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The Polarization of Waves in an Anisotropic Nonlinear-Elastic Medium

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Abstract

In an anisotropic linear-elastic medium, polarization of waves due to anisotropy of the medium is known to occur. In a way similar to phenomena in an anisotropic linear-elastic medium, polarization of waves takes place even in an anisotropic nonlinear-elastic medium. Polarized waves in a nonlinear-elastic medium are also *soliton*-like or *step*-shaped waves named *simple waves*.

In an isotropic nonlinear-elastic medium, the simple waves are separated into two categories, *non-coupled* and *coupled* simple waves. The former are dilatational waves, while the latter are coupled waves with dilatational (u-component) and distorsional (v- and w-components) properties, where u and $\{v, w\}$ are longitudinal and transverse components, respectively.

In the case of anisotropic nonlinear-elastic medium, two kinds of simple waves mentioned above also appear with some modification due to the presence of anisotropy. The secondary waves produced by anisotropy are exact simple waves instead of only disturbance-type waves.

Equations are numerically evaluated by use of an extended finite difference method expanded in a Taylor series. The wave source then has the form of a mountain ridge with a width $-4<hx<4$, where hx is a distance x normalized by wave number h of P waves in the linear theory.

Introduction

With the increased resolution of seismic data, there is a growing awareness that an isotropic description of the Earth may no longer be adequate. Anisotropy appears to be a ubiquitous property of earth materials.

Against the background of the 21-constant elastic theory, isotropy presents a highly degenerate case, and the study of waves in an anisotropic medium throws a more realistic light on the algebraic structure of waves. The study of waves in an anisotropic linear-elastic medium is summarized in the papers by BAMFORD (1977), CRAMPIN (1984) and KANESHIMA (1991).

Most past theoretical investigations of seismic anisotropy assumed linear-elastic models. The nonlinear theory of wave propagation has been developed thoroughly in different fields of physics, such as optics, fluid and gas dynamics. Particularly, impressive results have been obtained in nonlinear acoustics (RUDENKO and SOLUYAN, 1977; LJAMOV, 1983). As one of the theoretical studies, the work done by TSVANKIN *et al.* (1987) is interesting, since they discussed possible nonlinear distortions of well-known characteristics which are diagnostic of anisotropy such as shear-wave splitting.

The study of wave propagation in a nonlinear elastic medium is of importance to seismology in the near-field of a wave origin. In the case of an isotropic nonlinear-elastic medium, the present author (1990) carried out numerical experiments on the generation of nonlinear waves due to a local wave source. He then found specific type of waves named *Simple Waves* which are separated into two categories: (i) *non-coupled* and (ii) *coupled simple waves*, respectively. The former is a dilatational wave and the latter are coupled waves having both dilatational and distorsional properties.

The object of this paper is to study the mechanism of polarization of waves in an anisotropic nonlinear-elastic medium and how the *simple waves* found by the author (1990) in the isotropic medium will be modulated by the presence of anisotropy. Such a study is considered to be useful in exploration of the source mechanism.

The theory was developped using *computer algebra* installed on an NEC 9800 computer.

Notations

Since expressions concerning elastic coefficients are widespread throughout this paper, the notations of global expressions are summarized. For local expressions, the definitions for them will be given in the associated sections.

λ, μ: Lame's constants associated with linear terms in isotropic medium.

A, B, C: elastic coefficients associated with the second order terms in isotropic medium.
For the coefficients λ, μ, A, B, C, the previous paper by the present author (MOMOI, 1990) should be referred to; this paper will be referred to as *paper M* in the subsequent discussions.

c_{JP}: elastic coefficients in an anisotropic medium.

C_{JP}: c_{JP} normalized by μ.

$L_{am}=\lambda+\mu$.

$L_{2m}=\lambda+2\cdot\mu$.

$L_{mABC}=3\cdot\lambda+6\cdot\mu+2\cdot A+6\cdot B+2\cdot C$.

$P_{MD}=A+2\cdot B+2\cdot(\lambda+2\cdot\mu)$,

$F_{guc}=\lambda+2\cdot\mu+A+4\cdot B+2\cdot C$.

$G_{Unc}=2\cdot(\lambda+2\cdot\mu-\rho\cdot v_r^2)/L_{mABC}$
with ρ and v_r being the density of the medium and the velocity of moving axes, respectively.

$G_{uc}=(\mu-\rho\cdot v_r^2)/(2\cdot D_{vw})$
with $D_{vw}=A/4+B/2+\lambda/2+\mu$.

1. Expression of Energy

In conservative (nondissipative) thermoelastic media, the strain energy function E_n can be expanded about the state of zero strain by use of a strain tensor component U_J (for details, the reader should refer to BRUGGER (1964)) as the generalized Taylor expansion

$$E_n=(1/2!)c_{JP}\cdot U_J\cdot U_P\,, \tag{1.1}$$

where summations over repeated indices $\{J, P\}$ are implied (this convention will be used, unless stated otherwise), and c_{JP} the elastic coefficients defined by K. BRUGGER (1964), i.e.,

$$c_{JP}=\partial^2 E_n/\partial U_J \partial U_P\,. \tag{1.2}$$

It must be noted here that no restriction concerning the elastic coefficient, particularly about *weak* anisotropy, is assumed in the derivation of expressions (1.1) and (1.2) (see BRUGGER, 1964), so that these expressions can be used even in the case of *strong* anisotropy.

In this paper, the terms of expression (1.1) are taken into account up to third order in U_J. In the equations of motion, this order is, therefore, reduced to second order in U_J.

In the above, Voigt Notation $\{ij\sim J\}$

$$\{11\sim 1,\ 22\sim 2,\ 33\sim 3,\ 23\sim 4,\ 13\sim 5,\ 12\sim 6\}$$

and the convention

$$U_{ij}=(1/2)\cdot(1+\delta_{ij})\cdot U_J \tag{1.3}$$

with

$$U_{ij}=(u_{ij}+u_{ji}+u_{ki}\cdot u_{kj})/2 \tag{1.4}$$

are used, where $u_{ij}=\partial u_i/\partial x_j$ and $\{u_1, u_2, u_3\}$ are displacement components in the directions of the coordinate axes $\{x_1, x_2, x_3\}$. In later analyses, $\{u_1, u_2, u_3\}$ and $\{x_1, x_2, x_3\}$ will be alternatively expressed by $\{u, v, w\}$ and $\{x, y, z\}$. δ_{ij} is the Kronecker delta with suffixes i, j. The elastic coefficients C_{JP} normalized by μ (rigidity in an isotropic medium) will be used, i.e.,

$$C_{JP}=c_{JP}/\mu\,. \tag{1.5}$$

2. Equations

First of all, waves will be assumed to be propagated in the x-direction.

The stress tensor S_{ij} is related to the energy function

$$S_{ij}=\partial E_n/\partial u_{ij}\,. \tag{2.1}$$

On the other hand, the governing equations can be expressed (LANDAU and LIFSHITZ, 1985) by

$$\rho\partial^2 u_i/\partial t^2 = \partial S_{ij}/\partial x_j \qquad (i=1, 2, 3)\,, \tag{2.2}$$

where ρ and t are the density of the medium and the time factor, respectively.

By use of (2.1) and the expressions for energy in the foregoing section, the above equations (2.2) are reduced to the following.

$$\rho u_{t2} = q_1\,, \quad \rho v_{t2} = q_2\,, \quad \rho w_{t2} = q_3\,, \tag{2.3}$$

where

$$\begin{aligned} q_1 = {} & c_{16}\cdot u_x\cdot v_{x2} + c_{16}\cdot u_{x2}\cdot v_x + c_{16}\cdot v_{x2} + c_{116}\cdot u_x\cdot v_{x2} + c_{116}\cdot u_{x2}\cdot v_x \\ & + c_{15}\cdot u_x\cdot w_{x2} + c_{15}\cdot u_{x2}\cdot w_x + c_{15}\cdot w_{x2} + c_{115}\cdot u_x\cdot w_{x2} + c_{115}\cdot u_{x2}\cdot w_x \\ & + 3\cdot c_{11}\cdot u_x\cdot u_{x2} + c_{11}\cdot u_{x2} + c_{11}\cdot v_x\cdot v_{x2} + c_{11}\cdot w_x\cdot w_{x2} + c_{156}\cdot v_x\cdot w_{x2} \\ & + c_{156}\cdot v_{x2}\cdot w_x + c_{166}\cdot v_x\cdot v_{x2} + c_{155}\cdot w_x\cdot w_{x2} + c_{111}\cdot u_x\cdot u_{x2}\,, \end{aligned} \tag{2.3.1}$$

$$\begin{aligned} q_2 = {} & c_{16}\cdot u_x\cdot u_{x2} + c_{16}\cdot u_{x2} + 3\cdot c_{16}\cdot v_x\cdot v_{x2} + c_{16}\cdot w_x\cdot w_{x2} + c_{116}\cdot u_x\cdot u_{x2} \\ & + c_{15}\cdot v_x\cdot w_{x2} + c_{15}\cdot v_{x2}\cdot w_x + c_{11}\cdot u_x\cdot v_{x2} + c_{11}\cdot u_{x2}\cdot v_x + c_{156}\cdot u_x\cdot w_{x2} \\ & + c_{156}\cdot u_{x2}\cdot w_x + c_{166}\cdot u_x\cdot v_{x2} + c_{166}\cdot u_{x2}\cdot v_x + c_{66}\cdot v_{x2} + c_{56}\cdot w_{x2} \\ & + c_{666}\cdot v_x\cdot v_{x2} + c_{566}\cdot v_x\cdot w_{x2} + c_{566}\cdot v_{x2}\cdot w_x + c_{556}\cdot w_x\cdot w_{x2} \end{aligned} \tag{2.3.2}$$

$$\begin{aligned} q_3 = {} & c_{16}\cdot v_x\cdot w_{x2} + c_{16}\cdot v_{x2}\cdot w_x + c_{15}\cdot u_x\cdot u_{x2} + c_{15}\cdot u_{x2} + c_{15}\cdot v_x\cdot v_{x2} \\ & + 3\cdot c_{15}\cdot w_x\cdot w_{x2} + c_{115}\cdot u_x\cdot u_{x2} + c_{11}\cdot u_x\cdot w_{x2} + c_{11}\cdot u_{x2}\cdot w_x \\ & + c_{156}\cdot u_x\cdot v_{x2} + c_{156}\cdot u_{x2}\cdot v_x + c_{155}\cdot u_x\cdot w_{x2} + c_{155}\cdot u_{x2}\cdot w_x \\ & + c_{56}\cdot v_{x2} + c_{566}\cdot v_x\cdot v_{x2} + c_{556}\cdot v_x\cdot w_{x2} + c_{556}\cdot v_{x2}\cdot w_x + c_{55}\cdot w_{x2} \\ & + c_{555}\cdot w_x\cdot w_{x2}\,, \end{aligned} \tag{2.3.3}$$

with

$$\begin{aligned} & u_{t2} = \partial^2 u/\partial t^2\,, \quad u_x = \partial u/\partial x\,, \quad u_{x2} = \partial^2 u/\partial x^2\,, \\ & v_{t2} = \partial^2 v/\partial t^2\,, \quad v_x = \partial v/\partial x\,, \quad v_{x2} = \partial^2 v/\partial x^2\,, \\ & w_{t2} = \partial^2 w/\partial t^2\,, \quad w_x = \partial w/\partial x\,, \quad w_{x2} = \partial^2 w/\partial x^2\,. \end{aligned}$$

3. Finite Difference Equation by Taylor Method

In this section, finite difference equations will be introduced by use of the theory to second order in the derivative of displacements u, v and w with respect to x.

The variables t, x, u, v and w are here normalized by the wave number h of the P wave in the linear theory, i.e.,

$$\tau = h\cdot v_p\cdot t\,, \quad \chi = h\cdot x\,, \quad \xi = h\cdot u\,, \quad \eta = h\cdot v \quad \text{and} \quad \zeta = h\cdot w\,, \tag{3.1}$$

where v_p and v_s (the latter will be used later) are the velocities of P and S waves in the linear theory.

Equations (2.3) are reduced to

$$\partial\xi/\partial\tau=U\,,\quad \partial\eta/\partial\tau=V\,,\quad \partial\zeta/\partial\tau=W\,, \tag{3.2}$$

and

$$\partial U/\partial\tau=q_1/v_{ps}^2\,,\quad \partial V/\partial\tau=q_2/v_{ps}^2\,,\quad \partial W/\partial\tau=q_3/v_{ps}^2\,, \tag{3.3}$$

where

$$v_{ps}=v_p/v_s\,.$$

In order to evaluate the displacements $\{\xi, \eta, \zeta\}$ and velocities $\{U, V, W\}$ at a time $\tau+d\tau$ ($d\tau$: increment of time τ), we will use Taylor expansion in terms of τ to second order in $d\tau$ such that

$$\begin{aligned}\xi&=\xi_0+D_{\xi1}\cdot d\tau+D_{\xi2}\cdot(d\tau)^2/2\,,\\ \eta&=\eta_0+D_{\eta1}\cdot d\tau+D_{\eta2}\cdot(d\tau)^2/2\,,\\ \zeta&=\zeta_0+D_{\zeta1}\cdot d\tau+D_{\zeta2}\cdot(d\tau)^2/2\,,\end{aligned} \tag{3.4}$$

$$\begin{aligned}U&=U_0+D_{U1}\cdot d\tau+D_{U2}\cdot(d\tau)^2/2\,,\\ V&=V_0+D_{V1}\cdot d\tau+D_{V2}\cdot(d\tau)^2/2\,,\\ W&=W_0+D_{W1}\cdot d\tau+D_{W2}\cdot(d\tau)^2/2\,,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}&D_{\xi1}=(\partial\xi/\partial\tau)_0\,, && D_{\eta1}=(\partial\eta/\partial\tau)_0\,, && D_{\zeta1}=(\partial\zeta/\partial\tau)_0\,,\\ &D_{\xi2}=(\partial^2\xi/\partial\tau^2)_0\,, && D_{\eta2}=(\partial^2\eta/\partial\tau^2)_0\,, && D_{\zeta2}=(\partial^2\zeta/\partial\tau^2)_0\,,\\ &D_{U1}=(\partial U/\partial\tau)_0\,, && D_{V1}=(\partial V/\partial\tau)_0\,, && D_{W1}=(\partial W/\partial\tau)_0\,,\end{aligned} \tag{3.6}$$

and

$$D_{U2}=(\partial^2 U/\partial\tau^2)_0\,,\quad D_{V2}=(\partial^2 V/\partial\tau^2)_0\,,\quad D_{W2}=(\partial^2 W/\partial\tau^2)_0\,, \tag{3.7}$$

with suffix 0 indicating the evaluation at a time τ.

The above coefficients in (3.6) are expressed, by use of (3.2) and (3.3), as

$$\begin{aligned}&D_{\xi1}=U\,, && D_{\xi2}=D_{U1}=q_1/v_{ps}^2\,,\\ &D_{\eta1}=V\,, && D_{\eta2}=D_{V1}=q_2/v_{ps}^2\,,\\ &D_{W1}=W\,, && D_{\zeta2}=D_{W1}=q_3/v_{ps}^2\,.\end{aligned}$$

In order to obtain expressions (3.7), these expressions will be transcribed, by use of (3.3), as follows.

$$D_{U2}=q_{1\tau}/v_{ps}^2\,,\quad D_{V2}=q_{2\tau}/v_{ps}^2\,,\quad D_{W2}=q_{3\tau}/v_{ps}^2\,, \tag{3.7.1}$$

with

$$q_{1\tau}=\partial q_1/\partial\tau\,,\quad q_{2\tau}=\partial q_2/\partial\tau\,,\quad q_{3\tau}=\partial q_3/\partial\tau\,. \tag{3.7.2}$$

The expressions (3.7.2) can be obtained by differentiation of (2.3.1), (2.3.2) and (2.3.3) with respect to τ, i.e.,

$$q_{1\tau}=C_{115}\cdot e_{a1}+C_{116}\cdot e_{a5}+C_{156}\cdot e_{a8}+C_{11}\cdot e_{a3}+C_{111}\cdot e_{a4}+C_{16}\cdot e_{a6}$$
$$+C_{15}\cdot e_{a2}+C_{166}\cdot e_{a9}+C_{155}\cdot e_{a7}\,,$$
$$q_{2\tau}=C_{116}\cdot e_{b2}+C_{156}\cdot e_{b1}+C_{11}\cdot e_{b4}+C_{16}\cdot e_{b3}+C_{15}\cdot e_{b7}+C_{166}\cdot e_{b5}$$
$$+C_{556}\cdot e_{b6}+C_{566}\cdot e_{b8}+C_{666}\cdot e_{b10}+C_{66}\cdot e_{b11}+C_{56}\cdot e_{b9}\,,$$
$$q_{3\tau}=C_{115}\cdot e_{c1}+C_{156}\cdot e_{c5}+C_{11}\cdot e_{c2}+C_{16}\cdot e_{c7}+C_{15}\cdot e_{c4}+C_{155}\cdot e_{c3}$$
$$+C_{556}\cdot e_{c8}+C_{566}\cdot e_{c10}+C_{56}\cdot e_{c11}+C_{555}\cdot e_{c6}+C_{55}\cdot e_{c9}\,,$$

where

$$e_{a1}=U_x\cdot\zeta_{x2}+\xi_{x2}\cdot W_x+U_{x2}\cdot\zeta_x+\xi_x\cdot W_{x2}\,,$$
$$e_{a2}=W_{x2}+U_x\cdot\zeta_{x2}+\xi_{x2}\cdot W_x+U_{x2}\cdot\zeta_x+\xi_x\cdot W_{x2}\,,$$
$$e_{a3}=U_{x2}+3\cdot U_x\cdot\xi_{x2}+\zeta_{x2}\cdot W_x+3\cdot U_{x2}\cdot\xi_x+\zeta_x\cdot W_{x2}+\eta_{x2}\cdot V_x+\eta_x\cdot V_{x2}\,,$$
$$e_{a4}=U_x\cdot\xi_{x2}+U_{x2}\cdot\xi_x\,,$$
$$e_{a5}=U_x\cdot\eta_{x2}+\xi_{x2}\cdot V_x+U_{x2}\cdot\eta_x+\xi_x\cdot V_{x2}\,,$$
$$e_{a6}=V_{x2}+U_x\cdot\eta_{x2}+\xi_{x2}\cdot V_x+U_{x2}\cdot\eta_x+\xi_x\cdot V_{x2}\,,$$
$$e_{a7}=\zeta_{x2}\cdot W_x+\zeta_x\cdot W_{x2}\,,$$
$$e_{a8}=\zeta_{x2}\cdot V_x+W_x\cdot\eta_{x2}+\zeta_x\cdot V_{x2}+W_{x2}\cdot\eta_x\,,$$
$$e_{a9}=\eta_{x2}\cdot V_x+\eta_x\cdot V_{x2}\,,$$

$$e_{b1}=U_x\cdot\zeta_{x2}+\xi_{x2}\cdot W_x+U_{x2}\cdot\zeta_x+\xi_x\cdot W_{x2}\,,$$
$$e_{b2}=U_x\cdot\xi_{x2}+U_{x2}\cdot\xi_x\,,$$
$$e_{b3}=U_{x2}+U_x\cdot\xi_{x2}+\zeta_{x2}\cdot W_x+U_{x2}\cdot\xi_x+\zeta_x\cdot W_{x2}+3\cdot\eta_{x2}\cdot V_x+3\cdot\eta_x\cdot V_{x2}\,,$$
$$e_{b4}=U_x\cdot\eta_{x2}+\xi_{x2}\cdot V_x+U_{x2}\cdot\eta_x+\xi_x\cdot V_{x2}\,,$$
$$e_{b5}=U_x\cdot\eta_{x2}+\xi_{x2}\cdot V_x+U_{x2}\cdot\eta_x+\xi_x\cdot V_{x2}\,,$$
$$e_{b6}=\zeta_{x2}\cdot W_x+\zeta_x\cdot W_{x2}\,,$$
$$e_{b7}=\zeta_{x2}\cdot V_x+W_x\cdot\eta_{x2}+\zeta_x\cdot V_{x2}+W_{x2}\cdot\eta_x\,,$$
$$e_{b8}=\zeta_{x2}\cdot V_x+W_x\cdot\eta_{x2}+\zeta_x\cdot V_{x2}+W_{x2}\cdot\eta_x\,,$$
$$e_{b9}=W_{x2}\,,$$
$$e_{b10}=\eta_{x2}\cdot V_x+\eta_x\cdot V_{x2}\,,$$
$$e_{b11}=V_{x2}\,,$$

$$e_{c1}=U_x\cdot\xi_{x2}+U_{x2}\cdot\xi_x\,,$$
$$e_{c2}=U_x\cdot\zeta_{x2}+\xi_{x2}\cdot W_x+U_{x2}\cdot\zeta_x+\xi_x\cdot W_{x2}\,,$$
$$e_{c3}=U_x\cdot\zeta_{x2}+\xi_{x2}\cdot W_x+U_{x2}\cdot\zeta_x+\xi_x\cdot W_{x2}\,,$$
$$e_{c4}=U_{x2}+U_x\cdot\xi_{x2}+3\cdot\zeta_{x2}\cdot W_x+U_{x2}\cdot\xi_x+3\cdot\zeta_x\cdot W_{x2}+\eta_{x2}\cdot V_x+\eta_x\cdot V_{x2}\,,$$
$$e_{c5}=U_x\cdot\eta_{x2}+\xi_{x2}\cdot V_x+U_{x2}\cdot\eta_x+\xi_x\cdot V_{x2}\,,$$
$$e_{c6}=\zeta_{x2}\cdot W_x+\zeta_x\cdot W_{x2}\,,$$
$$e_{c7}=\zeta_{x2}\cdot V_x+W_x\cdot\eta_{x2}+\zeta_x\cdot V_{x2}+W_{x2}\cdot\eta_x\,,$$

$$e_{c8} = \zeta_{x2} \cdot V_x + W_x \cdot \eta_{x2} + \zeta_x \cdot V_{x2} + W_{x2} \cdot \eta_x ,$$
$$e_{c9} = W_{x2} ,$$
$$e_{c10} = \eta_{x2} \cdot V_x + \eta_x \cdot V_{x2} ,$$
$$e_{c11} = V_{x2} ,$$

with

$$\xi_x = \partial\xi/\partial\chi , \qquad \eta_x = \partial\eta/\partial\chi , \qquad \zeta_x = \partial\zeta/\partial\chi ,$$
$$\xi_{x2} = \partial^2\xi/\partial\chi^2 , \qquad \eta_{x2} = \partial^2\eta/\partial\chi^2 , \qquad \zeta_{x2} = \partial^2\zeta/\partial\chi^2 ,$$
$$U_x = \partial U/\partial\chi , \qquad V_x = \partial V/\partial\chi , \qquad W_x = \partial W/\partial\chi ,$$
$$U_{x2} = \partial^2 U/\partial\chi^2 , \qquad V_{x2} = \partial^2 V/\partial\chi^2 , \qquad W_{x2} = \partial^2 W/\partial\chi^2 .$$

In numerical computation, the above derivatives are replaced by the difference expressions, say;

$$\xi_x = (\xi_{+1} - \xi_{-1})/(2h_x) ,$$

and

$$\xi_{x2} = (\xi_{+1} + \xi_{-1} - 2\xi_0)/h_x^{\,2} ,$$

where $\{\xi_{-1}, \xi_{+1}\}$ are the displacements at points just left and right of a reference point with displacement ξ_0, and h_x is a mesh interval. The above description also applies for η, ζ, U, V and W.

In the following sections, numerical computations will be carried out by use of the above-mentioned procedure. The wave source is then assumed to have the form

$$Q = A(Q)/2 \cdot \{1 + \cos(\chi \cdot \pi/4)\} \qquad (-4 < \chi < 4) . \tag{3.8}$$

where $Q = u, v, w, U, V$ or W and $A(Q)$ indicates the magnitude of displacement or velocity. Since χ is normalized by h (wave number of P wave), the actual span x of the wave source must be divided by h.

Before going to the next section, we will mention the accuracy of numerical computation. Since our theory is developed to second order in the derivatives of displacements, numerical error due to the truncation of terms can be evaluated approximately by third order terms, i.e.,

$$(\partial\xi/\partial\chi)^3 , \quad (\partial\xi/\partial\chi)^2(\partial\eta/\partial\chi) , \quad (\partial\xi/\partial\chi)(\partial\eta/\partial\chi)^2 , \quad \text{etc.}$$

Among the above terms, the first term $(\partial\xi/\partial\chi)^3$ for the χ-component is the most significant as compared with other coupled terms, since the ξ, η and ζ components are generally propagated at different velocity and hence the order of the coupled terms is smaller than that of the non-coupled terms.

In the present computation, the half width of the wave source and height of the wave are 4 and 0.1, respectively. Derivatives $\partial\xi/\partial\chi$ and $(\partial\xi/\partial\chi)^3$ become the order of 0.025 ($=0.1/4$) and $0.156 \cdot 10^{-4}$, respectively. Quantitative discussion can, therefore, explain the physical behavior of waves.

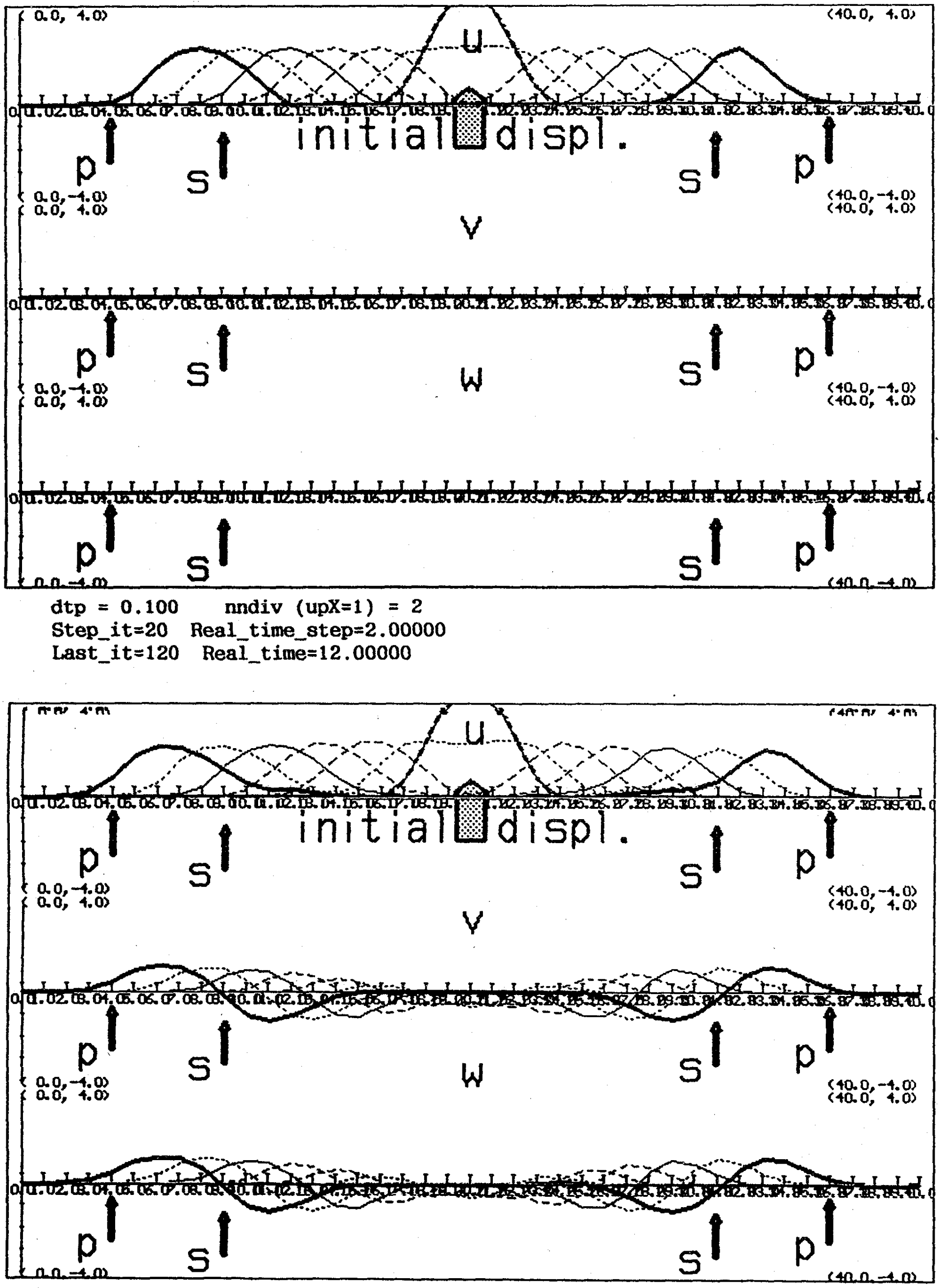


Fig. 1–1. Waves at the time $\tau=12$ produced by local wave origin.
Above: isotropic case. Below: anisotropic case.
Elastic coefficients:
$Lm=Am=Bm=Cm=1.0$ for isotropic ones and $d_{ij}=d_{ijk}=0.5$ for anisotropic ones.
Initial condition:
$Q=A(Q)/2\cdot\{1+\cos(x\cdot\pi/4)\}$ $(-4<x<4)$ with $A(Q)=0.1$ for $Q=u$ and $A(Q)=0$ for $Q=\{v,w,U,V,W\}$.
Scales: $\times 50$ for u and $\times 100$ for v and w.

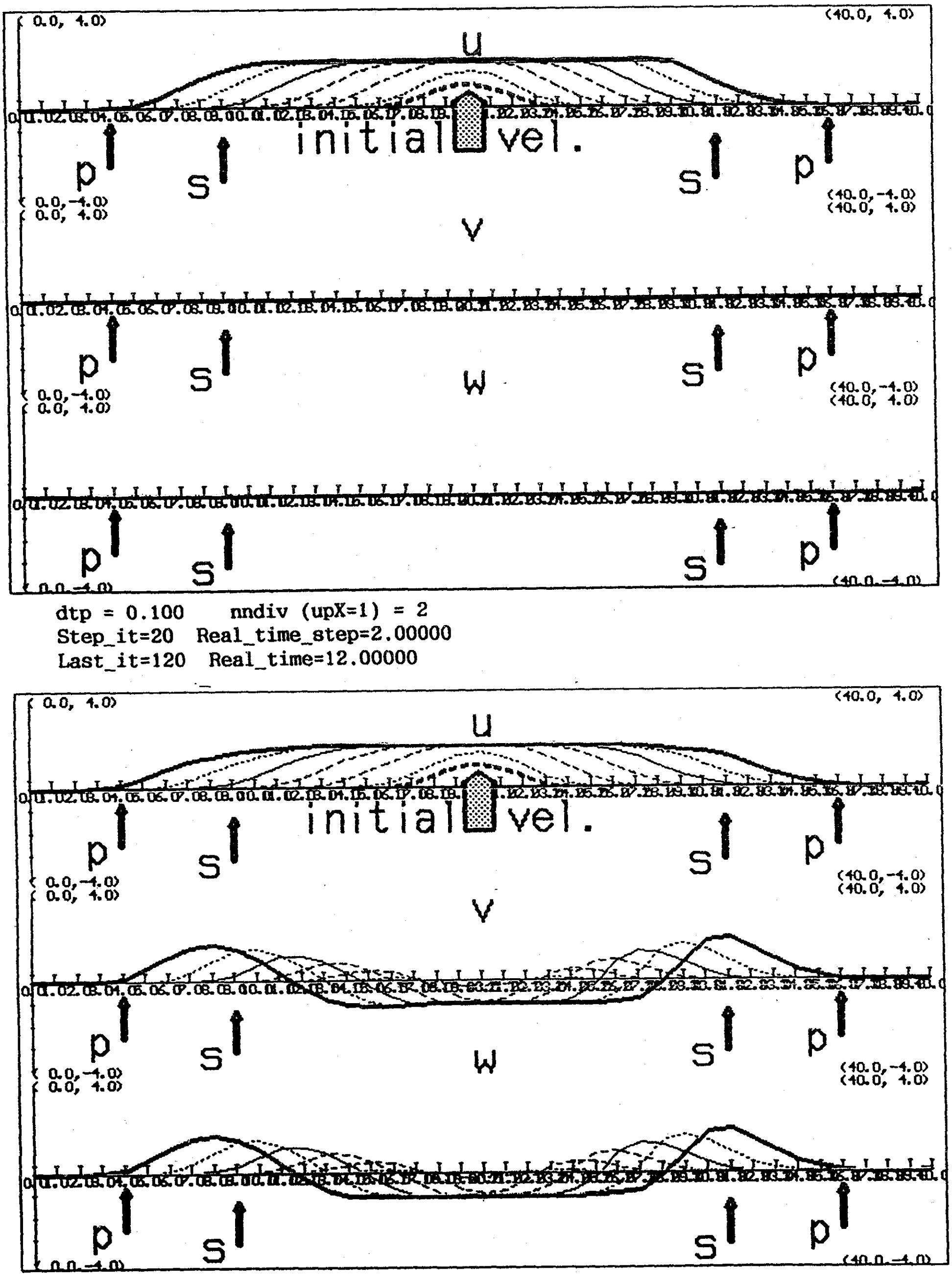
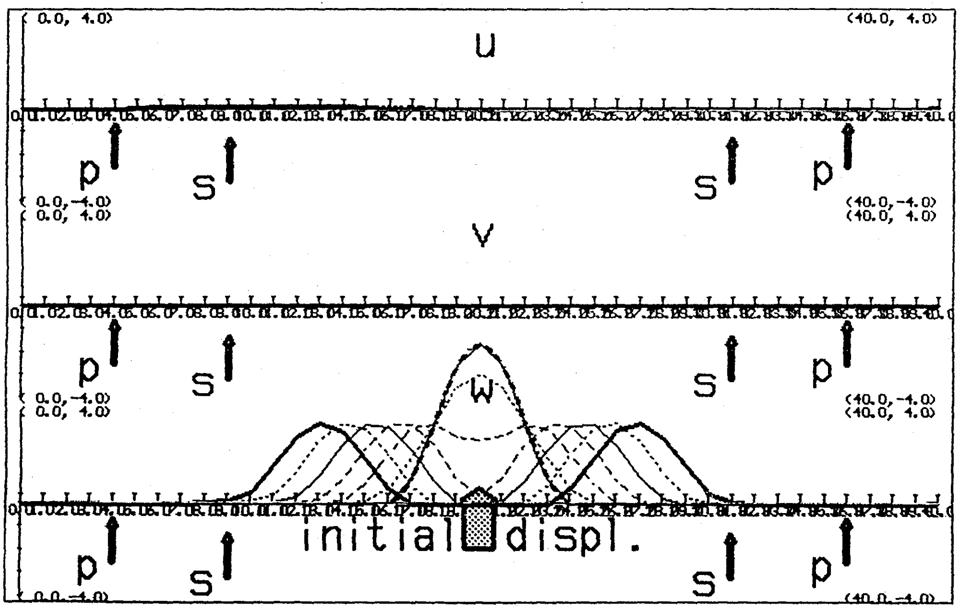


Fig. 1–2. Waves at the time $\tau=12$ produced by local wave origin.
Above: isotropic case. Below: anisotropic case.
Elastic coefficients:
$Lm=Am=Bm=Cm=1.0$ for isotropic ones and $d_{ij}=d_{ijk}=0.5$ for anisotropic ones.
Initial condition:
$Q=A(Q)/2\cdot\{1+\cos(x\cdot\pi/4)\}$ $(-4<x<4)$ with $A(Q)=0.1$ for $Q=U$ and $A(Q)=0$ for $Q=\{u,v,w,V,W\}$.
Scales: $\times 10$ for u and $\times 50$ for v and w.



dtp = 0.100 nndiv (upX=1) = 2
Step_it=20 Real_time_step=2.00000
Last_it=120 Real_time=12.00000

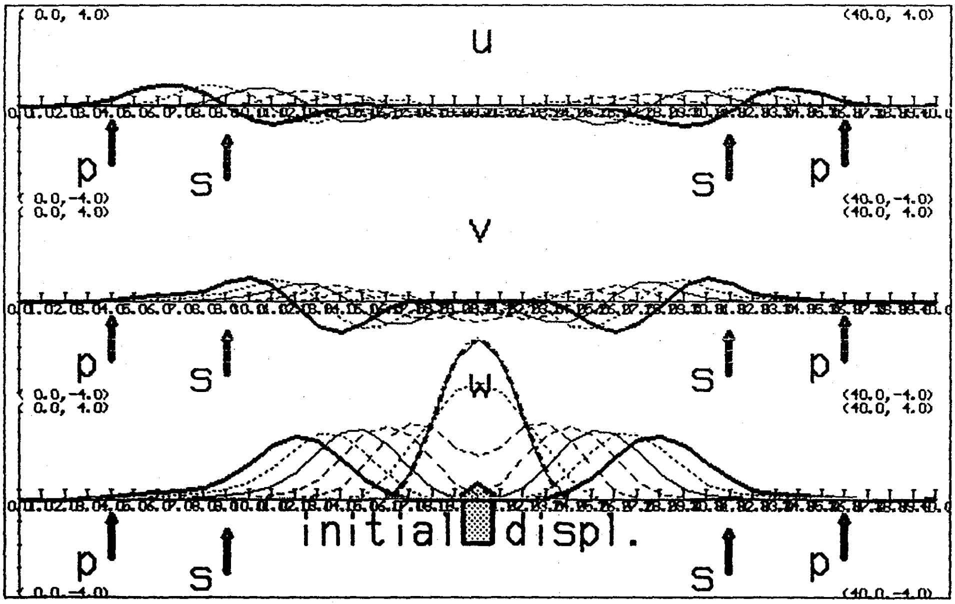
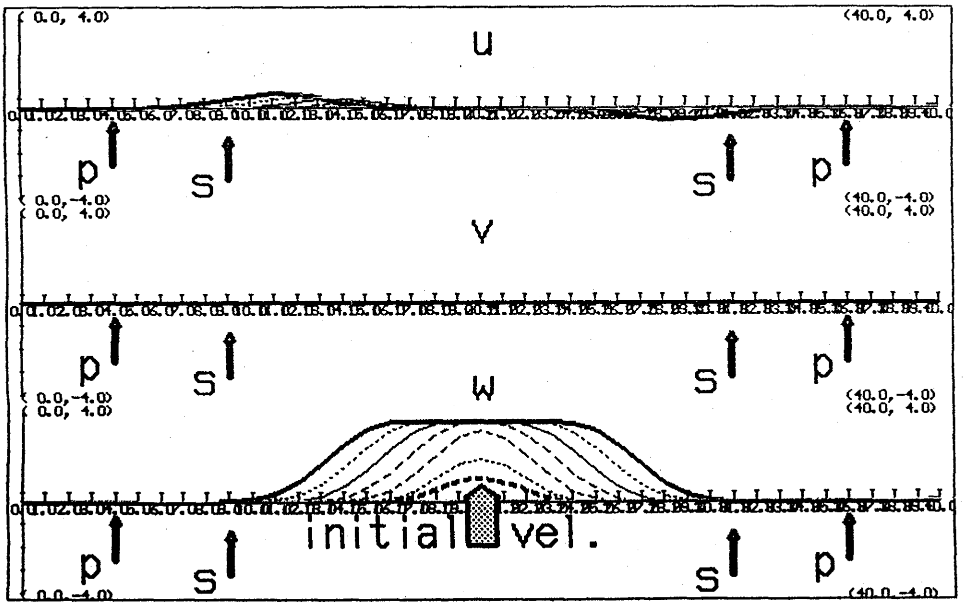


Fig. 2–1. Waves at the time $\tau=12$ produced by local wave origin.
Above: isotropic case. Below: anisotropic case.
Elastic coefficients:
$Lm=Am=Bm=Cm=1.0$ for isotropic ones and $d_{ij}=d_{ijk}=0.5$ for anisotropic ones.
Initial condition:
$Q=A(Q)/2\cdot\{1+\cos(x\cdot\pi/4)\}$ $(-4<x<4)$ with $A(Q)=0.1$ for $Q=w$ and $A(Q)=0$ for $Q=\{u,v,U,V,W\}$.
Scales: $\times 70$ for u, v and w.



dtp = 0.100 nndiv (upX=1) = 2
Step_it=20 Real_time_step=2.00000
Last_it=120 Real_time=12.00000

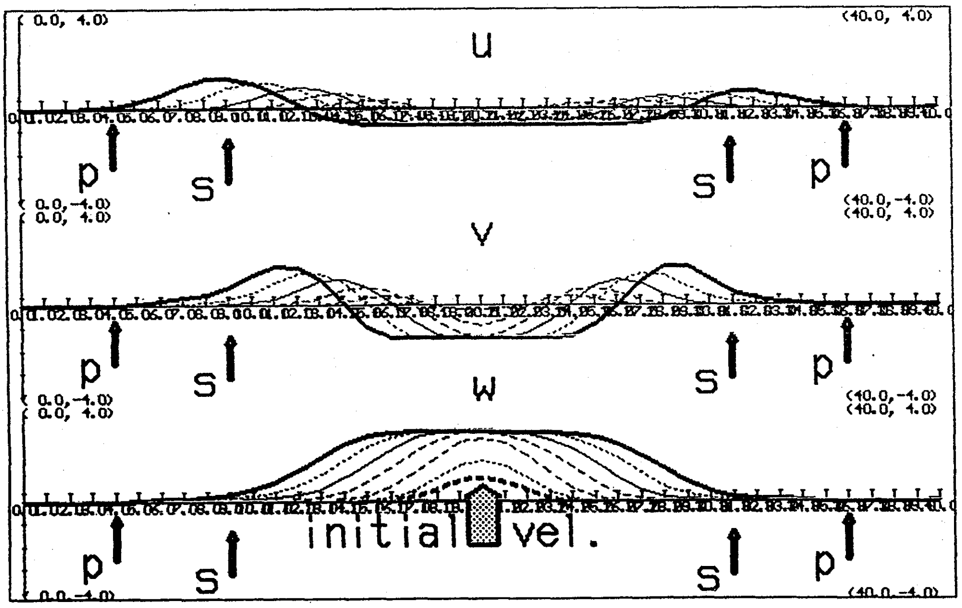


Fig. 2–2. Waves at the time $\tau=12$ produced by local wave origin.
Above: isotropic case. Below: anisotropic case.
Elastic coefficients:
$Lm=Am=Bm=Cm=1.0$ for isotropic ones and $d_{ij}=d_{ijk}=0.5$ for anisotropic ones.
Initial condtion:
$Q=A(Q)/2\cdot\{1+\cos(x\cdot\pi/4)\}\ (-4<x<4)$ with $A(Q)=0.1$ for $Q=W$ and $A(Q)=0$ for $Q=\{u,v,w,U,V\}$.
Scales: $\times 30$ for u, v and $\times 10$ for w.

4. Characteristic Equations for Simple Waves

In order to obtain stationary waves, second order equations (2.3) are transcribed as follows by use of the moving axes

$$t_r = v_r \cdot t , \qquad k_r = v_r \cdot t - x , \tag{4.1}$$

where t_r and k_r are variables with respect to the time and coordinate moving at a velocity v_r toward positive x. After the above reduction, terms with respect to time t_r are put to zero in order to obtain stationary waves; further, the equations are integrated over k_r. The following chracteristic equations, are obtained, which can explain characteristic behaviors of waves in the present nonlinear anisotropic medium, are obtained. Similar characteristic equations were described in (6.4) and (6.5) in *paper M*.

$$-\rho \cdot v_r^2 \cdot u_{kr} + c_{11} \cdot u_{kr} + c_{16} \cdot v_{kr} + c_{15} \cdot w_{kr} - q_{11} \cdot u_{kr} \cdot v_{kr} - q_{12} \cdot u_{kr} \cdot w_{kr}$$
$$- c_{156} \cdot v_{kr} \cdot w_{kr} - q_{15} \cdot u_{kr}^2 - q_{13} \cdot v_{kr}^2 - q_{14} \cdot w_{kr}^2 = 0 , \tag{4.2.1}$$

$$-\rho \cdot v_r^2 \cdot v_{kr} + c_{66} \cdot v_{kr} + c_{16} \cdot u_{kr} - c_{156} \cdot u_{kr} \cdot w_{kr} + c_{56} \cdot w_{kr} - q_{21} \cdot u_{kr} \cdot v_{kr}$$
$$- q_{24} \cdot v_{kr} \cdot w_{kr} - q_{22} \cdot v_{kr}^2 - q_{25} \cdot u_{kr}^2 - q_{23} \cdot w_{kr}^2 = 0 , \tag{4.2.2}$$

$$-\rho \cdot v_r^2 \cdot w_{kr} + c_{55} \cdot w_{kr} + c_{15} \cdot u_{kr} - c_{156} \cdot u_{kr} \cdot v_{kr} + c_{56} \cdot v_{kr} - q_{31} \cdot u_{kr} \cdot w_{kr}$$
$$- q_{32} \cdot v_{kr} \cdot w_{kr} - q_{33} \cdot w_{kr}^2 - q_{34} \cdot u_{kr}^2 - q_{35} \cdot v_{kr}^2 = 0 , \tag{4.2.3}$$

where

$$u_{kr} = \partial u / \partial k_r , \qquad v_{kr} = \partial v / \partial k_r , \qquad w_{kr} = \partial w / \partial k_r ,$$
$$u_{kr2} = \partial^2 u / \partial k_r^2 , \qquad v_{kr2} = \partial^2 v / \partial k_r^2 , \qquad w_{kr2} = \partial^2 w / \partial k_r^2 .$$

$$q_{11} = c_{16} + c_{116} , \qquad q_{12} = c_{15} + c_{115} ,$$
$$q_{13} = c_{11}/2 + c_{166}/2 , \qquad q_{14} = c_{11}/2 + c_{155}/2 ,$$
$$q_{15} = 3/2 \cdot c_{11} + c_{111}/2 , \qquad q_{31} = c_{11} + c_{155} ,$$
$$q_{32} = c_{16} + c_{556} , \qquad q_{33} = 3/2 \cdot c_{15} + c_{555}/2 ,$$
$$q_{34} = c_{15}/2 + c_{115}/2 , \qquad q_{35} = c_{15}/2 + c_{566}/2 ,$$
$$q_{21} = c_{11} + c_{166} , \qquad q_{22} = 3/2 \cdot c_{16} + c_{666}/2 ,$$
$$q_{23} = c_{16}/2 + c_{556}/2 , \qquad q_{24} = c_{15} + c_{566} ,$$
$$q_{25} = c_{16}/2 + c_{116}/2 ,$$

Characteristic equation for simple waves in an isotropic medium

In the case of an isotropic medium, elastic coefficients in an anisotropic medium are reduced to

$$c_{11} = c_{22} = c_{33} = \lambda + 2 \cdot \mu ,$$
$$c_{12} = c_{13} = c_{23} = \lambda ,$$
$$c_{44} = c_{55} = c_{66} = \mu ,$$

$$
\begin{aligned}
&c_{155}=c_{166}=c_{244}=c_{266}=c_{355}=c_{344}=A/2+B\,,\\
&c_{112}=c_{113}=c_{122}=c_{133}=c_{233}=c_{223}=2\cdot B+2\cdot C\,,\\
&c_{111}=c_{222}=c_{333}=2\cdot A+6\cdot B+2\cdot C\,,\\
&c_{456}=A/4\,,\\
&c_{366}=c_{255}=c_{144}=B\,,\\
&c_{123}=2\cdot C\,,
\end{aligned}
\tag{4.3.1}
$$

and the other coefficients are zero.

By use of (4.3.1), the above equations (4.2.1), (4.2.2) and (4.2.3) are simplified as

$$
\begin{aligned}
&(\rho\cdot v_r{}^2-L_{2m})\cdot u_{kr}+Q_1\cdot u_{kr}^2+Q_2\cdot(w_{kr}^2+v_{kr}^2)=0\,,\\
&(\rho\cdot v_r{}^2-\mu)\cdot v_{kr}+Q_3\cdot u_{kr}\cdot v_{kr}=0\,,\\
&(\rho\cdot v_r{}^2-\mu)\cdot w_{kr}+Q_3\cdot u_{kr}\cdot w_{kr}=0\,,
\end{aligned}
\tag{4.3.2}
$$

where

$$
\begin{aligned}
&L_{2m}=(\lambda+2\cdot\mu)\,,\\
&Q_1=A+3\cdot B+C+3/2\cdot L_{2m}\,,\\
&Q_2=A/4+B/2+L_{2m}/2\,,\\
&Q_3=A/2+B+L_{2m}\,.
\end{aligned}
\tag{4.3.3}
$$

By using an alternative derivation, the same characteristic equations for simple waves were obtained in *paper M*.

5. Simple Waves in an Isotropic Medium

Though simple waves in an isotropic medium were already discussed in *paper M*, the expressions for these waves are summarized here, since the notations used here differ from those in *paper M*.

Solving equations (4.3.2), we have two kinds of solutions.

(*i*) *Non-coupled simple wave in isotropic medium*

$$
\begin{aligned}
&u=G_{Unc}\cdot k_r \quad \text{or} \quad u=\text{const}_U\,,\\
&v=w=0\,,
\end{aligned}
\tag{5.1}
$$

with G_{Unc} and L_{mABC} given in the *notations*.

It will readily be found that equations (4.3.2) hold for the direct substitution of (5.1) into (4.3.2).

(*ii*) *Coupled simple wave in isotropic medium*

From the last two equations in (4.3.2), we have

$$
u=G_{uc}\cdot k_r \tag{5.2.1}
$$

with G_{uc} described in the *notations*.

Substitution of (5.2.1) into the first equation in (4.3.2) gives

$$v_{kr}^2 + w_{kr}^2 = N_{vw}/D_{vw} \tag{5.2.2}$$

with

$$N_{vw} = (2\cdot(\lambda+\mu) - (\lambda + 2\cdot\mu + A + 4\cdot B + 2\cdot C)\cdot G_{uc})\cdot G_{uc}/2 .$$

The torsional components v and w have some relative freedom, being related through relation (5.2.2), with notations $v_{kr} = \partial v/\partial k_r$ and $w_{kr} = \partial w/\partial k_r$. The discussions in *paper M* were carried out in the case $v=0$ in order to make clear typical physical characteristics.

Some mention will be made of the velocity of the coupled simple waves, though it was done in *paper M* for the particular case $v_{kr}=0$.

Solving equation (5.2.2) with respect to G_{uc}, we have

$$G_{uc} = L_{am}/F_{guc} \pm g_{uc}^{(1/2)}/F_{guc} , \tag{5.3}$$

with

$$g_{uc} = (-F_{guc}\cdot P_{MD}\cdot S_{vwkr} + 2\cdot L_{am}^2)/2 ,$$
$$S_{vwkr} = v_{kr}^2 + w_{kr}^2 ,$$

where, for $\{G_{uc}, L_{am}, F_{guc}, P_{MD}\}$, the *notations* should be referred to.

In order to interpret the physical meaning of expressions (5.3), weak nonlinearity of the elastic medium is assumed, i.e., $\{A, B, C\}$ are small as compared to $\{\lambda, \mu\}$.

When $\{v_{kr}, w_{kr}\}$ are small, expressions (5.3) are reduced to

$$\rho\cdot v_r{}^2 = -(2\cdot\lambda+\mu) + (A + 6\cdot B + 4\cdot C)\cdot L_{am}/L_{2m} + P_{MD}^2\cdot S_{vwkr}/(8\cdot L_{am}) , \tag{5.3.1}$$

$$\rho\cdot v_r{}^2 = \mu - P_{MD}^2\cdot S_{vwkr}/(8\cdot L_{am}) . \tag{5.3.2}$$

The right-hand side of (5.3.1) is negative, since we have assumed weak nonlinearity; hence, the first term is dominant as compared with other terms, while the left-hand side is always positive. (5.3.1) is, therefore, found to be a meaningless solution.

Solution (5.3.2) indicates that the moving velocity of the coupled simple waves is smaller than that of S waves in the linear theory.

Although this reduction has been carried out under the assumption of *weak* nonlinearity of the medium, many numerical experiments show that the above result is valid even in the range of *strong* nonlinearity.

6. Simple Waves in a Weakly Anisotropic Medium

Let d_{ij} or d_{ijk} be small deviations of elastic coefficients c_{ij} or c_{ijk} from those in an isotropic medium. In the case of a weakly anisotropic medium, the elastic coefficients are then expressed as:

$$
\begin{array}{lll}
c_{11}=L_{2m}+d_{11}\,, & c_{22}=L_{2m}+d_{22}\,, & c_{33}=L_{2m}+d_{33}\,,\\
c_{12}=\lambda+d_{12}\,, & c_{13}=\lambda+d_{13}\,, & c_{23}=\lambda+d_{23}\,,\\
c_{44}=\mu+d_{44}\,, & c_{55}=\mu+d_{55}\,, & c_{66}=\mu+d_{66}\,,\\
c_{155}=A_{2B}+d_{155}\,, & c_{166}=A_{2B}+d_{166}\,, & c_{244}=A_{2B}+d_{244}\,,\\
c_{266}=A_{2B}+d_{266}\,, & c_{355}=A_{2B}+d_{355}\,, & c_{344}=A_{2B}+d_{344}\,,\\
c_{112}=B_{2C}+d_{112}\,, & c_{113}=B_{2C}+d_{113}\,, & c_{122}=B_{2C}+d_{122}\,,\\
c_{133}=B_{2C}+d_{133}\,, & c_{233}=B_{2C}+d_{233}\,, & c_{223}=B_{2C}+d_{223}\,,\\
c_{111}=A_{BC}+d_{111}\,, & c_{222}=A_{BC}+d_{222}\,, & c_{333}=A_{BC}+d_{333}\,,\\
c_{456}=A/4+d_{456}\,, & & \\
c_{366}=B+d_{366}\,, & c_{255}=B+d_{255}\,, & c_{144}=B+d_{144}\,,\\
c_{123}=2\cdot C+d_{123}\,, & &
\end{array}
$$

Except for the above coefficients,

$$\{c_{ij}=d_{ij},\ c_{ijk}=d_{ijk}\} \tag{6.1}$$

where

$$A_{2B}=A/2+B\,,\quad B_{2C}=2\cdot B+2\cdot C\,,\quad A_{BC}=2\cdot A+6\cdot B+2\cdot C\,.$$

Let $\{u_0, v_0, w_0\}$ and $\{du, dv, dw\}$ be displacements in an isotropic medium and deviations from them due to the presence of anisotropy in the medium, respectively, i.e.,

$$u=u_0+du\,,\quad v=v_0+dv\,,\quad w=w_0+dw\,. \tag{6.2}$$

By use of (6.1) and (6.2) and by taking first order in $\{d_{ij}, d_{ijk}, du, dv, dw\}$, characteristic equations (4.2.1), (4.2.2) and (4.2.3) become

$$
\begin{aligned}
&q_a+(L_{2mvr}+L_{mABC}\cdot u_{0kr})\cdot du_{kr}+(v_{0kr}\cdot dv_{kr}+w_{0kr}\cdot dw_{kr})\cdot P_{MD}/2=0\,,\\
&q_b+m_{uvr}\cdot dv_{kr}+(v_{0kr}\cdot du_{kr}+u_{0kr}\cdot dv_{kr})\cdot p_{MD}/2=0\,,\\
&q_c+m_{uvr}\cdot dw_{kr}+(w_{0kr}\cdot du_{kr}+u_{0kr}\cdot dw_{kr})\cdot P_{MD}/2=0\,,
\end{aligned} \tag{6.3}
$$

where

$$
\begin{aligned}
q_a=&(4\cdot(d_{15}\cdot w_{0kr}+d_{16}\cdot v_{0kr})\cdot(u_{0kr}-1)+2\cdot(3\cdot u_{0kr}^2-2\cdot u_{0kr}+v_{0kr}^2\\
&+w_{0kr}^2)\cdot d_{11}+(v_{0kr}^2+w_{0kr}^2)\cdot P_{MD}+2\cdot d_{111}\cdot u_{0kr}^2+4\cdot d_{115}\cdot u_{0kr}\cdot w_{0kr}\\
&+4\cdot d_{116}\cdot u_{0kr}\cdot v_{0kr}+2\cdot d_{155}\cdot w_{0kr}^2+4\cdot d_{156}\cdot v_{0kr}\cdot w_{0kr}+2\cdot d_{166}\cdot v_{0kr}^2\\
&+4\cdot L_{2mvr}\cdot u_{0kr}+2\cdot L_{mABC}\cdot u_{0kr}^2)/4\,,
\end{aligned}
$$

$$
\begin{aligned}
q_b=&((u_{0kr}^2-2\cdot u_{0kr}+3\cdot v_{0kr}^2+w_{0kr}^2)\cdot d_{16}+d_{116}\cdot u_{0kr}^2+2\cdot d_{11}\cdot u_{0kr}\cdot v_{0kr}\\
&+2\cdot d_{156}\cdot u_{0kr}\cdot w_{0kr}+2\cdot d_{15}\cdot v_{0kr}\cdot w_{0kr}+2\cdot d_{166}\cdot u_{0kr}\cdot v_{0kr}\\
&+d_{556}\cdot w_{0kr}^2+2\cdot d_{566}\cdot v_{0kr}\cdot w_{0kr}-2\cdot d_{56}\cdot w_{0kr}+d_{666}\cdot v_{0kr}^2-2\cdot d_{66}\cdot v_{0kr}\\
&+2\cdot m_{uvr}\cdot v_{0kr}+P_{MD}\cdot u_{0kr}\cdot v_{0kr})/2\,,
\end{aligned}
$$

$$q_c = ((u_{0kr}^2 - 2 \cdot u_{0kr} + v_{0kr}^2 + 3 \cdot w_{0kr}^2) \cdot d_{15} + d_{115} \cdot u_{0kr}^2 + 2 \cdot d_{11} \cdot u_{0kr} \cdot w_{0kr}$$
$$+ 2 \cdot d_{155} \cdot u_{0kr} \cdot w_{0kr} + 2 \cdot d_{156} \cdot u_{0kr} \cdot v_{0kr} + 2 \cdot d_{16} \cdot v_{0kr} \cdot w_{0kr} + d_{555} \cdot w_{0kr}^2$$
$$+ 2 \cdot d_{556} \cdot v_{0kr} \cdot w_{0kr} - 2 \cdot d_{55} \cdot w_{0kr} + d_{566} \cdot v_{0kr}^2 - 2 \cdot d_{56} \cdot v_{0kr}$$
$$+ 2 \cdot m_{uvr} \cdot w_{0kr} + P_{MD} \cdot u_{0kr} \cdot w_{0kr})/2 , \tag{6.3.1}$$

$$L_{2mvr} = -L_{2m} + \rho \cdot v_r^2 , \quad m_{uvr} = -m_u + \rho \cdot v_r^2 .$$
$$u_{0kr} = \partial u_0 / \partial k_r , \quad v_{0kr} = \partial v_0 / \partial k_r , \quad w_{0kr} = \partial w_0 / \partial k_r ,$$
$$du_{kr} = \partial(du)/\partial k_r , \quad dv_{kr} = \partial(dv)/\partial k_r , \quad dw_{kr} = \partial(dw)/\partial k_r) .$$

(*i*) *Non-coupled simple wave in an anisotropic medium*

In this section, wave forms are given by

$$u = u_0 + du , \qquad v = dv , \qquad w = dw , \tag{6.4}$$

where u_0 is the non-coupled simple wave in an isotropic medium, which is expressed in (5.1).

As described in (5.1), u_0 has two expressions, i.e., $u_0 = G_{Unc} \cdot k_r$ or $u_0 = \text{const}_U$ ($v_0 = w_0 = 0$ for both cases). The former and latter will be treated in sections (*ia*) and (*ib*), respectively.

(*ia*) *The case $u_0 = G_{Unc} \cdot k_r$ and $v_0 = w_0 = 0$*

Substitution of (6.4) into (6.3) with u_0 described above yields

$$du_{kr} = G_{Unc} \cdot (2 \cdot (-L_{2mvr} + d_{11}) - G_{Unc} \cdot (L_{mABC} + d_{111} + 3 \cdot d_{11}))/D_u ,$$
$$dv_{kr} = G_{Unc} \cdot (2 \cdot d_{16} - G_{Unc} \cdot (d_{16} + d_{116}))/D_{vw} ,$$
$$dw_{kr} = G_{Unc} \cdot (2 \cdot d_{15} - G_{Unc} \cdot (d_{15} + d_{115}))/D_{vw} , \tag{6.5.1}$$

with

$$D_u = 2 \cdot L_{2mvr} + 2 \cdot G_{Guc} \cdot L_{mABC} ,$$
$$D_{vw} = 2 \cdot m_{uvr} + G_{Unc} \cdot P_{MD} , \tag{6.5.2}$$

where, for L_{mABC}, P_{MD} and G_{Unc}, the *notations* should be referred to.

In the above expressions (6.5.1), when $D_u = D_{vw} = 0$, du_{kr}, dv_{kr} and dw_{kr} become infinitely large, that is, we have a resonance condition. This result contradicts the assumption that du_{kr}, dv_{kr} and dw_{kr} are of order d_{ij} or d_{ijk}. This contradiction comes from the linearization of equations in terms of $\{d_{ij}, d_{ijk}, du, dv, dw\}$. If we take higher order terms into account, this difficulty will disappear. At any rate, when $D_u = D_{vw} = 0$, the magnitude of du_{kr}, dv_{kr} and dw_{kr} become large.

Evaluating $D_u = D_{vw} = 0$, we have

$$v_r^2 = v_p^2 - G_{Unc} \cdot L_{mABC}/\rho \qquad \text{(from } du_{kr} \text{ expression)} ,$$
$$v_r^2 = v_s^2 - G_{Unc} \cdot P_{MD}/(2 \cdot \rho) \qquad \text{(from } dv_{kr} \text{ and } dw_{kr} \text{ expressions)} . \tag{6.5.3}$$

The above results indicate that near the velocities of P and S waves in linear theory, the polarization of waves due to anisotropy of the medium occurs for the components u and $\{v, w\}$, respectively.

Since G_{Unc} is the gradient of the non-coupled simple waves $u_0 = G_{Unc} \cdot k_r$, we have the following results.

a) If the incident simple wave is 'Push' ($G_{Unc} > 0$), the polarizations occur at velocities lower than v_p and v_s, and

b) If the incident simple wave is 'Pull' ($G_{Unc} < 0$), the polarizations occur at the velocities larger than v_p and v_s, where 'Push' and 'Pull' imply forward and backward dsplacements in the advancing direction of the simple waves, and $\{v_p, v_s\}$ are the velocities of P and S waves in linear theory.

In the previous *paper M*, we obtained similar conclusions for simple waves in an isotropic medium. That is to say, when $v_r < v_p$ or $v_r > v_p$, 'Push' or 'Pull' non-coupled simple waves occur.

Numerical example for this case are given in Fig. 1–1 and Fig. 1–2. It should be noted that the values of anisotropic elastic coefficients used here are of an order exceeding the range of a *weakly* anisotropic medium, i.e., d_{ij} or $d_{ijk} \sim 0.5$. In these figures, the time and abscissa used are the normalized time and x-component, i.e., $\tau = h \cdot v_p \cdot t$ and $\chi = h \cdot x$, respectively. The inserted characters '*p*' and '*s*' indicate the arrival points of P and S waves in linear theory. These conventions will be followed in the figures appearing later. The outstanding chracteristics exposed in these figures are that the produced polarized waves are also typical simple waves.

(*ib*) *The case* $u_0 = \mathrm{const}_U$ *and* $v_0 = w_0 = 0$

Substitution of (6.4) into (6.3) with u_0 describd above yields

$$v_r^2 \cdot du_{kr} - v_p^2 \cdot du_{kr} = 0 \,,$$
$$v_r^2 \cdot dv_{kr} - v_s^2 \cdot dv_{kr} = 0 \,,$$
$$v_r^2 \cdot dw_{kr} - v_s^2 \cdot dw_{kr} = 0 \,. \tag{6.6}$$

When $v_r^2 \neq v_p^2$, the first equation gives the solution

$$u = \text{constant} \,. \tag{6.6.1}$$

From the last two equations in (6.6), we can take the solution:

$$\text{any values for } dv_{kr} \text{ and } dw_{kr} \text{ when } v_r^2 = v_s^2 \,. \tag{6.6.2}$$

Expressions $u_0 = \mathrm{const}_U$ and $u = \text{constant}$ indicate the horizontal top or bottom of the u simple waves. The solutions (6.6.2) show the existence of the v and w components accompanying the above u simple waves, which are propagated at the velocity of S waves in linear theory. An example for this case is shown in Fig. 1–2. These figures shows that the waves produced by the anisotropy are of a type of simple waves.

(*ii*) *Coupled simple waves in an anisotropic medium*

In this section, the wave forms are given by

$$u = u_0 + du \,, \quad v = v_0 + dv \,, \quad w = w_0 + dw \,, \tag{6.7}$$

where $\{u_0, v_0, w_0\}$ are coupled simple waves in an isotropic medium, which are represented by (5.2.1) and (5.2.2), with the substitution $\{u=u_0, v=v_0, w=w_0\}$. $\{du, dv, dw\}$ are small deviations of waves due to anisotropy from those in an isotropic medium. Though the same notation u_0 has been used in the foregoing section (*i*) in the case of *non-coupled* simple waves, the valid range of the definition of the notation is restricted in that section. The use of the notations $\{u_0, v_0, w_0\}$ as defined here is restricted to this section (*ii*).

In order to clarify the physical interpretation, we will assume $v_0=0$.

By using (6.7), equations (6.3) yield the following solutions:
For the v component

$$dv_{kr}=(2\cdot(u_{0kr}\cdot(d_{16}-w_{0kr}\cdot d_{156})+w_{0kr}\cdot d_{56}) \\ -(u_{0kr}^2\cdot(d_{16}+d_{116})+w_{0kr}^2\cdot(d_{16}+d_{556})))/D_{vv}\,, \tag{6.8.1}$$

with

$$D_{vv}=2\cdot m_{uvr}+u_{0kr}\cdot P_{MD}\,, \tag{6.8.2}$$

and for the u and w components

$$du_{kr}=2\cdot(P_{MD}\cdot w_{0kr}\cdot q_c-q_a\cdot(2\cdot m_{uvr}+u_{0kr}\cdot P_{MD}))/D_{uw}\,, \\ dw_{kr}=2\cdot(P_{MD}\cdot w_{0kr}\cdot q_a-2\cdot q_c\cdot(L_{2mvr}+L_{mABC}\cdot u_{0kr}))/D_{uw}\,, \tag{6.9.1}$$

with

$$D_{uw}=2\cdot(L_{2mvr}+L_{mABC}\cdot u_{0kr})\cdot(2\cdot m_{uvr}+P_{MD}\cdot u_{0kr})-P_{MD}^2\cdot w_{0kr}^2\,, \tag{6.9.2}$$

and

$$q_a=L_{2mvr}\cdot u_{0kr}+L_{mABC}\cdot u_{0kr}^2/2+u_{0kr}\cdot w_{0kr}\cdot d_{15} \\ +u_{0kr}\cdot w_{0kr}\cdot d_{115}-u_{0kr}\cdot d_{11}-w_{0kr}\cdot d_{15}+3/2\cdot u_{0kr}^2\cdot d_{11} \\ +u_{0kr}^2\cdot d_{111}/2+w_{0kr}^2\cdot P_{MD}/4+w_{0kr}^2\cdot d_{11}/2+w_{0kr}^2\cdot d_{155}/2\,,$$

$$q_c=u_{0kr}\cdot w_{0kr}\cdot P_{MD}/2+u_{0kr}\cdot w_{0kr}\cdot d_{11}+u_{0kr}\cdot w_{0kr}\cdot d_{155} \\ -u_{0kr}\cdot d_{15}+w_{0kr}\cdot m_{uvr}-w_{0kr}\cdot d_{55}+u_{0kr}^2\cdot d_{15}/2 \\ +u_{0kr}^2\cdot d_{115}/2+3/2\cdot w_{0kr}^2\cdot d_{15}+w_{0kr}^2\cdot d_{555}/2\,,$$

In order to obtain the velocities in a resonant state, the denominators in (6.8.1) and (6.9.1) are put equal to zero, i.e., $D_{vv}=0$ and $D_{uw}=0$. Solving these equations, we have for the v component:

$$v_r{}^2=v_s{}^2-u_{0kr}\cdot P_{MD}/(2\rho)\,, \tag{6.10.1}$$

and for the u and w components:

$$v_r{}^2=v_p{}^2-F_{r0}\cdot u_{0kr}/\rho+(\lambda+\mu)\cdot u_{0kr}\cdot w_{0kr}/(2\rho)+(u_{0kr}^2+w_{0kr}^2)/(8\rho)\,, \\ v_r{}^2=v_s{}^2-F_{r0}\cdot u_{0kr}/\rho-(\lambda+\mu)\cdot u_{0kr}\cdot w_{0kr}/(2\rho)-(u_{0kr}^2+w_{0kr}^2)/(8\rho)\,, \tag{6.10.2}$$

with

$$F_{r0}=(5/4\cdot A+7/2\cdot B+C+2\cdot\lambda+4\cdot\mu)\,.$$

From expressions (6.10.1) and (6.10.2), it is found that a maximum amount of polarization of waves appear near the velocities of P and S waves in linear theory. Examples are given in Fig. 2–1 and Fig. 2–2. As found from these figures, the waves produced by the presence of anisotropy have the properties of simple waves.

7. Conclusion

In a manner similar to phenomena in anisotropic *linear*-elastic media, the polarization of waves takes place even in anisotropic *nonlinear*-elastic media. Polarized waves in nonlinear-elastic media are also *soliton*-like or *step*-shaped waves named *simple waves.*

In the case of the isotropic nonlinear-elastic media, the simple waves are separated into two categories, *non-coupled* and *coupled* simple waves. The former are dilatational waves, while the latter are the coupled waves with the dilatational (u-component) and distorsional (v- and w-components) properties, where u and $\{v, w\}$ are longitudinal and transverse components, respectively.

In the case of the anisotropic nonlinear-elastic media, the two kinds of the simple waves mentioned above also appear with some modification due to the presence of the anisotropy. This modification is similar to the polarization of waves which is prevailing in the linear anisotropic media.

In the case of the anisotropic media, the non-coupled simple waves which have approximately the velocity of P waves in the linear theory are accompanied by the transverse components which are propagated approximately at the velocity of S waves. It must be noted here that the accompanying transverse components also behave as simple waves in a nonlinear elastic medium instead of only '*noise*' waves.

In the case of the weakly anisotropic media, if the incident non-coupled simple waves are 'Push' or 'Pull', the polarizations occur at the velocities lower or higher than those of P and S waves in the linear theory, respectively. This conclusion coincides with that for the non-coupled simple waves in an isotropic medium which advance as 'Push' or 'Pull' waves when the velocities are smaller or larger than that of P waves.

The coupled simple waves in an anisotropic media also cause the polarized components of the simple waves. These polarized waves also the property of the simple waves and are propagated nearly at the velocities of P and S waves in the linear theory.

References

Brugger, K., 1964, Thermodynamic Definition of Higher Order Elastic Coefficients, *Phys. Rev.*, **133**, 6A, A1611–A1612.

Bamford, D. and S. Crampin, 1977, Seismic anisotropy—the state of the art, *Geophys. J. R. astr. Soc.*, **49**, 1–8.

Crampin, S., E. M. Chesnokov and R. G. Hipkin, 1984, Seismic anisotropy—the state of the art: II, *Geophys. J. R. astr. Soc.*, **76**, 1–16.

KANESHIMA, S., 1991, Shear-wave Splitting Induced by Siesmic Anisotropy in the Earth, *Zisin* 2, **44**, 71–83.

LANDAU, L. D. and E. M. LIFSHITZ, 1985, (佐藤常三訳), 弾性理論, 東京図書, 149−150.

LJAMOV, V. E. 1983, Polarization effects and anisotropy of interaction of acoustic waves in crystals, MGU, Moscow.

MOMOI, T., 1990, Wave Propagation in Nonlinear-elastic Isotropic Media, *Bull. Earthq. Res. Inst.*, **65**, 413–432. (*this article is referred to as paper M*)

RUDENKO, O. V. and S. I. SOLUYAN, 1977, Theoretical foundations of nonlinear acoustics, Consultants Bureau, New York and London.

TSVANKIN, I. D. and E. M. CHESNOKOV, 1987, Plane wave propagation in nonlinear-elastic anisotropic media, *Geophys. J. R. astr. Soc.*, **91**, 413–427.

非線形異方性媒質における波の偏光

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線形の異方性媒質におけると同様に非線形異方性媒質においてもまた波の偏光がみられる.

非線形等方性媒質において単純波 (**Simple waves**) は二つの部類に分けられる, すなわち非結合単純波 (**Noncoupled simple waves**) および結合単純波 (**Coupled simple waves**) である. 前者は伸縮波 (u 成分) で後者はねじれ波 (v, w 成分) である.

非線形異方性媒質においてもまた上述の二つの単純波は現れる. しかしこれらの単純波に加えて媒質の異方性に基ずく二次的な波が現れる. この二次的な波によって等方性の媒質に基ずく基本的な単純波は, いわゆる偏光 (**Polarization**) 現象を起こす. この偏光現象は線形波のそれと類似しているが, 唯一の相違点は二次的な偏光波自身が単なる乱れの波ではなくまた非線形媒質における単純波そのものであるということです.

非線形異方性媒質において近似的に P 波の速度で伝播する非結合単純波は近似的に線形媒質における P 波 S 波の速度で伝わる二次的な偏光単純波を発生させる.

また特に弱い非線形媒質において入射非結合波が起こす偏光波は押し波のとき P 波 S 波の速度より遅い速度で, 引き波のとき速い速度で進行する.

また弱い非線形媒質において近似的に S 波の速度で伝播する結合波は同様に P 波および S 波の速度で伝わる単純波を発生させる.