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15. *Ray-Theoretical Approach to Frequency Equations of Spheroidal Oscillations of a Spherical Earth with Uniform or Non-Uniform Mantle and Core.*

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Abstract

In Section II, we discuss from a viewpoint of ray theory the frequency equations of the spheroidal oscillations of the Earth consisting of a uniform solid mantle and a uniform liquid core. It is shown that the equations which yield discrete eigenfrequencies are derived from a certain interference condition of body waves traveling in the Earth. The equations are expressed in different forms corresponding to different ray situations in the Earth and they are proved to be identical with the asymptotic frequency equations obtained in terms of the normal mode theory. The interference condition thus proved to be valid then enables us to interpret the free oscillation in terms of ray theory.

It is shown that the wave conversion of P to S or S to P at a boundary in the Earth induces the solotone effect in the distribution of eigenfrequencies of the modes for a fixed phase velocity.

In Section III, discussion is extended to the Earth in which wave velocities change continuously as functions of the radius in the mantle and in the core respectively. We then formally get frequency equations identical to those for the above homo./homo. case. The validity of the equations is confirmed by the numerical computation for a realistic Earth model. Employing these equations, we can easily evaluate the approximate eigenfrequencies of high radial modes of spheroidal oscillations of a radially heterogeneous Earth.

I. Introduction

In the past few years some theoretical investigations have been made on asymptotic properties of free oscillations of the Earth at high frequency and large angular order [SATO and LAPWOOD (1977a, b), KENNETT and WOODHOUSE (1978), KENNETT and NOLET (1979), ODAKA (1980b)]. These investigations were mainly made on the basis of the normal mode theory. The principal terms in the asymptotic frequency equations obtained there should be derived (or interpreted) in terms of ray theory because the equations are gained under high-frequency approximations. However, the ray-theoretical approach to the frequency equation of free oscillations of the spherical Earth has not yet been completely established.

Such attempts for the normal modes of plane stratified media were made by OFFICER (1951) and TOLSTOY and USDIN (1953). BRUNE (1966) applied their method to the spheroidal oscillations of a homogeneous spherical Earth. ODAKA (1978) made a similar attempt and he further treated the radial oscillations of two layered Earth. The ray-theoretical method employed by Odaka is in principle applicable to other more complicated cases such as the spheroidal oscillations of the Earth consisting of a homogeneous solid mantle and a homogeneous liquid core (homo./homo. model). The asymptotic frequency equations for this case have recently been obtained by ODAKA (1980b) on the basis of the normal mode theory. They are expressed in different forms corresponding to different ray situations in the Earth and are denoted in terms of reflection coefficients and intercept times of relevant P and S rays.

In Section II of this paper we try to construct the frequency equations for the spheroidal oscillations of the above-mentioned homo./homo. Earth in terms of ray theory. Then, the results can directly be compared with the formulas obtained by ODAKA (1980b), which leads to confirmation of the validity of the method developed here. This attempt will serve to make clear the physical meaning of the asymptotic frequency equations and to interpret free oscillations in terms of the ray theory. Asymptotic properties of the distribution of eigenfrequencies will be discussed on the basis of reduced simple frequency equations.

In Section III we try to extend the method developed in Sec. II to a radially heterogeneous Earth and perform numerical computations to confirm the validity of the formulas thus obtained.

II. Frequency equations for the Earth with uniform mantle and core

2.1. Ray parameter and phase velocity

We define the angles of incidence of P and S rays as i_i and f_i respectively as shown in Fig. 1 (subscript $i=0$ referring to the free surface, $i=2$ and 1 to the two sides of the mantle/core boundary). Then we have the following relation connecting Snell's law (ray parameter; p) and

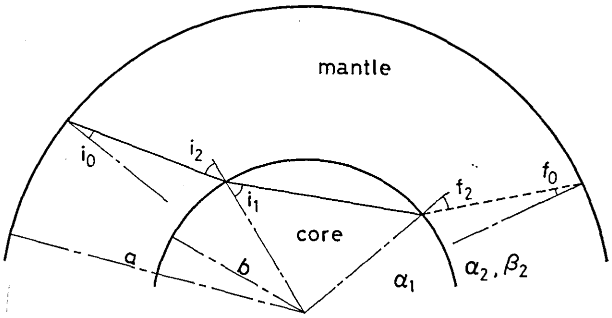

Fig. 1. Angles of incidence of P-ray (solid line) and S-ray (dashed line) on the free surface and the mantle-core boundary (both media are assumed homogeneous).

Jean's formula (inverse phase velocity of normal modes ; $1/c_0$) as employed by ODAKA (1980b).

$$\nu(=n+1/2)/\omega=a/c_0=p=(a/\alpha_2)\sin i_0=(a/\beta_2)\sin f_0=b/c$$
$$=(b/\alpha_2)\sin i_2=(b/\beta_2)\sin f_2=(b/\alpha_1)\sin i_1\,. \tag{1.1}$$

Here ω is the angular eigenfrequency of a spheroidal mode associated with the colatitudinal (angular) order number n, a and b the radii of the Earth and the core respectively, c the apparent velocity along the mantle-core boundary, and α_i, β_i the P- and S-wave velocities in the i-th medium ($i=1$ and 2 referring to the core and the mantle respectively).

According to whether each of the angles of incidence, i_0, f_0 etc. is defined as real or complex quantity, we have different ray geometry in the Earth. We will discuss each case separately in the following sections.

2.2. Case when the angles i_0, i_1, i_2, f_0 and f_2 are real

Imagine a ray situation where both P and S rays in the mantle with a given ray parameter reach the core and they are reflected back or refracted into the core (only as P rays). In such a case, all the angles of incidence of the rays in Eq. (1.1) are defined as real quantities.

Extending the idea of ODAKA (1978), we can formulate the interference condition of body waves equivalent to the steady vibration of the Earth. The situation is schematically illustrated in Fig. 2. Should the Earth be subject to free vibration with an angular frequency ω, then the motion at any given point R in the Earth can be expressed as $\exp(i\omega t)$ by taking the amplitude there as unity. If we limit ourselves to a spherically symmetrical Earth, the amplitude of the vibration at any point with the same radius and azimuth as the point R can identically

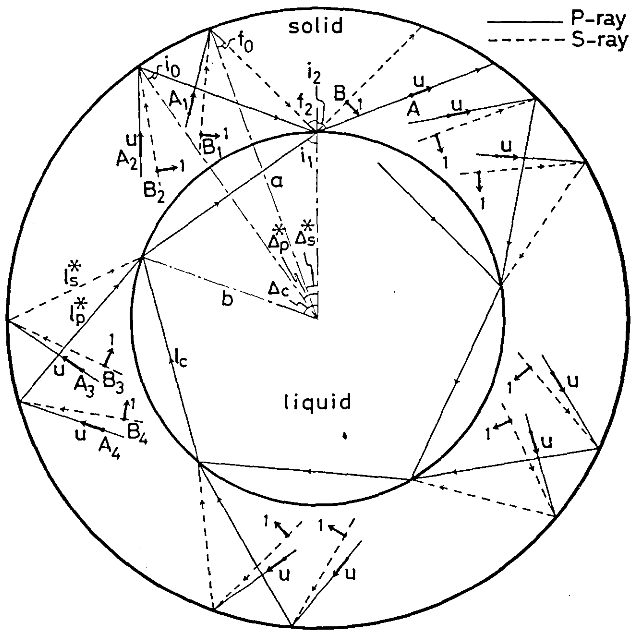

Fig. 2. Schematic illustration of the interference condition of body waves in the Earth with uniform solid mantle and liquid core when all the angles of incidence of the waves, i_0, i_1, i_2, f_0 and f_2 are real.

be denoted as unity, excluding the dependence on the colatitudinal angle θ. In this sense, all these points are equivalent to the point R. When we regard the free vibration as the superposition of two traveling waves propagating in opposite directions ($\pm\theta$) to each other along the surface of the Earth, this colatitudinal dependence is simply denoted by the multiplication of the two effects, the amplitude change caused by the geometrical spreading of waves and the phase shift due to the propagation. In the following discussion, we are only concerned with the disturbance traveling in a clockwise direction ($+\theta$) in the Earth, There will be no need to take into account the effect of the geometrical spreading factor in our problem because the effect is common for both types of wave propagation, the surface-wave type and body-wave type, on the spherical Earth.

Now, denote any point on a P-ray as A and set the amplitude of the disturbance associated with the P wave at its point as u (the positive direction of the displacement is defined by an arrow in Fig. 2). Then, all the points such as A_1, A_2, A_3, that are located at the same radial distance as point A are equivalent to A and thus the amplitude of the disturbance there can be set in common as u. In a similar manner, we put the amplitude of an S wave at any given point B as unity (the positive direction is defined by an arrow in the figure). This standardization of amplitude is made only for convenience' sake. Then, at any point such as B_1, B_2, B_3 equivalent to the point B, the amplitude of the S waves is given in common as unity. These amplitudes, u and 1, of course result from the interference of all body-wave phases that contribute to them. Here, it is convenient for obtaining the amplitudes of the disturbance at A and B to set the initial points of these interfering waves at those points that are equivalent to A and B. Then, we can formally denote the amplitudes there as follows, employing the notations similar to the conventional ones used for body-wave phases read on seismograms (*cf.* BULLEN; 1963).

$$\begin{aligned} u = & u[pPcP + pPKP + pPKKP + pPKKKP + \cdots] \\ & + [sPcP + sPKP + sPKKP + sPKKKP + \cdots] \\ & + u[pScP + pSKP + pSKKP + pSKKKP + \cdots] \\ & + [sScP + sSKP + sSKKP + sSKKKP + \cdots], \\ 1 = & [sScS + sSKS + sSKKS + sSKKKS + \cdots] \\ & + u[pScS + pSKS + pSKKS + pSKKKS + \cdots] \\ & + [sPcS + sPKS + sPKKS + sPKKKS + \cdots] \\ & + u[pPcS + pPKS + pPKKS + pPKKKS + \cdots]. \end{aligned} \tag{2.1}$$

In the upper formula which denotes the amplitude at A, the first series represents the contributions from the waves that pass through the points A_{2i} as P rays, and the second, third and fourth series describe the contributions from the waves that cross the points B_{2i} as S rays, A_{2i-1} as P and B_{2i-1} as S respectively, i being the integers, 1, 2, 3 etc. In the second formula which is concerned with the amplitude at B, each series denotes the contributions from the waves of which the initial positions are set at B_{2i-1}, A_{2i-1}, B_{2i} and A_{2i} ($i=1, 2, \cdots$) respectively.

Equation (2.1) is the formal representation of the interference condition of body waves equivalent to the steady vibration of the Earth. On the other hand, that Eq. (2.1) is satisfied by body waves traveling in the Earth ensures their persistent and steady propagation in it, and thus the situation is in conformity with the state of free vibration.

It will be easily found that the contributions from the phases contained in Eq. (2.1) are independent of one another and the contributions from the other phases are all included in them. The simple way of finding these independent rays is to pick up all the rays that cross only at their initial points the spherical surface with the same radius as the reference point A or B (in a given direction). This selection criterion will be valid for other ray situations.

The formal expression (2.1) can be translated into an explicit expression by considering the effect of reflection and transmission of the waves at the free surface and the mantle-core boundary and the phase shift due to the wave propagation. For the sake of simplifying numerical expressions, we imagine the extreme situation that the reference points A and B (and other points A_1, B_1, A_2, B_2 etc. as well) are taken sufficiently close to the free surface. Then, we get

$$
\begin{aligned}
u = & \, uR^0_{pp}e^{-2i\delta\bar{P}}[R^2_{pp}+T^2_{pp}T^1_{pp}e^{-2i\bar{P}_c}+T^2_{pp}R^1_{pp}T^1_{pp}e^{-4i\bar{P}_c}+\cdots] \\
& +R^0_{sp}e^{-2i\delta\bar{P}}[\qquad \text{〃} \qquad] \\
& +uR^0_{ps}e^{-i(\delta\bar{P}+\delta\bar{Q})}[R^2_{sp}+T^2_{sp}T^1_{pp}e^{-2i\bar{P}_c}+T^2_{sp}R^1_{pp}T^1_{pp}e^{-4i\bar{P}_c}+\cdots] \\
& +R^0_{ss}e^{-i(\delta\bar{P}+\delta\bar{Q})}[\qquad \text{〃} \qquad], \\
1 = & \, R^0_{ss}e^{-2i\delta\bar{Q}}[R^2_{ss}+T^2_{sp}T^1_{ps}e^{-2i\bar{P}_c}+T^2_{sp}R^1_{pp}T^1_{ps}e^{-4i\bar{P}_c}+\cdots] \\
& +uR^0_{ps}e^{-2i\delta\bar{Q}}[\qquad \text{〃} \qquad] \\
& +R^0_{sp}e^{-i(\delta\bar{P}+\delta\bar{Q})}[R^2_{ps}+T^2_{pp}T^1_{ps}e^{-2i\bar{P}_c}+T^2_{pp}R^1_{pp}T^1_{ps}e^{-4i\bar{P}_c}+\cdots] \\
& +uR^0_{pp}e^{-i(\delta\bar{P}+\delta\bar{Q})}[\qquad \text{〃} \qquad],
\end{aligned}
\tag{2.2}
$$

Here the symbols R^i_{xy} and T^i_{xy} stand for the reflection and transmission coefficients respectively and are specified in the Appendix. The phase terms are given by

$$
\begin{aligned}
\delta\bar{P} &= \omega\{(l^*_p/\alpha_2)-(a\Delta^*_p/c_0)\}, \\
\delta\bar{Q} &= \omega\{(l^*_s/\beta_2)-(a\Delta^*_s/c_0)\}, \\
2\bar{P}_c &= \omega\{(l_c/\alpha_1)-(a\Delta_c/c_0)\}-(\pi/2),
\end{aligned}
\tag{2.3}
$$

where the symbols $l^*_p(=a\cos i_0-b\cos i_2)$, $l^*_s(=a\cos f_0-b\cos f_2)$, $l_c(=2b\cos i_1)$ and Δ^*_p, Δ^*_s, Δ_c denote the length of each ray segment and the corresponding angular distance respectively (see Fig. 2). The first term in each braces of the above formulas means the phase delay due to the traveling of a corresponding body-wave phase. The second term means the phase shift attributable to the difference in colatitudinal coordinates, and the constant term, $\pi/2$ is caused by the passage of the deepest point of a ray [SHIMAMURA and SATO (1965)].

The infinite series that appears in common in each bracket of Eq.

(2.2) can be evaluated as

$$1+R^1_{pp}\mathrm{e}^{-2i\bar{P}_c}+(R^1_{pp}\mathrm{e}^{-2i\bar{P}_c})^2+(R^1_{pp}\mathrm{e}^{-2i\bar{P}_c})^3+\cdots$$
$$=(1-R^1_{pp}\mathrm{e}^{-2i\bar{P}_c})^{-1}. \qquad (2.4)$$

Then, employing the following symbols

$$\bar{e}_p=\mathrm{e}^{-i\delta\bar{P}}, \quad \bar{e}_s=\mathrm{e}^{-i\delta\bar{Q}}, \quad e_c=\mathrm{e}^{-2i\bar{P}_c},$$
$$\sigma=\mathrm{e}^{-2i\bar{P}_c}(1-R^1_{pp}\mathrm{e}^{-2i\bar{P}_c})^{-1}, \qquad (2.5)$$

we can simplify Eq. (2.2) as

$$u=\bar{e}_p^2(uR^0_{pp}+R^0_{sp})(R^2_{pp}+\sigma T^2_{pp}T^1_{pp})+\bar{e}_p\bar{e}_s(uR^0_{ps}+R^0_{ss})(R^2_{sp}+\sigma T^2_{sp}T^1_{pp}),$$
$$1=\bar{e}_s^2(R^0_{ss}+uR^0_{ps})(R^2_{ss}+\sigma T^2_{sp}T^1_{ps})+\bar{e}_p\bar{e}_s(R^0_{sp}+uR^0_{pp})(R^2_{ps}+\sigma T^2_{pp}T^1_{ps}). \qquad (2.6)$$

Eliminating the parameter u from these equations and using some relations in Eqs. (A.1) and (A.3), we get

$$(1-e_cR^1_{pp})+\bar{e}_p^2\bar{e}_s^2(R^1_{pp}-e_c)-\bar{e}_p^2R^0_{pp}(R^2_{pp}-e_cR^2_{ss})-\bar{e}_s^2R^0_{ss}(R^2_{ss}-e_cR^2_{pp})$$
$$-\bar{e}_p\bar{e}_s(1-e_c)(R^0_{ps}R^2_{sp}+R^0_{sp}R^2_{ps})=0. \qquad (2.7)$$

The above formulation is made in situations where the reference points A and B are set very close to the free surface. In another case as shown in Fig. 2, some additional phase terms appear in the equations. These terms, however, cancel each other in the process of eliminating the parameter u and we attain to the same equation as (2.7). Alternatively, we will be able to set a reference point on a P ray in the core. Then, we will get one equation from the first instead of the two equations in Eq. (2.1) or (2.2).

Rewriting Eq. (2.7) by use of Eq. (2.5), we gain

$$\{1-\mathrm{e}^{-2i(\delta\bar{P}+\delta\bar{Q}+\bar{P}_c)}\}+R^1_{pp}\{\mathrm{e}^{-2i(\delta\bar{P}+\delta\bar{Q})}-\mathrm{e}^{-2i\bar{P}_c}\}$$
$$+R^0_{pp}R^2_{ss}\{\mathrm{e}^{-2i(\delta\bar{P}+\bar{P}_c)}-\mathrm{e}^{-2i\delta\bar{Q}}\}+R^0_{pp}R^2_{pp}\{\mathrm{e}^{-2i(\delta\bar{Q}+\bar{P}_c)}-\mathrm{e}^{-2i\delta\bar{P}}\}$$
$$+(R^0_{ps}R^2_{sp}+R^0_{sp}R^2_{ps})\{\mathrm{e}^{-i(\delta\bar{P}+\delta\bar{Q}+2\bar{P}_c)}-\mathrm{e}^{-i(\delta\bar{P}+\delta\bar{Q})}\}=0. \qquad (2.8)$$

This equation is identical to the asymptotic frequency equation of the spheroidal oscillations of the Earth derived by ODAKA [1980b; Eqs. (8.5), (8.6)] on the basis of the normal mode theory. The agreement of the two equations ensures the validity of the ray-theoretical method (or the ray-theoretical interpretation of the free oscillations) developed in this article.

It is convenient for the numerical computation to express Eq. (2.8) in a form of a real function. This is easily done by multiplying by $\exp\{i(\delta\bar{P}+\delta\bar{Q}+\bar{P}_c)\}$ and we get

$$\sin(\delta\bar{P}+\delta\bar{Q}+\bar{P}_c)-R^1_{pp}\sin(\delta\bar{P}+\delta\bar{Q}-\bar{P}_c)-R^0_{pp}R^2_{ss}\sin(\delta\bar{P}-\delta\bar{Q}+\bar{P}_c)$$
$$+R^0_{pp}R^2_{pp}\sin(\delta\bar{P}-\delta\bar{Q}-\bar{P}_c)-(R^0_{ps}R^2_{sp}+R^0_{sp}R^2_{ps})\sin\bar{P}_c=0\,. \tag{2.9}$$

Note that all the reflection coefficients in the above equation take real values under the present ray situation. This equation yields discrete eigenfrequencies for a given ray parameter p (or phase velocity c_0).

It will be an interesting problem to get an equation valid for the extreme case when $c_0\to\infty$. In this case, from Eq. (1.1), we have the approximations

$$i_0\simeq f_0\simeq i_2\simeq f_2\simeq i_1\simeq 0\,, \tag{2.10}$$

indicating the radial propagation of all the waves.

Then, the following quantities appearing in Eq. (2.9) are approximated as

$$\begin{aligned}
&l^*_p\simeq l^*_s\simeq a-b=d_2,\quad l_c\simeq 2b\,,\\
&\Delta^*_p\simeq\Delta^*_s\simeq 0,\quad \Delta_c\simeq\pi\,,\\
&\delta\bar{P}\simeq h_2d_2,\quad \delta\bar{Q}\simeq k_2d_2,\quad 2\bar{P}_c\simeq 2h_1b-(n+1)\pi\,,\\
&R^0_{pp}=R^0_{ss}\simeq R^2_{ss}\simeq -1,\quad R^1_{pp}\simeq -R^2_{pp}=R^p_1\,,\\
&R^0_{ps}\simeq R^0_{sp}\simeq R^2_{ps}\simeq R^2_{sp}\simeq 0\,,
\end{aligned} \tag{2.11}$$

where h_i and k_i are the wave numbers of the P and S waves, $i=1$ and 2 referring to the core and mantle respectively. Then, Eq. (2.9) is reduced to

$$\sin k_2d_2[\sin(h_2d_2+h_1b-n\pi/2)+R^p_1\sin(h_2d_2-h_1b+n\pi/2)]=0\,. \tag{2.12}$$

This yields two independent equations and is equivalent to the Eq. (3.15) of ODAKA (1980a), who derived it by means of an asymptotic expansion (valid for relatively small angular order n) of an exact frequency equation. This fact implies that Eq. (2.9) holds for any values of n (originally it is expected to be valid for the large value of n).

Here, imagine the ray situation suggested by Eq. (2.10). Then, in Fig. 2, the points A_{4i-2}'s ($i=1, 2, 3, \cdots$) and A converge into a common single point and the points A_{4i}'s into the spherically symmetrical point to it. As for the S rays, the two points B and B_1 are reduced to one. All other points become meaningless because the conversion of the waves from P to S or vice versa does not occur at the core boundary and at the free surface [see the last relation of Eq. (2.11)]. It is found from Eqs. (2.9) and (2.11) that the dependence of Eq. (2.12) on the angular order n comes from the term $\bar{P}_c$, which is related to the phase shift caused by the traveling of the P rays across the core.

2.3. Case when the angles i_0, i_1, f_0 and f_2 are real

Imagine the ray situation that the P rays in the mantle do not reach the core and thus the angle i_2 is not defined as a real quantity (see Fig. 3).

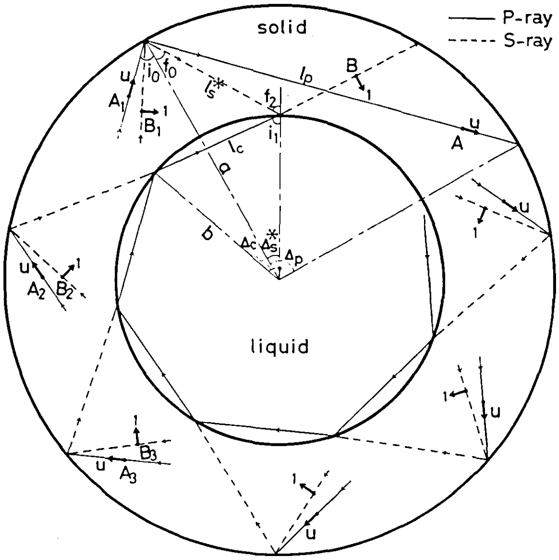

Fig. 3. Schematic illustration of the interference condition of body waves in the Earth when the angles of incidence of the waves, i_0, i_1, f_0 and f_2 are real.

Like in the preceding case, we set the amplitude of the P wave at any given point A on the P ray as u and that of the S wave at other given point B as unity. The points A_i's and B_i's ($i=1, 2, 3, \cdots$) in Fig. 3 are taken at the same depth as the reference points A and B and thus are equivalent to A and B respectively. It will be easily found that, by help of the selection criterion for independent rays explained in the preceding section, the interference condition of the body waves can be formally denoted as

$$u=u[pP]+[sP],$$

$$1=[sScS+sSKS+sSKKS+sSKKKS+\cdots] \qquad (3.1)$$

$$+u[pScS+pSKS+pSKKS+pSKKKS+\cdots],$$

The first simple relation comes from the fact that there are only two phases that contribute to the P wave at point A (see Fig. 3).

Assuming, for sake of convenience, the extreme situation that the two reference points A and B are taken sufficiently close to the surface of the Earth, we can readily transform Eq. (3.1) into an explicit form that

$$u = e^{-2i\bar{P}_a}[uR^0_{pp} + R^0_{sp}] ,$$
$$1 = R^0_{ss}e^{-2i\delta\bar{Q}}[R^2_{ss} + T^2_{sp}T^1_{ps}e^{-2i\bar{P}_c} + T^2_{sp}R^1_{pp}T^1_{ps}e^{-4i\bar{P}_c} + \cdots]$$
$$+ uR^0_{ps}e^{-2i\delta\bar{Q}}[\qquad \text{''} \qquad] , \tag{3.2}$$

where the phases $\delta\bar{Q}$ and $2\bar{P}_c$ are defined in Eq. (2.3) and $\bar{P}_a$ is given by

$$2\bar{P}_a = \omega\{(l_p/\alpha_2) - a(\Delta_p/c_0)\} - (\pi/2) , \tag{3.3}$$

Here the symbols $l_p(=2a\cos i_0)$ and Δ_p mean the length of a P-ray segment in the mantle and a corresponding angular distance. The physical meaning of each term is similar to that explained in the preceding section.

The infinite series in Eq. (3.2) can be simplified with the help of Eq. (2.4). Then, eliminating the parameter u, we get

$$\{1 - e_pR^0_{pp} - e_cR^1_{pp} + e_pe_cR^0_{pp}R^1_{pp}\} - \bar{e}_s(R^0_{pp} - e_p)\{R^2_{ss} - e_c(R^1_{pp}R^2_{ss} - T^2_{sp}T^1_{ps})\} = 0 , \tag{3.4}$$

where

$$e_p = e^{-2i\bar{P}_a} . \tag{3.5}$$

Other symbols $\bar{e}_s$ and e_c are defined in Eq. (2.5). Further manipulation with the use of some relations in Eqs. (A.1) and (A.3) yields

$$1 - R^0_{pp}e^{-2i\bar{P}_a} - R^1_{pp}e^{-2i\bar{P}_c} + R^2_{ss}e^{-2i(\bar{P}_a+\delta\bar{Q})} - R^2_{pp}e^{-2i(\bar{P}_a+\bar{P}_c+\delta\bar{Q})}$$
$$- R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}} + R^0_{pp}R^1_{pp}e^{-2i(\bar{P}_a+\bar{P}_c)} + R^0_{ss}R^2_{pp}e^{-2i(\bar{P}_c+\delta\bar{Q})} = 0 . \tag{3.6}$$

This is identical to the Eq. (9.5) of ODAKA (1980b), the asymptotic frequency equation obtained in terms of the normal mode theory. This success again proves the validity of the present ray-theoretical approach to the frequency equation of the free oscillations of the Earth.

It is found from Fig. 3 that there exist no actual rays corresponding to the reflection coefficient R^2_{pp} in Eq. (3.6). This originally appears in Eq. (3.4) in the form of $R^1_{pp}R^2_{ss} - T^2_{sp}T^1_{ps}$ [see Eq. (A.3)]. Then we have actual rays corresponding to each of these coefficients.

In the present case, all reflection coefficients except for $R^0_{pp}(=R^0_{ss})$ take complex values and thus, for the convenience of numerical computation, we define their real parts and imaginary parts separately as given in Appendix (A.4). Then, multiplying Eq. (3.6) by $\exp\{i(\bar{P}_a + \bar{P}_c + \delta\bar{Q})\}$, we get

$$R_1^2 \cos \chi_1 + (R_2^2 + R_3^2) \sin \chi_1 - R_1^2 \cos \chi_2 - (R_2^2 - R_3^2) \sin \chi_2$$

$$+ R_{pp}^0 R_1^2 \cos \chi_3 - R_{pp}^0 (R_2^2 - R_3^2) \sin \chi_3 - R_{pp}^0 R_1^2 \cos \chi_4 + R_{pp}^0 (R_2^2 + R_3^2) \sin \chi_4 = 0\,, \tag{3.7}$$

where

$$\chi_1 = \bar{P}_a + \bar{P}_c + \delta\bar{Q}, \quad \chi_2 = \bar{P}_a - \bar{P}_c + \delta\bar{Q},$$
$$\chi_3 = \bar{P}_a + \bar{P}_c - \delta\bar{Q}, \quad \chi_4 = \bar{P}_a - \bar{P}_c - \delta\bar{Q}\,. \tag{3.8}$$

2.4. Case when the angles i_1, f_0 and f_2 are real

Imagine the ray situation where no P rays are seen in the mantle for a given ray parameter as shown in Fig. 4. Then, the angles i_0 and i_2 are defined as complex quantities and the P waves in the mantle are permitted to exist only as inhomogeneous waves (BREKHOVSKIKH ; 1960), which contribute nothing to the formation of other body-wave phases in the ray theory.

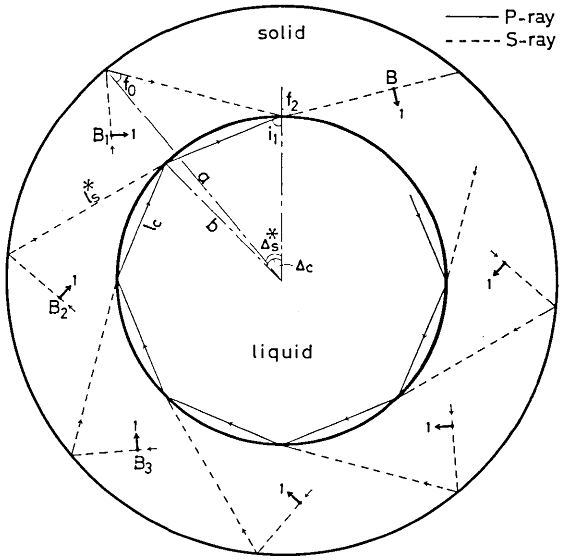

Fig. 4. Schematic illustration of the interference condition of body waves in the Earth when the angles of incidence, i_1, f_0 and f_2 are real. The points B, B_1, B_2 etc. are set at the same depth in the mantle.

Set the reference point B on a S ray in the mantle where the amplitude of the S wave is standardized as unity. Then, the interference condition is formally denoted as

$$1 = [sScS + sSKS + sSKKS + sSKKKS + \cdots]\,. \tag{4.1}$$

This is readily translated into

$$1 = R^0_{ss} e^{-2i\delta\bar{Q}} [R^2_{ss} + T^2_{sp} T^1_{ps} e^{-2i\bar{P}_c} + T^2_{sp} R^1_{pp} T^1_{ps} e^{-4i\bar{P}_c} + \cdots]. \tag{4.2}$$

Further reduction is possible with the aid of Eqs. (2.4) and (A.3), and we get

$$1 - R^1_{pp} e^{-2i\bar{P}_c} - R^0_{ss} R^2_{ss} e^{-2i\delta\bar{Q}} + R^0_{ss} R^2_{pp} e^{-2i(\bar{P}_c + \delta\bar{Q})} = 0. \tag{4.3}$$

This is identical to the Eq. (10.3) of ODAKA (1980b). The coefficient R^2_{pp} in Eq. (4.3) arises from the same circumstances as mentioned in the preceding section.

In Fig. 4, we set the reference point B on the S ray in the mantle. Alternatively, we can set it on the P ray in the core. Then, corresponding to Eqs. (4.1) and (4.2), we get equations different in appearance. But the final equation is shown to be equivalent to Eq. (4.3).

It will be convenient for numerical computation to express Eq. (4.3) in the form

$$\begin{aligned} &\{R^0_1 R^2_1 - R^0_2(R^2_2 + R^2_3)\} \cos(\bar{P}_c + \delta\bar{Q}) + \{R^0_1(R^2_2 + R^2_3) + R^0_2 R^2_1\} \sin(\bar{P}_c + \delta\bar{Q}) \\ &- \{R^0_1 R^2_1 - R^0_2(R^2_2 - R^2_3)\} \cos(\bar{P}_c - \delta\bar{Q}) + \{R^0_1(R^2_2 - R^2_3) + R^0_2 R^2_1\} \sin(\bar{P}_c - \delta\bar{Q}) = 0, \end{aligned} \tag{4.4}$$

where the coefficients R^0_1, R^2_1 etc. are defined in Eq. (A.4).

2.5. Case when the angles i_0, i_2, f_0 and f_2 are real

Imagine the ray situation that both P and S rays in the mantle with a given ray parameter strike the core boundary but are totally reflected there (see Fig. 5). This is the special case of Sec. 2.2 in appearance.

Because of the simple ray-geometry, we easily find that there are

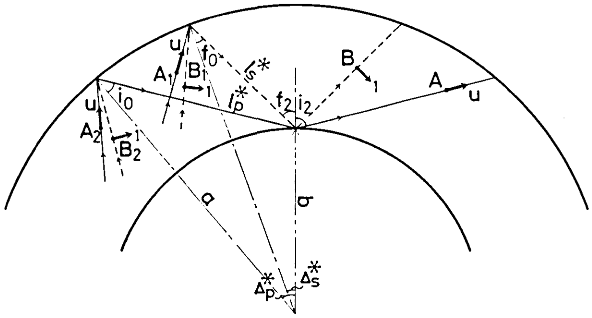

Fig. 5. Schematic illustration of the interference condition of body waves in the Earth when the angles of incidence, i_0, i_2, f_0 and f_2 are real.

only four kinds of independent rays that contribute to the interference condition of body waves. Then, we get the formal equation

$$\begin{aligned} u&=u[pPcP+pScP]+[sPcP+sScP]\,, \\ 1&=[sScS+sPcS]+u[pScS+pPcS]\,. \end{aligned} \tag{5.1}$$

The right-hand sides of the above equations consist of the first terms in the brackets of Eq. (2.1) and thus, following the similar steps to (2.2) through (2.8), we get

$$\begin{aligned} &1-(R^0_{pp}R^2_{pp}e^{-i\delta\bar{P}}+R^0_{ps}R^2_{sp}e^{-i\delta\bar{Q}})e^{-i\delta\bar{P}}-(R^0_{ss}R^2_{ss}e^{-i\delta\bar{Q}}+R^0_{sp}R^2_{ps}e^{-i\delta\bar{P}})e^{-i\delta\bar{Q}} \\ &\quad+(R^0_{pp}R^0_{ss}-R^0_{ps}R^0_{sp})(R^2_{pp}R^2_{ss}-R^2_{ps}R^2_{sp})e^{-2i(\delta\bar{P}+\delta\bar{Q})}=0\,, \end{aligned} \tag{5.2}$$

which, with the aid of Eq. (A.3), is further reduced to

$$\begin{aligned} &1+R^1_{pp}e^{-2i(\delta\bar{P}+\delta\bar{Q})}-R^0_{pp}R^2_{pp}e^{-2i\delta\bar{P}}-R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}} \\ &\quad-(R^0_{ps}R^2_{sp}+R^0_{sp}R^2_{ps})e^{-i(\delta\bar{P}+\delta\bar{Q})}=0\,. \end{aligned} \tag{5.3}$$

This equation is identical to the Eq. (11.5) of ODAKA (1980b) [combined with Eq. (8.6)]. The reflection coefficient R^1_{pp} in Eq. (5.3), to which we do not have any corresponding rays, comes from the substitution of it for $R^2_{pp}R^2_{ss}-R^2_{ps}R^2_{sp}$ in Eq. (5.2) [see Eq. (A.3)].

A further reduction of Eq. (5.3) yields

$$\begin{aligned} &(R^2_1+R^2_2)\cos(\delta\bar{P}+\delta\bar{Q})-R^2_3\sin(\delta\bar{P}+\delta\bar{Q})+R^0_{pp}(R^2_1-R^2_2)\cos(\delta\bar{P}-\delta\bar{Q}) \\ &\quad-R^0_{pp}R^2_3\sin(\delta\bar{P}-\delta\bar{Q})+(1/2)(R^0_{ps}R^2_5-R^0_{sp}R^2_4)=0\,, \end{aligned} \tag{5.4}$$

where all the coefficients are defined as real quantities.

2.6. Case when the angles i_0, f_0 and f_2 are real

Imagine the ray situation where P rays in the mantle do not reach the core boundary while the corresponding S rays do and are totally reflected there (see Fig. 6). This is the special case of Sec. 2.3 in appearance, and we immediately obtain the formal equation as

$$\begin{aligned} u&=u[pP]+[sP]\,, \\ 1&=[sScS]+u[pScS]\,. \end{aligned} \tag{6.1}$$

Then, by following the same steps from Eq. (3.1) to Eq. (3.6), we get

$$1-R^0_{pp}e^{-2i\bar{P}_a}+R^2_{ss}e^{-2i(\bar{P}_a+\delta\bar{Q})}-R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}}=0\,. \tag{6.2}$$

This is identical to Eq. (12.2) of ODAKA (1980b).

Further simplification is possible by agreeing that $|R^2_{ss}|=1$ and

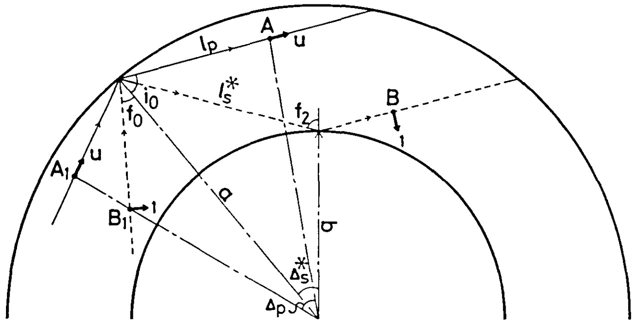

Fig. 6. Schematic illustration of the interference condition of body waves in the Earth when the angles of incidence, i_0, f_0 and f_2 are real.

$R^0_{pp}(=R^0_{ss})$ is real, and we attain

$$\cos(\bar{P}_a+\delta\bar{Q}+\varepsilon_2)-R^0_{pp}\cos(\bar{P}_a-\delta\bar{Q}-\varepsilon_2)=0\,, \tag{6.3}$$

where the phase ε_2 is defined by

$$R^2_{ss}=\{R^2_2-i(R^2_1+R^2_3)\}/\{R^2_2+i(R^2_1+R^2_3)\}=\mathrm{e}^{-2i\varepsilon_2}. \tag{6.4}$$

The coefficients R^2_1, R^2_2 etc. are given in Appendix (A. 4).

Equation (6.3) is similar in form to the one of Eq. (2.12). The asymptotic behavier of eigenvalues in this form of equation has been investigated in detail by ODAKA (1980a), which shows the systematic variation called the "solotone effect". Hence, we can readily conclude that the roots (eigenfrequencies) of Eq. (6.3) show the same kind of systematic variasion in their distribution for a fixed ray parameter. The solotone effect in the case of Eq. (2.12) is caused by the existence of the internal discontinuity surface, that is, the core boundary where the transmission and reflection of the P waves, in other words the wave division, occurs. In the present case, a similar phenomenon does not occur at the core boundary but the wave conversion of P to S or S to P arises at the free surface, which induces the solotone effect.

2.7. Case when the angles f_0 and f_2 are real.

Imagine a very simple ray situation as shown in Fig. 7, which corresponds to the case where the S rays in the mantle in Fig. 4 (Sec. 2.4) are totally reflected at the core boundary. Then we get the formal expression of the interference condition as

$$1=[sScS]\,. \tag{7.1}$$

This is explicitly denoted as

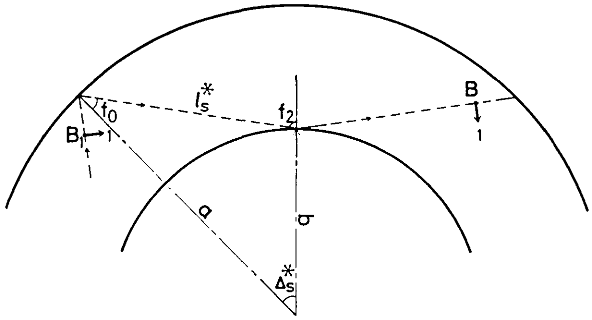

Fig. 7. Schematic illustration of the interference condition of body waves in the Earth when the angles of incidence f_0 and f_2 are real.

$$1 = R_{ss}^0 R_{ss}^2 e^{-2i\delta\bar{Q}}\,, \tag{7.2}$$

which is identical to the Eq. (13.2) of ODAKA (1980b).

Now that $|R_{ss}^0|=1$ and $|R_{ss}^2|=1$ in the present case, Eq. (7.2) can be reduced to

$$\cos(\delta\bar{Q}-\varepsilon_0+\varepsilon_2)=0\,, \tag{7.3}$$

where the phase ε_2 is defined by Eq. (6.4) and ε_0 by

$$R_{ss}^0 = -(R_1^0+iR_2^0)/(R_1^0-iR_2^0) = -e^{2i\varepsilon_0}\,. \tag{7.4}$$

From Eqs. (2.3) and (7.3), we can directly obtain the ray-theoretical (or asymptotic) eigenfrequencies as

$$f_j=(1/2)\{j+(1/2)+(\varepsilon_0-\varepsilon_2)/\pi\}/\{(l_s^*/\beta_2)-(a\Delta_s^*/c_0)\}\,, \tag{7.5}$$

where j is the integer. These frequencies are distributed with equal intervals for a given ray parameter (or phase velocity). Unlike the preceding case (Sec. 2.6) the wave conversion does not occur at the free surface in this case, and thus the solotone effect is not induced in their distribution.

2.8a. Case when the angles i_0 and f_0 are real

Imagine the ray situation shown in Fig. 8A. In this case (and in the next case as well), the core is ray-theoretically not concerned with the problem at all. Hence we have no differences between the present model and the homogeneous Earth model.

It is easy from Fig. 8A to formulate the interference condition formally as

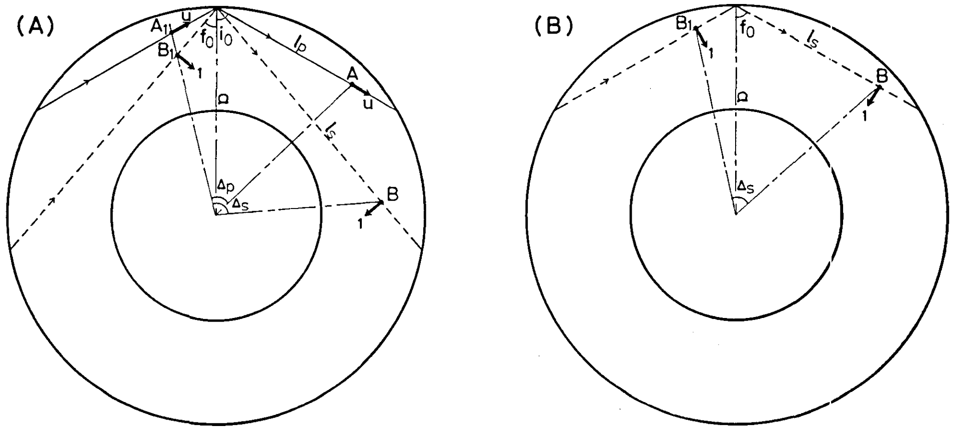

Fig. 8. Schematic illustration of the interference condition of body waves in the Earth when the angles of incidence i_0 and f_0 are real (case A) and the angle f_0 is real (case B). A solid line denotes a P-ray and a dashed line an S-ray. The points A and A_1 are associated with the P-ray and B and B_1 with the S-ray. The two points in each group are set at the same depth in the mantle.

$$\begin{aligned} u &= u[pP]+[sP]\,, \\ 1 &= [sS]+u[pS]\,. \end{aligned} \tag{8.1}$$

Imagine, for simplicity's sake, the limiting case where the reference points A and B tend to the surface of the Earth. Then, Eq. (8.1) can be translated into

$$\begin{aligned} u &= [uR^0_{pp}+R^0_{sp}]e^{-2i\bar{P}_a}\,, \\ 1 &= [R^0_{ss}+uR^0_{ps}]e^{-2i\bar{Q}_a}\,, \end{aligned} \tag{8.2}$$

where the phase term $\bar{P}_a$ is given in Eq. (3.3) and $\bar{Q}_a$ by

$$2\bar{Q}_a=\omega\{(l_s/\beta_2)-(a\Delta_s/c_0)\}-(\pi/2)\,. \tag{8.3}$$

By eliminating the parameter u from Eq. (8.2), we get

$$1-R^0_{pp}e^{-2i\bar{P}_a}-R^0_{ss}e^{-2i\bar{Q}_a}+e^{-2i(\bar{P}_a+\bar{Q}_a)}=0\,. \tag{8.4}$$

This is identical to the Eq. (14.4) of ODAKA (1980b) and to the asymptotic frequency equation for the uniform Earth as well.

Multiplying Eq. (8.4) by $\exp\{i(\bar{P}_a+\bar{Q}_a)\}$, we get

$$\cos(\bar{P}_a+\bar{Q}_a)-R^0_{pp}\cos(\bar{P}_a-\bar{Q}_a)=0\,, \tag{8.5}$$

where the reflection coefficient R^0_{pp} is a real quantity. This equation is similar in form to Eq. (6.3), and thus we can expect the solotone effect in the distribution of its solutions (eigenfrequencies) for a given ray parameter. In this case it is caused by the conversion of the P and S waves at the free surface of the Earth like in the case of Sec. 2.6.

Here we will give some comments on the parameter u. This is generally defined as the amplitude of a P wave when that of an S wave is standardized as unity. A ray-theoretical frequency equation is obtained by eliminating u from two simultaneous equations denoting the interference condition of body waves. Each solution (eigenfrequency) of that equation in turn assigns the value of u through either of these two equations. This fact indicates that each normal mode is connected with the P and S waves with a specific amplitude ratio and apparent velocity (in addition there of cource are modes with which only S waves are concerned). Then, it is natural to consider from the viewpoint of ray theory that the radial dependence of a normal mode vibration (radial eigenfunction) results from the interference between the upgoing and downgoing components of these specific body waves. In fact, ODAKA and USAMI (1978) confirmed the close agreement between the surface displacements calculated for the incidence on the free surface of plane P and S waves with the specific amplitude ratio and angle of incidence and the surface values of the radial eigenfunctions (displacement) for the spheroidal oscillations of a homogeneous Earth.

2.8b. Case when the angle f_0 is real

The ray situation is shown in Fig. 8B, and we immediately get the formal equation as

$$1=[sS], \tag{8.6}$$

which is explicitly represented as

$$1=R^0_{ss}e^{-2i\bar{Q}_a}, \tag{8.7}$$

where the phase term $\bar{Q}_a$ is defined in Eq. (8.3). This equation is identical to the Eq. (15.2) of ODAKA (1980b). Noting that $|R^0_{ss}|=1$, we can reduce Eq. (8.7) to

$$\cos(\bar{Q}_a-\varepsilon_0)=0, \tag{8.8}$$

where the phase ε_0 is defined in Eq. (7.4).

The solutions (eigenfrequencies) for the above equation can directly be obtained as

$$f_j=\{j-(1/4)+(\varepsilon_0/\pi)\}/\{(l_s/\beta_2)-(a\Delta_s/c_0)\}, \tag{8.9}$$

where j is the integer. Hence the frequencies are equally spaced for a given ray parameter and thus, unlike in the preceding case, we cannot expect the solotone effect in their distribution. This is because no conversion of the waves occurs at the free surface.

2.9a. Case when the angles i_0, i_1 and f_0 are real

2.9b. Case when the angles i_1 and f_0 are real

2.9c. Case when the angle i_1 is real

Let us suppose that the P-wave velocity in the core is lower than the S-wave velocity in the mantle. This Earth model is unrealistic but possible theoretically and experimentally. Then there exist another three cases as shown in Fig. 9(A), (B) and (C) respectively. In the first two cases, we have the ray propagation in the mantle and in the core independently for a given ray parameter and, in the third case, we have only the P wave propagation in the core.

The ray geometries in the mantle of the cases (A) and (B) have already been met in Sec. 2.8a and 2.8b respectively and they were discussed there. As for the P-wave propagation in the core, which is similar to the S-wave propagation in the mantle treated in Sec. 2.8b, we get the same form of equation as Eq. (8.7), that is,

$$1-R^1_{pp}e^{-2i\bar{P}_c}=0\,, \tag{9.1}$$

where the phase term $\bar{P}_c$ is the same as that defined in Eq. (2.3). This equation is identical to the one of Eq. (16.8) of ODAKA (1980b) who derived it on the basis of normal mode theory.

Noting that $|R^1_{pp}|=1$ and putting

$$R^1_{pp}=\{(R^2_1-R^2_2)+iR^2_3\}/\{(R^2_1-R^2_2)-iR^2_3\}=e^{2i\varepsilon_1}\,, \tag{9.2}$$

we can simplify Eq. (9.1) as

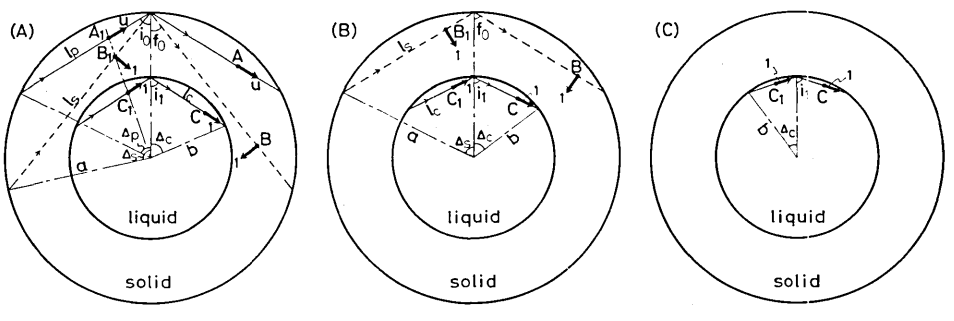

Fig. 9. Schematic illustration of the interference condition of body waves in the Earth when the angles of incidence i_0, f_0 and i_1 are real (case A), the angles f_0 and i_1 are real (case B) and the angle i_1 is real (case C).

$$\sin(\bar{P}_c - \varepsilon_1) = 0\,. \tag{9.3}$$

Hence, its solutions (eigenfrequencies) are given by

$$f_j = \{j + (1/4) + (\varepsilon_1/\pi)\}/\{(l_c/\alpha_1) - (a\Delta_c/c_0)\}\,, \tag{9.4}$$

where j is the integer. These frequencies are distributed with equal spacing for a given ray parameter. In the present case, the waves are totally reflected at the boundary and, moreover no conversion of the waves occurs there like in the previous cases (Sections 2.7, 2.8b).

The solution (9.4) yields the approximate eigenfrequencies of the spheroidal oscillations of the Earth that are attributable to multiple total reflections of the P waves in the liquid core. We will be able to find similar wave propagation in low velocity zones in the upper mantle and in the ocean where the trapping of the elastic waves is expected to occur.

2.10. Numerical computation of eigenfrequencies

It has been proved in previous sections that a certain interference condition of body waves reflected multiply in the Earth leads to the ray-theoretical frequency equations that are identical to the asymptotic forms of the exact frequency equation of the free oscillation of the Earth. There is no doubt that those equations generally yield good approximations to true eigenfrequencies when they are applied to the modes of very high frequency and large order numbers. It is, however, difficult to estimate their accuracy over a wide range of the modes. The easy way to do this is to compare the two kinds of solutions, the exact and approximate ones, by numerical computation.

Here we assume an Earth model employed by SATÔ and USAMI (1964), which is defined by the constants

$$a = 6370\text{ km}, \quad b = 3470\text{ km}, \quad \rho_1/\rho_2 = 2.2,$$

$$\alpha_1 = 10.0, \quad \alpha_2 = 11.55, \quad \beta_2 = 6.667\ \text{(km/sec)}.$$

The procedure for computing ray-theoretical (or asymptotic) eigenfrequencies is as follows. First, we set the value of f_0, the angle of incidence of an S wave on the surface of the Earth. Then, other angles of incidence, i_0, i_1 etc., ray parameter and phase velocity are reduced to known quantities through Eq. (1.1). Then according to whether each of these angles of incidence is real or not, we can determine which ray-theoretical equation is available for the present ray situation among Eqs. (2.9), (3.7), (4.4) etc. Solving this equation numerically, we get a sequence of discrete eigenfrequencies for a fixed value of f_0 (or ray parameter p).

The corresponding order numbers are in turn determined from Eq. (1.1), which do not generally take integral values. We repeat this procedure for a new value of f_0. When an increment Δf_0 is taken sufficiently small, we get a nearly continuously varying frequency-order number curve for each radial mode.

The solid curves in Fig. 10 show the eigenfrequencies thus obtained as plotted against the fractional order number n and joined by solid lines. It is found that these frequencies all pertain to radial higher modes ($i \geqq 2$), $i=1$ denoting the fundamental modes after SATÔ and USAMI (1964). Circles mean the solutions of the exact frequency equation [Eq. (2.4) of ODAKA (1980b)] computed for the order number $n=1$, 2, 5(5)30, 40. As for the modes $i=1$ and $1'$, computation is made for every n, where $i=1'$, means the modes associated with boundary waves between the mantle and the core [SATÔ and USAMI (1964)]. The solutions for these modes can not be obtained in terms of the ray-theoretical method as long as we consider only realistic rays. This does not mean that the ray-theoretical

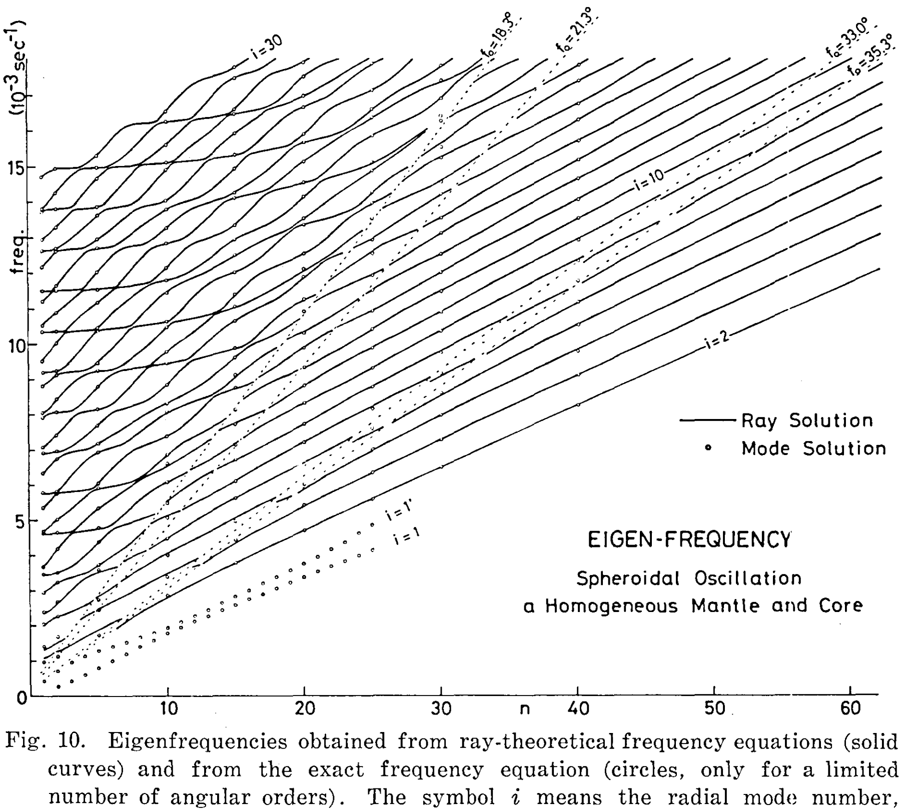

Fig. 10. Eigenfrequencies obtained from ray-theoretical frequency equations (solid curves) and from the exact frequency equation (circles, only for a limited number of angular orders). The symbol i means the radial mode number, $i=1$ being the fundamental modes. A dashed line denotes a line of a constant phase velocity.

interpretation of these modes is impossible. This kind of problem will not occur in the case of torsional oscillations.

It is found from the figure that the agreement between the exact solutions and approximate solutions is, in general, very good over the whole range of i and n exhibited here (except for the modes $i=1$ and $1'$). The excellent agreement observed for small order numbers ($n=1, 2, 5$ etc.) proves the validity of Eq. (2.9) for such modes, as suggested in Sec. 2.2.

Discontinuities in the frequency-order number (f-n) curves (solid lines) occur when they cross the dashed lines which represent the lines of certain constant phase velocities (the values of the corresponding incidence angle, f_0 are given in the figure) and separate the different ray situations in the f-n space. These five regions from upper left to lower right correspond to those shown in Figs. 2, 3, 6, 8A and 8B respectively. The above discontinuities result from the fact that the different equations are employed in the calculation of the frequencies in respective regions, and thus we can state that the ray-theoretical frequency equations do not, in general, give good approximations for nearly critical ray situations (any one of the incidence angles, i_0, i_1 etc., approaches the grazing angle, 90°).

III. Frequency equations for a radially heterogeneous Earth

3.1. Frequency equations

In Sec. II, we investigated in detail the ray-theoretical approach to the frequency equation of the spheroidal oscillations of the Earth with a uniform solid mantle and a uniform liquid core. The method developed there is entirely available for a more realistic Earth in which P- and S-wave velocities vary continuously with the radius.

Figure 11 illustrates schematically the interference condition of P and

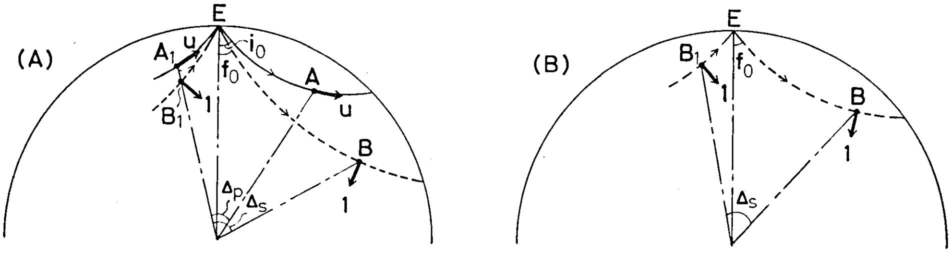

Fig. 11. Schematic illustration of the interference condition of body waves in the Earth with a continuously varying velocity structure with the radius. Case A is where the angles of incidence i_0 and f_0 are real and Case B is where the angle f_0 is real.

S waves traveling in a radially heterogeneous Earth. Here the two points A and A_1 which are taken at the same depths on the P rays are equivalent to each other as explained in Sec. 2.2. The points B and B_1 on the S rays are in a similar situation. Then, the discussion made in Sec. 2.2 still applies to the present case, and we get the following formal equations for Case (A),

$$\begin{aligned} u&=u[pP]+[sP]\,, \\ 1&=[sS]+u[pS]\,, \end{aligned} \tag{1.1}$$

and for Case (B),

$$1=[sS]\,. \tag{1.2}$$

These equations are the same as Eqs. (II. 8.1) and (II. 8.6) respectively and are, like in the previous case, transformed into

$$\begin{aligned} u&=[uR^0_{pp}+R^0_{sp}]\,\mathrm{e}^{-2i\bar{P}_a}\,, \\ 1&=[R^0_{ss}+uR^0_{ps}]\,\mathrm{e}^{-2i\bar{Q}_a}\,, \end{aligned} \tag{1.3}$$

for Case (A), and

$$1=R^0_{ss}\mathrm{e}^{-2i\bar{Q}_a}\,, \tag{1.4}$$

for Case (B). The reflection coefficients at the free surface can be defined in a similar manner to the homogeneous case. The phase terms $\bar{P}_a$ and $\bar{Q}_a$ are also given in similar forms to Eqs. (II. 3.3) and (II. 8.3) respectively. However, in the present case, the rays do not, in general, make straight lines, and thus the terms related to the intercept time (the first two terms in these equations) have to be defined in integral forms along the curved P and S rays respectively, which can further be transformed into integrals with respect to the radius r. Consequently, we get

$$\begin{aligned} 2\bar{P}_a&=2\omega\int_{r_p}^{a}(1/r)\{(r/\alpha_2)^2-p^2\}^{1/2}dr-(\pi/2)\,, \\ 2\bar{Q}_a&=2\omega\int_{r_s}^{a}(1/r)\{(r/\beta_2)^2-p^2\}^{1/2}dr-(\pi/2)\,, \end{aligned} \tag{1.5}$$

where $\alpha_2(r)$ and $\beta_2(r)$ mean the P- and S-wave velocities in the Earth respectively, and r_p and r_s the radial distances from the center of the Earth to the deepest points (turning points) of the P and S rays specified by a given ray parameter p respectively.

The same manipulation as developed in Sec. 2.8 leads to

$$\cos(\bar{P}_a+\bar{Q}_a)-R^0_{pp}\cos(\bar{P}_a-\bar{Q}_a)=0\,, \tag{1.6}$$

for Cose (A), and

$$\cos(\bar{Q}_a - \varepsilon_0) = 0\,, \tag{1.7}$$

for Case (B), where ε_0 is defind in the same relationship as in Eq. (II. 7.4). KENNETT and WOODHOUSE (1978) have obtained an equation equivalent to Eq. (1.6) on the basis of the asymptotic theory for the free oscillations of the Earth. It is of interest to point out the fact that in our theory, when the parameter u is given (for a fixed value of the ray parameter), the phases $\bar{P}_a$ and $\bar{Q}_a$ are reduced to known quantities, whereas in their theory, when the mode solution $U(a)/V(a)$, the ratio of radial to tangential surface displacements, is given, their functions corresponding to $\bar{P}_a$ and $\bar{Q}_a$ are reduced to known quantities. This fact reveals a close connection between the two quantities, u and $U(a)/V(a)$, as discussed in Sec. 2.8a.

The solutions of Eq. (1.7) can explicitly be obtained as, corresponding to Eq. (II. 8.9),

$$f_j = \{j - (1/4) + (\varepsilon_0/\pi)\}/(2\chi_s)\,, \tag{1.8}$$

where j is the integer and χ_s is given by

$$\chi_s = \int_{r_s}^{a} (1/r)\{(r/\beta_2)^2 - p^2\}^{1/2} dr\,. \tag{1.9}$$

In the above discussion, we have presupposed an Earth without a discontinuity. Next, we assume an Earth consisting of a non-uniform solid mantle and a non-uniform liquid core, in each of which the wave velocities vary continuously with the radius. The above equations are still applicable to a ray situation in which neither the P or S rays in the mantle reach the core boundary. In other cases we have to develop the same discussions as in Sec. 2.2 through 2.7. Then, we are sure to be led to formally identical frequency equations to those obtained there. The only difference is that the phase terms in the equations have to be defined in integral forms as, corresponding to Eq. (II. 2.3),

$$\begin{aligned} \delta\bar{P} &= \omega\int_b^a (1/r)\{(r/\alpha_2)^2 - p^2\}^{1/2} dr\,, \\ \delta\bar{Q} &= \omega\int_b^a (1/r)\{(r/\beta_2)^2 - p^2\}^{1/2} dr\,, \\ 2\bar{P}_c &= 2\omega\int_{r_c}^{b} (1/r)\{(r/\alpha_1)^2 - p^2\}^{1/2} dr - (\pi/2)\,, \end{aligned} \tag{1.10}$$

where $\alpha_1(r)$ and $\alpha_2(r)$ are the P-wave velocities in the core and mantle respectively as functions of the radius, $\beta_2(r)$ the S-wave velocity in the mantle, a and b the radii of the Earth and the core and r_c is the radial distance from the center of the Earth to the deepest point (turning point) of the P ray in the core.

Once the phase terms are provided in the integral forms as in Eqs. (1.5) and (1.10), all the frequency equations that are obtained in Sec. II become applicable for the Earth with a radially heterogeneous mantle and core.

3.2. Numerical computation of eigenfrequencies

For the ray-theoretical approach to be valid for the wave propagation in a non-uniform elastic medium, it is required that the relative change of the elastic parameters is very small over a wavelength. Then we can transform the general equation of motion in the elastic medium into the so-called wave equation [*e. g.*, BEN-MENAHEM (1960)], which will be further reduced to the eikonal equation yielding the fundamentals of ray theory [*e. g.*, OFFICER (1958)]. From a practical point of view, however, it is convenient to execute a numerical computation for rough estimations of the accuracy of the ray-theoretical frequency equations. Here we do the computation for two Earth models with realistic mantle structures by

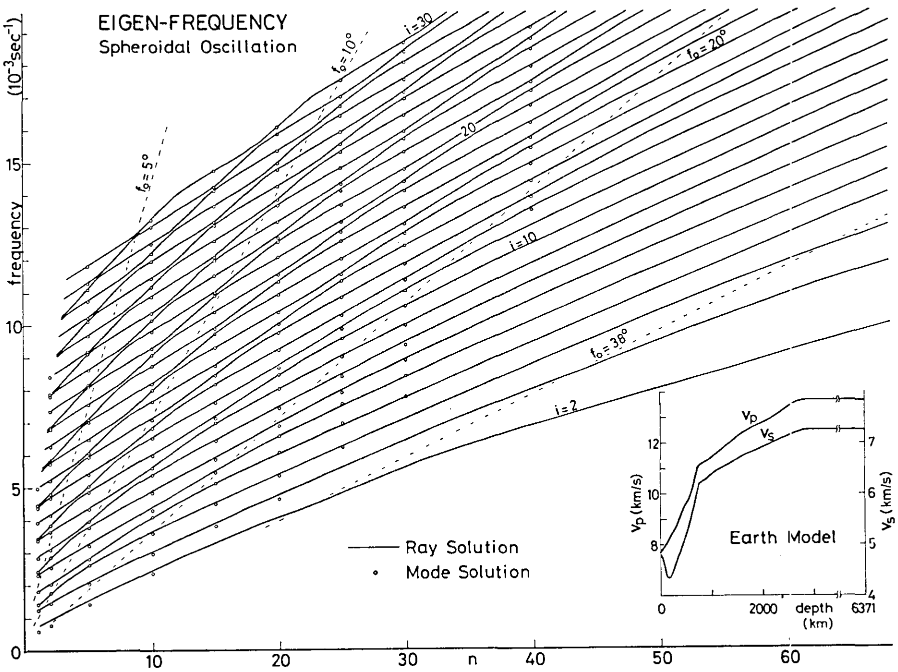

Fig. 12. Eigenfrequencies obtained from the ray-theoretical frequency equations (solid curves). The velocity structure of the Earth is given in the inset where the uniform solid core is assumed. Circles denote the mode solutions computed by means of the matrix method. A dashed line means a line of a constant phase velocity.

restricting our interest to the modes in a relatively low frequency range.

Figure 12 exhibits the frequency-order number curves for the Earth consisting of a solid mantle of Model 1066A [GILBERT and DZIEWONSKI (1975), with a slight modification] and a solid uniform core. The velocity structure is shown in the inset. This model is designed so that it contains no first order discontinuity. Hence we can compute all the ray-theoretical frequencies on the basis of the two simple equations, (1.6) and (1.7). Then, following the procedure similar to that mentioned in Sec. 2.10, we get the solid curves in the figure. For reference, the lines of some constant phase velocities are denoted by the straight dashed lines (the values of the corresponding incidence angle, f_0, are given in the figure). Equations (1.6) or (1.7) are employed according to whether or not f_0 is smaller than 38.4°. The symbol i signifies the radial mode number, $i=2$ being the first higher modes. As in the previous case (Fig. 10), the curve for the fundamental modes is missing.

Circles denote the eigenfrequencies computed in terms of the matrix method [ODAKA (1980a)]. Then the model was approximated by the stack of 92 uniform spherical layers in welded contact. These solutions stand for the exact eigenfrequencies and are obtained only for a limited number of modes. Generally speaking, the agreement between the two kinds of solutions is very good and, in most cases, the differences are within 1.5 percent.

An attempt to obtain eigenfrequencies on the basis of an asymptotic frequency equation has already been made by PEKERIS (1965) for the torsional oscillations of the Earth.

Here it should be noted that the ray-theoretical method requires only the velocity distribution in the Earth while the method based on the normal mode theory requires both velocity and density distributions (in the above case, we assumed the density distribution analogous to that of Model 1066A). This point has been brought out by PEKERIS (1965) and NOLET and KENNETT (1978) as the asymptotic property of the normal modes.

The next numerical example is concerned with a more realistic Earth in which the velocity structure is given by the Gutenberg model [see the inset in Fig. 13 or in more detail USAMI *et al.* (1965)]. The computation is made on the basis of the equations derived in Sec. II [Eqs. (II. 2.9), (II. 3.7), (II. 4.4) etc.]. The phases are evaluated in terms of Eqs. (1.5) and (1.10). We then overlook the effect of the reflection of waves at the Moho discontinuity and the inner core (liquid) boundary. The density ratio at the mantle-core boundary is taken from the Bullen A′ model,

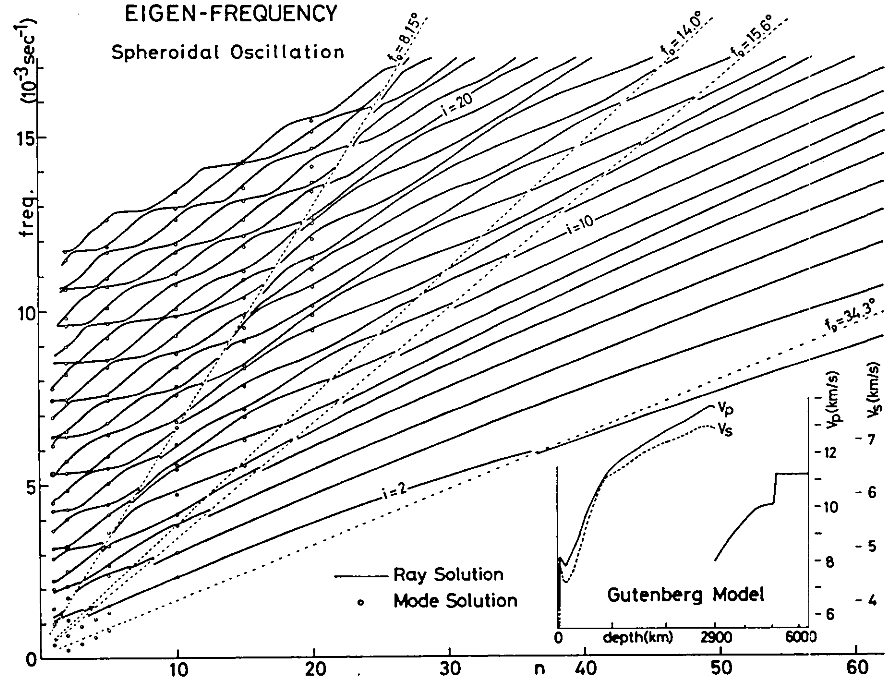

Fig. 13. Eigenfrequencies obtained from the ray-theoretical frequency equations (solid curves). The velocity structure of the Earth is given in the inset. Circles denote the mode solutions computed by means of the matrix method. A dashed line means a line of a constant phase velocity.

which is required for the computation of the reflection coefficients there. The eigenfrequencies thus obtained are shown by solid curves in Fig. 13, where i is the radial mode number ($i=2$ being the first higher modes). Dashed lines denote the lines of certain constant phase velocities (the values of the corresponding incidence angle, f_0, are given in the figure), which separate the different ray situations in the f-n (frequency-order number) space. The five regions from upper left to lower right correspond to those shown in Figs. 2, 3, 6, 8A and 8B respectively. Discontinuities in the solid curves on both sides of the dashed lines suggest the invalidity of the ray-theoretical frequency equations for nearly critical ray situations.

Circles represent the mode solutions computed in terms of the matrix method. In applying it, we have substituted the mantle and the core by 43 and 27 homogeneous spherical layers respectively and assumed the Bullen A′ model as the density distribution in the Earth. They are obtained only for a limited number of modes. But the agreement between the ray-theoretical solutions and mode solutions are, in general, very good and the differences are in most cases within 1.5 percent.

In view of the satisfactory result attained in the above two numerical examples, we can positively state that the ray-theoretical frequency equations are applicable for a radially heterogeneous Earth. These equations are very simple to express, and thus it is quite easy to obtain approximate spheroidal eigenfrequencies of a realistic Earth. Equations (II. 2.9) and (II. 3.7) are especially applicable with comparatively high accuracy, because these equations are concerned with high radial modes with high phase velocities and the effect of Moho discontinuity is small while the effect of the mantle-core boundary is rather great. As for relatively low radial modes with large order numbers, the effect of the Moho discontinuity will not be very small and thus the equations obtained in this paper will not be so effective. Hence it is desirable for such modes to construct ray-theoretical frequency equations anew by allowing for the effect of a solid-solid interface.

IV. Summary

The derivation of frequency equations in terms of ray theory has been accomplished for the spheroidal oscillations of the Earth consisting of a uniform solid mantle and a uniform liquid core. The formulation was made on the basis of a certain interference condition of body waves subjected to multiple reflections in the Earth and it is shown that the equations thus obtained are identical to the asymptotic frequency equations derived in terms of the normal mode theory. Then, by virtue of the interference condition thus proved to be valid, we can interpret a free oscillation (eigenvibration) of the Earth from the viewpoint of ray theory as; it is a state in which the amplitudes of P and S waves inherent in respective modes are kept constant at any given depth in the Earth, and the numerical formulation of this situation (or the condition which ensures this situation) is nothing but a frequency equation.

The wave conversion of P to S or S to P at a boundary in the Earth induces the solotone effect in the distribution of eigenfrequencies of the modes for a fixed phase velocity.

The ray-theoretical method developed above is also available for the Earth of which elastic parameters vary continuously as functions of the radius in the mantle and in the core respectively. Then we get formally identical frequency equations to those for the previous homo./homo. case. The validity of the equations is confirmed by the numerical computation for a realistic Earth model. Employing those equations, we can evaluate with great ease the approximate eigenfrequencies of the spheroidal oscil-

lations of a radially heterogeneous Earth. The equations then will be more effective for high radial modes with small angular orders.

In view of the present satisfactory results, we can state that the ray-theoretical approach to the frequency equations of free oscillations of a spherical Earth has in principle been established in this paper. This method will be applicable more easily to torsional oscillations because only S waves are concerned with the problem and no conversion of the waves occurs at boundaries in the Earth.

Recently, some attempts have been made to apply the asymptotic frequency equations of free oscillations to the inversion problem to infer the velocity structure of the Earth from free oscillation (or dispersion) data [KENNETT and WOODHOUSE (1978), BRODSKIĬ and LEVSHIN (1979)]. We will make a similar attempt in a future paper on the torsional oscillations by taking into account the effect of Mono discontinuity.

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Appendix

Reflection and transmission coefficients

If we suppose that a plane P or S wave is incident on a plane boundary between two homogeneous media, then we can get the amplitudes of reflected and refracted waves with ease by use of the conventional method [*e. g.*, EWING *et al.* (1957)]. Here we are interested in a free solid boundary (the free surface of the Earth) and a solid-liquid interface (the mantle-core boundary).

Reflection and transmission coefficients are defined as the amplitudes of displacement of reflected and refracted waves as against the unit amplitude (in displacement) of an incident P or S wave on a boundary. In Fig. A are shown the notations of the coefficients by R^i_{xy} (reflection) and T^i_{xy} (transmission) and the positive directions of displacement of respective waves are defined by bold arrows. Solid and dashed lines denote P and S rays respectively. The two media, 1 and 2, correspond to the core and

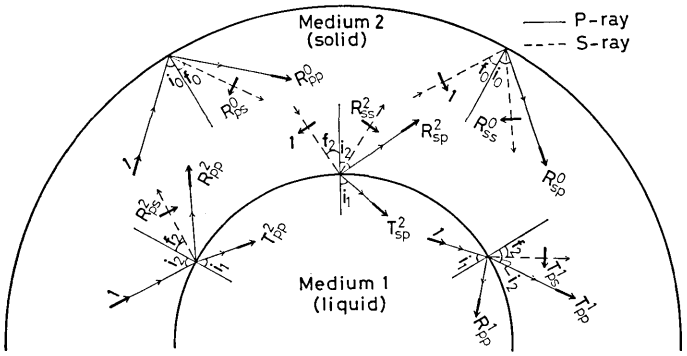

Fig. A. Reflection and transmission coefficients defined for plane P- and S-wave incidence on the plane boundaries, free surface and mantle-core boundary, which are shown by curved surfaces for convenience of comparison with the spherical Earth.

the mantle respectively and, for sake of convenience, those boundaries are delineated as curved surfaces. The numerical superscripts, 0, 1 and 2, associated with the coefficients refer to the waves incident upon the free surface and upon the mantle-core boundary from inside the core (medium 1) and from inside the mantle (medium 2) respecsively. Angles of incidence (reflection, refraction) are written as i_i (subscript $i=0, 1, 2$) for the P waves and f_i ($i=0, 2$) for the S waves.

The density, rigidity, P- and S-wave velocities are noted as ρ_i, μ_i, α_i and β_i respectively ($i=1$ and 2 referring to the medium 1 (core) and the medium 2 (mantle) respectively). Then we get the following formulas.

$$R^0_{pp}=R^0_{ss}=[-\{(c_0/\beta_2)^2-2\}^2+4\xi_0\eta_0]/\Delta_0\,,$$

$$R^0_{ps}=-(\alpha_2/\beta_2)\cdot 4\xi_0\{(c_0/\beta_2)^2-2\}/\Delta_0\,,$$

$$R^0_{sp}=(\beta_2/\alpha_2)\cdot 4\eta_0\{(c_0/\beta_2)^2-2\}/\Delta_0\,,$$

$$\Delta_0=\{(c_0/\beta_2)^2-2\}^2+4\xi_0\eta_0\,,$$

$$R^2_{pp}=\{\mu_2\xi_1[-\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]+\rho_1\xi_2(c^2/\beta_2)^2\}/\Delta_2\,,$$

$$R^2_{ss}=\{\mu_2\xi_1[-\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]-\rho_1\xi_2(c^2/\beta_2)^2\}/\Delta_2\,,$$

$$R^1_{pp}=\{\mu_2\xi_1[\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]-\rho_1\xi_2(c^2/\beta_2)^2\}/\Delta_2\,,$$

$$R^2_{ps}=(\alpha_2/\beta_2)\cdot 4\mu_2\xi_1\xi_2\{(c/\beta_2)^2-2\}/\Delta_2\,,$$

$$R^2_{sp}=-(\beta_2/\alpha_2)\cdot 4\mu_2\xi_1\eta_2\{(c/\beta_2)^2-2\}/\Delta_2\,,$$

$$T^2_{pp}=(\alpha_2/\alpha_1)\cdot 2\mu_2\xi_2(c/\beta_2)^2\{(c/\beta_2)^2-2\}/\Delta_2\,,$$

$$T^2_{sp}=(\beta_2/\alpha_1)\cdot 4\mu_2\xi_2\eta_2(c/\beta_2)^2/\Delta_2\,,$$

$$T^1_{pp}=(\alpha_1/\alpha_2)\cdot 2\rho_1\xi_1c^2\{(c/\beta_2)^2-2\}/\varDelta_2\,,$$

$$T^1_{ps}=-(\alpha_1/\beta_2)\cdot 4\rho_1\xi_1\xi_2c^2/\varDelta_2\,,$$

$$\varDelta_2=\{\mu_2\xi_1[\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]+\rho_1\xi_2(c^2/\beta_2)^2\}\,, \tag{A.1}$$

where

$$\xi_0=\sqrt{(c_0/\alpha_2)^2-1}\,,\quad \xi_2=\sqrt{(c/\alpha_2)^2-1}\,,\quad \xi_1=\sqrt{(c/\alpha_1)^2-1}\,,$$

$$\eta_0=\sqrt{(c_0/\beta_2)^2-1}\,,\quad \eta_2=\sqrt{(c/\beta_2)^2-1}\,, \tag{A.2}$$

$$c_0=\alpha_2/\sin i_0=\beta_2/\sin f_0\,,$$

$$c=\alpha_2/\sin i_2=\beta_2/\sin f_2=\alpha_1/\sin i_1\,.$$

The coefficients in Eq. (A.1) are related to one another with the relations,

$$(R^0_{pp})^2-R^0_{ps}R^0_{sp}=1\,,$$

$$R^2_{pp}R^2_{ss}-R^2_{ps}R^2_{sp}=R^1_{pp}\,,$$

$$R^2_{pp}T^1_{ps}-R^2_{ps}T^1_{pp}=T^1_{ps}\,,$$

$$R^2_{ss}T^1_{pp}-R^2_{sp}T^1_{ps}=-T^1_{pp}\,,$$

$$R^1_{pp}R^2_{pp}-T^1_{pp}T^2_{pp}=R^2_{ss}\,, \tag{A.3}$$

$$R^1_{pp}R^2_{ss}-T^2_{sp}T^1_{ps}=R^2_{pp}\,,$$

$$R^1_{pp}R^2_{sp}-T^2_{sp}T^1_{pp}=R^2_{sp}\,,$$

$$R^1_{pp}R^2_{ps}-T^2_{pp}T^1_{ps}=R^2_{ps}\,,$$

$$T^1_{pp}T^2_{pp}-T^1_{ps}T^2_{sp}=(1+R^1_{pp})(1-R^1_{pp})\,.$$

The coefficients in Eq. (A.1) are not always defined in a real domain when any one of the angles of incidence i_0, i_1 etc. takes a complex value. Hence we factor them into several terms and define the following coefficients which take real values.

$$R^0_1=\{(c_0/\beta_2)^2-2\}^2\,,\quad R^0_2=4|\xi_0\eta_0|\,,$$

$$R^2_1=\mu_2|\xi_1|\{(c/\beta_2)^2-2\}^2\,,\quad R^2_2=4\mu_2|\xi_1\xi_2\eta_2|\,,$$

$$R^2_3=\rho_1|\xi_2|(c^2/\beta_2)^2\,, \tag{A.4}$$

$$R^2_4=(\alpha_2/\beta_2)\cdot 4\mu_2|\xi_1\xi_2|\{(c/\beta_2)^2-2\}\,,$$

$$R^2_5=(\beta_2/\alpha_2)\cdot 4\mu_2|\xi_1\eta_2|\{(c/\beta_2)^2-2\}\,.$$

15. 波線理論による弾性球の伸び縮み振動の特性方程式の導出

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近年，自由振動の特性方程式の漸近的性質に関する研究が盛んであるが，殆んどの場合がノーマル・モード理論に基づいた議論である．しかし，それが短周期近似による議論である以上，得られる主要項は波線理論的に解釈されるべき量の筈である．にもかかわらずこの方面からの研究は十分には行われていない．

本研究の第II節では，均質なマントルと均質な流体核より成る地球を仮定し，その伸び縮み振動の特性方程式を波線理論的に導くことを試みる．先ず，球内部で多重反射する実体波の或る種の干渉条件より特性方程式を導く．その際種々の波線伝播状態に対応して異なった式が得られるが，それらはノーマル・モード理論によって既に得られている漸近的特性方程式に完全に一致する．この事は，用いられた実体波の干渉条件の妥当性を証明するものである．それは同時に，自由（固有）振動の波線理論的解釈を明らかにするものでもある．言い換えると，自由（固有）振動とは，各モードに固有の P 波と S 波の振幅が球内のある深さで常に一定に保たれている状態ということが出来る．またその状態を数式化したもの（その状態を保証するもの）が特性方程式である．

第III節では，上の波線理論による方法が，弾性常数が深さの関数として連続的に変化している球モデルにも適用できることを示す．その際得られる特性方程式は，形式的には上の均質モデルの場合に導かれた式に一致し，その有効性は数値計算によって証明される．これらの式は非常に単純な表現を有しており，それを用いて不均質球モデルに対する近似的固有振動数が容易に得られる．それは特に，angular order の比較的小さな高次のモードに対して有効である．この波線理論的特性方程式の導出は，ノーマル・モード理論に基づいて漸近式を得る方法に比べ，ここで取扱った問題に関して言えば著しく簡単である．