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The Waves Excited near the Free Surface in the Half-infinite Nonlinear-elastic Isotropic Medium

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Abstract

There exist two kinds of simple waves, i.e., non-coupled simple waves (only longitudinal component) and coupled simple waves (both longitudinal and transverse components included) in a nonlinear elastic medium. When these two simple waves impinge vertically on a free surface, the behavior of the reflected waves is elucidated.

In a nonlinear elastic medium, the stress condition is also nonlinear, so that the stress-free surface condition yields usually a variety of the reflected waves for the incidence of the wave. These solutions of the reflected simple waves are obtained for the incidence of the simple waves.

Whether the incident simple waves are non-coupled or coupled, the *excited* reflected wave is a *non-coupled* simple wave. This excited wave has slow velocity as compared with that of the incident simple waves.

The obtained result is very significant in order to interpret the wave behavior at the free surface on the occasion of a *direct-hit* earthquake.

Introduction

When a wave source is located right below the free surface, i.e., in the case of a *direct-hit earthquake* (this term will be used in the following discussion), the theory of waves in a linear elastic medium does not be applied, so that the theory in a *nonlinear* elastic medium will be used in this paper, focusing attention on the response of the incident waves at the free surface.

In previous papers (Momoi, 1990 and 1992; these papers will be referred to as *paper M1* and *M2*, respectively), we have found two kinds of simple waves, i.e., a *non-coupled* simple wave (only longitudinal component) and *coupled* simple waves (longitudinal and transverse components coupled) in a nonlinear elastic medium. In this paper, when these two simple waves impinge vertically on the free surface, the reflected waves are evaluated at the free surface.

The theory was developed using *computer algebra* installed on an NEC 9800 computer.

1. Expression of Energy

In conservative (nondissipative) thermoelastic isotropic media, the strain energy function E_n can be expressed by use of three strain tensor invariants (for details, the reader should refer to BLAND (1969)) in the generalized Taylor expansion

$$E_n = I_1^2 \cdot \lambda/2 + I_2 \cdot \mu + A \cdot I_3/3 + B \cdot I_2 \cdot I_1 + C \cdot I_1^3/3\,, \qquad (1.1)$$

where λ, μ, A, B, C are elastic coefficients, and I_1, I_2, I_3 (U_j as strain tensor component) are strain tensor invariants expressed as

$$\begin{aligned} I_1 &= U_1 + U_2 + U_3\,, \\ I_2 &= U_1^2 + U_2^2 + U_3^2 + U_4^2/2 + U_5^2/2 + U_6^2/2\,, \\ I_3 &= U_1^3 + U_2^3 + U_3^3 + 3/4 \cdot U_1 \cdot U_5^2 + 3/4 \cdot U_1 \cdot U_6^2 + 3/4 \cdot U_2 \cdot U_4^2 + 3/4 \cdot U_2 \cdot U_6^2 \\ &\quad + 3/4 \cdot U_3 \cdot U_4^2 + 3/4 \cdot U_3 \cdot U_5^2 + 3/4 \cdot U_4 \cdot U_5 \cdot U_6\,. \end{aligned} \qquad (1.2)$$

In this paper, nonlinearity of the medium is taken into account up to third order in U_J. In the equations of motion, this order is, therefore, reduced to second order in U_J.

In the above, Voigt Notation $\{ij \sim J\}$

$$\{11\sim1,\ 22\sim2,\ 33\sim3,\ 23\sim4,\ 13\sim5,\ 12\sim6\}$$

and the convention

$$U_{ij} = (1/2) \cdot (1 + \delta_{ij}) \cdot U_J \qquad (1.3)$$

with

$$U_{ij} = (u_{ij} + u_{ji} + u_{ki} \cdot u_{kj})/2 \qquad (1.4)$$

are used, where $u_{ij} = \partial u_i/\partial x_j$ and $\{u_1, u_2, u_3\}$ are displacement components in the directions of the Cartesian coordinate axes $\{x_1, x_2, x_3\}$. Then the x- and z-axes are positive rightward and downward, respectively, and the y-axis positive forward. In later discussions, $\{u_1, u_2, u_3\}$ and $\{x_1, x_2, x_3\}$ will be alternatively expressed by $\{u, v, w\}$ and $\{x, y, z\}$. δ_{ij} is the Kronecker delta with suffixes i, j.

In the later discussion, the elastic coefficients $\{\lambda, A, B, C\}$ normalized by μ (rigidity in an isotropic medium) will be used, i.e.,

$$L_m = \lambda/\mu\,, \quad A_m = A/\mu\,, \quad B_m = B/\mu\,, \quad C_m = C/\mu\,. \qquad (1.5)$$

2. Model, Equations and Surface Conditions

The stress tensor S_{ij} (i-component on j-plane) is related to the energy function

$$S_{ij} = \partial E_n/\partial u_{ij}\,, \qquad (2.1)$$

By use of the above relation, the governing equations can be expressed (LANDAU and LIFSHITZ, 1985) by

$$\rho \partial^2 u_i/\partial t^2 = \partial S_{ij}/\partial x_j \quad (i = 1, 2, 3)\,, \qquad (2.2)$$

where ρ and t are the density of the medium and the time factor, respectively.

Let the problem be assumed to be two-dimensional. Since the study is focused on the problem of a direct-hit earthquake, the displacements are u and w, varying only in the z-direction, independent of x and y. Only moving waves in the direction of the z-axis are considered. By use of (2.1) and the expressions for energy in the foregoing section, the above two equations are reduced to the following.

$$\rho\cdot u_{t2}=q_1 , \quad \rho\cdot w_{t2}=q_3 \tag{2.3}$$

where

$$\begin{aligned} q_1&=\mu\cdot u_{z2}+f_{19}\cdot u_{z2}\cdot w_z+f_{19}\cdot w_{z2}\cdot u_z , \\ q_3&=L_{2m}\cdot w_{z2}+f_{19}\cdot u_{z2}\cdot u_z+f_{37}\cdot w_z\cdot w_{z2} , \end{aligned} \tag{2.3.1}$$

with

$$\begin{aligned} &u_{t2}=\partial^2 u/\partial t^2 , \quad u_z=\partial u/\partial z , \quad u_{z2}=\partial^2 u/\partial z^2 , \\ &w_{t2}=\partial^2 w/\partial t^2 , \quad w_z=\partial w/\partial z , \quad w_{z2}=\partial^2 w/\partial z^2 , \end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned} L_{2m}&=\lambda+2\cdot\mu , \\ f_{19}&=(A+2\cdot B+2\cdot\lambda+4\cdot\mu)/2 , \\ f_{37}&=2\cdot A+6\cdot B+2\cdot C+3\cdot\lambda+6\cdot\mu . \end{aligned} \tag{2.3.3}$$

By use of (1.1) and (2.1), the nonlinear stress-free conditions at the free surface ($z=0$) become

$$\begin{aligned} S_{13}&=u_z+u_z\cdot f_1 w_z=0 , \\ S_{33}&=w_z\cdot L_{2mm}+u_z^2\cdot f_1/2+w_z^2\cdot g_1/2=0 , \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} L_{2mm}&=L_{2m}/\mu=2+L_m , \\ f_1&=2+A_m/2+B_m+L_m , \\ g_1&=6+2\cdot A_m+6\cdot B_m+2\cdot C_m+3\cdot L_m , \end{aligned} \tag{2.4.1}$$

where the stress conditions are assumed to depend on only z (independent of x and y), since the simple waves are assumed normally incident on the free surface.

Solving equations (2.4), we have two kinds of solutions.

Case-I

$$\begin{aligned} &u_z=0 && \text{(from eq. } S_{13}) , \\ &w_z=-2\cdot L_{2mm}/g_1 && \text{(from eq. } S_{33}) , \\ \text{or } &w_z=0 && \text{(from eq. } S_{33}) . \end{aligned} \tag{2.5}$$

Case-II

$$w_z = -1/f_1 \quad \text{(from eq. } S_{13}\text{)}\,,$$
$$u_z^2 = -g_1/f_1^3 + 2\cdot L_{2mm}/f_1^2 \quad \text{(from eq. } S_{33}\text{)}\,. \tag{2.6}$$

3. Simple Waves

As discussed in the previous papers *M1* and *M2*, equations (2.3) are transcribed as follows by use of the moving axes

$$t_r = v_r \cdot t\,, \quad k_r = v_r \cdot t + s_z \cdot z \quad (s_z = \pm)\,, \tag{3.1}$$

where t_r and k_r are variables with respect to time and coordinates moving at a velocity v_r in the z-direction. The symbol s_z refers to the moving axis associated with the waves moving upward (for $+$) and downward (for $-$), respectively. After the above reduction, terms with respect to the derivative of time t_r are put equal to zero in order to obtain stationary waves, and, further, the equations are integrated over k_r. The following equations, which are obtained, govern the characteristic behavior of waves in a conservative nonlinear-elastic medium.

$$s_z \cdot f_1 \cdot u_{kr} \cdot w_{kr} + u_{kr} \cdot (1 - v_r^2/v_s^2) = 0\,,$$
$$s_z \cdot f_1 \cdot u_{kr}^2 + s_z \cdot g_1 \cdot w_{kr}^2 + 2\cdot (v_p^2 - v_r^2)/v_s^2 \cdot w_{kr} = 0\,, \tag{3.2}$$

where

$$u_{kr} = \partial u/\partial k_r\,, \quad w_{kr} = \partial w/\partial k_r\,. \tag{3.2.1}$$

and $\{v_p, v_s\}$ are the velocities of $\{P, S\}$ waves in the linear theory, respectively.

Solving the above equations, we have two kinds of solutions.

(i) Non-coupled simple wave

$$u = 0\,,$$
$$w = -2\cdot s_z \cdot V_{pm1}/g_1 \cdot k_r \quad \text{or} \quad w = \text{const}\,, \tag{3.3}$$

with

$$V_{pm1} = (v_p^2 - v_{r1}^2)/v_s^2\,, \tag{3.3.1}$$

where $v_{r1}(=v_r)$ is the velocity of the non-coupled simple wave.

(ii) Coupled simple wave

$$u = s_z \cdot s_u \cdot U^{1/2}/(v_s \cdot f_1) \cdot k_r \quad (s_z = s_u = \pm)\,,$$
$$w = -s_z \cdot V_{sm2}/f_1 \cdot k_r\,, \tag{3.4}$$

with

$$U = V_{sm2} \cdot (2\cdot v_p^2 + v_s^2 \cdot (-2 + V_{sm2} \cdot (2 - g_1/f_1)))\,,$$

$$V_{sm2} = 1 - v_{r2}^2/v_s^2 , \tag{3.4.1}$$

where $v_{r2}(=v_r)$ is the velocity of the coupled simple wave.

4. Expressions for Incident and Reflected Simple Waves

4.1. *Incident simple waves*

For a non-coupled simple wave, from (3.3),

$$\begin{aligned} &u_{iN} = 0 , \\ &w_{iN} = -2 \cdot V_{pm1i} \cdot (z + t \cdot v_{r1i})/g_1 \quad \text{or} \quad w_{iN} = \text{const} , \end{aligned} \tag{4.1}$$

with

$$V_{pm1i} = (v_p^2 - v_{r1i}^2)/v_s^2 , \tag{4.1.1}$$

where $\{u_{iN}, w_{iN}\}$ and v_{r1i} are displacement components $\{u, w\}$ and the velocity associated with the non-coupled incident simple wave.

For the coupled simple wave, from (3.4),

$$\begin{aligned} &u_{iC} = s_I \cdot U_{iC}^{1/2} \cdot (v_{r2i} \cdot t + z)/(v_s \cdot f_1) \quad (s_I = \pm) , \\ &w_{iC} = -V_{sm2i} \cdot (z + t \cdot v_{r2i})/f_1 , \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} &U_{iC} = V_{sm2i} \cdot (2 \cdot v_p^2 + v_s^2 \cdot (-2 + V_{sm2i} \cdot (2 - g_1/f_1))) , \\ &V_{sm2i} = 1 - v_{r2i}^2/v_s^2 , \end{aligned} \tag{4.2.1}$$

where $\{u_{iC}, w_{iC}\}$ and v_{r2i} are displacement components $\{u, w\}$ and the velocity associated with the coupled incident simple wave.

4.2. *Reflected simple waves*

For simple waves incident on a free surface, we have two kinds of reflected simple waves.

For a non-coupled simple wave, from (3.3),

$$\begin{aligned} &u_{rN} = 0 , \\ &w_{rN} = 2 \cdot V_{pm1r} \cdot (-z + t \cdot v_{r1r})/g_1 \end{aligned} \tag{4.3}$$

with

$$V_{pm1r} = (v_p^2 - v_{r1r}^2)/v_s^2 , \tag{4.3.1}$$

where $\{u_{rN}, w_{rN}\}$ and v_{r1r} are displacement components $\{u, w\}$ and the velocity associated with the non-coupled reflected simple wave.

For a coupled simple wave, from (3.4),

$$\begin{aligned} &u_{rC} = s_R \cdot U_{rC}^{1/2} \cdot (z - v_{r2r} \cdot t)/(v_s \cdot f_1) \quad (s_R = \pm) , \\ &w_{rC} = V_{sm2r} \cdot (-z + t \cdot v_{r2r})/f_1 , \end{aligned} \tag{4.4}$$

with

$$U_{rC}=V_{sm2r}\cdot(2\cdot v_p^2+v_s^2\cdot(-2+V_{sm2r}\cdot(2-g_1/f_1)))\,,$$
$$V_{sm2r}=1-v_{r2r}^2/v_s^2\,, \tag{4.4.1}$$

where $\{u_{rC}, w_{rC}\}$ and v_{r2r} are displacement components $\{u, w\}$ and the velocity associated with the coupled reflected simple wave.

5. Waves at the Free Surface for the Incidence of Non-coupled Simple Waves

In this section, behavior of the waves at the free surface will be discussed for the incidence of the non-coupled simple waves described in (4.1). In this case, the total waves are expressed as

$$u=u_{iN}+u_{rN}+u_{rC}\,,$$
$$w=w_{iN}+w_{rN}+w_{rC}\,, \tag{5.1}$$

where $\{u_{iN}, w_{iN}\}$, $\{u_{rN}, w_{rN}\}$ and $\{u_{rC}, w_{rC}\}$ are given in (4.1), (4.3) and (4.4), respectively.

It is noted here that the total waves are expressed by linear combination of three terms as in linear theory. In nonlinear problem, such a formulation of the expressions is, in general, impossible, since coupling of two terms will be expected. In the case of simple waves, this formulation of the expressions is possible. In derivation of the expressions for simple waves, the equation is integrated one time with respect to k_r, so that the expression for simple waves becomes linear with respect to t and x. On the other hand, each term of the original equations (2.3) and (2.3.1) involves second derivative of t or z. As a result, the above expressions (5.1) always satisfy equations (2.3). This feature is similar to that in the case of the nonlinear multi-*soliton* problem. In this case, the solutions behave like linear waves in spite of nonlinear waves.

It must be noted here that, in the usual case in linear theory, expression (5.1) might be described in the form:

$$u=u_{iN}+A_{uN}\cdot u_{rN}+A_{uC}\cdot u_{rC}\,,$$
$$w=w_{iN}+A_{wN}\cdot w_{rN}+A_{wC}\cdot w_{rC}\,,$$

with A_{uN}, A_{uC}, A_{wN} and A_{wC} as unknown coefficients to be determined from boundary conditions. In the case of a nonlinear elastic medium, the total displacements do not include such unknown coefficients as shown in (5.1). In this case, the unknown coefficients are replaced by undetermined velocities v_{r1r} (for the reflected non-coupled simple wave) and v_{r2r} (for the reflected coupled simple wave). Substituting (5.1) into the stress conditions (2.5) and (2.6), the velocities v_{r1r} and v_{r2r} are determined as follows.

5.1. Case-I: Surface condition for non-coupled simple wave incidence

By use of (2.5), i.e., *Case-I*, and (5.1), we have the following solutions, the derivation of which is detailed in Appendix A.

For a reflected coupled simple wave,

$$u_{rC}=w_{rC}=0\,. \quad \text{(from (A.3) of Appendix A)}$$

The above indicates that the coupled reflected simple wave disappears when the incident wave is a non-coupled simple wave.

For a reflected non-coupled simple wave,

$$v_{r1r}=v_s\cdot V_{pm1i}{}^{1/2}\,. \quad \text{(from (A.4) of Appendix A)} \tag{5.2.1}$$

In the usual case, V_{pm1i} $(=(v_p{}^2-v_{r1i}{}^2)/v_s{}^2)$ is small, since the velocity of the incident non-coupled simple wave is near the velocity of the *P* wave in the linear theory. The above expression indicates that the reflected non-coupled simple wave has very slow velocity, and is nearly trapped near the surface.

The ratio of reflected to incident simple waves at the surface ($z=0$) becomes

$$w_{rN}/w_{iN}=-v_p{}^2/(V_{pm1i}{}^{1/2}\cdot v_s\cdot v_{r1i})\,, \quad \text{(from (A.6) of Appendix A)} \tag{5.2.2}$$

where v_{r1i} is nearly equal to v_p in usual case.

The above ratio is very large, since the denominator on the right-hand side of the equation includes the small factor V_{pm1i}. This result shows that, on the occasion of incidence of a non-coupled simple wave into the free surface, the velocity of the reflected non-coupled simple wave becomes very small and, as a result, the amplitude of the reflected non-coupled simple wave becomes very large, causing a quasi-trapping of the waves near the surface.

There exists another reflected non-coupled simple wave with velocity v_{r1r} which is the solution of the equation

$$v_{r1r}{}^2/v_s{}^2=V_{pm1i}+v_p{}^2/v_s{}^2\,. \quad \text{(from (A.7) of Appendix A)} \tag{5.3.1}$$

As found from the above, the velocity v_{r1r} is nearly equal to the velocity of the *P* wave in linear theory, since V_{pm1i} is usually small. By use of the above, the ratio of reflected to incident simple waves at the surface ($z=0$) is, when V_{pm1i} is small,

$$w_{rN}/w_{iN}=1+V_{pm1i}\cdot v_s{}^2/v_p{}^2\,. \quad \text{(from (A.9) of Appendix A)} \tag{5.3.2}$$

Since V_{pm1i} is small, the reflected simple wave in this case occurs nearly with the amplitude of the incident wave.

5.2. Case-II: Surface condition for non-coupled simple wave incidence

By use of (2.6) and (5.1), we have the following solutions, the reduction of which is detailed in Appendix B.

For a reflected coupled simple wave, the velocity becomes zero, so that this kind of simple wave does not exist, since the velocity is required to be positive (see the Appendix B).

For a reflected non-coupled simple wave, we have

$$v_{r1r}{}^2/v_s{}^2 = V_{pm1i} + v_p{}^2/v_s{}^2 . \quad \text{(from (B.4) of the Appendix B)} \qquad (5.4.1)$$

The ratio of reflected to incident simple waves at the surface is then

$$w_{rN}/w_{iN} = 1 + V_{pm1i} \cdot v_s{}^2/v_p{}^2 . \quad \text{(from (B.5) of Appendix B)} \qquad (5.4.2)$$

The above two expressions (5.4.1) and (5.4.2) are completely the same as those in (5.3.1) and (5.3.2). In this case (*section 5.2*), no amplification of the reflected waves occurs.

6. Waves at the Free Surface for the Incidence of Coupled Simple Waves

In this section, behavior of the waves at the free surface will be discussed for the incidence of the coupled simple waves described in (4.2). In this case, the total waves are expressed as

$$u = u_{iC} + u_{rN} + u_{rC} ,$$
$$w = w_{iC} + w_{rN} + w_{rC} , \qquad (6.1)$$

where $\{u_{iC}, w_{iC}\}$, $\{u_{rN}, w_{rN}\}$ and $\{u_{rC}, w_{rC}\}$ are given in (4.2), (4.3) and (4.4), respectively.

Substituting (6.1) into the stress conditions (2.5) and (2.6), the velocities v_{r1r} and v_{r2r} will be determined as follows.

6.1. Case-I: Surface condition for coupled simple wave incidence

By use of (2.5) and (6.1), we have the following solutions, the reduction of which is detailed in Appendix C.

For the reflected coupled simple waves, the velocity is

$$v_{r2r} = v_{r2i} . \quad \text{(from (C.3.1) of Appendix C)} \qquad (6.2.1)$$

The ratios of reflected to incident simple waves at the surface are then

$$u_{rC}/u_{iC} = 1 \quad \text{and} \quad w_{rC}/w_{iC} = -1 . \quad \text{(from (C.3.2) of Appendix C)} \qquad (6.2.2)$$

As shown above, the coupled waves are reflected directly without any amplification.

For reflected non-coupled simple waves, there exists a noticeable solution, the velocity of which satisfies the equation

$$v_{r1r} = v_s \cdot V_{sm2i}{}^{1/2} \cdot (g_1/f_1)^{1/2} . \quad \text{(from (C.4) of Appendix C)} \qquad (6.3.1)$$

The ratio of the reflected to incident simple waves at the surface becomes

$$w_{rN}/w_{iC} = -2 \cdot v_p{}^2 \cdot f_1{}^{1/2}/(V_{sm2i}{}^{1/2} \cdot v_s \cdot g_1{}^{1/2} \cdot v_{r2i}) . \qquad (6.3.2)$$
$$\text{(from (C.6) of Appendix C)}$$

Since the value of V_{sm2i} is usually small, the velocity of the reflected non-coupled simple wave is small from (6.3.1). In (6.3.2), the denominator on the right-hand side is also small because of the factor V_{sm2i} (then v_{r2i} is of the order of

v_p), so that the incident *coupled* simple wave is reflected at the free surface as a *non-coupled* simple wave which is significantly amplified.

In this case, there exists another solution for the reflected non-coupled simple wave

$$v_{r1r}{}^2/v_s{}^2 = V_{sm2i} \cdot g_1/f_1 + v_p{}^2/v_s{}^2 . \quad \text{(from (C.7.1) of Appendix C)} \qquad (6.4.1)$$

The ratio of reflected to incident simple waves at the surface then becomes

$$w_{rN}/w_{iC} = 2 \cdot v_p/v_s . \quad \text{(from (C.7.2) of Appendix C)} \qquad (6.4.2)$$

From (6.4.1), the reflected non-coupled wave has a velocity nearly equal to that of the P wave in linear theory, since V_{sm2i} is small. Then the amplitude of the reflected wave is slightly amplified (see (6.4.2)).

6.2. *Case-II: Surface condition for coupled simple wave incidence*

By use of (2.6) and (6.1), we have the following solutions, the reduction of which is detailed in Appendix D.

For the reflected coupled simple waves,

$$v_{r2r} = v_s \cdot f_i^{1/2} \cdot V_{sm2i}{}^{1/4} , \quad \text{(from (D.1) of Appendix D)} \qquad (6.5.1)$$

with

$$f_i = (2 \cdot f_1 \cdot (2 \cdot f_1 \cdot v_p{}^2 - g_1 \cdot v_s{}^2) \cdot (v_p{}^2 - v_s{}^2))^{1/2} / (f_1 \cdot v_p{}^2 + v_s{}^2 \cdot (f_1 - g_1)) .$$

The ratio of the reflected to incident simple waves at the surface then becomes

$$u_{rC}/u_{iC} = -s_R \cdot F_c \cdot (2 \cdot f_1 \cdot v_p{}^2 - v_s{}^2 \cdot g_1)^{1/2} / (f_1{}^{1/4} \cdot V_{sm2i}{}^{1/4}) ,$$
$$w_{rC}/u_{iC} = v_s \cdot F_c \cdot f_1{}^{1/4} / V_{sm2i}{}^{1/4} \quad \text{(from (D3) of Appendix D)} \qquad (6.5.2)$$

with

$$F_c = v_s \cdot (-v_s{}^2 + v_p{}^2)^{1/4} \cdot (2 \cdot f_1 \cdot v_p{}^2 - v_s{}^2 \cdot g_1)^{1/4} / (2^{1/4} \cdot v_{r2i} \cdot (f_1 \cdot v_p{}^2 + v_s{}^2 \cdot (f_1 - g_1))^{1/2}) .$$

In the usual case, V_{sm2i} is small, so that the reflected coupled simple waves stagnate near the surface. The denomonators on the right-hand sides of (6.5.2) include the factor V_{sm2i}. As the result, the ratio of the reflected to incident simple waves becomes large, as found from (6.5.2).

In the usual case, V_{pm1i} and V_{sm2i} are smaller than 1, so that the inequality

$$V_{pm1i}{}^{1/2} < V_{sm2i}{}^{1/4}$$

seems to be valid, for instance,

$$\text{when} \quad V_{pm1i} \sim V_{sm2i} \sim 0.1 , \quad V_{pm1i}{}^{1/2} \sim 0.3 \quad \text{and} \quad V_{sm2i}{}^{1/4} \sim 0.6 .$$

From this result, the significance of the reflected waves (6.3.2) is considered to be larger than that of (6.5.2).

For a reflected non-coupled smiple wave, there exists a solution. The velocity v_{r1r} and wave w_{rN} are given by

$$v_{r1r}{}^2/v_s{}^2 = (-C_C \cdot g_1 \cdot U_U \cdot v_s{}^2 + 2 \cdot f_1 \cdot v_p{}^2)/(2 \cdot f_1 \cdot v_s{}^2) ,$$

$$w_{rN} = C_C \cdot U_U \cdot (-z + t \cdot v_{r1r})/f_1 \,. \quad \text{(from (D.4.1) of Appendix D)} \qquad (6.6.1)$$

The ratio of the reflected to incident simple waves at the surface then becomes

$$w_{rN}/\dot{u}_{iC} = C_C \cdot v_{r1r} \cdot v_s/(s_I \cdot v_{r2i}) \,. \quad \text{(from (D.4.2) of Appendix D)} \qquad (6.6.2)$$

No amplification occurs for this wave.

7. Summary of the Most Significant Reflected Waves

Among the reflected simple waves described in the foregoing sections, the excited modes of the waves are very significant on the occasion of a *direct-hit* earthquake, so that the obtained results are summarized here.

Case-i In the case of *non-coupled* simple wave incidence, the excited reflected waves are *non-coupled* simple waves given by (5.2.1) and (5.2.2).

Case-ii In the case of *coupled* simple wave incidence, the excited reflected waves are *non-coupled* simple waves given by (6.3.1) and (6.3.2).

We have now arrived at the conclusion that the most significant reflected waves are *non-coupled* simple waves, irrespective of the kind of the incident simple waves, i.e., non-coupled or coupled ones.

8. Numerical Experiments

In the theory developed in the foregoing sections, the total waves are expressed by (5.1) and (6.1). These expressions can be written:

$$u = u_{iN} + u_{rN} + u_{rC} + u_{\text{other}} \,,$$
$$w = w_{iN} + w_{rN} + w_{rC} + w_{\text{other}}$$

and so on. In the above expressions, the terms u_{other} and w_{other} involve the effects of reflections other than the type of simple waves and also diffraction near the surface.

In order to confirm the effectiveness of the expressions of the total waves (5.1) and (6.1), which do not have the terms u_{other} and w_{other}, and the results obtained from these simplified expressions (in the foregoing sections), numerical experiments will be carried out by use of the extended finite difference equations. The results based on the finite difference method involve the effects of the terms u_{other} and w_{other}.

8.1. Finite difference equation by Taylor Method

In this section, finite difference equations will be introduced by use of equations (2.3). The variables t, x, u, and w are here normalized by the wave number h of the P wave in linear theory, i.e.,

$$\tau = h \cdot v_p \cdot t \,, \quad \chi = h \cdot z \,, \quad \xi = h \cdot u \,, \quad \zeta = h \cdot w \,, \qquad (8.1.1)$$

where v_p and v_s (the latter will be used later) are the velocities of P and S waves in linear theory.

Equations (2.3) are reduced to

$$\partial\xi/\partial\tau=U\,,\quad \partial\zeta/\partial\tau=W\,,$$
$$\partial U/\partial\tau=E_U\,,\quad \partial W/\partial\tau=E_W\,, \tag{8.1.2}$$

where

$$E_U=1/v_{ps}{}^2\cdot(\xi_{\chi 2}+f_1\cdot\zeta_\chi\cdot\xi_{\chi 2}+f_1\cdot\xi_\chi\cdot\zeta_{\chi 2})\,,$$
$$E_W=\zeta_{\chi 2}+1/v_{ps}{}^2\cdot(f_1\cdot\xi_{\chi 2}\cdot\xi_\chi+g_1\cdot\zeta_\chi\cdot\zeta_{\chi 2})\,, \tag{8.1.3}$$

with

$$v_{ps}=v_p/v_s\,,$$
$$\xi_\chi=\partial\xi/\partial\chi\,,\qquad \zeta_\chi=\partial\zeta/\partial\chi\,,$$
$$\xi_{\chi 2}=\partial^2\xi/\partial\chi^2\,,\quad \zeta_{\chi 2}=\partial^2\zeta/\partial\chi^2\,. \tag{8.1.4}$$

In order to evaluate displacements $\{\xi,\zeta\}$ and velocities $\{U,W\}$ at a time $\tau+d\tau$ ($d\tau$: increment of time τ), we will use a Taylor expansion in terms of τ to second order in $d\tau$ such that:

$$\xi=\xi_0+D_{\xi 1}\cdot d\tau+D_{\xi 2}\cdot(d\tau)^2/2\,,$$
$$\zeta=\zeta_0+D_{\zeta 1}\cdot d\tau+D_{\zeta 2}\cdot(d\tau)^2/2\,, \tag{8.1.5}$$

$$U=U_0+D_{U1}\cdot d\tau+D_{U2}\cdot(d\tau)^2/2\,,$$
$$W=W_0+D_{W1}\cdot d\tau+D_{W2}\cdot(d\tau)^2/2\,, \tag{8.1.6}$$

where

$$D_{\xi 1}=(\partial\xi/\partial\tau)_0\,,\qquad D_{\zeta 1}=(\partial\zeta/\partial\tau)_0\,,$$
$$D_{\xi 2}=(\partial^2\xi/\partial\tau^2)_0\,,\quad D_{\zeta 2}=(\partial^2\zeta/\partial\tau^2)_0\,,$$
$$D_{U1}=(\partial U/\partial\tau)_0\,,\quad D_{W1}=(\partial W/\partial\tau)_0\,, \tag{8.1.7}$$

and

$$D_{U2}=(\partial^2 U/\partial\tau^2)_0\,,\quad D_{W2}=(\partial^2 W/\partial\tau^2)_0\,. \tag{8.1.8}$$

with suffix 0 indicating the evaluation at time τ.

The above coefficients in (8.1.7) are expressed, by use of (8.1.2), as

$$D_{\xi 1}=U\,,\quad D_{\xi 2}=D_{U1}=E_U\,,$$
$$D_{\zeta 1}=W\,,\quad D_{\zeta 2}=D_{W1}=E_W\,.$$

In order to obtain expressions (8.1.8), these expressions will be transcribed, by use of (8.1.2), as follows.

$$D_{U2}=\partial E_U/\partial\tau\,,\quad D_{W2}=\partial E_W/\partial\tau\,. \tag{8.1.8a}$$

The expressions (8.1.8a) can be obtained by differentiation of (8.1.3) with respect to τ, i.e.,

$$D_{U2}=1/v_{ps}{}^2\cdot(U_{\chi2}+f_1\cdot W_\chi\cdot\xi_{\chi2}+f_1\cdot W_{\chi2}\cdot\xi_\chi+f_1\cdot U_\chi\cdot\zeta_{\chi2}+f_1\cdot U_{\chi2}\cdot\zeta_\chi)\,,$$
$$D_{W2}=W_{\chi2}+1/v_{ps}{}^2\cdot(f_1\cdot\xi_{\chi2}\cdot U_\chi+f_1\cdot\xi_\chi\cdot U_{\chi2}+W_\chi\cdot\zeta_{\chi2}\cdot g_1+W_{\chi2}\cdot\zeta_\chi\cdot g_1)\,, \quad (8.1.8b)$$

with

$$U_\chi=\partial U/\partial\chi\,, \quad W_\chi=\partial W/\partial\chi\,,$$
$$U_{\chi2}=\partial^2U/\partial\chi^2\,, \quad W_{\chi2}=\partial^2W/\partial\chi^2\,. \quad (8.1.9)$$

In numerical computation, the derivatives of ξ are replaced by difference expressions, say:

$$\xi_\chi=(\xi_{+1}-\xi_{-1})/(2h_\chi)\,,$$

and

$$\xi_{\chi2}=(\xi_{+1}+\xi_{-1}-2\xi_0)/h_\chi{}^2\,, \quad (8.1.10)$$

where $\{\xi_{-1},\xi_{+1}\}$ are the displacements at points just above and under a reference point with displacement ξ_0, and h_χ is a mesh interval. The above description also applies for ζ, U, and W. It must be noted here that the above expressions (8.1.10) can be applied only for the mesh points *inside* the medium. A different difference method will be used in the region next to the surface boundary where uneven intervals are used (see Appendix E).

8.2. Combinations of surface conditions

The surface conditions (2.5) and (2.6) are transcribed as follows by use of the finite difference method in normalized form.

Case I

$$\xi_s=\xi_{s+1}\,,$$
$$\zeta_s=\zeta_{s+1}+2\cdot v_{ps}{}^2/g_1\cdot h_s\,,$$
$$\text{or}\ \ \zeta_s=\zeta_{s+1}\,, \quad (8.2.1a)$$

Case II

$$\zeta_s=\zeta_{s+1}+h_s/f_1\,,$$
$$\xi_s=\xi_{s+1}+h_s\cdot H_h{}^{1/2}/f_1{}^{3/2}\,,$$
$$\text{or}\ \ \xi_s=\xi_{s+1}-h_s\cdot H_h{}^{1/2}/f_1{}^{3/2}\,, \quad (8.2.1b)$$

with

$$H_h=-g_1+2\cdot f_1\cdot v_{ps}{}^2\,,$$

where the suffixes s and $s+1$ refer to the values at the free surface and the mesh point neighboring the surface, respectively, and h_s is the interval between the above two points (the interval inside the medium is h_χ, as shown in (8.1.10)).

From the above conditions, we also have the conditions for the velocities U and W in finite difference form.

$$U_s = U_{s+1}\,,$$
$$W_s = W_{s+1}\,. \tag{8.2.1c}$$

From (8.2.1a) and (8.2.1b), we have the following combinations of the surface conditions.

Case [1]

$$\xi_s = \xi_{s+1}\,,$$
$$\zeta_s = \zeta_{s+1}\,,$$

Case [2]

$$\xi_s = \xi_{s+1}\,,$$
$$\zeta_s = \zeta_{s+1} + 2\cdot v_{ps}{}^2/g_1 \cdot h_s\,,$$

Case [3]

$$\xi_s = \xi_{s+1} + h_s \cdot H_h{}^{1/2}/f_1{}^{3/2}\,,$$
$$\zeta_s = \zeta_{s+1} + h_s/f_1\,,$$

Case [4]

$$\xi_s = \xi_{s+1} - h_s \cdot H_h{}^{1/2}/f_1{}^{3/2}\,,$$
$$\zeta_s = \zeta_{s+1} + h_s/f_1\,. \tag{8.2.2}$$

In the application of the above four conditions, a priority cannot be found among them. As one of the possible procedures, we selected one of the four surface conditions (8.2.2) and applied the *random function* installed on the computer. The computation of velocity was carried out by use of (8.2.1c). In the computation, the mesh interval h_s near the surface was calculated by

$$h_s = h_\chi/2\,, \tag{8.2.3}$$

i.e., half of the mesh interval inside the medium. The reason is described in Appendix E.

8.3. *Initial condition and accuracy*

In the following experiments, numerical computations will be carried out by use of the above-mentioned procedure. The wave source is then given by the expression

$$Q = A(Q)/2\cdot\{1+\cos(\chi\cdot\pi/4)\} \quad (-16<\chi<24)\,. \tag{8.3.1}$$

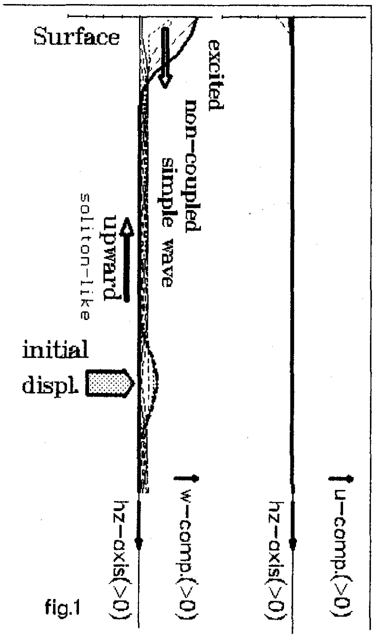

Fig. 1

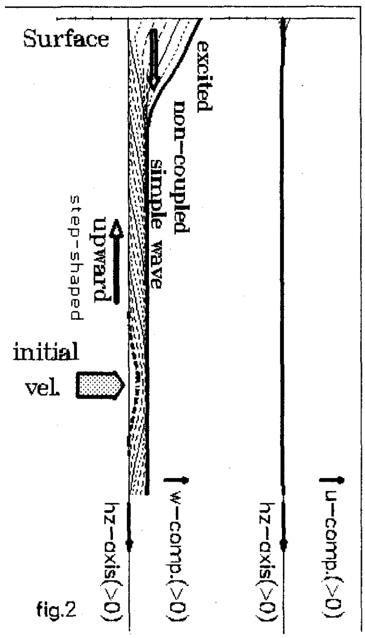

Fig. 2

Fig. 1 (left). Incidence of the non-coupled simple waves (soliton-like) onto the surface. The specifications of the elastic coefficients are $L_m = A_m = B_m = C_m = 1.0$. These elastic coefficients will be also used in the following Figs. 2, 3 and 4.

Initial displacement (longitudinal),

$$w = 0.1/2 \cdot \{1 + \cos(hz \cdot \pi/4)\} \quad (-16 < hz < 24)$$
$$= 0 \quad \text{(otherwise)}.$$

and $u = U = W = 0$.

Thick broken, thin broken and solid lines indicate the initial conditions, displacements at the successive transient stages and the displacements at the final stage of the computation near the surface. These conventions will be used in Figs. 2, 3 and 4.

Fig. 2 (right). Incidence of the non-coupled simple waves (step-shaped) onto the surface.

Initial velocity (longitudinal),

$$W = 0.1/2 \cdot \{1 + \cos(hz \cdot \pi/4)\} \quad (-16 < hz < 24)$$
$$= 0 \quad \text{(otherwise)}.$$

and $u = U = w = 0$.

where $Q = u, w, U$ or W and $A(Q)$ indicates the magnitude of displacement or velocity. In the numerical experiment, $A(Q) = 0.1$ was assumed. The wave source was situated at the depth $\chi (= hz) = 20$. This value is irrespective of actual events of the earthquake. The value is determined only from the fact that the generated waves traveling through such a distance become nearly stable simple waves according to numerical experiments.

Before going to the next section, we will mention the accuracy of numerical computation. Since our theory is developed to second order in the derivatives of

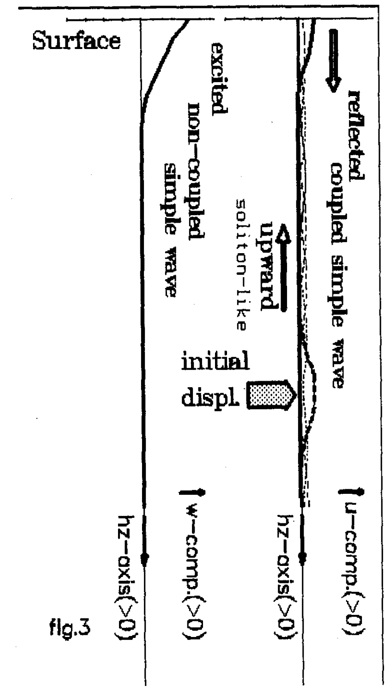

Fig. 3

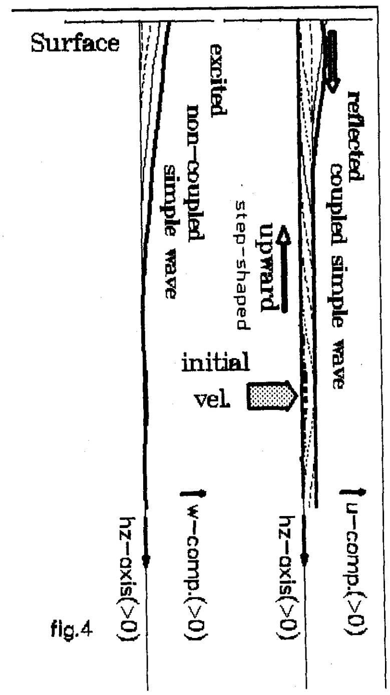

Fig. 4

Fig. 3 (left). Incidence of the coupled simple waves (soliton-like) onto the surface.
Initial displacement (transverse),

$$u=0.1/2\cdot\{1+\cos(hz\cdot\pi/4)\} \quad (-16<hz<24)$$
$$=0 \quad \text{(otherwise)}.$$

and $U=w=W=0$.

Fig. 4 (right). Incidence of the non-coupled simple waves (step-shaped) onto the surface.
Initial velocity (transverse),

$$U=0.1/2\cdot\{1+\cos(hz\cdot\pi/4)\} \quad (-16<hz<24)$$
$$=0 \quad \text{(otherwise)}.$$

and $u=w=W=0$.

displacements, numerical error due to the truncation of terms can be evaluated approximately by third order terms, i.e.,

$$(\partial\xi/\partial\chi)^3, \quad (\partial\xi/\partial\chi)^2(\partial\zeta/\partial\chi), \quad (\partial\xi/\partial\chi)(\partial\zeta/\partial\chi)^2, \quad \text{etc.}$$

Among the above terms, the first term $(\partial\xi/\partial\chi)^3$ for the χ-component is the most significant as compared with other coupled terms, since the ξ and ζ components are generally propagated at different velocities and hence the order of the coupled terms is smaller than that of the non-coupled terms.

In the present computation, the half width of the wave source and the height of the wave are 4 and 0.1, respectively. Derivatives $\partial\xi/\partial\chi$ and $(\partial\xi/\partial\chi)^3$ are of the order of 0.025 ($=0.1/4$) and $0.156\cdot10^{-4}$, respectively. Quantitative discussion can, therefore, explain the physical behavior of waves.

8.4. Computation procedure

In this section, the computation procedure will be outlined.

First, by use of the finite difference equations described in section 8.1, the displacements and velocities at the mesh points *inside* the medium (points at the surface excluded) at a time $\tau+d\tau$ were computed by those at the mesh points (including the surface boundary) at a time τ under the initial conditions given in section 8.3.

Second, the displacements and velocities at the surface at time $\tau+d\tau$ were computed by use of the set of surface conditions (8.2.2), (8.2.1c) with the displacements and velocities (the values at time $\tau+d\tau$) inside the medium obtained as in the foregoing paragraph. In this computation, before the head of the waves arrives at the surface, the conditions involving the nonlinear effect were not applied, but only the condition indicating direct reflection, i.e., *Case [1]* in (8.2.2) was used. After the nonlinearity of the surface conditions is effective (after the head arrives), one condition among the four conditions (8.2.2) was selected and applied by use of the *random function* on the computer.

As for the time of application of the nonlinearity surface conditions, *Cases [2], [3]* and *[4]* in (8.2.2), some mention will be made here. Since the non-coupled and coupled simple waves are propagated nearly at velocities of P and S waves in the linear theory, respectively, the nonlinearity conditions are assumed to become effective after the travel of P (non-coupled case) and S (coupled case) waves over the distance from the wave source to the free surface.

In the numerical experiments, the elastic coefficients are specified as $L_m=A_m=B_m=C_m=1.0$. The mesh size h_χ and time step $d\tau$ are taken as 0.5 and 0.05, respectively. Since the non-linear problem is treated, we cannot obtain the analytical expression of stability condition such as *Neumann condition* in the linear case. Therefore, the stability of the computation was confirmed by doing a number of numerical experiments, varying the values of h_χ *and* $d\tau$.

8.5. Case of non-coupled simple wave incidence

In order to generate a non-coupled simple wave, the following initial conditions are considered for the longitudinal component.

Case of initial displacement,

$$A(w)=0.1\,,$$
$$A(Q)=0 \quad \text{for } Q=u,\ U \text{ and } W. \tag{8.5.1}$$

Case of initial velocity,

$$A(W)=0.1\,,$$
$$A(Q)=0 \quad \text{for } Q=u,\ U \text{ and } w. \tag{8.5.2}$$

The numerical examples are given in Figs. 1 and 2. These figures are examples of the excited wave in *Case-i* in section 7 and indicate the generation of the quasi-trapped non-coupled simple waves near the surface.

8.6. Case of coupled simple wave incidence

In order to generate a coupled simple wave, the following initial conditions are considered for the transverse component.

Initial displacement,

$$A(u)=0.1\,,$$
$$A(Q)=0 \quad \text{for } Q=U,\ w \text{ and } W. \tag{8.6.1}$$

Initial velocity,

$$A(U)=0.1\,,$$
$$A(Q)=0 \quad \text{for } Q=u,\ w \text{ and } W. \tag{8.6.2}$$

Figs. 3 and 4 are numerical examples for the above two initial conditions, and also examples of the excited waves in *Case-ii* in section 7. As expected from the theory, the *non-coupled* simple wave is excited near the surface in these figures, even though the incident wave is of *coupled* type.

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Appendix A

Detailed Reduction in the Case-I surface condition for the incidence of the non-coupled simple wave

By use of the first equation $u_z=0$ in (2.5) and (5.1), we have

$$v_{r2r}{}^2=v_s{}^2\,, \tag{A.1}$$
$$v_{r2r}{}^2=v_s{}^2\cdot f_g\,, \tag{A.2}$$

where

$$f_g=-(2\cdot L_m+1)+f_{g1}\,,$$
$$f_{g1}=(A_m+6\cdot B_m+4\cdot C_m)\cdot(L_m+1)/(A_m+4\cdot B_m+2\cdot C_m+L_m+2)\,. \tag{A.2.1}$$

When A_m, B_m and C_m are small (case of *weak nonlinearity*), f_{g1} is small compared with the first term $-(2\cdot L_m+1)$ in f_g so that f_g is negative. As seen in (A.2), the left- and right-hand sides become positive and negative, respectively, so (A.2) is an invalid solution.

By use of (A.1) and (4.4.1), $V_{sm2r}=0$, so that the coupled reflected simile waves (4.4) disappear, i.e.,

$$u_{rC}=w_{rC}=0\,. \tag{A.3}$$

Substitution of (A.1) into the second equation $w_z=-2\cdot L_{2mm}/g_1$ in (2.5), after the substitution of (5.1), yields

$$v_{r1r}=v_s\cdot V_{pm1i}{}^{1/2}\,. \tag{A.4}$$

Then the expression of the reflected non-coupled simple wave at the surface ($z=0$) becomes

$$w_{rN}=2\cdot t\cdot V_{pm1i}{}^{1/2}\cdot(-V_{pm1i}\cdot v_s^2+v_p^2)/(v_s\cdot g_1)\,.$$

In the usual case, V_{pm1i} $(=(v_p^2-v_{r1i}^2)/v_s^2)$ is small, since the velocity of the incident non-coupled simple wave is near the velocity of the p wave in linear theory. The above expression is reduced to

$$w_{rN}=2\cdot t\cdot V_{pm1i}{}^{1/2}\cdot v_p^2/(v_s\cdot g_1)\,. \tag{A.5}$$

The ratio of reflected to incident simple waves at the surface becomes

$$w_{rN}/w_{iN}=-v_p^2/(V_{pm1i}{}^{1/2}\cdot v_s\cdot v_{r1i})\,. \tag{A.6}$$

Substitution of (A.1) into the third equation $w_z=0$ in (2.5), after the substitution of (5.1), gives

$$v_{r1r}^2/v_s^2=V_{pm1i}+v_p^2/v_s^2\,. \tag{A.7}$$

The expression of the reflected non-coupled simple wave then becomes

$$w_{rN}=2\cdot V_{pm1i}\cdot(z-t\cdot v_{r1r})/g_1\,. \tag{A.8}$$

The ratio of reflected to incident simple waves at the surface ($z=0$) is, when V_{pm1i} is small,

$$w_{rN}/w_{iN}=1+V_{pm1i}\cdot v_s^2/v_p^2\,. \tag{A.9}$$

Appendix B

Detailed Reduction in the Case-II surface condition for the incidence of a non-coupled simple wave

By use of the second equation

$$u_z^2=-g_1/f_1^3+2\cdot L_{2mm}/f_1^2$$

in (2.6) and (5.1), we have two solutions

$$V_{sm2r}=(g_1-2\cdot v_p^2/v_s^2\cdot f_1)/(-g_1+2\cdot f_1)\,, \tag{B.1}$$

$$V_{sm2r}=1\,. \tag{B.2}$$

In the case of (B.1), the equation for the velocity becomes

$$v_{r2r}^2/v_s^2=-2\cdot L_m+f_{g1}\,,$$

where f_{g1} is given in (A.2.1) of the foregoing Appendix A. As already mentioned in Appendix A, f_{g1} is small in case of *weak nonlinearity.* Then the left- and right-hand sides of the above equation become positive and negative, respectively, so that the above equation does not hold valid in the case of weak nonlinearity.

In the case of (B.2), the velocity becomes

$$v_{r2r}=0\,,$$

so that the reflected coupled simple wave does not exist, since $v_{r2r}>0$ is required.

By use of (B.2) and the first expression $w_z=-1/f_1$ of (2.6), after the substitution of (5.1), we have

$$v_{r1r}^2/v_s^2=V_{pm1i}+v_p^2/v_s^2\,. \tag{B.4}$$

This expression is the same as that in (A.7). Therefore, the ratio of reflected to incident simple waves at the surface is the same as that in (A.9), i.e.,

$$w_{rN}/w_{iN}=1+V_{pm1i}\cdot v_s^2/v_p^2\,. \tag{B.5}$$

Appendix C

Detailed Reduction in the Case-I surface condition for the incidence of the coupled simple wave

By use of the first equation $u_z=0$ in (2.5) and (6.1), we have two solutions for V_{sm2r}

$$V_{sm2r}=v_0-V_{sm2i} \tag{C.1}$$

with

$$v_0=-2\cdot(v_p^2-v_s^2)\cdot f_1/((2\cdot f_1-g_1)\cdot v_s^2)$$

and

$$V_{sm2r}=V_{sm2i}\,. \tag{C.2}$$

Transcribing (C.1), this expression is reduced to

$$v_{r2r}^2/v_s^2=-(2\cdot L_m+1)+V_{sm2i}+f_{g1}\,,$$

where f_{g1} is given in (A.2.1). As discussed in Appendix A, when the nonlinearity of the medium is weak, both sides of the above equation have different signs, so that the above equation does not hold.

By use of (C.2), the velocity of the reflected coupled simple wave becomes

$$v_{r2r}=v_{r2i}\,. \tag{C.3.1}$$

Then the expressions of the wave are

$$u_{rC}=s_I\cdot U_{iC}^{1/2}\cdot(-z+t\cdot v_{r2r})/(v_s\cdot f_1)\,,$$
$$w_{rC}=V_{sm2i}\cdot(-z+t\cdot v_{r2r})/f_1\,.$$

The ratios of reflected to incident simple waves at the surface ($z=0$) are

$$u_{rC}/u_{iC}=1 \quad \text{and} \quad w_{rC}/w_{iC}=-1\,. \tag{C.3.2}$$

By use of (C.3.1) and the second equation $w_z=-2\cdot L_{2mm}/g_1$ in (2.5), after the substitution of (6.1), yields

$$v_{r1r}=v_s\cdot V_{sm2i}{}^{1/2}\cdot(g_1/f_1)^{1/2}\,. \tag{C.4}$$

Then the expression of the reflected non-coupled simple wave at the surface ($z=0$) becomes

$$w_{rN}=2\cdot t\cdot V_{sm2i}{}^{1/2}\cdot v_s\cdot(-V_{sm2i}\cdot g_1+v_p{}^2/v_s{}^2\cdot f_1)/(f_1{}^{3/2}\cdot g_1{}^{1/2})\,.$$

When V_{sm2i} is small (usual case), the above is simplified into

$$w_{rN}=2\cdot t\cdot v_p{}^2\cdot V_{sm2i}{}^{1/2}/(v_s\cdot g_1{}^{1/2}\cdot f_1{}^{1/2})\,. \tag{C.5}$$

The ratio of reflected to incident simple waves at the surface becomes

$$w_{rN}/w_{iC}=-2\cdot v_p{}^2\cdot f_1{}^{1/2}/(V_{sm2i}{}^{1/2}\cdot v_s\cdot g_1{}^{1/2}\cdot v_{r2i})\,. \tag{C.6}$$

By use of (C.3.1) and the third equation $w_z=0$ in (2.5), after the substitution of (6.1), we have another solution for the reflected non-coupled simple wave

$$v_{r1r}{}^2/v_s{}^2=V_{sm2i}\cdot g_1/f_1+v_p{}^2/v_s{}^2\,. \tag{C.7.1}$$

Then the expression of the reflected non-coupled simple wave is

$$w_{rN}=2\cdot V_{sm2i}\cdot(z-t\cdot v_p)/f_1\,.$$

The ratio of reflected to incident simple waves at the surface becomes

$$w_{rN}/w_{iC}=2\cdot v_p/v_s\,. \tag{C.7.2}$$

Appendix D

Detailed Reduction in the Case-II surface condition for the incidence of the coupled simple wave

By use of the second equation

$$u_z{}^2=-g_1/f_1{}^3+2\cdot L_{2mm}/f_1{}^2$$

in (2.6) and (6.1), we have two solutions

$$v_{r2r}{}^2/v_s{}^2=C_C\cdot U_U\,, \tag{D.1}$$

$$C_C=f_1{}^{1/2}\cdot(2\cdot f_1\cdot v_p{}^2-g_1\cdot v_s{}^2)^{1/2}/(f_1\cdot v_p{}^2+v_s{}^2\cdot(f_1-g_1))\,,$$

$$U_U=2^{1/2}\cdot V_{sm2i}{}^{1/2}\cdot(v_p{}^2-v_s{}^2)^{1/2}\,,$$

$$v_{r2r}{}^2/v_s{}^2=-2\cdot L_m+f_{g1}-C_C\cdot U_U\,, \tag{D.2}$$

where f_{g1} given in (A.2.1).

In the above equation, the term $C_C\cdot U_U$ is small, since U_U is small owing to the existence of V_{sm2i}, and in the case of weak nonlinearity of the medium (f_{g1} is then small), the above equation (D.2) does not hold, since the left- and right-hand sides are positive and negative, respectively.

By use of (D.1), the reflected coupled simple waves at the surface become

$$u_{rC} = -s_R \cdot t \cdot f_c \cdot (2 \cdot f_1 \cdot v_p^2 - v_s^2 \cdot g_1)^{1/2} / f_1^{1/2} ,$$
$$w_{rC} = v_s \cdot t \cdot f_c ,$$

where

$$f_c = 2^{1/4} \cdot V_{sm2i}^{1/4} \cdot (2 \cdot f_1 \cdot v_p^2 - v_s^2 \cdot g_1)^{1/4} \cdot (v_p^2 - v_s^2)^{1/4} / (f_1^{3/4} \cdot (f_1 \cdot v_p^2 + v_s^2 \cdot (f_1 - g_1))^{1/2}) .$$

The ratio of the reflected to incident simple waves at the surface then becomes

$$u_{rC} / u_{iC} = -s_R \cdot F_c \cdot (2 \cdot f_1 \cdot v_p^2 - v_s^2 \cdot g_1)^{1/2} / (f_1^{1/4} \cdot V_{sm2i}^{1/4}) ,$$
$$w_{rC} / u_{iC} = v_s \cdot F_c \cdot f_1^{1/4} / V_{sm2i}^{1/4} \qquad \text{(D.3)}$$

where

$$F_c = v_s \cdot (-v_s^2 + v_p^2)^{1/4} \cdot (2 \cdot f_1 \cdot v_p^2 - v_s^2 \cdot g_1)^{1/4} / (2^{1/4} \cdot v_{r2i} \cdot (f_1 \cdot v_p^2 + v_s^2 \cdot (f_1 - g_1))^{1/2}) .$$

For the reflected non-coupled simple wave, there exits a solution by use of the first condition $w_z = -1/f_1$ in (2.6) after the substitution of (6.1). The velocity v_{r1r} and wave w_{rN} are given by

$$v_{r1r}^2 / v_s^2 = (-C_C \cdot g_1 \cdot U_U \cdot v_s^2 + 2 \cdot f_1 \cdot v_p^2) / (2 \cdot f_1 \cdot v_s^2) ,$$
$$w_{rN} = C_C \cdot U_U \cdot (-z + t \cdot v_{r1r}) / f_1 . \qquad \text{(D.4.1)}$$

The ratio of the reflected to incident simple waves at the surface then becomes

$$w_{rN} / u_{iC} = C_C \cdot v_{r1r} \cdot v_s / (s_I \cdot v_{r2i}) . \qquad \text{(D.4.2)}$$

Appendix E

In the region adjacent to the surface boundary, the mesh intervals are unevenly divided. In the following discussions, the values with suffix s, $s+1$ and $s+2$ indicate the values at the surface, the first and second points from the surface, respectively. The first and second derivatives in difference form are then approximated by,

at the free surface,

$$(\partial \xi / \partial \chi)_{\text{surf}} = (\xi_{s+1} - \xi_s) / h_s , \qquad \text{(E.1)}$$

at the mesh point next to the free surface,

$$(\partial \xi / \partial \chi)_{\text{next}} = (\xi_{s+2} - \xi_s) / (h_s + h_\chi) , \qquad \text{(E.2.1)}$$

$$(\partial^2 \xi / \partial \chi^2)_{\text{next}} = ((\xi_{s+2} - \xi_{s+1}) / h_\chi - (\xi_{s+1} - \xi_s) / h_s) / ((h_s + h_\chi) / 2) , \qquad \text{(E.2.2)}$$

where the discussion also can be applied for the cases of ζ, U and W, though the following developments are done only for the case of ξ.

In order to evaluate the accuracy due to the approximation based on the above finite difference, the differences $D^{(1)}{}_{surf}$ (for (E.1)), $D^{(1)}{}_{next}$ (for (E.2.1)) and $D^{(2)}{}_{next}$ (for (E.2.2)) of both sides of the above equations will be evaluated. When the right-hand sides of the equations are evaluated, the following Taylor expansions are used.

At the free surface,

$$\xi_{s+1}=\xi_{surf}+(\partial\xi/\partial\chi)_{surf}\cdot h_s+(\partial^2\xi/\partial\chi^2)_{surf}\cdot h_s{}^2/2\,,\quad (\xi_{surf}=\xi_s)$$

and, at the point next to the free surface,

$$\xi_s=\xi_{next}-(\partial\xi/\partial\chi)_{next}\cdot h_s+(\partial^2\xi/\partial\chi^2)_{next}\cdot h_s{}^2/2-(\partial^3\xi/\partial\chi^3)_{next}\cdot h_s{}^3/3!\,,$$

$$\xi_{s+2}=\xi_{next}+(\partial\xi/\partial\chi)_{next}\cdot h_\chi+(\partial^2\xi/\partial\chi^2)_{next}\cdot h_\chi{}^2/2+(\partial^3\xi/\partial\chi^3)_{next}\cdot h_\chi{}^3/3!\,,\quad (\xi_{next}=\xi_{s+1})$$

where it must be noted here that ξ_s and ξ_{s+2} are expanded in terms of h_s and h_χ, respectively.

The evaluated results are as follows.

$$D^{(1)}{}_{surf}=-(\partial^2\xi/\partial\chi^2)_{surf}\cdot h_s/2\,, \tag{E.3.1}$$

and

$$D^{(1)}{}_{next}=(\partial^2\xi/\partial\chi^2)_{next}\cdot(h_s-h_\chi)/2\,,$$
$$D^{(2)}{}_{next}=(\partial^3\xi/\partial\chi^3)_{next}\cdot(h_s-h_\chi)/3\,. \tag{E.3.2}$$

The above results indicate that, if $(\partial^2\xi/\partial\chi^2)_{surf}$, $(\partial^2\xi/\partial\chi^2)_{next}$ and $(\partial^3\xi/\partial\chi^3)_{next}$ are assumed to be of the same order, the accuracies of the three derivatives based on the finite difference become appropriately small when

$$h_s=h_\chi/2\,. \tag{E.3.3}$$

半無限非線形等方性媒質の自由表面に励起される波について

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無限非線形等方性媒質において二種類の単純波すなわち非結合単純波（noncoupled simple wave: 縦成分のみの波）と結合単純波（coupled simple wave: 縦成分と横成分の結合波）が発生することを筆者は以前の論文で論じた．本論文では，自由表面のある半無限非線形媒質において自由表面直下から真上に上述の単純波が突入したときに自由表面で何が起こるかを解明した．この現象の解明は直下型地震を理解するのに極めて有意義だと考えられる．

非線形媒質では自由表面の応力ゼロ (stress free) の条件が非線形となり，自由表面からの反射波は複数個の解となる．エネルギーの受容力の観点から反射波の振幅が一番大きい波がもっとも注目すべき波と考えられる．この注目すべき波について次のことが判明した．

媒質内部から自由表面に垂直に突入した単純波，これが非結合単純波であれ結合単純波であれ，それに関わり無く，大きい振幅の波として非結合型の単純波が反射される，その反射単純波の速度は非常に遅くなる．その結果として直下型地震のとき地表では大災害が予想される．