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Wave Propagation in Nonlinear-elastic Isotropic Media

—Two-Dimensional Case—

Takao Momoi
Earthquake Research Institute, University of Tokyo

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Abstract

Wave propagation in the nonlinear-elastic isotropic media was analyzed in a two-dimensional case. In the analysis, governing equations take into account both the physical nonlinearity caused by the stress-strain relation and the geometric nonlinearity resulting from the quadratic strain-displacement relation. The equations obtained are numerically evaluated by use of the extended finite difference method expanded in Taylor series.

The wave sources have a form of mountain ridge with a width $-2<hx<2$, where hx is a distance x normalized by wave number h of P waves in the linear theory. The waves generated are then *aperiodic*. Only soliton-like or step-shaped *simple waves* (after gas-dynamics) are found numerically. Existence of these waves are also confirmed analytically by use of the second order theory. Unlike the linear theory, the velocity of the simple waves in the nonlinear theory is not exactly the same as that of P or S waves in the linear theory relying on the elastic media. Advancing speed of the waves depends on the gradient of the front simple wave.

In simple waves with large amplitude, the u component (in the direction of the propagation) is more remarkably dispersed than the transverse component. This phenomenon is likely to be observed at a great distance as the P wave dispersion instead of simple wave dispersion.

Introduction

Most of the previous theoretical investigations into seismic waves used linear-elastic models. This approach was natural, because the general adopted view had been that the seismic ground strains were too weak to excite any nonlinear phenomenon. Meanwhile, the nonlinearity of the waves seems to be still significant in big earthquakes, particularly near the seismic source.

The nonlinear theory of wave propagation has been developed fully in different fields of physics, such as optics, fluid and gas dynamics. Impressive results, both theoretical and experimental, have been obtained in nonlinear acoustics (RUDENKO and SOLUYAN, 1977; LJAMOV, 1983). In the last few years, experimental investigations by means of powerful vibrators has revealed some unusual seismic effects, such as the generation of combinations of frequencies, which cannot be explained in terms of linear theory (NIKOLAEV *et al.*, 1983).

Recent theoretical studies concerning nonlinear-elastic media have used perturbation method (WHITAM, 1974; LJAMOV, 1983; TSVANKIN and CHESNOKOV, 1987). Assuming the nonlinear waves to be relatively small, they represented the displacement as a sum: the main term having the transient plane wave satisfying the linear equations of motion and the secondary term superimposing the main term. The perturbation method is only useful in the evaluation of the nonlinear *effect* on the linear periodic waves. In the study we report in this paper, it will be found that the waves in the nonlinear-elastic media are *never* periodic, as the *Kdv* nonlinear equation possesses steady progressing wave solution in the form of solitary waves or solitons (SCOTT, 1973).

In this paper, we developed the theory of wave propagation in nonlinear-elastic isotropic two-dimensional media without assuming the periodicity. The development of the theory will be made by use of *computer algebra* installed on NEC 9800 computer (NEC COMPANY, Japan).

1. Notation

The following notations will be used in this paper.

- t : time,
- ρ : density of the medium,
- x, z : Cartesian coordinates, x being horizontal and z vertically downwards,
- xk : alternative notation of Cartesian coordinates with k (as suffix) running 1 to 2, i.e., x1=x and x2=z,
- u, w : displacement components in the direction of the coordinate x, z,
- ui : alternative notation of displacements with i (as suffix) running 1 to 2, i.e., u1=u and u2=w,
- La, mu: λ, μ in Greek,
- vp : velocity of P waves, $((\lambda+2\mu)/\rho^{1/2}$, in the linear theory,
- vs : velocity of S waves, $(\mu/\rho)^{1/2}$, in the linear theory,

h : wave number of P wave in the linear theory,
La, mu, A, B, C, D, E, F, G: elastic coefficient,
Lm, Am, Bm, Cm, Dm, Em, Fm, Gm: elastic coefficient normalized by mu, i.e., La/mu, A/mu, B/mu, C/mu, D/mu, E/mu, F/mu and G/mu,
Uik : strain tensor (i, k: suffices), i.e.,
Uik = (uik + uki + uji*ujk)/2
with uik = ∂(ui)/∂(xk),
summation over repeated index j being implied,
I1, I2, I3: strain tensor invariants, i.e.,
I1 = Uii, I2 = Uik^2, I3 = Uij*Ujk*Uki,
suffices i, j, k following the summation convention,

2. Expression of Energy

In isotropic media, it is known that the strain energy function *En* is expressed by use of strain tensor invariants I1, I2, I3, in a series of *generalized* Taylor expansion (BLAND, 1969; LANDAU-LIFSHITZ, 1965). In this paper, the expansion will be taken up to the fourth order of strain tensor and the phenomenon will be limited to a two-dimensional case, i.e.,

$$En = En1 + En2\,, \tag{2.1}$$

$$En1 = e1 + e2 + e3 + e4 + e5\,, \tag{2.2}$$

$$En2 = e6 + e7 + e8 + e9\,, \tag{2.3}$$

$$e1 = \mathrm{I2\ mu}\,, \quad e2 = \mathrm{I1^2\ La/2}\,, \quad e3 = \mathrm{A\ I3/3}\,,$$
$$e4 = \mathrm{B\ I1\ I2}\,, \quad e5 = \mathrm{C\ I1^3/3}\,, \tag{2.4}$$

$$e6 = \mathrm{D\ I1\ I3}\,, \quad e7 = \mathrm{E\ I2^2}\,, \quad e8 = \mathrm{F\ I1^2\ I2}\,, \quad e9 = \mathrm{I1^4\ G}\,. \tag{2.5}$$

In equation (2.1), the first *En*1 and second *En*2 indicate the terms up to third and fourth order of strain tensor, respectively.

In case the displacements u1 and u2 are only x1-dependent, the above energy is expressed, after substitution of I1, I2 and I3, by

$$En = En1 + En2\,, \tag{2.6}$$

$$En1 = \mathrm{u11^2\ Lamu + u21^2\ mu/2 + u11^3\ F1 + u11\ u21^2\ F2}\,, \tag{2.7}$$

$$En2 = \mathrm{u11^4\ F3 + u21^4\ F4 + u11^2\ u21^2\ F5}\,, \tag{2.8}$$

where

$$\mathrm{Lamu = mu + La/2}\,,$$
$$\mathrm{F1 = A/3 + B + C/3 + Lamu}\,,$$
$$\mathrm{F2 = A/4 + B/2 + Lamu}\,,$$

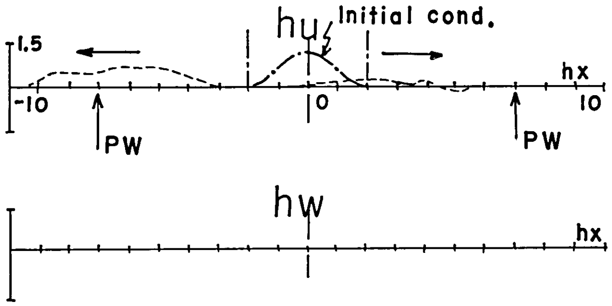

Fig. 1-1

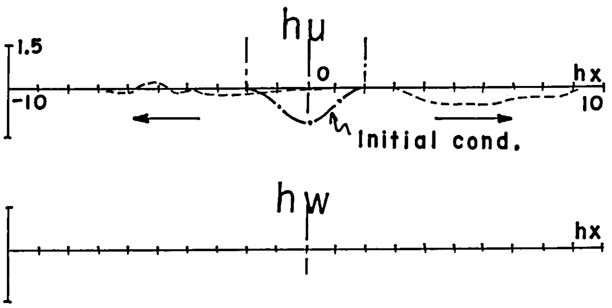

Fig. 1-2

Figs. 1-1, 1-2. Nonlinear waves at tp=5.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0. Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} (−2<hx<2) with A(hu)=0.6 (Fig. 1-1), −0.6 (Fig. 1-2), A(hw)=A(U)=A(W)=0. Vertical scale: ×2. Horizintal scale: ×1. 'PW' in Fig. 1-1 indicates the supposed arrival point of P waves in the linear theory.

$$F3=A/2+3/2\,B+C/2+E+F+Lamu/4+D+G\,,$$
$$F4=A/8+B/4+E/4+mu/4+La/8\,,$$
$$F5=5/8\,A+7/4\,B+C/2+E+F/2+Lamu/2+3/4\,D\,,$$

3. Equation

Stress tensor Sij (i, j: suffices) is related to energy function (Seeger and Buck, 1860)

$$Sij=\partial En/\partial uij\,. \qquad (3.1)$$

By use of the above relation, governing equations for nonlinear isotropic media can be expressed (Landau and Lifshitz, 1985) by

$$\rho\ \partial^2 u/\partial t^2=\partial S11/\partial x\,,$$
$$\rho\ \partial^2 w/\partial t^2=\partial S21/\partial x\,.$$

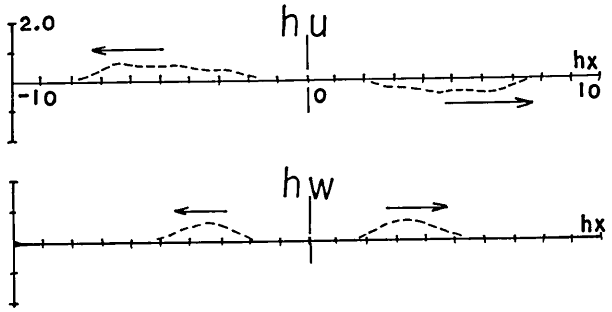

Fig. 2-1

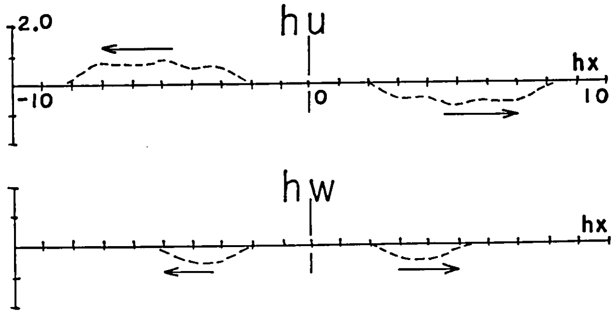

Fig. 2-2

Figs. 2-1, 2-2. Nonlinear waves at tp=6.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0. Initial condition: Q=A(Q)/2*{1+cos(hx*π/2)} (−2<hx<2) with A(hw)=0.6 (Fig. 2-1), −0.6 (Fig. 2-2), A(hu)=A(U)=A(W)=0. Vertical scale: ×10 for hu and ×2 for hw. Horizintal scale: ×1 for both.

By use of (3.1) and the expressions of energy in the foregoing section, the above equations are reduced to the following.

$$(\partial^2 u/\partial t^2)/vs^2 = eq11 + eq12\,, \tag{3.2}$$

$$(\partial^2 w/\partial t^2)/vs^2 = eq21 + eq22\,, \tag{3.3}$$

with

$$eq11 = Lam\ \partial^2 u/\partial x^2 + 1/2\ \partial\,(G1(\partial w/\partial x)^2 + G2\,(\partial u/\partial x)^2)/\partial x\,, \tag{3.4}$$

$$eq21 = \partial^2 w/\partial x^2 + G1\ \partial((\partial u/\partial x)(\partial w/\partial x))/\partial x\,, \tag{3.5}$$

$$eq12 = 12\ G4\ (\partial u/\partial x)^2(\partial^2 u/\partial x^2) + 2\ G5\ \partial((\partial u/\partial x)(\partial w/\partial x)^2)/\partial x\,, \tag{3.6}$$

$$eq22 = 12\ G3\ (\partial w/\partial x)^2(\partial^2 w/\partial x^2) + 2\ G5\ \partial((\partial u/\partial x)^2(\partial w/\partial x))/\partial x\,, \tag{3.7}$$

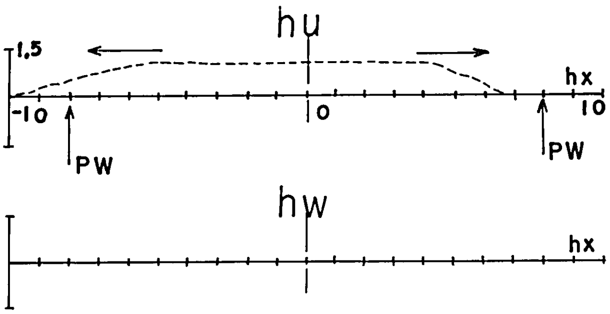

Fig. 3-1

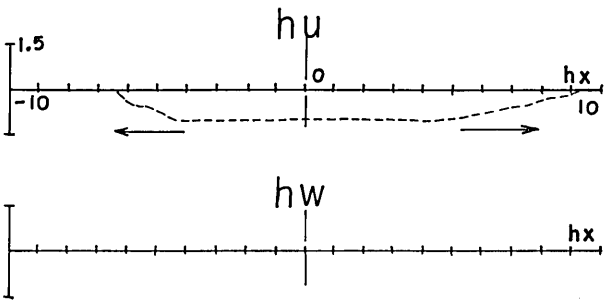

Fig. 3-2

Figs. 3-1, 3-2. Nonlinear waves at tp=6.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0. Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} ($-2<$hx<2) with A(U)=0.6 (Fig. 3-1), -0.6 (Fig. 3-2), A(hu)=A(hw)=A(W)=0. Vertical scale: $\times 2$. Horizintal scale: $\times 1$. 'PW' in Fig. 3-1 indicates the supposed arrival point of P waves in the linear theory.

where vs is the velocity of S waves in the linear theory, the terms on the right-hand sides of (3.2) and (3.3) are to be taken up to

(i) eq11 and eq21 to the second order of the derivative of (u, w).

(ii) eq11+eq12 and eq21+eq22 to the third order of the derivative of (u, w),

constants Lam, G1, G2, G3, G4, G5 are expressed as the sum of elastic coefficient normalized by mu (see Section 1. Notation),

Lam=Lm+2 ,

G1=Lam+Am/2+Bm ,

G2=3 Lam+2 Am+6 Bm+2 Cm ,

G3=Am/8+Bm/4+Em/4+Lam/8 ,

G4=Am/2+3/2 Bm+Cm/2+Em+Fm+Lam/8+Dm+Gm ,

G5=5/8 Am+7/4 Bm+Cm/2+Em+Fm/2+Lam/4+3/4 Dm ,

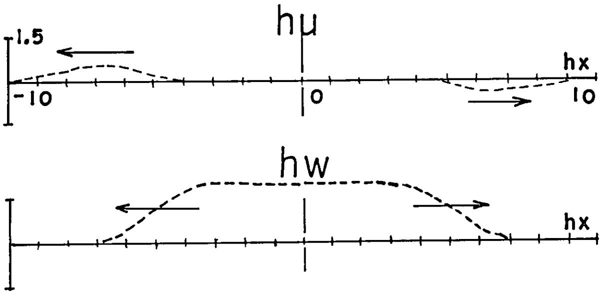

Fig. 4-1

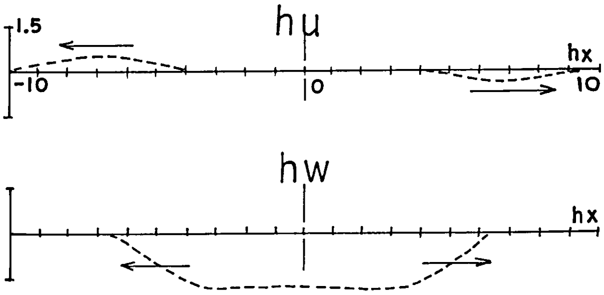

Fig. 4-2

Figs. 4-1, 4-2. Nonlinear waves at tp=8.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0. Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} (−2<hx<2) with A(W)=0.6 (Fig. 4.1), −0.6 (Fig. 4-2), A(hu)=A(hw)=A(U)=0. Vertical scale: ×2. Horizintal scale: ×1.

4. Finite Difference Equation by Taylor Method

In this section, finite difference scheme will be introduced by use of the theory to the third order of the derivative of the displacements u and w.

By use of wave number h of P wave in the linear theory as normalization factor, equations (3.2) and (3.3) are reduced to

$$\partial \mathrm{U}/\partial \mathrm{tp} = \mathrm{Ru}/\mathrm{Lam}\,, \tag{4.1}$$

$$\mathrm{U} = \partial \mathrm{hu}/\partial \mathrm{tp}\,, \tag{4.2}$$

$$\partial \mathrm{W}/\partial \mathrm{tp} = \mathrm{Rw}/\mathrm{Lam}\,, \tag{4.3}$$

$$\mathrm{W} = \partial \mathrm{hw}/\partial \mathrm{tp}\,, \tag{4.4}$$

$$\begin{aligned} \mathrm{Ru} = {} & \mathrm{Lam\ ux2} + \mathrm{G2\ ux2\ ux} + \mathrm{G1\ wx\ wx2} \\ & + 2\,\mathrm{G5\ ux2\ wx}^2 + 12\,\mathrm{G4\ ux2\ ux}^2 + 4\,\mathrm{G5\ wx\ wx2\ ux}\,, \end{aligned} \tag{4.5}$$

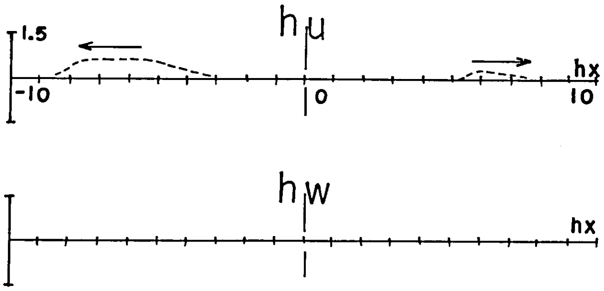

Fig. 5-1

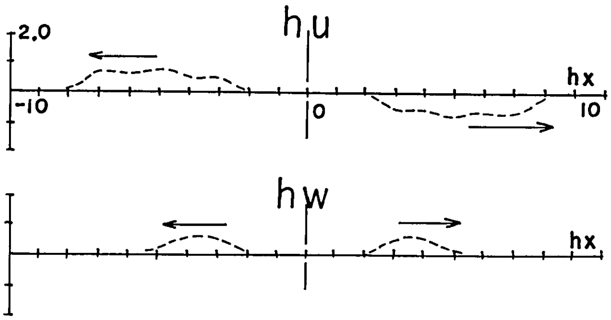

Fig. 5-2

Figs. 5-1, 5-2. Nonlinear waves at tp=6.0 generated by local wave origin. Elastic coefficient: Lm=1.0, Am=Bm=Cm=Dm=Em=Fm=Gm=0, Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} (−2<hx<2) with {A(hu)=0.6 A(hw)=A(U)=A(W)=0} (Fig. 5-1) and {A(hw)=0.6, A(hu)=A(U)=A(W)=0} (Fig. 5-2). Vertical scale: ×2 (Fig. 5-1) and ×10 (Fig. 5-2) for hu, and ×2 (both) for hw. Horizintal scale: ×1.

$$\begin{aligned}\mathrm{Rw} = {} & \mathrm{wx2} + \mathrm{G1\ ux2\ wx} + \mathrm{G1\ wx2\ ux} \\ & + 2\ \mathrm{G5\ wx2\ ux}^2 + 12\ \mathrm{G3\ wx}^2\ \mathrm{wx2} + 4\ \mathrm{G5\ ux2\ wx\ ux}\,, \end{aligned} \tag{4.6}$$

where

$$\mathrm{hu} = h\ u\,, \quad \mathrm{hw} = h\ w\,, \quad \mathrm{tp} = h\ vp\ t\,, \quad \mathrm{hx} = h\ x\,,$$
$$\mathrm{ux} = \partial \mathrm{hu}/\partial \mathrm{hx}\,, \quad \mathrm{ux2} = \partial^2 \mathrm{hu}/\partial \mathrm{hx}^2\,,$$
$$\mathrm{wx} = \partial \mathrm{hw}/\partial \mathrm{hx}\,, \quad \mathrm{wx2} = \partial^2 \mathrm{hw}/\partial \mathrm{hx}^2\,.$$

In order to evaluate the displacements (hu, hw) and velocities (U, W) at a time tp+dtp (dtp: increment of time tp), we will use Taylor expansion in terms of tp to the second order of dtp such that

$$\mathrm{hu} = \mathrm{hu0} + \mathrm{dhu}[1]\ \mathrm{dtp} + \mathrm{dhu}[2]\ (\mathrm{dtp})^2/2\,, \tag{4.7}$$

$$\mathrm{hw} = \mathrm{hw0} + \mathrm{dhw}[1]\ \mathrm{dtp} + \mathrm{dhw}[2]\ (\mathrm{dtp})^2/2\,, \tag{4.8}$$

$$\mathrm{U} = \mathrm{U0} + \mathrm{dU}[1]\ \mathrm{dtp} + \mathrm{dU}[2]\ (\mathrm{dtp})^2/2\,, \tag{4.9}$$

$$\mathrm{W} = \mathrm{W0} + \mathrm{dW}[1]\ \mathrm{dtp} + \mathrm{dW}[2]\ (\mathrm{dtp})^2/2\,, \tag{4.10}$$

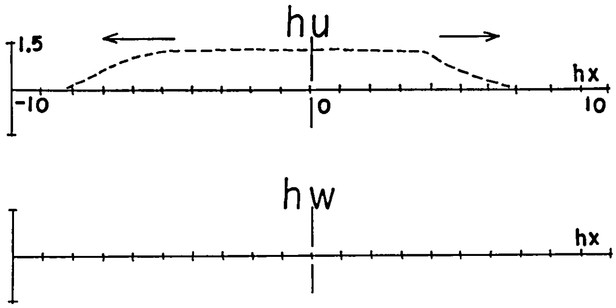

Fig. 5-3

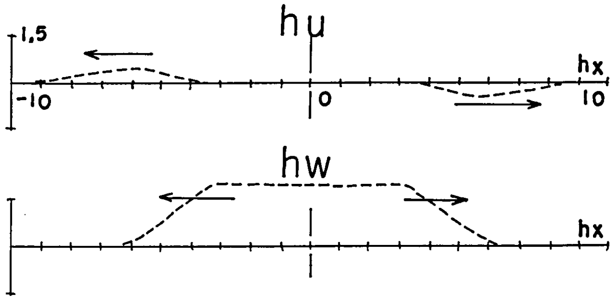

Fig. 5-4

Figs. 5-3, 5-4. Nonlinear waves at tp=8.0 generated by local wave origin. Elastic coefficient: Lm=1.0, Am=Bm=Cm=Dm=Em=Fm=Gm=0. Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} ($-2<$hx<2) with {A(U)=0.6, A(hu)=A(hw)=A(W)=0} (Fig. 5-3) and {A(W)=0.6, A(hu)=A(hw)=A(U)=0} (Fig. 5-4). Vertical scale: $\times 2$. Horizintal scale: $\times 1$.

with

hu0 and hw0 are hu and hw at the time tp,

$$\text{dhu}[1]=\partial\text{hu}/\partial\text{tp}\,, \qquad \text{dhu}[2]=\partial^2\text{hu}/\partial\text{tp}^2\,, \tag{4.11}$$

$$\text{dhw}[1]=\partial\text{hw}/\partial\text{tp}\,, \qquad \text{dhw}[2]=\partial^2\text{hw}/\partial\text{tp}^2\,, \tag{4.12}$$

$$\text{dU}[1]=\partial\text{U}/\partial\text{tp}\,, \qquad \text{dU}[2]=\partial^2\text{U}/\partial\text{tp}^2\,, \tag{4.13}$$

$$\text{dW}[1]=\partial\text{W}/\partial\text{tp}\,, \qquad \text{dW}[2]=\partial^2\text{W}/\partial\text{tp}^2\,, \tag{4.14}$$

where the coefficients of power dtp are evaluated at the time tp.

The coefficients in (4.11) to (4.14) are expressed as follows by using (4.1) to (4.6).

$$\text{dhu}[1]=\text{U}\,, \qquad \text{dhu}[2]=\text{Ru}/\text{Lam}\,, \tag{4.15}$$

$$\text{dhw}[1]=\text{W}\,, \qquad \text{dhw}[2]=\text{Rw}/\text{Lam}\,, \tag{4.16}$$

$$\text{dU}[1]=\text{Ru}/\text{Lam}\,, \qquad \text{dW}[1]=\text{Rw}/\text{Lam}\,, \tag{4.17}$$

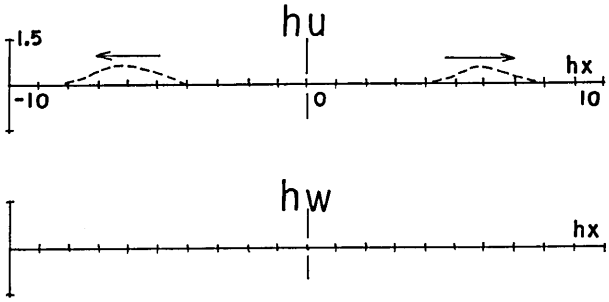

Fig. 6-1

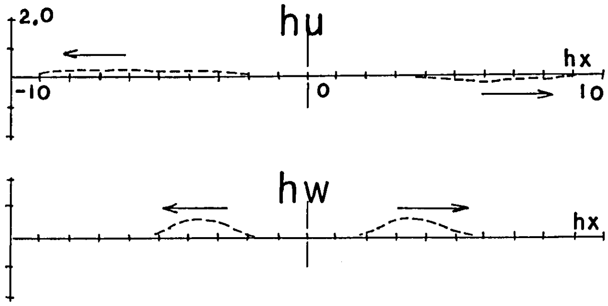

Fig. 6-2

Figs. 6-1, 6-2. Nonlinear waves at tp=8.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=3.0. Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} (−2<hx<2) with {A(hu)=0.06, A(hw)=A(U)=A(W)=0} (Fig. 6-1) and {A(hw)=0.06, A(hu)=A(U)=A(W)} (Fig. 6-2). Vertical scale: ×20 (Fig. 6-1) and ×200 (Fig. 6-2) for hu, and ×20 (both) for hw. Horizintal scale: ×1.

$$\mathrm{dU[2]=Rutp/Lam}\,, \qquad \mathrm{dW[2]=Rwtp/Lam}\,, \tag{4.18}$$

with

$$\mathrm{Rutp}=\partial\mathrm{Ru}/\partial\mathrm{tp} \qquad \text{and} \qquad \mathrm{Rwtp}=\partial\mathrm{Rw}/\partial\mathrm{tp}\,.$$

The last expressions Rutp and Rwtp are obtained explicitly by direct differentiation of (4.5) and (4.6) with respect to tp and substitution of U and W.

$$\begin{aligned}\mathrm{Rutp}=&\ \mathrm{Lam\ Ux2}\\ &+\mathrm{G2\ Ux2\ ux}+\mathrm{G2\ Ux\ ux2}\\ &+\mathrm{G1\ Wx\ wx2}+\mathrm{G1\ Wx2\ wx}\\ &+12\ \mathrm{G4\ Ux2\ ux^2}+24\ \mathrm{G4\ Ux\ ux2\ ux}\\ &+2\ \mathrm{G5\ Ux2\ wx^2}+4\ \mathrm{G5\ wx2\ wx\ Ux}\\ &+4\ \mathrm{G5\ Wx\ wx2\ ux}+4\ \mathrm{G5\ Wx\ wx\ ux2}+4\ \mathrm{G5\ Wx2\ wx\ ux}\,,\end{aligned} \tag{4.19}$$

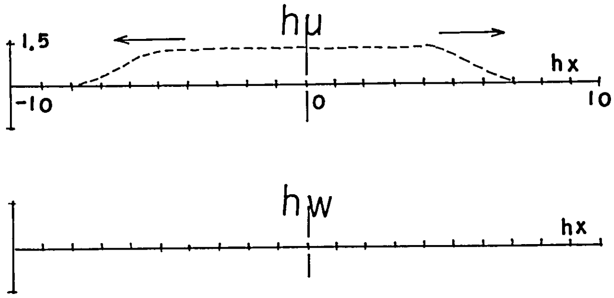

Fig. 6-3

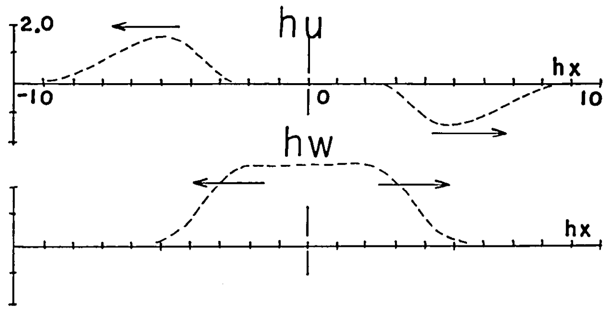

Fig. 6-4

Figs. 6-3, 6-4. Nonlinear waves at tp=8.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=3.0. Initial condition: $Q=A(Q)/2*\{1+\cos(hx*\pi/2)\}$ $(-2<hx<2)$ with {A(U)=0.06, A(hu)=A(hw)=A(W)=0} (Fig. 6-3) and {A(W)=0.06, A(hu)=A(hw)=A(U)=0} (Fig. 6-4). Vertical scale: ×20 (Fig 6-3) and ×200 (Fig. 6-4) for hu, and ×20 (both) for hw. Horizintal scale: ×1.

$$
\begin{aligned}
\mathrm{Rwtp} = \mathrm{Wx2} & \\
& + \mathrm{G1\ Ux2\ wx} + \mathrm{G1\ wx2\ Ux} \\
& + \mathrm{G1\ Wx\ ux2} + \mathrm{G1\ Wx2\ ux} \\
& + 2\ \mathrm{G5\ Wx2\ ux}^2 + 4\ \mathrm{G5\ Wx\ ux2\ ux} \\
& + 12\ \mathrm{G3\ Wx2\ wx}^2 + 24\ \mathrm{G3\ Wx\ wx2\ wx} \\
& + 4\ \mathrm{G5\ Ux2\ wx\ ux} + 4\ \mathrm{G5\ wx2\ Ux\ ux} + 4\ \mathrm{G5\ wx\ Ux\ ux2}\,, \qquad (4.20)
\end{aligned}
$$

where

$$\mathrm{ux} = \partial \mathrm{hu}/\partial \mathrm{hx}\,, \qquad \mathrm{ux2} = \partial^2 \mathrm{hu}/\partial \mathrm{hx}^2\,, \tag{4.21}$$

$$\mathrm{wx} = \partial \mathrm{hw}/\partial \mathrm{hx}\,, \qquad \mathrm{wx2} = \partial^2 \mathrm{hw}/\partial \mathrm{hx}^2\,, \tag{4.22}$$

$$\mathrm{Ux} = \mathrm{U}\partial/\partial \mathrm{hx}\,, \qquad \mathrm{Ux2} = \partial^2 \mathrm{U}/\partial \mathrm{hx}^2\,, \tag{4.23}$$

$$\mathrm{Wx} = \partial \mathrm{W}/\partial \mathrm{hx}\,, \qquad \mathrm{Wx2} = \partial^2 \mathrm{W}/\partial \mathrm{hx}^2\,. \tag{4.24}$$

In occasion of numerical calculation, (4.21) to (4.24) are computed

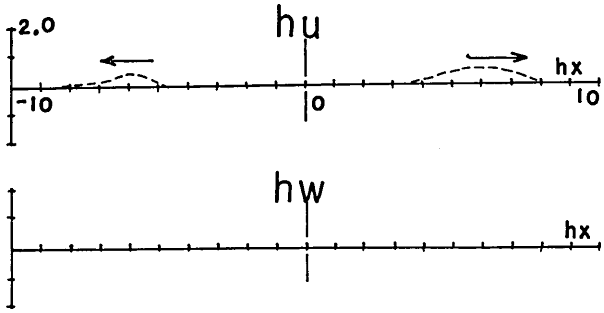

Fig. 7-1

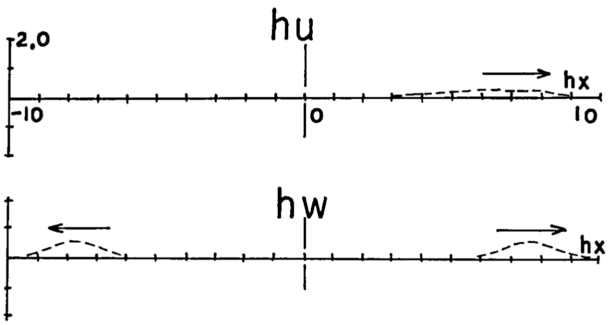

Fig. 7-2

Figs. 7-1, 7-2. Nonlinear waves at tp=6.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=−1.4. Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} (−2<hx<2) with {A(hu)=0.05, A(hw)=A(U)=A(W)=0} (Fig. 7-1) and {A(hw)=0.05, A(hu)=A(U)=A(W)=0} (Fig. 7-2). Vertical scale: ×20 (Fig. 7-1) and ×200 (Fig. 7-2) for hu, and ×20 (both) for hw. Horizintal scale: ×1.

by the difference

$$\begin{aligned} \mathrm{Qx} &= \partial \mathrm{QQ}/\partial \mathrm{hx} \\ &= (-\mathrm{Q1}+\mathrm{Q3})/(2\,\mathrm{dhx})\,, \\ \mathrm{Qx2} &= \partial^2 \mathrm{QQ}/\partial \mathrm{hx}^2 \\ &= (-2\,\mathrm{Q2}+\mathrm{Q1}+\mathrm{Q3})/\mathrm{dhx}^2\,, \end{aligned}$$

where Q in {Qx, Qx2} indicates variables {u, w, U, W}, QQ in the derivatives variables {hu, hw, U, W}, dhx a mesh interval, {Q1, Q2, Q3} the variables {hu, hw, U, W} at successive mesh points, respectively.

5. Numerical Computation

Numerical computation will be carried out by use of Taylor series (4.7) to (4.10) in section 4.

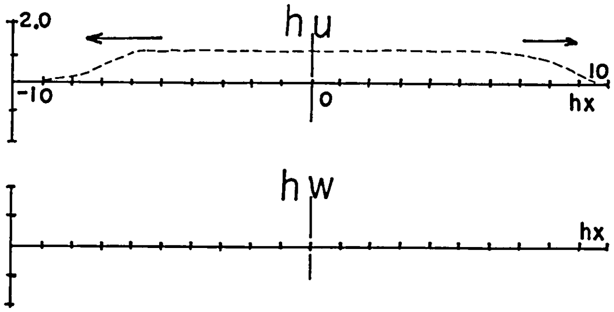

Fig. 7-3

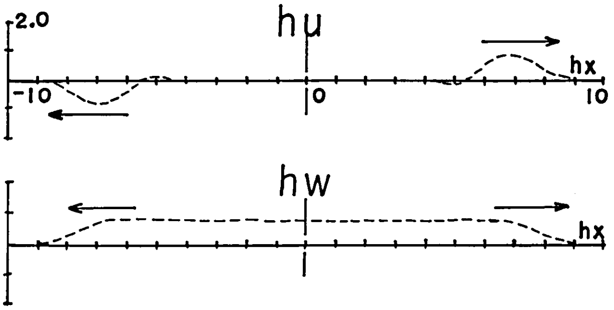

Fig. 7-4

Figs. 7-3, 7-4. Nonlinear waves at tp=6.0 generated by local wave origin. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=−1.4. Initial condition: Q=A(Q)/2*{1+cos (hx*π/2)} (−2<hx<2) with {A(U)=0.05, A(hu)=A(hw)=A(W)=0} (Fig. 7-3) and {A(W)=0.05, A(hu)=A(hw)=A(U)=0} (Fig. 7-4). Vertical scale: ×20 (Fig. 7-3) and ×1000 (Fig. 7-4) for hu, and ×20 (both) for hw. Horizintal scale: ×1.

Model Assumed

At the origin of the coordinates, the initial (tp=0) displacement or velocity

$$Q = A(Q)/2\ \{1+\cos(hx\,\pi/2)\} \qquad \text{in the range} \quad -2<hx<2 \qquad (5.1)$$

is given, where Q=hu, hw, U or W. The expression (5.1) indicates a local mountain ridge (since two-dimensional) with height A(Q).

In computation, mesh size dhx=1/3 and time step dtp=0.002 will be assumed. The convergence or stability of numerical computation is then confirmed by double increase of dividing points. Computed range of hx is −10 to 10. The computations are terminated before the head of the advancing waves reach one of the either end of the computed range (hx=−10 or +10) in order to avoid the effects of the artificial boundaries at both ends.

As for the effect of truncated terms of the computation, it is

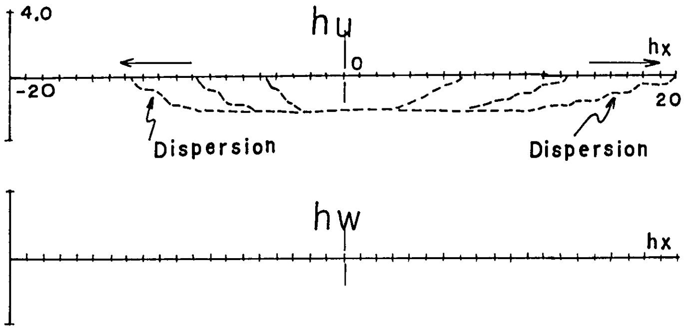

Fig. 8. Typical dispersion of high-amplitude simple waves. Time step: dtp=4.0. Last time: tp=12.0. Elastic coefficient: Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=3.0. Initial condition: $Q=A(Q)/2*\{1+\cos(hx*\pi/2)\}$ $(-2<hx<2)$ with A(U)=−0.6, A(hu)=A(hw)=A(W)=0. Vertical scale: ×4. Horizintal scale: ×1.

evaluated approximately by the first one of the truncated terms. In this paper, the fourth order term become the case, say, for x-derivative,

$$(\partial u/\partial x)^4\,,$$
$$(\partial u/\partial x)^3(\partial w/\partial x)\,,\qquad (\partial u/\partial x)^2(\partial w/\partial x)^2\,,\qquad (\partial u/\partial x)^2(\partial w/\partial x)\,,$$
$$(\partial w/\partial x)^4\,,\quad \text{etc.}$$

Among the above terms, the first term $(\partial u/\partial x)^4$ is the most signi ficant as compared with the coupled terms, since the u and w components are generally propagated at different velocity and hence the order of the coupled terms is smaller than that of the non-coupled terms.

In the present computation, half width of the wave source is 2 and the height of the wave 0.6 (=A(Q)) in the case of the highest amplitude. The derivatives $\partial u/\partial x$ and $(\partial u/\partial x)^4$ then become the order of 0.3 (=0.6/2) and 0.008, respectively. That is to say, the effect of the truncated terms are, at most, the order of the line width of the curves depicted.

It must be noted here that the above discussion is based on the assumption that the elastic coefficients of the above terms are the same order.

Numerical Instances

In numerical computations, the following numerical values are used for elastic coefficient and the amplitude A(Q) of the wave origin.

instance 1.

Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0 ,

A(hu)=0.6 (Fig. 1-1) and −0.6 (Fig. 1-2) ,

$A(hw)=A(U)=A(W)=0$,

instance 2.

$Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0$,

$A(hw)=0.6$ (Fig. 2-1) and -0.6 (Fig. 2-2) ,

$A(hu)=A(U)=A(W)=0$,

instance 3.

$Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0$,

$A(U)=0.6$ (Fig. 3-1) and -0.6 (Fig. 3-2) ,

$A(hu)=A(hw)=A(W)=0$,

instance 4.

$Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=1.0$,

$A(W)=0.6$ (Fig. 4-1) and -0.6 (Fig. 4-2) ,

$A(hu)=A(hw)=A(U)=0$.

instance 5.

$Lm=1.0$, $Am=Bm=Cm=Dm=Em=Fm=Gm=0$,

$A(hu)=0.6$, $A(hw)=A(U)=A(W)=0$, (Fig. 5-1)

$A(hw)=0.6$, $A(hu)=A(U)=A(W)=0$, (Fig. 5-2)

$A(U)=0.6$, $A(hu)=A(hw)=A(W)=0$, (Fig. 5-3)

$A(W)=0.6$, $A(hu)=A(hw)=A(U)=0$, (Fig. 5-4)

instance 6.

$Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=3.0$,

$A(hu)=0.06$, $A(hw)=A(U)=A(W)=0$, (Fig. 6-1)

$A(hw)=0.06$, $A(hu)=A(U)=A(W)=0$, (Fig. 6-2)

$A(U)=0.06$, $A(hu)=A(hw)=A(W)=0$, (Fig. 6-3)

$A(W)=0.06$, $A(hu)=A(hw)=A(U)=0$, (Fig. 6-4)

instance 7.

$Lm=Am=Bm=Cm=Dm=Em=Fm=Gm=-1.4$,

$A(hu)=0.05$, $A(hw)=A(U)=A(W)=0$, (Fig. 7-1)

$A(hw)=0.05$, $A(hu)=A(U)=A(W)=0$, (Fig. 7-2)

$A(U)=0.05$, $A(hu)=A(hw)=A(W)=0$, (Fig. 7-3)

$A(W)=0.05$, $A(hu)=A(hw)=A(U)=0$, (Fig. 7-4)

Classification of Wave Source

For the convenience of later reference, the type of wave source is classified into the following two categories (i) and (ii).

(i) *Displacement-type*: only initial displacement hu or hw are non-zero while initial velocities U and W being zero. Figures of this type are Figs. 1-1, 1-2, 2-1, 2-2, 5-1, 5-2, 6-1, 6-2, 7-1 and 7-2.

(ii) *Velocity-type*: only initial velocity U or W are non-zero while initial displacements hu and hw being zero. Figures of this type are Figs. 3-1, 3-2, 4-1, 4-2, 5-3, 5-4, 6-3, 6-4, 7-3 and 7-4.

The above classification of wave source type is very reasonable since solitary waves, say solitons, are generated by the wave source of displacement-type while typical step-shaped waves by the source of velocity-type.

6. Simple Waves

Throughout all the graphs in Fig. 1-1 to Fig. 7-4, it is seen that the nonlinear waves generated from the local wave origin are not periodic. Typical simple waves (BLAND, 1969; JEFFREY and TANIUCHI, 1964) are found. The equations of waves to the second order of the derivative of the displacement are very useful in explaining this feature. The second order equations are given by (3.2) and (3.3) with terms eq11 and eq21 only.

Transcribing the second order equations (3.2) to (3.5) with the moving axis with the velocity vr,

$$\mathrm{tr}=\mathrm{vr}\,t\,,\qquad \mathrm{kr}=\mathrm{vr}\,t-x\,, \tag{6.1}$$

where tr and kr are new independent variables on the moving axis, we have

$$\begin{aligned}\partial^2 u/\partial \mathrm{tr}^2&+2\,(\partial^2 u/\partial \mathrm{tr}\partial \mathrm{kr})+\partial^2 u/\partial \mathrm{kr}^2\\&=\mathrm{vp}^2\,(\partial^2 u/\partial \mathrm{kr}^2)/\mathrm{vr}^2\\&\quad-\mathrm{vs}^2\,\mathrm{G2}\,(\partial u/\partial \mathrm{kr})(\partial^2 u/\partial \mathrm{kr}^2)/\mathrm{vr}^2\\&\quad-\mathrm{vs}^2\,\mathrm{G1}\,(\partial w/\partial \mathrm{kr})(\partial^2 w/\partial \mathrm{kr}^2)/\mathrm{vr}^2\,,\end{aligned} \tag{6.2}$$

$$\begin{aligned}\partial^2 w/\partial \mathrm{tr}^2&+2\,(\partial^2 w/\partial \mathrm{tr}\partial \mathrm{kr})+\partial^2 w/\partial \mathrm{kr}^2\\&=\mathrm{vs}^2\,(\partial^2 w/\partial \mathrm{kr}^2)/\mathrm{vr}^2\\&\quad-\mathrm{vs}^2\,\mathrm{G1}\,(\partial u/\partial \mathrm{kr})(\partial^2 w/\partial \mathrm{kr}^2)/\mathrm{vr}^2\\&\quad-\mathrm{vs}^2\,\mathrm{G1}\,(\partial^2 u/\partial \mathrm{kr}^2)(\partial w/\partial \mathrm{kr})/\mathrm{vr}^2\,.\end{aligned} \tag{6.3}$$

In order to obtain the stable solutions on the moving axis, we assume the displacements u and w to be independent of tr and integrate with respect to kr. Equations (6.2) and (6.3) are simplified as follows.

$$\mathrm{Vrp}\,(\partial u/\partial \mathrm{kr})+\mathrm{G2}\,(\partial u/\partial \mathrm{kr})^2/2+\mathrm{G1}\,(\partial w/\partial \mathrm{kr})^2/2=0\,, \tag{6.4}$$

$$\mathrm{Vrs}\,(\partial w/\partial \mathrm{kr})+\mathrm{G1}\,(\partial u/\partial \mathrm{kr})(\partial w/\partial \mathrm{kr})=0\,, \tag{6.5}$$

where

$$\mathrm{Vrp} = (\mathrm{vr}^2 - \mathrm{vp}^2)/\mathrm{vs}^2\,, \qquad \mathrm{Vrs} = -1 + \mathrm{vr}^2/\mathrm{vs}^2\,. \tag{6.6}$$

The above two equations (6.4) and (6.5) are very important, since these equations explain the characteristic behaviors in Figs. 1-1 to 7-4. It must be noted here that eqs. (6.4) and (6.5), from the derivation, are only significant for the positive-ward advancing waves instead of negative-ward.

Non-coupled Simple Waves

The 'non-coupled' implies that the components u and w are not interrelated. In case of simple waves, it will be found that 'non-coupled' is actually only u-component.

By assuming that

$$\mathrm{w} = \mathrm{w0} \quad (const)\,, \tag{6.7}$$

eq. (6.5) is found always zero and eq. (6.4) yields the solutions

$$\mathrm{u} = \mathrm{Gu\,kr} \qquad \text{with} \quad \mathrm{Gu} = -2\,\mathrm{Vrp}/\mathrm{G2}\,, \tag{6.8}$$

and

$$\mathrm{u} = \mathrm{u0} \quad (const)\,. \tag{6.9}$$

Expressions (6.7), (6.8) and (6.9) indicate the generation of the step-shaped *stand-alone* u simple waves which is not coupled with w-component. Instances in this case are Figs. 1-1, 1-2, 3-1, 3-2, 5-1, 5-3, 6-1, 6-3, 7-1 and 7-3. As shown in these figures, existence of non-coupled simple waves does not depend on the values of elastic coefficient. Expression (6.8) implies that the advancing velocity (vr) of stand-alone u simple waves depends on the gradient Gu of the front wave through the relation (6.6). It must be noted that these simple waves are not propagated with the velocity of P waves. Instances in Figs. 1-1, 1-2, 3-1, 3-2 and etc. indicate that the stand-alone u simple waves on both sides are propagated at different velocities depending on the gradient.

As readily seen from (6.8) with the help (6.6), it is found that,

$$\text{when} \quad \mathrm{vr} > \mathrm{vp}\,, \quad \text{'pull' wave occurs}\,, \tag{6.10}$$

while,

$$\text{when} \quad \mathrm{vr} < \mathrm{vp}\,, \quad \text{'push' wave occurs}\,, \tag{6.11}$$

where G2 is assumed to be positive (in ordinary case), 'pull' and 'push' imply the negative and positive displacements in the advancing direction of the waves. The typical instance is given in Fig. 3-2.

As typical instances, Figs. 1-1 and 3-1 indicate that the waves propagated leftwards and rightwards move with a velocity faster and slower than that of P waves, respectively, where the situations with 'PW' in the figures are the estimated arrival points of P waves in the linear theory.

Coupled Simple Waves

The term 'coupled' implies that the components u and w are closely interrelated in the governing equations. In the case of simple waves, it will be found that the wave form of u-component is determined by the equation with the acceleration term $\partial^2 w/\partial t^2$ while that of the w-component by the equation with the term $\partial^2 u/\partial t^2$. Solving eq. (6.5), which originally has the acceleration term $\partial^2 w/\partial t^2$, we have

$$u = Gu\,kr \qquad \text{with} \quad Gu = -Vrs/G1\,. \tag{6.12}$$

Substitution of (6.12) into (6.4), which originally has the term $\partial^2 u/\partial t^2$, yields

$$w = \pm\sqrt{2}\,Gw\,kr \qquad \text{with} \quad Gw = \sqrt{Gw2}\,, \tag{6.13}$$

where

$$Gw2 = Vrs\,\{(1 - vp^2/vs^2) + Vrs\,(1 - G2/(2\,G1))\}/(G1^2)\,. \tag{6.14}$$

As shown above, each component of the displacement is determined by the counterpart equation to it. Expressions (6.12) and (6.13) indicate coupled simple waves. Instances of these expressions are given in Figs. 2-1, 2-2, 4-1, 4-2, 5-2, 5-4, 6-2, 6-4, 7-2 and 7-4.

In order to obtain more clear physical interpretation for the above figures, the quantity Vrs in (6.14) is assumed to be small. In other words, the moving velocity (vr) of the simple waves is assumed to have a value similar to that (vs) of S waves. Expression Gw in (6.13) and (6.14) is then reduced to

$$Gw = \sqrt{-Vrs}\,\sqrt{vp^2/vs^2 - 1}\ /\ |G1|\,. \tag{6.15}$$

From the above expression, when Vrs is small, existence of the coupled simple waves require $-Vrs > 0$, i.e., $vr < vs$; the velocity (vr) of the coupled simple waves must be smaller than that (vs) of S waves, and also, by comparing Gu in (6.12) and Gw in (6.15), it is found that the intensity of the u-component ($\sim -Vrs$) is the second order of that w-component ($\sim\sqrt{-Vrs}$).

In Fig. 8, the computation range of hx is extended to -20 to 20. This figure shows a typical dispersion of u simple waves, as the waves advance. This phenomenon may be observed at a great distance from the wave source as the P wave dispersion instead of simple wave dispersion.

On the other hand, w simple waves are very stable, and no difinite dispersion like u simple waves is found throughout all the figures concerning w component.

Summary

In nonlinear elastic media, dynamic equations to the third order of the derivative of the displacement are introduced by use of the expression of energy function, the stress-strain relation and the quadratic strain-displacement relation. The equations obtained are numerically solved by use of the extended finite difference method expanded in Taylor series.

The wave sources are assumed to have a form of mountain ridge with a width $-2<hx<2$, where hx is the distance x normalized by wave number h of P waves in the linear theory. In such a wave source, no periodic waves are generated in nonlinear theory. Only Soliton-like or step-shaped *simple waves* (after gas-dynamics) are then found numerically. For the former, the waves are produced by a wave source of a type of initial displacement (*displacement-type*) while, for the latter, the wave source is of a type of initial velocity (*velocity-type*). Existence of simple waves are also confirmed analytically by use of the second order thory.

Unlike the linear theory, the velocity of the simple waves in nonlinear media depends on the amplitude, literally gradient, of the front simple waves. These waves are not propagated at a velocity of P or S waves in the linear theory, though near these values.

As for coupled simple waves (u and w components co-existent), the w-component of these waves is propagated at a velocity lower than that of S waves in the linear theory.

In simple waves with large amplitude, the u component (in the direction of the propagation) is more remarkably dispersed than the transverse component. This phenomenon may be observed at a great distance from the wave source as the P wave dispersion instead of simple wave dispersion.

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非線形等方性弾性体における波の伝播

東京大学地震研究所　桃 井 高 夫

非線形等方性弾性体において，エネルギー関数，非線形歪-応力テンソル，非線形歪テンソルを用い，変位微分の三次項まで含む非線形動的方程式 (Nonlinear dynamic equation) が導入された.

得られた方程式はテイラー展開を用いて拡張された差分法によって数値的に解いた．このとき初期条件としては変位または変位速度を幅 $-2<hx<2$ の山嶺の形で与える．ただし，hx は線形理論の P 波の波数 h によって正規された距離 x とする．上記の初期条件で発生する波は非周期性の波で，ソリトン波型の，あるいは段型の *Simple Wave* (気体力学に倣って) が見いだされる．前者の波は初期値が変位型 (*displacement-type*) の波源によって，後者の波は初期値が速度型 (*velocity-type*) の波源によって起こされる.

Simple Wave の存在は変位の微分が二次 (second-order) までの方程式の解析解によっても理論的に確かめられる．線形理論と異なり，*Simple Wave* の速度は波の前面傾斜に依存する．線型理論の P 波 S 波の速度に近いが，これらの速度では進行しない.

Coupled Simple Wave (u, w 両成分が共存する *Simple Wave*) で w 成分は線形理論の S 波の速度より遅い速度で伝播する.

大振幅の *Simple Wave* では波の進行方向に沿う u 成分は横成分に比較してより著しく分散を受ける．震源域より遠いところで，これは *Simple Wave* の分散としてではなく P 波の分散として考えられてしまう恐れがある.