}{dh_1}f(t)-\frac{\lambda_s^2}{h_1}A_s\int_{h_1}^{h_2}b(\xi)\chi_s^2(\xi)d\xi}{\int_{h_1}^{h_2}b(\xi)\chi_s^2(\xi)\,d\xi}. \quad \ldots\ldots\ldots\ldots\ldots(29)$$

Now $\frac{\partial^2\eta}{\partial t^2}$ is also expanded in series of $\chi_s(x)$, thus:

$$\frac{\partial^2\eta}{\partial t^2}=\sum_{s=1}^{\infty}\frac{d^2A_s}{dt^2}\chi_s(x). \quad \ldots\ldots\ldots\ldots\ldots\ldots(30)$$

Whence (3), (29), and (30) give

$$\frac{d^2A_s}{dt^2}+g\frac{\lambda_s^2}{h_1}A_s=\frac{gb(h_1)h(h_1)\dfrac{d\chi_s(h_1)}{dh_1}}{\displaystyle\int_{h_1}^{h_2}b(\xi)\chi_s^2(\xi)\,d\xi}f(t). \qquad\qquad (31)$$

To solve this equation by the conditions that

$$t=0:\!-\qquad A_s=0, \qquad\qquad (32)$$

$$\text{and}\quad \frac{dA_s}{dt}=0, \qquad\qquad (33)$$

which correspond to (4) and (5), we obtain

$$A_s=\frac{h_1^{1/2}g^{1/2}b(h_1)h(h_1)\chi_s'(h_1)}{\lambda_s\displaystyle\int_{h_1}^{h_2}b(\xi)\chi_s^2(\xi)\,d\xi}\int_0^t f(\xi)\sin\left\{g^{1/2}\frac{\lambda_s}{\sqrt{h_1}}(t-\xi)\right\}d\xi. \qquad (34)$$

Then the final solution that satisfies (3), (4), (5), (6), (7) is easily written by means of (23) and (34), such that

$$\eta=(h_1g)^{1/2}b(h_1)h(h_1)\sum_{s=1}^{\infty}\frac{\chi_s'(h_1)\chi_s(x)}{\lambda_s\displaystyle\int_{h_1}^{h_2}b(\xi)\chi_s^2(\xi)\,d\xi}\int_0^t f(\xi)\sin\left\{g^{1/2}\frac{\lambda_s}{\sqrt{h_1}}(t-\xi)\right\}d\xi, \qquad (35)$$

where λ_s is the sth root of

$$\frac{d\chi_s(h_2)}{dh_2}-k\chi_s(h_2)=0, \qquad\qquad (36)$$

and $\chi_s(x)$ is the normal function satisfies (15).

In the following study we take for simplicity, $k=0$, that is

$$\frac{\partial\eta}{\partial x}=0 \quad \text{at} \quad x=h_2, \qquad\qquad (37)$$

$$\text{and}\quad \frac{d\chi_s(h_2)}{dh_2}=0. \qquad\qquad (38)$$

Using this general expression (35), we shall study the surface elevation of water in bays of special forms.

2a. A Straight Bay, with Horizontal Bed, and Vertical Sides.

Let us suppose that the breadth of a bay is constantly B, and that the depth is equal to D, then

$$b(x)=B, \quad h(x)=D. \qquad\qquad (39)$$

Equation (15) in this case then reduces to

$$\frac{d^2\chi_s(x)}{dx^2}+\frac{\lambda_s^2}{h_1D}\chi_s(x)=0, \quad \dots\dots\dots\dots\dots\dots(40)$$

of which the solution is

$$\chi_s(x)=\frac{\sqrt{x}}{\sqrt{h_1}}\left\{Y_{1/2}\left(\frac{\lambda_s\sqrt{h_1}}{\sqrt{D}}\right)J_{1/2}\left(\frac{\lambda_s}{\sqrt{h_1D}}x\right)-J_{1/2}\left(\frac{\lambda_s\sqrt{h_1}}{\sqrt{D}}\right)Y_{1/2}\left(\frac{\lambda_s}{\sqrt{h_1D}}x\right)\right\} \dots(41)$$

where $J_{1/2}(\alpha)$ and $Y_{1/2}(\alpha)$ are Bessel functions.

Expression (39) naturally satisfies the condition $\chi_s(x)$ at $x=h_1$ such that $\chi_s(h_1)=0$ expressed by (16), and λ_s is the sth root of

$$\frac{d}{dh_2}\left[\sqrt{h_2}\left\{Y_{1/2}\left(\frac{\lambda_s\sqrt{h_1}}{\sqrt{D}}\right)J_{1/2}\left(\frac{\lambda_sh_2}{\sqrt{h_1D}}\right)-J_{1/2}\left(\frac{\lambda_s\sqrt{h_1}}{\sqrt{D}}\right)Y_{1/2}\left(\frac{\lambda_sh_2}{\sqrt{h_1D}}\right)\right\}\right]=0, \quad ..(42)$$

which is obtained from the condition expressed in (38).

Now (41) reduces to

$$\chi_s(x)=-\frac{2\sqrt{D}}{\pi\lambda_s\sqrt{h_1}}\left\{\cos\left(\frac{\lambda_s\sqrt{h_1}}{\sqrt{D}}\right)\sin\left(\frac{\lambda_s}{\sqrt{h_1D}}x\right)-\sin\left(\frac{\lambda_s\sqrt{h_1}}{\sqrt{D}}\right)\cos\left(\frac{\lambda_s}{\sqrt{h_1D}}x\right)\right\},$$

$$\dots\dots\dots\dots\dots(43)$$

where λ_s is equal to the expression

$$\lambda_s=\left(s-\frac{1}{2}\right)\frac{\pi\sqrt{h_1D}}{a}, \quad \dots\dots\dots\dots\dots\dots(44)$$

where s is any positive integer such that 1, 2, 3, and a is equal to the distance from $x=h_1$ to $x=h_2$ that is the length of the canal. Therefore from expression (35)

$$\eta=\frac{4h_1\sqrt{g}}{\pi h_2^{3/2}}\sum_{s=1}^{\infty}\frac{\lambda_s\sin\left\{\dfrac{\lambda_s(x-h_1)}{\sqrt{h_1D}}\right\}}{\left\{\dfrac{2\lambda_sa}{\sqrt{h_1D}}-\sin\left(\dfrac{2\lambda_sa}{\sqrt{h_1D}}\right)\right\}}\int_0^t f(\xi)\sin\left\{\frac{\sqrt{g}\lambda_s}{\sqrt{h_1}}(t-\xi)\right\}d\xi.\dots(45)$$

To obtain this expression we use the integral formula

$$\int_{h_1}^{h_2}\chi_s^2(\xi)d\xi=\frac{B\sqrt{D^3}}{\pi\lambda_s^3\sqrt{h_1}}\left\{\frac{2\lambda_sa}{\sqrt{h_1D}}-\sin\left(\frac{2\lambda_sa}{\sqrt{h_1D}}\right)\right\}. \quad \dots\dots\dots(46)$$

The surface elevation at the closed end of the bay is given by

$$\eta_{x=h_2}=\frac{4h_1\sqrt{g}}{\pi h_2^{3/2}}\sum_{s=1}^{\infty}\frac{\lambda_s\sin\left\{\dfrac{\lambda_sa}{\sqrt{h_1D}}\right\}}{\left\{\dfrac{2\lambda_sa}{\sqrt{h_1D}}-\sin\left(\dfrac{2\lambda_sa}{\sqrt{h_1D}}\right)\right\}}\int_0^t f(\xi)\sin\left\{\frac{\sqrt{g}\lambda_s}{\sqrt{h_1}}(t-\xi)\right\}d\xi.$$

$$\dots\dots\dots\dots\dots\dots(47)$$

2b. When the elevation $f(t)$ at the open end of the rectangular bay is given by

$$f(t)=\gamma\eta_0\frac{\sqrt{gD}}{a}te^{-\beta\frac{\sqrt{gD}}{a}t}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots(48)$$

where γ and β are merely numerical parameters by which the form of $f(t)$ varies, the surface elevation at the closed end $x=h_2$ using the integral formula

$$\int_0^t \xi e^{-p\xi}\sin\sqrt{\frac{g}{h_1}}\lambda_s(t-\xi)\,d\xi$$

$$=\frac{\sqrt{gh_1}\lambda_s}{g\lambda_s^2+h_1p^2}te^{-pt}+\frac{2p\sqrt{g}\sqrt{h_1^3}\lambda_s}{(g\lambda_s^2+h_1p^2)^2}e^{-pt}$$

$$-\frac{2p\sqrt{g}\sqrt{h_1^3}}{(g\lambda_s^2+h_1p^2)^2}\cos\frac{\sqrt{g}\lambda_s}{\sqrt{h_1}}t-\frac{h_1(g\lambda_s^2-h_1p^2)}{(g\lambda_s^2+h_1p^2)^2}\sin\frac{\sqrt{g}\lambda_s}{\sqrt{h_1}}t, \qquad \ldots\ldots(48)$$

is given by the expression

$$\eta_{x=h_2}=2\gamma\eta_0\sum_{s=1}^{\infty}\left[\frac{(-1)^{s+1}\left(s-\frac{1}{2}\right)}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+\beta^2\right\}}\frac{\sqrt{gD}}{a}te^{-\beta\frac{\sqrt{gD}}{a}t}\right.$$

$$+\frac{2(-1)^{s+1}\left(s-\frac{1}{2}\right)\beta}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+\beta^2\right\}^2}e^{-\beta\frac{\sqrt{gD}}{a}t}$$

$$-\frac{2(-1)^{s+1}\left(s-\frac{1}{2}\right)\beta}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+\beta^2\right\}^2}\cos\left\{\left(s-\frac{1}{2}\right)\pi\frac{\sqrt{gD}}{a}t\right\}$$

$$\left.-\frac{\frac{1}{\pi}\left\{\left(s-\frac{1}{2}\right)^2\pi^2-\beta^2\right\}}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+\beta^2\right\}^2}\sin\left\{\left(s-\frac{1}{2}\right)\pi\frac{\sqrt{gD}}{a}t\right\}\right]. \qquad \ldots\ldots(49)$$

From this expression we can see that the water surface at $x=h_2$ in the bay rises after time $\frac{a}{\sqrt{gD}}$, the time necessary for the wave to propagate from the open end to the head of the bay, and that after a certain interval of time, of which the magnitude is a function of $\sqrt{gD}$, a and β, the free oscillations of the water or seiches of the bay are excited in the bay, and the periods of the free oscillations are as usual equal to $\frac{2a}{\sqrt{gD}}\left(s-\frac{1}{2}\right)$.

Since the respective terms of the series expressed by (49) become very small and equal to zero (from the stand point of physics) when the value of s becomes more or less large, say for example, 7 or 8, it is possible to make numerical calculations of expression (49).

We shall give examples of two calculations, one numerical and the other graphical for the cases $\beta=1$ and $\beta=3$.

When $\beta=1$,

$$\eta_{x=h_1}=f(t)=\gamma\eta_0\frac{\sqrt{gD}}{a}te^{-\frac{\sqrt{gD}}{a}t}, \quad \ldots\ldots\ldots\ldots\ldots (50)$$

and

$$\begin{aligned}\eta_{x=h_2}=\gamma\eta_0\sum_{s=1}^{\infty}(-1)^{s+1}\Bigg[&-\frac{2\left(s-\frac{1}{2}\right)}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+1\right\}}\frac{\sqrt{gD}}{a}te^{-\frac{\sqrt{gD}}{a}t}\\&-\frac{4\left(s-\frac{1}{2}\right)}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+1\right\}^2}e^{-\frac{\sqrt{gD}}{a}t}\\&+\frac{4\left(s-\frac{1}{2}\right)}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+1\right\}^2}\cos\left\{\pi\left(s-\frac{1}{2}\right)\frac{\sqrt{gD}}{a}t\right\}\\&+\frac{2\left\{\left(s-\frac{1}{2}\right)^2\pi^2-1\right\}}{\pi\left\{\left(s-\frac{1}{2}\right)^2\pi^2+1\right\}^2}\sin\left\{\pi\left(s-\frac{1}{2}\right)\frac{\sqrt{gD}}{a}t\right\}\Bigg].\end{aligned} \quad (51)$$

of which the results of the numerical calculations are shown in Figs. 3 and 4. In Fig. 4, the thin lines correspond to the respective terms of the series when $s=1$, 2, 3, 4, 5, 6, 7 and 8, while the thick line show the final result which corresponds to (51) when the all terms from $s=1$ to $s=8$ are summed. These figures show that when the time interval of the elevation at the open end is longer than the fundamental period of the bay $\frac{4a}{\sqrt{gD}}$, the seich of the fundamental period predominates in the bay. Upon lapse of time $\frac{a}{\sqrt{gD}}$ after the initial time, the

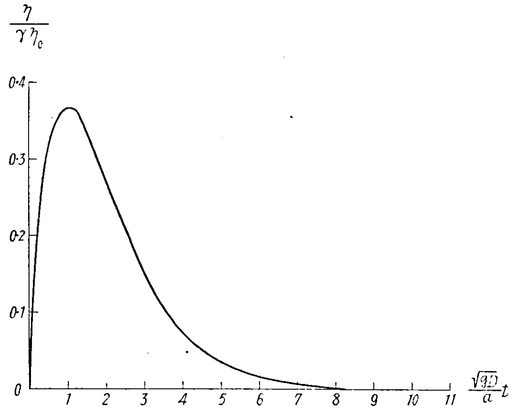

Fig. 3. $\eta_{x=h_1}$, $\beta=1$.

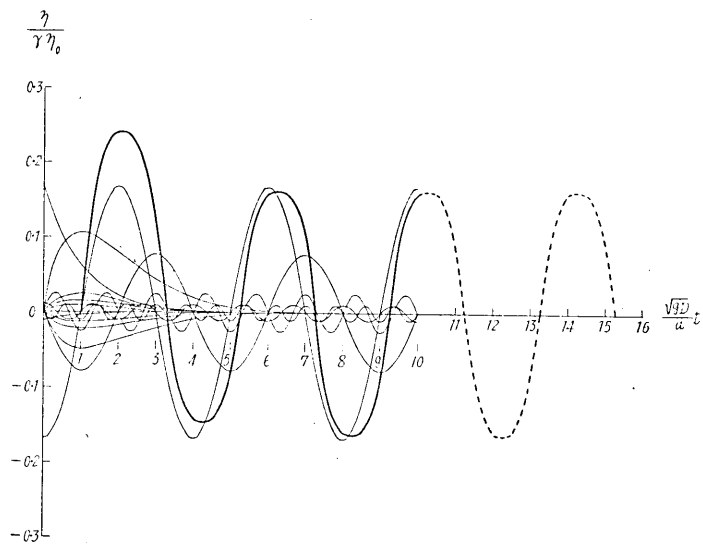

Fig. 4. $\eta_{x=h_2}$, $\beta=1$.

water surface at the closed end of course begins to rise.

When $\beta=3$,

$$\eta_{x=h_1}=f(t)=\gamma\eta_0\frac{\sqrt{gD}}{a}\,te^{-3\frac{\sqrt{gD}}{a}t},$$

and $$\eta_{x=h_2}=\gamma\eta_0\sum_{s=1}^{\infty}(-1)^{s+1}\left[-\frac{2\left(s-\frac{1}{2}\right)}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+9\right\}}\frac{\sqrt{gD}}{a}te^{-3\frac{\sqrt{gD}}{a}t}\right.$$

$$-\frac{12\left(s-\frac{1}{2}\right)}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+9\right\}^2}e^{-3\frac{\sqrt{gD}}{a}t}$$

$$+\frac{12\left(s-\frac{1}{2}\right)}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2+9\right\}^2}\cos\left\{\pi\left(s-\frac{1}{2}\right)\frac{\sqrt{gD}}{a}t\right\}$$

$$+\frac{2\left\{\left(s-\frac{1}{2}\right)^2\pi^2-9\right\}}{\pi\left\{\left(s-\frac{1}{2}\right)^2\pi^2+9\right\}^2}\sin\left\{\pi\left(s-\frac{1}{2}\right)\frac{\sqrt{gD}}{a}t\right\}\Bigg]\quad\ldots\ldots(53)$$

The results of the numerical calculations are shown in Figs. 5 and 6. In Fig. 6 the thin lines have also the same meaning as in Fig. 4, while the thick line shows the final result corresponding to (53) when the summation of the respective terms in (53) are made from $s=1$ to $s=8$. From these figures we can understand that when the time interval of the elevation at the open end is nearly equal to the fundamental period of the bay $\frac{4a}{\sqrt{gD}}$, the seich of the fundamental period is naturally excited, and the seich of the third harmonics of which the period is $\frac{4a}{3\sqrt{gD}}$ is also excited. When the interval of time during which the excitation at the open end takes place is nearly equal to the fundamental period $\frac{4a}{\sqrt{gD}}$, the ratio of amplitude of the free oscillation of the fundamental period to the maximum height at the open end is larger than that in such cases as when the excitation time interval at the open end is longer than $\frac{4a}{\sqrt{gD}}$.

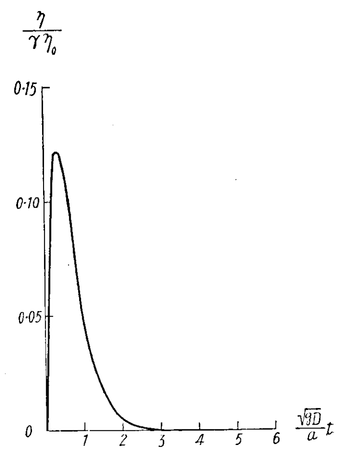

Fig. 5. $\eta_{x=h_1}$, $\beta=3$.

2c. When $f(t)$ at the open end of the rectangular bay is given by

$$f(t)=\tau\eta_0\sin\left(\alpha\frac{\sqrt{gD}}{a}t\right),\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(54)$$

where τ and α are dimensionless numbers which respectively change the amplitude of the elevation and the period of the oscillation.

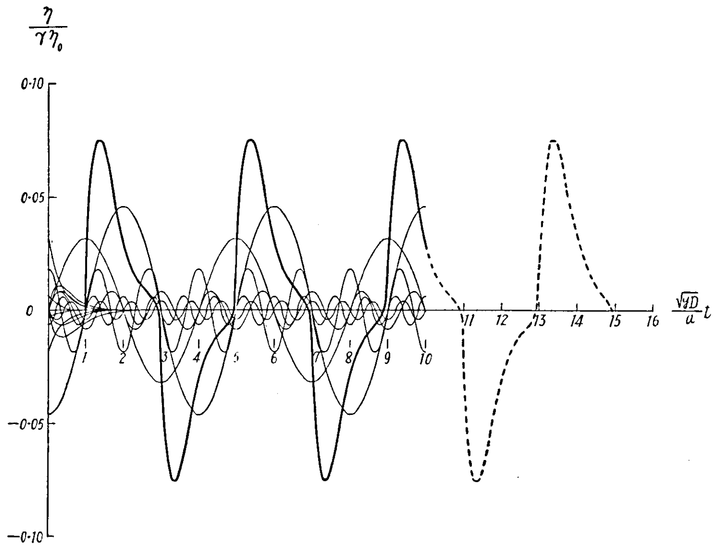

Fig. 6. $\eta_{x=h_2}$, $\beta=3$.

Now $\int_0^t \sin \kappa\xi \sin\sqrt{\frac{g}{h_1}}\lambda_s(t-\xi)\, d\xi$

$$=\frac{\sqrt{gh_1}\lambda_s}{(g\lambda_s^2-h_1\kappa^2)}\sin \kappa t-\frac{h_1\kappa}{(g\lambda_s^2-h_1n^2)}\sin\sqrt{\frac{g}{h_1}}\lambda_s t. \quad \ldots.(55)$$

Using the above integral formula, the surface elevation at the closed end $x=h_2$ due to (54) is formulated as follows:

$$\eta_{x=h_2}=2\tau\eta_0\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{\left\{\left(s-\frac{1}{2}\right)^2\pi^2-\alpha^2\right\}}\left[\left(s-\frac{1}{2}\right)\sin\alpha\frac{\sqrt{gD}}{a}t\right.$$

$$\left.-\frac{\alpha}{\pi}\sin\left\{\left(s-\frac{1}{2}\right)\pi\frac{\sqrt{gD}}{a}t\right\}\right]. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(56)$$

This expression shews that, as in the general theory of the forced harmonic oscillation of an elastic pendulum, when $\alpha=\left(s-\frac{1}{2}\right)\pi$, that is, when the period of elevation $f(t)$ at the open end is equal to the period of the free oscillation of water in the bay, the wave height in the estuary becomes infinite. That is to say, a resonance phenomenon takes place in the bay, to which case the theory of the present study is inapplicable. Free oscillations of periods $\frac{4a}{\sqrt{gD}\left(s-\frac{1}{2}\right)}$, when $s=1, 2, 3, \ldots\ldots$, are also excited to satisfy the initial condition in the bay.

When $\alpha=2\pi$,

$$f(t)=\tau\eta_0 \sin\left(2\pi \frac{\sqrt{gD}}{a}t\right),$$

and
$$\eta_{x=h_2}=2\tau\eta_0\sum_{s=1}^{\infty}\frac{(-1)^{s+1}}{\left\{\left(s-\frac{1}{2}\right)^2-4\right\}\pi^2}$$
$$\times\left\{\left(s-\frac{1}{2}\right)\sin\left(2\pi\frac{\sqrt{gD}}{a}t\right)-2\sin\left(\left(s-\frac{1}{2}\right)\pi\frac{\sqrt{gD}}{a}t\right)\right\}.\;.\;.\;.\;.\;(58)$$

Since the exciting period at the open end is $\frac{a}{\sqrt{gD}}$, the free oscillation of the third harmonics whose period is $\frac{4a}{3\sqrt{gD}}$ predominates in the bay. The periods and the amplitudes of the free oscillations are asfollows:

period	amplitude / $\tau\eta_0$	period	amplitude / $\tau\eta_0$
$\frac{4a}{\sqrt{gD}}$	0·1082	$\frac{4a}{11\sqrt{gD}}$	0·015
$\frac{4a}{3\sqrt{gD}}$	0·232	$\frac{4a}{13\sqrt{gD}}$	0·0705
$\frac{4a}{5\sqrt{gD}}$	0·180	$\frac{4a}{15\sqrt{gD}}$	0·008
$\frac{4a}{7\sqrt{gD}}$	0·049	$\frac{4a}{17\sqrt{gD}}$	0·006
$\frac{4a}{9\sqrt{gD}}$	0·025	$\frac{4a}{19\sqrt{gD}}$	0·005

The numerical results of (57) and (58) are shown in Figs. 7 and 8 respectively. In Fig. 8, the fine lines correspond to the elementary harmonic oscillations that correspond to the respective terms in (58) from $s=1$ to $s=10$. The thick line indicates the final oscillation of water at $x=h_2$, and shews that the oscillation whose period is equal to that of the forced oscillation $\frac{\sqrt{gD}}{a}$, and which continues for the interval 3·5 $\frac{\sqrt{gD}}{a}$, are repeated with period $\frac{4a}{\sqrt{gD}}$.

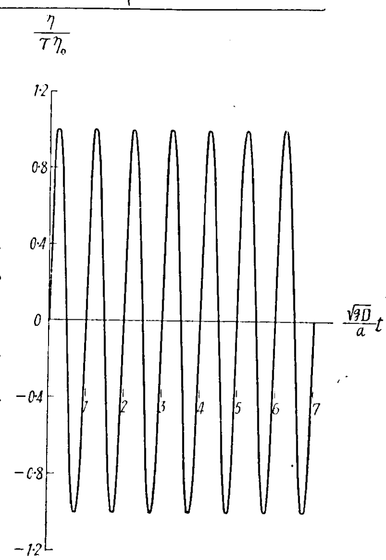

Fig. 7. $\eta_{x=h_1}$, $\alpha=2\pi$.

2d. When the elevation $f(t)$ at the open end of the bay is given by

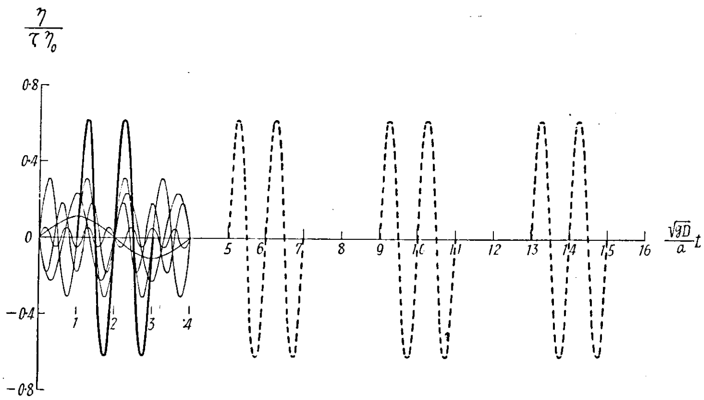

Fig. 8. $\eta_{x=h_2}$, $\alpha=2\pi$.

$$f(t)=\gamma\eta_0\frac{\sqrt{gD}}{a}te^{-\beta\frac{\sqrt{gD}}{a}t}+\tau\eta_0\sin\alpha\frac{\sqrt{gD}}{a}t, \qquad (59)$$

the water height at $x=h_2$ is given by

$$\eta_{x=h_2}=\eta_{x=h_2} \text{ expressed by (49)}$$
$$+\eta_{x=h_2} \text{ expressed by (58).} \qquad (60)$$

For $\frac{\tau}{\gamma}=$, $\beta=1$, and $\alpha=3$, the numerical results of (59) and (60) are shown in Figs. 9 and 10 respectively. These curves are obtained by superposing the two curves in Figs. 3 and 7.

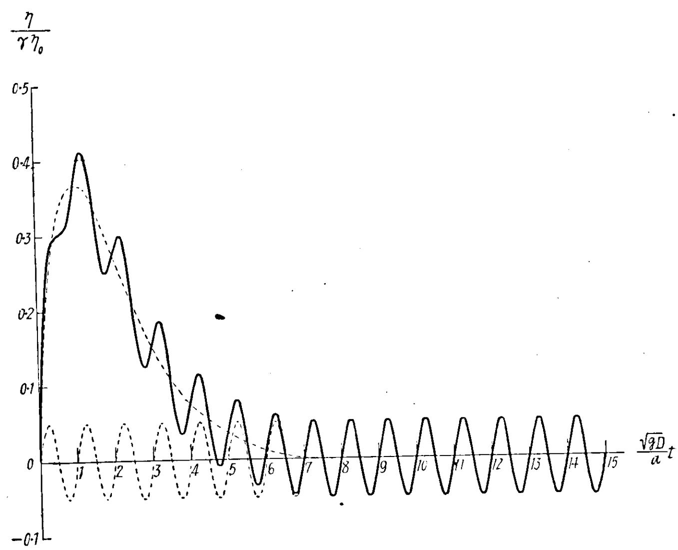

Fig. 9.

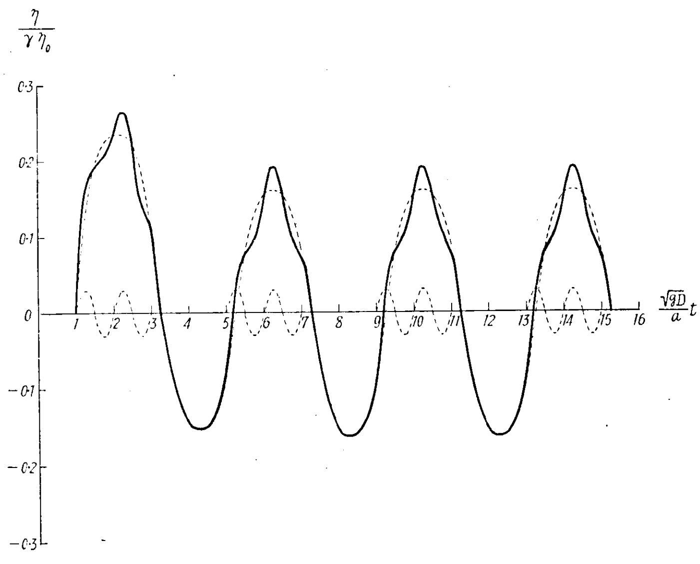

Fig. 10.

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12. 灣內に生ずる長波に關する研究 (其の 1)

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1 つの波が灣に入る場合, 灣頭で反射した波のエネルギの 1 部分は大洋へ擴つて行き次第に灣內の波のエネルギーは消失して濟ふものである. 昭和 3 年 3 月の三陸津浪の時得た檢潮記錄を見ると所謂津浪の爲め灣內に生じた海水の動搖は數日續いており, 灣口でのエネルギーの消失は可なり少ないものであるので, 本計算では全然灣口でのエネルギーの消失を考へに入れずに議論を進めてみた.

一般に切口が任意の形をしておる 1 つの灣に就て, その灣口の水位が何かの原因で時間的に任意な變化をする場合, その灣內には如何なる種類の長波が生ずるものであるかと云ふ事を明かにし, 灣口で起る波の形, 或はその周期等の爲め, 灣內の水の運動はどの樣になるかを一般的に研究した.

具體的には 1 つの長方形をした深さ一樣な灣に就て, 灣口での波高の變化を與へて, 計算を進め, その結果を圖をもつて論じた. 切口が色々の形をしてゐる場合の具體的な計算は次の機會に讓つておいた.
