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15. *Asymptotic Frequency Equations for Spheroidal Oscillations of a Spherical Earth with a Uniform Mantle and Core.*

— Modes of Finite Phase Velocity —

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Abstract

An attempt is made of deriving asymptotic frequency equations valid for the modes of very high frequency and finite phase velocity. The method is based on expanding an exact frequency equation into its asymptotic form by the use of the asymptotic formulas for the spherical Bessel functions and spherical Neumann functions which appear in it. An Earth is assumed to consist of a uniform solid mantle and a uniform liquid core. The equations are expressed in different forms corresponding to the different ray geometries in the Earth and are denoted in terms of reflection and transmission coefficients and intercept times of relevant P and S rays.

1. Introduction

SATO and LAPWOOD (1977a, b) have obtained asymptotic frequency equations of torsional oscillations of layered spherical shells valid for the modes of very high phase velocity and finite phase velocity respectively. The equations are expressed as functions of frequency and ray parameter including reflection coefficients of SH waves at internal discontinuities and times of transit of the waves through the layers.

As for spheroidal oscillations, ODAKA (1978) has derived asymptotic formulas of the frequency equation of a uniform Earth in terms of both normal mode theory and ray theory. Investigation into spheroidal modes of a layered Earth has been made by ODAKA (1980a), who has obtained frequency equations valid at very high phase velocity. Then he has shown that the exact frequency equation is decoupled corresponding to the decoupling of P and S waves in the Earth which occurs for the radial propagation of the waves in it.

As the next stage of investigation, we will try to get spheroidal frequency equations valid for the modes of very high frequency and finite phase velocity. In this case, not only pure P and S waves but also waves converted from one wave type to another at boundaries in the medium are concerned with the formation of the equations. This will make the problem quite complicated when we try to express them in terms of ray-geometrical parameters. Hence, in this paper, discussion will be made on the simplest model, that is, an Earth with a uniform solid mantle and liquid core. However, once a procedure is established, its extension to a model with one or two discontinuities in the mantle will be possible.

2. Frequency equation

We assume a spherically symmetric two-layered Earth consisting of a uniform solid mantle and a uniform liquid core. Then, a frequency equation of spheroidal oscillations of the Earth which determines their eigenfrequencies is obtained from the requirements that the radial and tangential stresses on the surface of the Earth and the tangential stress on the mantle-core boundary vanish and that the radial displacement and stress are continuous across the mantle-core boundary.

Following ODAKA (1980a), who formulated spheroidal frequency equations for multi-layered elastic spheres in terms of the matrix method, we can readily obtain the formal frequency equation relevant to the present case:

$$\mathfrak{F}=\begin{vmatrix} e_{11}^{1}(b) & -e_{11}^{2}(b) & -e_{12}^{2}(b) & -e_{13}^{2}(b) & -e_{14}^{2}(b) \\ e_{31}^{1}(b) & -e_{31}^{2}(b) & -e_{32}^{2}(b) & -e_{33}^{2}(b) & -e_{34}^{2}(b) \\ 0 & e_{41}^{2}(b) & e_{42}^{2}(b) & e_{43}^{2}(b) & e_{44}^{2}(b) \\ 0 & e_{31}^{2}(a) & e_{32}^{2}(a) & e_{33}^{2}(a) & e_{34}^{2}(a) \\ 0 & e_{41}^{2}(a) & e_{42}^{2}(a) & e_{43}^{2}(a) & e_{44}^{2}(a) \end{vmatrix}=0\,, \tag{2.1}$$

where $e_{1k}^{i}(r)$, $e_{3k}^{i}(r)$, $e_{4k}^{i}(r)$, $(k=1,2,3,4)$ are the functions with radial dependence associated with the radial displacement, radial and tangential stresses respectively, which are defined below. Superscripts $i=1$ and 2 refer to the core and mantle, and a and b refer to the radii of the Earth and the core respectively. The radial functions in Eq. (2.1) are expressed in terms of the spherical Bessel and spherical Neumann functions, $j_n(x)$ and $n_n(x)$, as follows.

$$e^1_{11}(r)=h_1 r j'_n(h_1 r)\,, \qquad e^1_{31}(r)=-\lambda_1 (h_1 r)^2 j_n(h_1 r)\,,$$

$$\begin{aligned}
&(e^2_{1k}(r),\, k=1,2,3,4)=(h_2 r j'_n(h_2 r),\, N^2 j_n(k_2 r),\, h_2 r n'_n(h_2 r),\, N^2 n_n(k_2 r))\,,\\
&(e^2_{3k}(r),\, k=1,2,3,4)=(\mu_2 g(j_n, h_2 r),\, \mu_2 N^2 f(j_n, k_2 r),\, \mu_2 g(n_n, h_2 r),\, \mu_2 N^2 f(n_n, k_2 r))\,,\\
&(e^2_{4k}(r),\, k=1,2,3,4)=(\mu_2 f(j_n, h_2 r),\, \mu_2 h(j_n, k_2 r),\, \mu_2 f(n_n, h_2 r),\, \mu_2 h(n_n, k_2 r))\,,
\end{aligned} \tag{2.2}$$

where h_i and k_i are the wavenumbers of P and S waves, λ_i and μ_i the Lamé elastic parameters, $i=1$ and 2 referring to the core and mantle respectively, and n denotes the order of the spherical Bessel function (equivalent to the colatitudinal order number). Other notations are defined by

$$\begin{aligned}
&N^2=n(n+1)\,, \quad z'_n(\zeta_i r)=dz_n(\zeta_i r)/d(\zeta_i r)\,,\\
&f(z_n, \zeta_i r)=2\zeta_i r z'_n(\zeta_i r)-2z_n(\zeta_i r)\,,\\
&g(z_n, \zeta_i r)=-4\zeta_i r z'_n(\zeta_i r)-\{(k_i r)^2-2N^2\} z_n(\zeta_i r)\,,\\
&h(z_n, k_i r)=f(z_n, k_i r)+g(z_n, k_i r)\,.
\end{aligned} \tag{2.3}$$

Equation (2.1) is rather formal and therefore, we rewrite it in a form convenient for further expansion:

$$\begin{aligned}
\mathfrak{F}=&\{e_{31}B_{12}-e_{11}A_{12}(b)\}A_{34}(a)-\{e_{31}B_{13}-e_{11}A_{13}(b)\}A_{24}(a)\\
&+\{e_{31}B_{14}-e_{11}A_{14}(b)\}A_{23}(a)+\{e_{31}B_{23}-e_{11}A_{23}(b)\}A_{14}(a)\\
&-\{e_{31}B_{24}-e_{11}A_{24}(b)\}A_{13}(a)+\{e_{31}B_{34}-e_{11}A_{34}(b)\}A_{12}(a)=0\,,
\end{aligned} \tag{2.4}$$

where

$$e_{11}=e^1_{11}(b)\,, \qquad e_{31}=e^1_{31}(b)\,,$$

$$A_{jk}(r)=\begin{vmatrix} e^2_{3j}(r) & e^2_{3k}(r) \\ e^2_{4j}(r) & e^2_{4k}(r) \end{vmatrix}, \qquad B_{jk}=\begin{vmatrix} e^2_{1j}(b) & e^2_{1k}(b) \\ e^2_{4j}(b) & e^2_{4k}(b) \end{vmatrix}. \tag{2.5}$$

Prior to the further reduction of Eq. (2.4) to its asymptotic forms, we shall bring in the basic equation connecting the mode scheme with the ray scheme, and some asymptotic formulas for $j_n(x)$ and $n_n(x)$.

3. Modes and rays

BEN-MENAHEM (1964) found the equivalency relation between the ray parameter (p) and the inverse phase velocity of normal modes ($1/c_0$). This relation will be expressed for our spheroidal case as follows:

$$\begin{aligned}
\nu(=n+1/2)/\omega=a/c_0=p=(a/\alpha_2)\sin i_0=(a/\beta_2)\sin f_0=b/c\\
=(b/\alpha_2)\sin i_2=(b/\beta_2)\sin f_2=(b/\alpha_1)\sin i_1\,,
\end{aligned} \tag{3.1}$$

where ω is the angular eigenfrequency of any given mode with the colatitudinal order n, c the apparent velocity along the mantle-core boundary, α_i, β_i the P- and S-wave velocities in the i-th medium ($i=1$ and 2 referring to the core and mantle respectively), i_i and f_i the angles of incidence of the P and S rays on the free surface (subscript $i=0$), and on the mantle-core boundary ($i=1$ and 2 referring to the two sides of the interface, the core and mantle sides respectively). In the following discussion, we will use the relation (3.1) as the definition of connecting p with ν and ω.

The five arguments which appear in Eq. (2.4) can be expressed as

$$h_2a=\nu c_0/\alpha_2,\quad h_2b=\nu c/\alpha_2,\quad h_1b=\nu c/\alpha_1,\quad k_2a=\nu c_0/\beta_2,\quad k_2b=\nu c/\beta_2\,. \tag{3.2}$$

Equations (3.1) and (3.2) are of great use for expressing asymptotic formulas for mode solutions in terms of ray-geometrical parameters. Then, asymptotic equations of Eq. (2.4) are expressed in different forms corresponding to different situations prescribed by relative magnitudes of the five arguments in Eq. (3.2) and the order number n because the function $j_n(x)$ (similarly $n_n(x)$) shows different asymptotic properties depending on whether its argument is larger or smaller than its order.

4. Asymptotic formulas for $j_n(x)$, $n_n(x)$ etc. when $x>n+1/2\gg 1$

From the asymptotic approximations for the Bessel functions of the first and the second kind in which the argument is larger than its order, both being large and positive [*e. g.*, WATSON (1952, p. 234)], we have

$$\begin{aligned} j_n(x)&\simeq(1/x)\cos\{\bar{X}(x)\}/\sqrt{\sin\bar{y}} \\ n_n(x)&\simeq(1/x)\sin\{\bar{X}(x)\}/\sqrt{\sin\bar{y}} \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} x&=\nu\sec\bar{y}\qquad (0<\bar{y}<\pi/2,\ \nu=n+1/2) \\ \bar{X}(x)&=\nu(\tan\bar{y}-\bar{y})-\pi/4\,. \end{aligned} \tag{4.2}$$

Then, for the functions in Eq. (2.3), we get

$$\begin{aligned} j_n'(x)&\simeq-(1/x)\sin\{\bar{X}(x)\}\sqrt{\sin\bar{y}}\,, \\ n_n'(x)&\simeq(1/x)\cos\{\bar{X}(x)\}\sqrt{\sin\bar{y}}\,, \\ f(j_n,x)&\simeq-2\sin\{\bar{X}(x)\}\sqrt{\sin\bar{y}}\,, \\ g(j_n,x)&\simeq-\{(x_k^2-2N^2)/x\}\cos\{\bar{X}(x)\}/\sqrt{\sin\bar{y}}\,, \\ f(n_n,x)&\simeq2\cos\{\bar{X}(x)\}\sqrt{\sin\bar{y}}\,, \\ g(n_n,x)&\simeq-\{(x_k^2-2N^2)/x\}\sin\{\bar{X}(x)\}/\sqrt{\sin\bar{y}}\,, \end{aligned} \tag{4.3}$$

where x stands for $\zeta_i r$ ($\zeta_i = h_i$ or k_i) and x_k for $k_i r$.

In accordance with Eq. (4.2), we define the following notations for individual arguments.

$$\begin{aligned} &h_2 a = \nu \sec \bar{p}_a,\ h_2 b = \nu \sec \bar{p}_b,\ h_1 b = \nu \sec \bar{p}_c, \\ &k_2 a = \nu \sec \bar{q}_a,\ k_2 b = \nu \sec \bar{q}_q \quad (0 < \bar{p}_a, \bar{p}_b, \bar{p}_c, \bar{q}_a, \bar{q}_b < \pi/2)\,, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\bar{X}(h_2 a) = \bar{P}_a = \nu(\tan \bar{p}_a - \bar{p}_a) - \pi/4,\ \bar{X}(h_2 b) = \bar{P}_b = \nu(\tan \bar{p}_b - \bar{p}_b) - \pi/4\,, \\ &\bar{X}(h_1 b) = \bar{P}_c = \nu(\tan \bar{p}_c - \bar{p}_c) - \pi/4\,, \\ &\bar{X}(k_2 a) = \bar{Q}_a = \nu(\tan \bar{q}_a - \bar{q}_a) - \pi/4,\ \bar{X}(k_2 b) = \bar{Q}_b = \nu(\tan \bar{q}_b - \bar{q}_b) - \pi/4\,. \end{aligned} \tag{4.5}$$

Then, from Eqs. (3.1), (3.2), (4.4) and (4.5), we can show

$$\sin \bar{p}_a = \sqrt{1-(\alpha_2/c_0)^2} = \cos i_0\,,$$

and

$$\begin{aligned} 2\bar{P}_a &= 2\nu\{\tan(\pi/2 - i_0) - (\pi/2 - i_0)\} - \pi/2 \\ &= \omega\{(2a/\alpha_2)\cos i_0 - a(\pi - 2i_0)/c_0\} - \pi/2\,. \end{aligned}$$

Similar manipulations are possible for other quantities, which lead to

$$\begin{aligned} &\bar{p}_a = \pi/2 - i_0,\ \bar{p}_b = \pi/2 - i_2,\ \bar{p}_c = \pi/2 - i_1,\ \bar{q}_a = \pi/2 - f_0,\ \bar{q}_b = \pi/2 - f_2, \\ &\sin \bar{p}_a = (\alpha_2/c_0)\xi_0,\ \sin \bar{p}_b = (\alpha_2/c)\xi_2,\ \sin \bar{p}_c = (\alpha_1/c)\xi_1, \\ &\sin \bar{q}_a = (\beta_2/c_0)\eta_0,\ \sin \bar{q}_b = (\beta_2/c)\eta_2\,, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &2\bar{P}_a = \omega\{T_p - a\Delta_p/c_0\} - \pi/2,\ 2\bar{Q}_a = \omega\{T_s - a\Delta_s/c_0\} - \pi/2\,, \\ &2\bar{P}_c = \omega\{T_c - a\Delta_c/c_0\} - \pi/2\,, \\ &\delta\bar{P} = \bar{P}_a - \bar{P}_b = \omega\{T_p^* - a\Delta_p^*/c_0\},\ \delta\bar{Q} = \bar{Q}_a - \bar{Q}_b = \omega\{T_s^* - a\Delta_s^*/c_0\}\,, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} &\xi_0 = \sqrt{(c_0/\alpha_2)^2 - 1}\,,\quad \xi_2 = \sqrt{(c/\alpha_2)^2 - 1}\,,\quad \xi_1 = \sqrt{(c/\alpha_1)^2 - 1}\,, \\ &\eta_0 = \sqrt{(c_0/\beta_2)^2 - 1}\,,\quad \eta_2 = \sqrt{(c/\beta_2)^2 - 1}\,, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &T_p = (2a/\alpha_2)\cos i_0,\quad T_s = (2a/\beta_2)\cos f_0,\quad T_c = (2b/\alpha_1)\cos i_1\,, \\ &T_p^* = (a\cos i_0 - b\cos i_2)/\alpha_2,\quad T_s^* = (a\cos f_0 - b\cos f_2)/\beta_2\,, \\ &\Delta_p = \pi - 2i_0,\quad \Delta_s = \pi - 2f_0,\quad \Delta_c = \pi - 2i_1,\quad \Delta_p^* = i_2 - i_0,\quad \Delta_s^* = f_2 - f_0\,. \end{aligned} \tag{4.9}$$

T, T^*, Δ and Δ^* are the travel times and the angular distances associated

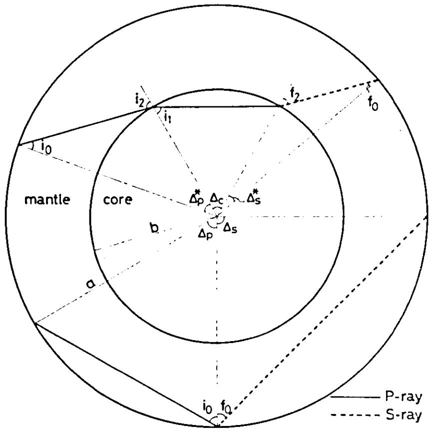

Fig. 1. Schematical illustration of possible rays in the Earth consisting of a uniform solid mantle and liquid core.

with the ray segments shown in Fig. 1. Now we have real angles of incidence corresponding to any arguments satisfying the present constraint, $x>\nu$. When both i_0 and i_2 are defined as real quantities, we use the term $\delta\bar{P}$. However, when i_0 is real but i_2 has to be defined as a complex number, the term $2\bar{P}_a$ will become useful. For S rays, similar situations occur and we will use two terms, $2\bar{Q}_a$ and $\delta\bar{Q}$, properly.

Allowing for the relation $p=a/c_0=dT/d\Delta$, we find that the quantity $T-a\Delta/c_0$ represents the intercept time of a tangent to a travel time curve (T, Δ). Hence, the quantities $2\bar{P}_a$, $2\bar{Q}_a$ etc. in Eq. (4.7) are closely connected with the intercept times for the corresponding rays.

5. Asymptotic formulas for $j_n(x)$, $n_n(x)$ etc. when $n+1/2>x\gg 1$

From the asymptotic expansions of the Bessel functions of the first and the second kind in which the argument is less than the order, both being large and positive [*e. g.*, WATSON (1952, p. 243)], we get

$$\begin{aligned} j_n(x) &\simeq (1/2x)\exp\{\hat{X}(x)\}/\sqrt{\sinh\hat{y}}\,, \\ n_n(x) &\simeq -(1/x)\exp\{-\hat{X}(x)\}/\sqrt{\sinh\hat{y}}\,, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} x &= \nu\,\mathrm{sech}\,\hat{y} \quad (\nu=n+1/2)\,, \\ \hat{X}(x) &= \nu(\tanh\hat{y}-\hat{y})\,. \end{aligned} \tag{5.2}$$

Then, we get the approximations

$$
\begin{aligned}
&j_n'(x) \simeq (1/2x) \exp\{\hat{X}(x)\} \sqrt{\sinh \hat{y}}\,, \\
&n_n'(x) \simeq (1/x) \exp\{-\hat{X}(x)\} \sqrt{\sinh \hat{y}}\,, \\
&f(j_n, x) \simeq \exp\{\hat{X}(x)\} \sqrt{\sinh \hat{y}}\,, \\
&g(j_n, x) \simeq -\{(x_k^2 - 2N^2)/2x\} \exp\{\hat{X}(x)\}/\sqrt{\sinh \hat{y}}\,, \\
&f(n_n, x) \simeq 2 \exp\{-\hat{X}(x)\} \sqrt{\sinh \hat{y}}\,, \\
&g(n_n, x) \simeq \{(x_k^2 - 2N^2)/x\} \exp\{-\hat{X}(x)\}/\sqrt{\sinh \hat{y}}\,,
\end{aligned}
\tag{5.3}
$$

where x stands for $\zeta_i r$ ($\zeta_i = h_i$ or k_i) and x_k for $k_i r$.

In accordance with Eq. (5.2), we define

$$
\begin{aligned}
&h_2 a = \nu \operatorname{sech} \hat{p}_a, \quad h_2 b = \nu \operatorname{sech} \hat{p}_b, \quad h_1 b = \nu \operatorname{sech} \hat{p}_c, \\
&k_2 a = \nu \operatorname{sech} \hat{q}_a, \quad k_2 b = \nu \operatorname{sech} \hat{q}_b\,,
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
&\hat{X}(h_2 a) = \hat{P}_a = \nu(\tanh \hat{p}_a - \hat{p}_a), \quad \hat{X}(h_2 b) = \hat{P}_b = \nu(\tanh \hat{p}_b - \hat{p}_b)\,, \\
&\hat{X}(h_1 b) = \hat{P}_c = \nu(\tanh \hat{p}_c - \hat{p}_c)\,, \\
&\hat{X}(k_2 a) = \hat{Q}_a = \nu(\tanh \hat{q}_a - \hat{q}_a), \quad \hat{X}(k_2 b) = \hat{Q}_b = \nu(\tanh \hat{q}_b - \hat{q}_b)\,.
\end{aligned}
\tag{5.5}
$$

Then, from Eqs. (3.1), (3.2) and (5.4), we have

$$
\sinh \hat{p}_a = \sqrt{(\alpha_2/c_0)^2 - 1} = i(\alpha_2/c_0)\xi_0\,,
$$

where i is the imaginary unit. Similar manipulations lead us to

$$
\begin{aligned}
&\sinh \hat{p}_a = i(\alpha_2/c_0)\xi_0, \quad \sinh \hat{p}_b = i(\alpha_2/c)\xi_2, \quad \sinh \hat{p}_c = i(\alpha_1/c)\xi_1, \\
&\sinh \hat{q}_a = i(\beta_2/c_0)\eta_0, \quad \sinh \hat{q}_b = i(\beta_2/c)\eta_2\,.
\end{aligned}
\tag{5.6}
$$

Now, the coefficients such as ξ_0, ξ_1 defined by Eq. (4.8) become imaginary numbers, and then we define them as

$$
\xi_j = -i|\xi_j| \quad (j = 0, 1, 2), \qquad \eta_j = -i|\eta_j| \quad (j = 0, 2)\,.
\tag{5.7}
$$

The angles of incidence i_0, f_0 etc. in Eq. (3.1) now have to be defined as complex numbers, and thus the corresponding body waves have a property of inhomogeneous plane waves [BREKHOVSKIKH (1960, p. 4)].

From the nature of the function

$$
f(y) = \nu(\tanh y - y)
$$

which appears in Eq. (5.5), that is, it is a monotonically decreasing function with y and $f(y) < 0$ for $y > 0$, we get the following relations for large ν:

$$
\begin{aligned}
&\exp\{-\hat{P}_b\} \gg 1\,, \qquad \exp\{-\hat{Q}_b\} \gg 1\,, \\
&\exp\{\delta \hat{P}\} \gg 1\,, \qquad \exp\{\delta \hat{Q}\} \gg 1\,,
\end{aligned}
\tag{5.8}
$$

where

$$\delta\hat{P}=\hat{P}_a-\hat{P}_b\,, \qquad \delta\hat{Q}=\hat{Q}_a-\hat{Q}_b\,. \tag{5.9}$$

In later sections, the above relations become very helpful in selecting the most predominant term in the frequency equation.

6. Asymptotic frequency equation

In expanding the frequency equation (2.4) into its asymptotic forms, we need to use different forms of asymptotic approximations for the functions $j_n(x)$, $n_n(x)$, $f(j_n, x)$ etc. depending on whether x is larger or smaller than $\nu(=n+1/2)$ as we studied in the previous two sections. We then have a lot of cases that have to be treated separately, each of which is specified by the relative magnitudes between the five arguments, h_2a, h_2b, h_1b, k_2a, k_2b, and the fractional order ν. This makes the problem rather troublesome. In order to avoid repeating tedious and lengthy manipulations for each case, we are required to reduce Eq. (2.4) to a form common to all cases as far as we can. To make this possible, asymptotic formulas for each function, $j_n(x)$, $n_n(x)$ etc. have to be expressed in a common form for two cases, $x>\nu$ and $x<\nu$. Hence, we write those for $j_n(x)$, $n_n(x)$ and their derivatives in common in a form

$$\begin{aligned}
&j_n(x)\simeq(1/2x)\{e^{\sigma X}+e^{-\tau X}\}/\sqrt{\mathrm{s}y}\,,\\
&n_n(x)\simeq\{1/(\sigma+\tau)x\}\{e^{\tau X}-e^{-\sigma X}\}/\sqrt{\mathrm{s}y}\,,\\
&j_n'(x)\simeq(\sigma/2x)\{e^{\sigma X}-e^{-\tau X}\}\sqrt{\mathrm{s}y}\,,\\
&n_n'(x)\simeq\{\sigma/(\sigma+\tau)x\}\{e^{\tau X}+e^{-\sigma X}\}\sqrt{\mathrm{s}y}\,,
\end{aligned} \tag{6.1}$$

with the convention that

$$\text{for } x>\nu\,, \qquad \sigma=\tau=i \text{ (imaginary unit)}, \; X=\bar{X}, \; \mathrm{s}y=\sin\bar{y}\,, \tag{6.2}$$

and

$$\text{for } x<\nu\,, \qquad \sigma=1, \; \tau=0, \; X=\hat{X}, \; e^{\pm\tau\hat{X}}=0, \; \mathrm{s}y=\sinh\hat{y}\,. \tag{6.3}$$

Then, we get

$$\begin{aligned}
&f(j_n, x)\simeq\sigma\{e^{\sigma X}-e^{-\tau X}\}\sqrt{\mathrm{s}y}\,,\\
&g(j_n, x)\simeq-\{(x_k^2-2N^2)/2x\}\{e^{\sigma X}+e^{-\tau X}\}/\sqrt{\mathrm{s}y}\,,\\
&f(n_n, x)\simeq\{2\sigma/(\sigma+\tau)\}\{e^{\tau X}+e^{-\sigma X}\}\sqrt{\mathrm{s}y}\,,\\
&g(n_n, x)\simeq-\{(x_k^2-2N^2)/(\sigma+\tau)x\}\{e^{\tau X}-e^{-\sigma X}\}/\sqrt{\mathrm{s}y}\,,\\
&h(z_n, x_k)\simeq g(z_n, x_k) \qquad (z_n=j_n \text{ or } n_n)\,.
\end{aligned} \tag{6.4}$$

It will be easily proved that the expressions (6.1) and (6.4) surely cover two different formulas, Eqs. (4.1), (4.3) and Eqs. (5.1), (5.3).

Further reduction of Eq. (2.4) will be made under these expressions, and then the convention (6.2) and (6.3) will be applied to the resulting equation to obtain asymptotic equations for individual cases prescribed by relative magnitudes between ν and h_2a, k_2b etc.

In the following expressions, we use the notations

$$E(\sigma X\pm)=e^{\sigma X}\pm e^{-\tau X}, \qquad E(\tau X\pm)=e^{\tau X}\pm e^{-\sigma X}, \tag{6.5}$$

where the upper and lower signs of both sides have a one to one correspondence to each other.

Substitution of Eqs. (6.1) and (6.4) into (2.5) leads to

$$\begin{aligned}
&A_{34}(a)\simeq\mu_2^2\{1/(\sigma_a+\tau_a)(\sigma_a'+\tau_a')\}\{\phi_1E(\tau_aP_a-)E(\tau_a'Q_a-)-4\phi_2E(\tau_aP_a+)E(\tau_a'Q_a+)\},\\
&A_{24}(a)\simeq2\mu_2^2N^2\{(k_2a)^2-2N^2\}/k_2a\,,\\
&A_{23}(a)\simeq\mu_2^2\{1/(\sigma_a+\tau_a)\}\{-(\phi_1/2)E(\tau_aP_a-)E(\sigma_a'Q_a+)+2\phi_2E(\tau_aP_a+)E(\sigma_a'Q_a-)\}\,,\\
&A_{14}(a)\simeq\mu_2^2\{1/(\sigma_a'+\tau_a')\}\{(\phi_1/2)E(\sigma_aP_a+)E(\tau_a'Q_a-)-2\phi_2E(\sigma_aP_a-)E(\tau_a'Q_a+)\}\,,\\
&A_{13}(a)\simeq-2\mu_2^2\{(k_2a)^2-2N^2\}/h_2a\,,\\
&A_{12}(a)\simeq\mu_2^2\{(\phi_1/4)E(\sigma_aP_a+)E(\sigma_a'Q_a+)-\phi_2E(\sigma_aP_a-)E(\sigma_a'Q_a-)\}\,,
\end{aligned}\tag{6.6}$$

where

$$\begin{aligned}
&\phi_1=\{(k_2a)^2-2N^2\}^2/\{h_2a\cdot k_2a\sqrt{\mathrm{s}p_a\mathrm{s}q_a}\}\,,\\
&\phi_2=N^2\sigma_a\sigma_a'\sqrt{\mathrm{s}p_a\mathrm{s}q_a}\,.
\end{aligned}\tag{6.7}$$

P_a stands for $\bar{P}_a$ or $\hat{P}_a$ and p_a for $\bar{p}_a$ or $\hat{p}_a$ (both defined in Eq. (4.5) or (5.5)) according to $h_2a>\nu$ or $h_2a<\nu$. Similarly Q_a stands for $\bar{Q}_a$ or $\hat{Q}_a$ and q_a for $\bar{q}_a$ or $\hat{q}_a$ according to $k_2a>\nu$ or $k_2a<\nu$. Hence, $\mathrm{s}p_a$ is short for $\sin\bar{p}_a$ or $\sinh\hat{p}_a$ and $\mathrm{s}q_a$ for $\sin\bar{q}_a$ or $\sinh\hat{q}_a$. σ_a and τ_a are associated with P_a, and σ_a' and τ_a' with Q_a. A pair (σ_a, τ_a) takes a value (i, i) and $(1, 0)$ corresponding to $h_2a>\nu$ and $h_2a<\nu$ respectively. σ_a' and τ_a' can be defined in a similar way. The coefficients $A_{jk}(b)$'s in Eq. (2.4) are readily obtained by simply replacing all the subscripts a's in Eqs. (6.6) and (6.7) by b's and the radial distance a by b.

Similarly we have for the coefficients B_{jk}'s

$$\begin{aligned}
&B_{12}\simeq-(\mu_2/4)k_2b\cdot\sigma_bE(\sigma_bP_b-)E(\sigma_b'Q_b+)\sqrt{\mathrm{s}p_b/\mathrm{s}q_b}\,,\\
&B_{14}\simeq-(\mu_2/2)k_2b\{\sigma_b/(\sigma_b'+\tau_b')\}E(\sigma_bP_b-)E(\tau_b'Q_b-)\sqrt{\mathrm{s}p_b/\mathrm{s}q_b}\,,\\
&B_{23}\simeq(\mu_2/2)k_2b\{\sigma_b/(\sigma_b+\tau_b)\}E(\tau_bP_b+)E(\sigma_b'Q_b+)\sqrt{\mathrm{s}p_b/\mathrm{s}q_b}\,,\\
&B_{34}\simeq-\mu_2\cdot k_2b\{\sigma_b/(\sigma_b+\tau_b)(\sigma_b'+\tau_b')\}E(\tau_bP_b+)E(\tau_b'Q_b-)\sqrt{\mathrm{s}p_b/\mathrm{s}q_b}\,,\\
&B_{13}\simeq B_{24}\simeq0\,.
\end{aligned}\tag{6.8}$$

P_b stands for $\bar{P}_b$ or $\hat{P}_b$, p_b for $\bar{p}_b$ or $\hat{p}_b$, and $\mathrm{s}p_b$ for $\sin\bar{p}_b$ or $\sinh\hat{p}_b$ according to $h_2b>\nu$ or $h_2b<\nu$. Similarly, Q_b stands for $\bar{Q}_b$ or $\hat{Q}_b$, q_b for $\bar{q}_b$ or $\hat{q}_b$, and $\mathrm{s}q_b$ for $\sin\bar{q}_b$ or $\sinh\hat{q}_b$ according to $k_2b>\nu$ or $k_2b<\nu$. (σ_b, τ_b) and (σ'_b, τ'_b) are associated with P_b and Q_b respectively and take the values (i, i) or $(1, 0)$ corresponding to two situations.

After substitution of the above formulas into Eq. (2.4) and some manipulations, we can get the asymptotic frequency equation in a rather lengthy form

$$
\begin{aligned}
\mathfrak{F}\simeq\mathfrak{F}_a=(\mu_2^3/4)[&\{(\sigma_a+\tau_a)(\sigma'_a+\tau'_a)\}^{-1}\{\phi_1E(\tau_aP_a-)E(\tau'_aQ_a-)\\
&\qquad -4\phi_2E(\tau_aP_a+)E(\tau'_aQ_a+)\}\\
&\cdot\{\phi_0E(\sigma_cP_c+)E(\sigma_bP_b-)E(\sigma'_bQ_b+)-\phi_1E(\sigma_cP_c-)E(\sigma_bP_b+)E(\sigma'_bQ_b+)\\
&\qquad +4\phi_2E(\sigma_cP_c-)E(\sigma_bP_b-)E(\sigma'_bQ_b-)\}\\
&-\{(\sigma_a+\tau_a)(\sigma'_b+\tau'_b)\}^{-1}\{\phi_1E(\tau_aP_a-)E(\sigma'_aQ_a+)-4\phi_2E(\tau_aP_a+)E(\sigma'_aQ_a-)\}\\
&\cdot\{\phi_0E(\sigma_cP_c+)E(\sigma_bP_b-)E(\tau'_bQ_b-)-\phi_1E(\sigma_cP_c-)E(\sigma_bP_b+)E(\tau'_bQ_b-)\\
&\qquad +4\phi_2E(\sigma_cP_c-)E(\sigma_bP_b-)E(\tau'_bQ_b+)\}\\
&-\{(\sigma_b+\tau_b)(\sigma'_a+\tau'_a)\}^{-1}\{\phi_1E(\sigma_aP_a+)E(\tau'_aQ_a-)-4\phi_2E(\sigma_aP_a-)E(\tau'_aQ_a+)\}\\
&\cdot\{\phi_0E(\sigma_cP_c+)E(\tau_bP_b+)E(\sigma'_bQ_b+)-\phi_1E(\sigma_cP_c-)E(\tau_bP_b-)E(\sigma'_bQ_b+)\\
&\qquad +4\phi_2E(\sigma_cP_c-)E(\tau_bP_b+)E(\sigma'_bQ_b-)\}\\
&+\{(\sigma_b+\tau_b)(\sigma'_b+\tau'_b)\}^{-1}\{\phi_1E(\sigma_aP_a+)E(\sigma'_aQ_a+)-4\phi_2E(\sigma_aP_a-)E(\sigma'_aQ_a-)\}\\
&\cdot\{\phi_0E(\sigma_cP_c+)E(\tau_bP_b+)E(\tau'_bQ_b-)-\phi_1E(\sigma_cP_c-)E(\tau_bP_b-)E(\tau'_bQ_b-)\\
&\qquad +4\phi_2E(\sigma_cP_c-)E(\tau_bP_b+)E(\tau'_bQ_b+)\}\\
&-\phi_3E(\sigma_cP_c-)]=0\,.
\end{aligned}
\tag{6.9}
$$

where

$$
\begin{aligned}
&\phi_0=(\lambda_1/2)h_1b\cdot k_2b\cdot\sigma_b\sqrt{\mathrm{s}p_b/\mathrm{s}p_c\mathrm{s}q_b}\,,\\
&\phi_1=(\mu_2/2)\{(k_2b)^2-2N^2\}^2(h_2b\cdot k_2b)^{-1}\sigma_c\sqrt{\mathrm{s}p_c/\mathrm{s}p_b\mathrm{s}q_b}\,,\\
&\phi_2=(\mu_2/2)N^2\sigma_b\sigma_c\sigma'_b\sqrt{\mathrm{s}p_b\mathrm{s}p_c\mathrm{s}q_b}\,,\\
&\phi_3=8\mu_2N^2\{(k_2a)^2-2N^2\}\{(k_2b)^2-2N^2\}\{(h_2a\cdot k_2b)^{-1}+(h_2b\cdot k_2a)^{-1}\}\sigma_c\sqrt{\mathrm{s}p_c}\,.
\end{aligned}
\tag{6.10}
$$

$\mathfrak{F}_a$ means the asymptotic formula of the exact frequency equation $\mathfrak{F}$. P_c stands for $\bar{P}_c$ or $\hat{P}_c$, p_c for $\bar{p}_c$ or $\hat{p}_c$, and $\mathrm{s}p_c$ for $\sin\bar{p}_c$ or $\sinh\hat{p}_c$ according to $h_1b>\nu$ or $h_1b<\nu$. Corresponding to this condition we have either case, $\sigma_c=\tau_c=i$ or $\sigma_c=1$, $\tau_c=0$. Further simplification of Eq. (6.9) will have to be made for individual cases which correspond to different ray situations.

7. Connection with reflection coefficients

Under the condition that both the arguments (or frequency) and the order are large, we can show that certain combinations of the coefficients ϕ_1, ψ_1 etc. defined by Eqs. (6.7) and (6.10) are closely connected with reflection coefficients of body waves incident on the free surface and the mantle-core boundary.

Suppose that a plane P- or S-wave with a unit amplitude (in displacement) impinges on a plane boundary between two homogeneous media. Then we define reflection coefficients as amplitudes of displacement of reflected waves (components are illustrated in Fig. 2). Following the conventional method [*e.g.*, EWING *et al.* (1957)], we get

$$\begin{aligned}
&R^0_{pp}=R^0_{ss}=[-\{(c_0/\beta_2)^2-2\}^2+4\xi_0\eta_0]/\Delta_0\,,\\
&R^0_{ps}=-4(\alpha_2/\beta_2)\xi_0\{(c_0/\beta_2)^2-2\}/\Delta_0\,,\\
&R^0_{sp}=4(\beta_2/\alpha_2)\eta_0\{(c_0/\beta_2)^2-2\}/\Delta_0\,,\\
&R^2_{pp}=\{\mu_2\xi_1[-\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]+\lambda_1\xi_2(c^2/\alpha_1\beta_2)^2\}/\Delta_2\,,\\
&R^2_{ss}=\{\mu_2\xi_1[-\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]-\lambda_1\xi_2(c^2/\alpha_1\beta_2)^2\}/\Delta_2\,,\\
&R^1_{pp}=\{\mu_2\xi_1[\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]-\lambda_1\xi_2(c^2/\alpha_1\beta_2)^2\}/\Delta_2\,,\\
&R^2_{ps}=4\mu_2(\alpha_2/\beta_2)\xi_1\xi_2\{(c/\beta_2)^2-2\}/\Delta_2\,,\\
&R^2_{sp}=-4\mu_2(\beta_2/\alpha_2)\xi_1\eta_2\{(c/\beta_2)^2-2\}/\Delta_2\,,\\
&\Delta_0=\{(c_0/\beta_2)^2-2\}^2+4\xi_0\eta_0\,,\\
&\Delta_2=\mu_2\xi_1[\{(c/\beta_2)^2-2\}^2+4\xi_2\eta_2]+\lambda_1\xi_2(c^2/\alpha_1\beta_2)^2\,,
\end{aligned} \tag{7.1}$$

where the numerical superscripts 0, 1 and 2 attached to the reflection coefficients refer to the reflections at the free surface and at the mantle-core boundary (incident from inside the core and inside the mantle) respectively, and the coefficients ξ_0, η_0 etc. are defined in Eq. (4.8).

Now we can approximate, by use of Eq. (3.2),

$$\begin{aligned}
&N^2\simeq\nu^2\,,\\
&(k_2a)^2-2N^2\simeq\nu^2\{(c_0/\beta_2\}^2-2\}\,,\\
&(k_2b)^2-2N^2\simeq\nu^2\{(c/\beta_2)^2-2\}\,.
\end{aligned} \tag{7.2}$$

Moreover, allowing for Eqs. (4.6), (5.6), (6.2) and (6.3), we can write

$$\begin{aligned}
&\mathrm{s}p_a\cdot\sigma_a=i(\alpha_2/c_0)\xi_0,\quad \mathrm{s}p_b\cdot\sigma_b=i(\alpha_2/c)\xi_2,\quad \mathrm{s}p_c\cdot\sigma_c=i(\alpha_1/c)\xi_1\,,\\
&\mathrm{s}q_a\cdot\sigma'_a=i(\beta_2/c_0)\eta_0,\quad \mathrm{s}q_b\cdot\sigma'_b=i(\beta_2/c)\eta_2\,.
\end{aligned} \tag{7.3}$$

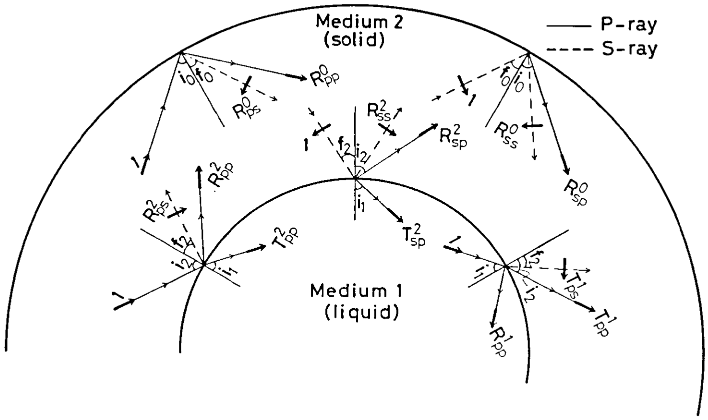

Fig. 2. Reflection and transmission coefficients defined for plane P- and S-wave incidence on the plane boundaries, free surface and mantle-core boundary, which are shown by curves for the convenience of comparing with the spherical Earth.

Here, for simplicity of later numerical expressions, we introduce a multiplicative factor K_0 defined by

$$K_0 = h_2a \cdot h_2b \cdot k_2a \cdot k_2b(c/\alpha_1)\nu^{-8}\sqrt{sp_a sp_b sp_c sq_a sq_b}\,. \tag{7.4}$$

Then, from Eqs. (3.2), (6.7), (6.10) and above formulas, we get

$$\begin{aligned}
2K_0\phi_1\psi_1 &\simeq i\{(c_0/\beta_2)^2-2\}^2\cdot\mu_2\xi_1\{(c/\beta_2)^2-2\}^2 &&= iR_{11}\,,\\
32K_0\phi_2\psi_2 &\simeq i\cdot 4\xi_0\eta_0\cdot 4\mu_2\xi_1\xi_2\eta_2 &&= iR_{12}\,,\\
8K_0\phi_2\psi_0 &\simeq -i\cdot 4\xi_0\eta_0\cdot\lambda_1\xi_2(c^2/\alpha_1\beta_2)^2 &&= -iR_2\,,\\
8K_0\phi_2\psi_1 &\simeq -i\cdot 4\xi_0\eta_0\cdot\mu_2\xi_1\{(c/\beta_2)^2-2\}^2 &&= -iR_{31}\,,\\
8K_0\phi_1\psi_2 &\simeq -i\cdot\{(c_0/\beta_2)^2-2\}^2\cdot 4\mu_2\xi_1\xi_2\eta_2 &&= -iR_{32}\,,\\
2K_0\phi_1\psi_0 &\simeq i\{(c_0/\beta_2)^2-2\}^2\cdot\lambda_1\xi_2(c^2/\alpha_1\beta_2)^2 &&= iR_4\,,\\
2K_0\psi_3 &\simeq i\cdot 32\{(c_0/\beta_2)^2-2\}\cdot\mu_2\xi_1\{(c/\beta_2)^2-2\}\sqrt{\xi_0\xi_2\eta_0\eta_2/\sigma_a\sigma_b\sigma'_a\sigma'_b} &&= iR_0
\end{aligned} \tag{7.5}$$

Referring to Eq. (7.1), we will find that the coefficients R_{11}, R_{12} etc. defined by Eq. (7.5) and the reflection coefficients consist of several common terms such as $\{(c_0/\beta_2)^2-2\}^2$, $4\xi_0\eta_0$. In fact, those linear combinations can be expressed in terms of the reflection coefficients, that is,

$$\begin{aligned}
R_{11}+R_{12}+R_2+R_{31}+R_{32}+R_4 &= \Delta_0\Delta_2 &&= R_x\,,\\
+\quad -\quad +\quad +\quad - \quad &= \Delta_0\Delta_2 R^1_{pp} &&= R_a\,,\\
+\quad -\quad -\quad -\quad + \quad &= \Delta_0\Delta_2 R^0_{pp}R^2_{ss} &&= R_b\,,
\end{aligned}$$

$$
\begin{array}{llllllll}
R_{11}+ & R_{12}+ & R_{2}- & R_{31}- & R_{32}- & R_{4} & =\Delta_0\Delta_2 R_{pp}^0 R_{pp}^2 & =R_c\,, \\
 & - & - & - & + & + & =-\Delta_0\Delta_2 R_{pp}^0 & =-R_d\,, \\
 & - & + & - & + & - & =-\Delta_0\Delta_2 R_{pp}^0 R_{pp}^1 & =-R_e\,, \\
 & - & + & + & - & + & =-\Delta_0\Delta_2 R_{ss}^2 & =-R_f\,, \\
 & - & - & + & - & - & =-\Delta_0\Delta_2 R_{pp}^2 & =-R_g\,,
\end{array}
$$

$$R_0=2\Delta_0\Delta_2\sqrt{R_{ps}^0 R_{sp}^0 R_{ps}^2 R_{sp}^2/\sigma_a\sigma_b\sigma_a'\sigma_b'}\,. \tag{7.6}$$

The coefficients R_x, R_a etc. are written only for definition.

In the following sections, we will derive the asymptotic frequency equation in an explicit form corresponding to each of all possible cases that are prescribed by combinations of relative magnitudes between the five arguments h_2a, h_2b, h_1b, k_2a, k_2b and the order ν.

8. Asymptotic frequency equation when $h_1b,\ h_2b>\nu$

This condition maintains that the other three arguments h_2a, k_2a and k_2b are also larger than ν, because $k_2a>k_2b$, $h_2a>h_2b>\nu$. Hence, from Eq. (3.1), we have real angles of incidence for all i_j's and f_j's; both P and S rays in the mantle with a given ray parameter strike the core and penetrate it (*cf.* Fig. 3a).

Now the approximations Eqs. (4.1) through (4.3) are available for all the arguments. Hence we have, from Eq. (6.2),

$$\sigma_a=\tau_a=\sigma_b=\tau_b=\sigma_c=\tau_c=\sigma_a'=\tau_a'=\sigma_b'=\tau_b'=i\,, \tag{8.1}$$

and thus, from Eq. (6.5),

$$E(\sigma X\pm)=E(\tau X\pm)=E(iX\pm)=e^{iX}\pm e^{-iX}\,, \tag{8.2}$$

where X stands for $\bar{P}_r$ and $\bar{Q}_r$ $(r=a, b, c)$ and (σ, τ) for (σ_r, τ_r) and (σ_r', τ_r').

Then, we have another helpful relations between any two X's, say, X_1 and X_2,

$$
\begin{aligned}
&E(iX_1+)E(iX_2+)-E(iX_1-)E(iX_2-)=2E\{i(X_1-X_2)+\}=2E(i\delta X+)\,,\\
&E(iX_1-)E(iX_2+)-E(iX_1+)E(iX_2-)=2E\{i(X_1-X_2)-\}=2E(i\delta X-)\,,
\end{aligned} \tag{8.3}
$$

where $\delta X=X_1-X_2$. These relations will be readily confirmed by substitution of the exponential functions (8.2) into (8.3).

With the aid of Eqs. (8.1) through (8.3), we can fairly simplify Eq. (6.9) as

$$-i(8K_0/\mu_2^3)\mathfrak{F}_a=(R_{11}+R_{12})E(i\bar{P}_c-)E(i\delta\bar{P}-)E(i\delta\bar{Q}-)$$
$$+R_2E(i\bar{P}_c+)E(i\delta\bar{P}-)E(i\delta\bar{Q}+)+(R_{31}+R_{32})E(i\bar{P}_c-)E(i\delta\bar{P}+)E(i\delta\bar{Q}+)$$
$$+R_4E(i\bar{P}_c+)E(i\delta\bar{P}+)E(i\delta\bar{Q}-)-R_0E(i\bar{P}_c-)=0\,, \tag{8.4}$$

where $\delta\bar{P}$ and $\delta\bar{Q}$ are defined in Eq. (4.7) and R_{11}, R_{12} etc. in (7.5). Multiplying by $\exp\{-i(\bar{P}_c+\delta\bar{P}+\delta\bar{Q})\}/\Delta_0\Delta_2$ and employing (7.6), we finally obtain the asymptotic frequency equation:

$$\{1-e^{-2i(\bar{P}_c+\delta\bar{P}+\delta\bar{Q})}\}+R_{pp}^1\{e^{-2i(\delta\bar{P}+\delta\bar{Q})}-e^{-2i\bar{P}_c}\}$$
$$+R_{pp}^0R_{ss}^2\{e^{-2i(\bar{P}_c+\delta\bar{P})}-e^{-2i\delta\bar{Q}}\}+R_{pp}^0R_{pp}^2\{e^{-2i(\bar{P}_c+\delta\bar{Q})}-e^{-2i\delta\bar{P}}\}$$
$$+2\sqrt{R_{ps}^0R_{sp}^0R_{ps}^2R_{sp}^2}\{e^{-i(2\bar{P}_c+\delta\bar{P}+\delta\bar{Q})}-e^{-i(\delta\bar{P}+\delta\bar{Q})}\}=0\,. \tag{8.5}$$

Here $\delta\bar{P}$, $\delta\bar{Q}$ and $\bar{P}_c$ are the functions of frequency as shown in Eq. (4.7), and thus the equation yields discrete eigenfrequencies for a given ray parameter.

The last term of the above equation seems to be not in harmony with the other terms in its expression. However, further approximation is possible for it, that is,

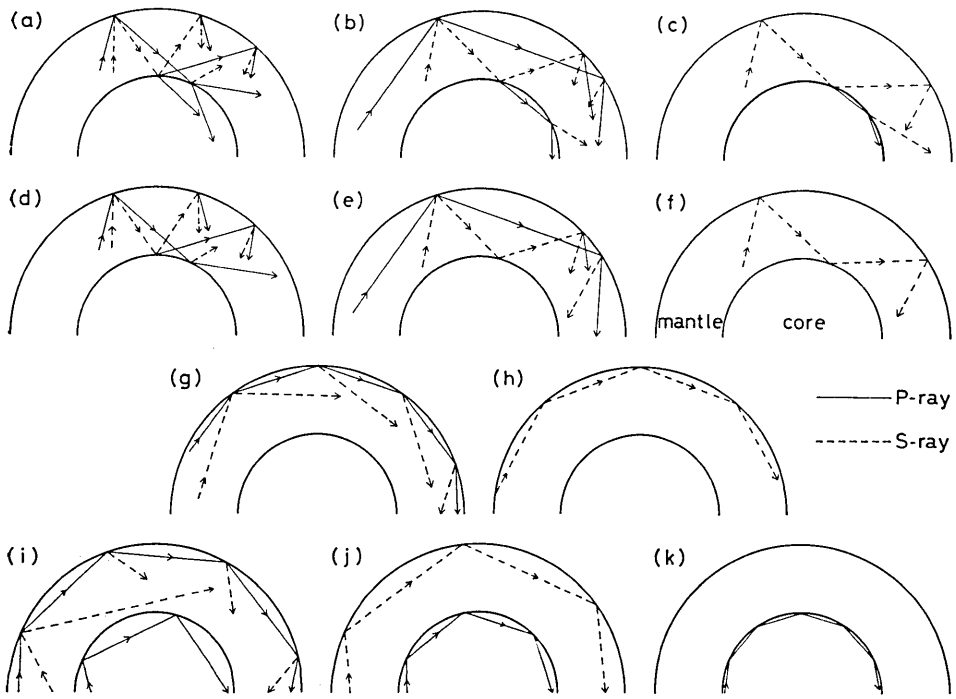

Fig. 3. 11 possible ray-geometries in the Earth consisting of a uniform mantle and core, which depend on ray parameter and velocity structure.

$$2\sqrt{R^0_{ps}R^0_{sp}R^2_{ps}R^2_{sp}} \simeq R^0_{ps}R^2_{sp}+R^0_{sp}R^2_{ps}\,. \tag{8.6}$$

This is true if we have

$$2\sqrt{\xi_0\eta_0\xi_2\eta_2} \simeq \xi_0\eta_2+\xi_2\eta_0\,, \tag{8.7}$$

because it is easily found from Eq. (7.1) that both conditions are identical. By expanding ξ_2 given by (4.8) in a power series of $(\alpha_2/c)^2$ where $c>\alpha_2$ and similarly ξ_0 etc., we will find that the approximation (8.7) is, in general, valid except for the case when $c\simeq\alpha_2$. When the phase velocity c approaches α_2, the approximation (4.1) becomes unavailable, and thus we have to discuss such a case separately.

Derivation of Eq. (8.5) in terms of ray theory and numerical calculation will be made in a future paper [ODAKA (1980b)].

9. Asymptotic frequency equation when h_2a, k_2b, $h_1b>\nu>h_2b$

From Eq. (3.1), we will find that the angles i_0, i_1, f_0 and f_2 are defined as real numbers but i_2 as a complex number. This means that P rays in the mantle do not reach the core as is illustrated in Fig. 3b.

Approximations (4.1) through (4.3) are available for the arguments larger than ν, but Eqs. (5.1) through (5.3) for h_2b. Then, we can put, from (6.2) and (6.3),

$$\begin{aligned}&\sigma_a=\tau_a=\sigma_c=\tau_c=\sigma'_a=\tau'_a=\sigma'_b=\tau'_b=i\,,\\ &\sigma_b=1,\quad \tau_b=0,\quad e^{\pm\tau_b\hat{P}_b}=0\,.\end{aligned} \tag{9.1}$$

Hence, we have, from Eq. (6.5),

$$E(\sigma_bP_b\pm)=e^{\hat{P}_b}\,,\qquad E(\tau_bP_b\pm)=\pm e^{-\hat{P}_b} \tag{9.2}$$

Employing the above formulas and Eq. (8.3), we can simplify Eq. (6.9) as

$$\begin{aligned}-i(8K_0/\mu_2^3)\mathfrak{F}_a=(e^{\hat{P}_b}/2)[&\{R_{11}E(i\bar{P}_a-)-R_{12}E(i\bar{P}_a+)\}E(i\bar{P}_c-)E(i\delta\bar{Q}-)\\ &+\{R_{31}E(i\bar{P}_a+)-R_{32}E(i\bar{P}_a-)\}E(i\bar{P}_c-)E(i\delta Q+)\\ &-\{R_2E(i\bar{P}_a+)E(i\delta\bar{Q}+)+R_4E(i\bar{P}_a-)E(i\delta\bar{Q}-)\}E(i\bar{P}_c+)]\\ +ie^{-\hat{P}_b}[&\{R_{11}E(i\bar{P}_a+)+R_{12}E(i\bar{P}_a-)\}E(i\bar{P}_c-)E(i\delta\bar{Q}-)\\ &+\{R_{31}E(i\bar{P}_a-)+R_{32}E(i\bar{P}_a+)\}E(i\bar{P}_c-)E(i\delta\bar{Q}+)\\ &+\{R_2E(i\bar{P}_a-)E(i\delta\bar{Q}+)+R_4E(i\bar{P}_a+)E(i\delta\bar{Q}-)\}E(i\bar{P}_c+)]\\ -R_0E(i\bar{P}_c-)=0\,,&\end{aligned} \tag{9.3}$$

where $\bar{P}_a$, $\bar{P}_c$, $\delta\bar{Q}$ are defined in Eqs. (4.5) and (4.7), $\hat{P}_b$ in Eq. (5.5).

Further reduction with the aid of Eqs. (8.2) and (7.6), and multiplication by $\exp\{-i(\bar{P}_c+\bar{P}_a+\delta\bar{Q})\}$ lead to

$$(e^{\hat{P}_b}/2)[-R_g+R_f e^{-2i\bar{P}_c}+R_e e^{-2i\delta\bar{Q}}-R_d e^{-2i(\bar{P}_c+\delta\bar{Q})}-R_c e^{-2i\bar{P}_a}$$
$$+R_b e^{-2i(\bar{P}_a+\bar{P}_c)}+R_a e^{-2i(\bar{P}_a+\delta\bar{Q})}-R_x e^{-2i(\bar{P}_a+\bar{P}_c+\delta\bar{Q})}]$$
$$+ie^{-\hat{P}_b}[R_x-R_a e^{-2i\bar{P}_c}-R_b e^{-2i\delta\bar{Q}}+R_c e^{-2i(\bar{P}_c+\delta\bar{Q})}-R_d e^{-2i\bar{P}_a}$$
$$+R_e e^{-2i(\bar{P}_a+\bar{P}_c)}+R_f e^{-2i(\bar{P}_a+\delta\bar{Q})}-R_g e^{-2i(\bar{P}_a+\bar{P}_c+\delta\bar{Q})}]$$
$$-R_0[e^{-i(\bar{P}_a+\delta\bar{Q})}-e^{-i(2\bar{P}_c+\bar{P}_a+\delta\bar{Q})}]=0\,. \tag{9.4}$$

From the relation (5.8), we find that the second term is predominant in the equation. Hence, we finally obtain by use of Eq. (7.6)

$$1-R^0_{pp}e^{-2i\bar{P}_a}-R^1_{pp}e^{-2i\bar{P}_c}+R^2_{ss}e^{-2i(\bar{P}_a+\delta\bar{Q})}-R^2_{pp}e^{-2i(\bar{P}_a+\bar{P}_c+\delta\bar{Q})}$$
$$+R^0_{pp}R^1_{pp}e^{-2i(\bar{P}_a+\bar{P}_c)}-R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}}+R^0_{ss}R^2_{pp}e^{-2i(\bar{P}_c+\delta\bar{Q})}=0\,. \tag{9.5}$$

In the last two terms, the coefficient R^0_{ss} is employed instead of R^0_{pp}, both being identical as shown at the top of Eq. (7.1).

In the present case, we have no rays corresponding to the reflection coefficient R^2_{pp}, which appears in the above equation. However, introducing transmission coefficients, T^1_{ps} and T^2_{sp} (*cf.* Fig. 2), we can replace R^2_{pp} by $R^1_{pp}R^2_{ss}-T^2_{sp}T^1_{ps}$. Then, we have rays corresponding to each of these coefficients. Ray-theoretical approach to Eq. (9.5) will elucidate this situation well [ODAKA (1980b)].

10. Asymptotic frequency equation when $k_2b,\ h_1b>\nu>h_2a$

We find from Eq. (3.1) that the angles i_1, f_0 and f_2 are defined as real numbers but i_0 and i_2 as complex numbers. This means that S rays incident on the free surface and on the mantle-core boundary are not followed by ordinary reflected P rays; the ray geometry is illustrated in Fig. 3c.

Now we can put, from Eqs. (6.2) and (6.3),

$$\begin{aligned}&\sigma_c=\tau_c=\sigma'_a=\tau'_a=\sigma'_b=\tau'_b=i\,,\\&\sigma_a=\sigma_b=1,\quad \tau_a=\tau_b=0,\quad e^{\pm\tau_a\hat{P}_a}=e^{\pm\tau_b\hat{P}_b}=0\,.\end{aligned} \tag{10.1}$$

Then, following the similar procedure as in the previous sections, we get

$$ie^{\delta\hat{P}}[R_x-R_a e^{-2i\bar{P}_c}-R_b e^{-2i\delta\bar{Q}}+R_c e^{-2i(\bar{P}_c+\delta\bar{Q})}]$$
$$-ie^{-\delta\hat{P}}[R_c-R_b e^{-2i\bar{P}_c}-R_a e^{-2i\delta\bar{Q}}+R_x e^{-2i(\bar{P}_c+\delta\bar{Q})}]$$
$$-R_0[e^{-i\delta\bar{Q}}-e^{-i(2\bar{P}_c+\delta\bar{Q})}]=0\,, \tag{10.2}$$

where $\bar{P}_c$ and $\delta\bar{Q}$ are defined in Eqs. (4.5) and (4.7), $\delta\hat{P}$ in (5.9) and R_x,

R_a etc. in (7.6).

From Eq. (5.8), contributions from the latter two terms are negligible compared with that from the first term, and thus we finally get, by the use of Eq. (7.6) and the relation $R^0_{pp}=R^0_{ss}$,

$$1-R^1_{pp}e^{-2i\bar{P}_c}-R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}}+R^0_{ss}R^2_{pp}e^{-2i(\bar{P}_c+\delta\bar{Q})}=0\,. \tag{10.3}$$

11. Asymptotic frequency equation when $h_2b>\nu>h_1b$

From Eq. (3.1), we find that the angles i_0, i_2, f_0 and f_2 are the real numbers but i_1 the complex number. This means that P and S waves incident on the mantle-core boundary from inside the mantle are totally reflected; the ray geometry is shown in Fig. 3d.

Now, we can put, from Eqs. (6.2) and (6.3),

$$\begin{aligned}&\sigma_a=\tau_a=\sigma_b=\tau_b=\sigma'_a=\tau'_a=\sigma'_b=\tau'_b=i\,,\\&\sigma_c=1,\quad \tau_c=0,\quad e^{\pm\tau_c\hat{P}_c}=0\,.\end{aligned} \tag{11.1}$$

Then from Eq. (6.5), we have

$$E(\sigma_cP_c+)=E(\sigma_cP_c-)=e^{\hat{P}_c}\,. \tag{11.2}$$

Referring to Eq. (6.9), we find that the above term can be put outside the bracket. In consequence, we have only to replace $E(i\bar{P}_c\pm)$ in Eq. (8.4) by $\exp\{\hat{P}_c\}$, and we immediately obtain

$$\begin{aligned}-i(8K_0/\mu_2^3)\mathfrak{F}_a=e^{\hat{P}_c}[&(R_{11}+R_{12})E(i\delta\bar{P}-)E(i\delta\bar{Q}-)\\&+R_2E(i\delta\bar{P}-)E(i\delta\bar{Q}+)+(R_{31}+R_{32})E(i\delta\bar{P}+)E(i\delta\bar{Q}+)\\&+R_4E(i\delta\bar{P}+)E(i\delta\bar{Q}-)-R_0]=0\,.\end{aligned} \tag{11.3}$$

Further reduction in terms of Eqs. (8.2) and (7.6) yields

$$e^{\hat{P}_c}[R_xe^{i(\delta\bar{P}+\delta\bar{Q})}+R_ae^{-i(\delta\bar{P}+\delta\bar{Q})}-R_be^{i(\delta\bar{P}-\delta\bar{Q})}-R_ce^{-i(\delta\bar{P}-\delta\bar{Q})}-R_0]=0\,. \tag{11.4}$$

Hence, multiplying by $\exp\{-i(\delta\bar{P}+\delta\bar{Q})\}$, we finally attain

$$\begin{aligned}&1+R^1_{pp}e^{-2i(\delta\bar{P}+\delta\bar{Q})}-R^0_{pp}R^2_{pp}e^{-2i\delta\bar{P}}-R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}}\\&\quad-2\sqrt{R^0_{ps}R^0_{sp}R^2_{ps}R^2_{sp}}\,e^{-i(\delta\bar{P}+\delta\bar{Q})}=0\,.\end{aligned} \tag{11.5}$$

where $\delta\bar{P}$ and $\delta\bar{Q}$ are defined in Eq. (4.7). Further approximation by means of Eq. (8.6) is possible for the last term of the above equation. There exists no rays which cause the reflection coefficient R^1_{pp}. However, we have the relation $R^1_{pp}=R^2_{pp}R^2_{ss}-R^2_{ps}R^2_{sp}$. Then, we have real rays corresponding to each coefficients in the right-hand side.

12. Asymptotic frequency equation when h_2a, $k_2b>\nu>h_2b$, h_1b

Equation (3.1) defines real angles for i_0, f_0 and f_2, and complex angles for i_1 and i_2. Ray geometry relevant to this situation is shown in Fig. 3e.

Reduction of Eq. (6.9) with the similar manipulations as developed in the previous sections provides

$$\begin{aligned}&-(e^{\hat{P}_b}/2)[R_g-R_e e^{-2i\delta\bar{Q}}+R_c e^{-2i\bar{P}_a}-R_a e^{-2i(\bar{P}_a+\delta\bar{Q})}]\\&+ie^{-\hat{P}_b}[R_x-R_b e^{-2i\delta\bar{Q}}-R_d e^{-2i\bar{P}_a}+R_f e^{-2i(\bar{P}_a+\delta\bar{Q})}]\\&-R_0 e^{-i(\bar{P}_a+\delta\bar{Q})}=0\,.\end{aligned}\tag{12.1}$$

Then, keeping only the leading term by means of Eq. (5.8) and employing the relations (7.6) and $R^0_{pp}=R^0_{ss}$, we get

$$1-R^0_{pp}e^{-2i\bar{P}_a}+R^2_{ss}e^{-2i(\bar{P}_a+\delta\bar{Q})}-R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}}=0\,.\tag{12.2}$$

13. Asymptotic frequency equation when $k_2b>\nu>h_2a$, h_1b

Equation (3.1) defines real angles for f_0 and f_2, and complex angles for i_0, i_1 and i_2. Hence, there exists no ordinary P rays as shown in Fig. 3f.

Reduction of Eq. (6.9) leads to

$$-ie^{-\delta\hat{P}}[R_c e^{i\delta\bar{Q}}-R_a e^{-i\delta\bar{Q}}]-ie^{\delta\hat{P}}[-R_x e^{i\delta\bar{Q}}+R_b e^{-i\delta\bar{Q}}]-R_0=0\,.\tag{13.1}$$

In view of the relation (5.8), we find that the second term is predominant in the equation. Hence, we get by the use of the relations (7.6) and $R^0_{pp}=R^0_{ss}$,

$$1-R^0_{ss}R^2_{ss}e^{-2i\delta\bar{Q}}=0\,.\tag{13.2}$$

14. Asymptotic frequency equation when $h_2a>\nu>k_2b$, h_1b

Equation (3.1) defines the angles i_0 and f_0 as real numbers and the angles i_1, i_2 and f_2 as complex numbers. Hence, P and S rays in the mantle do not reach the mantle-core boundary, as is illustrated in Fig. 3g.

Now, we can reduce Eq. (6.9) to

$$\begin{aligned}-i(8K_0/\mu_2^3)\mathfrak{F}_a=e^{\hat{P}_c}[&(1/4)e^{\hat{P}_b+\hat{Q}_b}\{R_aE(i\overline{P+Q}+)+R_eE(i\overline{P-Q}+)\}\\&+(i/2)e^{\hat{P}_b-\hat{Q}_b}\{-R_gE(i\overline{P+Q}-)+R_cE(i\overline{P-Q}-)\}\\&-(i/2)e^{-\hat{P}_b+\hat{Q}_b}\{R_fE(i\overline{P+Q}-)+R_bE(i\overline{P-Q}-)\}\\&-e^{-\hat{P}_b-\hat{Q}_b}\{R_xE(i\overline{P+Q}+)-R_dE(i\overline{P-Q}+)\}-R_0]=0\,,\end{aligned}\tag{14.1}$$

where

$$\overline{P\pm Q}=\bar{P}_a\pm\bar{Q}_a\tag{14.2}$$

and thus

$$E(i\overline{P\pm Q}\pm)=e^{i(\bar{P}_a\pm\bar{Q}_a)}\pm e^{-i(\bar{P}_a\pm\bar{Q}_a)}\,. \tag{14.3}$$

Then, we find from the relation (5.8) that the fourth term in the bracket of Eq. (14.1) makes a predominant contribution to the equation. Hence we get, with the aid of Eqs. (7.6), (14.3) and the relation $R^0_{pp}=R^0_{ss}$,

$$1-R^0_{pp}e^{-2i\bar{P}_a}-R^0_{ss}e^{-2i\bar{Q}_a}+e^{-2i(\bar{P}_a+\bar{Q}_a)}=0\,. \tag{14.4}$$

15. Asymptotic frequency equation when $k_2a>\nu>k_2b,\ h_2a,\ h_1b$

From Eq. (3.1) we find that the angle f_0 is real but all other angles, i_0, i_1, i_2 and f_2 are complex. Hence we have the simplest ray geometry as shown in Fig. 3h.

Now, Eq. (6.9) can be reduced to

$$\begin{aligned}-i(8K_0/\mu_2^3)\mathfrak{F}_a=e^{\hat{P}_c}[(i/2)e^{-\delta\hat{P}+\hat{Q}_b}\{R_ee^{i\bar{Q}_a}+R_ae^{-i\bar{Q}_a}\}\\ -(i/2)e^{\delta\hat{P}+\hat{Q}_b}\{R_fe^{i\bar{Q}_a}+R_be^{-i\bar{Q}_a}\}+e^{-\delta\hat{P}-\hat{Q}_b}\{R_ce^{i\bar{Q}_a}-R_ge^{-i\bar{Q}_a}\}\\ -e^{\delta\hat{P}-\hat{Q}_b}\{R_xe^{i\bar{Q}_a}-R_de^{-i\bar{Q}_a}\}-R_0]=0\,.\end{aligned} \tag{15.1}$$

It is found from Eq. (5.8) that the fourth term in the bracket of Eq. (15.1) makes a predominant contribution to the equation. Hence we get, with the help of the relations (7.6) and $R^0_{pp}=R^0_{ss}$,

$$1-R^0_{ss}e^{-2i\bar{Q}_a}=0\,. \tag{15.2}$$

16. Asymptotic frequency equation when $h_2a,\ h_1b>\nu>k_2b$

We find from Eq. (3.1) that real numbers are assigned to the angles i_0, i_1, f_0 but complex numbers to the angles i_2 and f_2. Ray geometry corresponding to this situation is shown in Fig. 3i. The figure indicates that P and/or S rays prescribed by any given ray parameter exist independently in the mantle and in the core respectively. When the P-wave velocity in the core is larger than the S-wave velocity in the mantle, as is expected for realistic Earth models, P rays in the core are always accompanied with converted S rays traveling into the mantle. Hence, the present case is not realistic but theoretically possible. The same can be said for the succeeding two cases.

Now we can put, from Eqs. (6.2) and (6.3),

$$\begin{aligned}&\sigma_a=\tau_a=\sigma_c=\tau_c=\sigma'_a=\tau'_a=i\,,\\ &\sigma_b=\sigma'_b=1,\quad \tau_b=\tau'_b=0,\quad e^{\pm\tau_b\hat{P}_b}=e^{\pm\tau'_b\hat{Q}_b}=0\,.\end{aligned} \tag{16.1}$$

Substituting these relations into Eq. (6.9), we obtain

$$(8/\mu_2^3)\mathfrak{F}_a=(1/4)\{\Phi_1E(i\overline{P+Q}+)+\Phi_2E(i\overline{P-Q}+)\}\{\Psi_3e^{i\bar{P}_c}-\Psi_0e^{-i\bar{P}_c}\}e^{\hat{P}_b+\hat{Q}_b}$$
$$-(i/2)\{\Phi_1E(i\overline{P+Q}-)-\Phi_2E(i\overline{P-Q}-)\}\{\Psi_1e^{i\bar{P}_c}-\Psi_2e^{-i\bar{P}_c}\}e^{\hat{P}_b-\hat{Q}_b}$$
$$-(i/2)\{\Phi_1E(i\overline{P+Q}-)+\Phi_2E(i\overline{P-Q}-)\}\{\Psi_2e^{i\bar{P}_c}-\Psi_1e^{-i\bar{P}_c}\}e^{-\hat{P}_b+\hat{Q}_b}$$
$$-\{\Phi_1E(i\overline{P+Q}+)-\Phi_2E(i\overline{P-Q}+)\}\{\Psi_0e^{i\bar{P}_c}-\Psi_3e^{-i\bar{P}_c}\}e^{-\hat{P}_b-\hat{Q}_b}$$
$$+8\psi_3\{e^{i\bar{P}_c}-e^{-i\bar{P}_c}\}=0\,, \tag{16.2}$$

where the function $E(i\overline{P\pm Q}\pm)$ is defined in Eq. (14.3) and

$$\begin{aligned}&\Phi_1=\phi_1-4\phi_2\,, \qquad \Phi_2=-\phi_1-4\phi_2\,,\\ &\Psi_0=2\psi_0+2\psi_1-8\psi_2\,, \qquad \Psi_1=2\psi_0-2\psi_1-8\psi_2\,,\\ &\Psi_2=-2\psi_0-2\psi_1-8\psi_2\,, \qquad \Psi_3=-2\psi_0+2\psi_1-8\psi_2\,.\end{aligned} \tag{16.3}$$

The coefficients ϕ_i and ψ_i are defined in Eqs. (6.7) and (6.10) respectively.

Here, we introduce two multiplicative factors K_a and K_b defined by

$$\begin{aligned}&K_a=h_2a\cdot k_2a\cdot\nu^{-4}\sqrt{\mathrm{s}p_a\mathrm{s}q_a}\,,\\ &K_b=h_2b\cdot k_2b\cdot(c/\alpha_1)\cdot\nu^{-4}\sqrt{\mathrm{s}p_b\mathrm{s}p_c\mathrm{s}q_b}\,.\end{aligned} \tag{16.4}$$

These factors are connected with K_0 in Eq. (7.4) by

$$K_0=K_a\cdot K_b\,. \tag{16.5}$$

Then we get, from Eqs. (3.2), (7.1), (7.2) and (7.3),

$$\begin{aligned}&K_a\Phi_1=\Delta_0\,, \quad K_a\Phi_2=\Delta_0R_{pp}^0\,,\\ &K_b\Psi_0=i\Delta_2, \quad K_b\Psi_1=i\Delta_2R_{pp}^2, \quad K_b\Psi_2=i\Delta_2R_{ss}^2, \quad K_b\Psi_3=i\Delta_2R_{pp}^1\,.\end{aligned} \tag{16.6}$$

Substitution of these formulas into Eq. (16.2) leads to

$$-i(8K_0/\mu_2^3)\mathfrak{F}_a=$$
$$\Delta_0\Delta_2[(1/4)\{E(i\overline{P+Q}+)+R_{pp}^0E(i\overline{P-Q}+)\}\{R_{pp}^1e^{i\bar{P}_c}-e^{-i\bar{P}_c}\}e^{\hat{P}_b+\hat{Q}_b}$$
$$-(i/2)\{E(i\overline{P+Q}-)-R_{pp}^0E(i\overline{P-Q}-)\}\{R_{pp}^2e^{i\bar{P}_c}-R_{ss}^2e^{-i\bar{P}_c}\}e^{\hat{P}_b-\hat{Q}_b}$$
$$-(i/2)\{E(i\overline{P+Q}-)+R_{pp}^0E(i\overline{P-Q}-)\}\{R_{ss}^2e^{i\bar{P}_c}-R_{pp}^2e^{-i\bar{P}_c}\}e^{-\hat{P}_b+\hat{Q}_b}$$
$$-\{E(i\overline{P+Q}+)-R_{pp}^0E(i\overline{P-Q}+)\}\{e^{i\bar{P}_c}-R_{pp}^1e^{-i\bar{P}_c}\}e^{-\hat{P}_b-\hat{Q}_b}]$$
$$-R_0\{e^{i\bar{P}_c}-e^{-i\bar{P}_c}\}=0\,. \tag{16.7}$$

Here, we find from Eq. (5.8) that the fourth term in the bracket makes a predominant contribution to the equation. Hence, we approximately have independent equations,

$$1-R^0_{pp}e^{-2i\bar{P}_a}-R^0_{ss}e^{-2i\bar{Q}_a}+e^{-2i(\bar{P}_a+\bar{Q}_a)}=0\,,$$
$$1-R^1_{pp}e^{-2i\bar{P}_c}=0\,, \tag{16.8}$$

where the relation $R^0_{pp}=R^0_{ss}$ is employed and the first equation is identical with Eq. (14.4).

17. Asymptotic frequency equation when $k_2a,\ h_1b>\nu>h_2a,\ k_2b$

From Eq. (3.1), the angles i_1 and f_0 are assigned as real numbers and the angles i_0, i_2 and f_2 as complex numbers; the ray geometry is illustrated in Fig. 3j.

Then we have, instead of Eq. (16.7),

$$\begin{aligned}-i(8K_0/\mu_2^3)\mathfrak{F}_a=\Delta_0\Delta_2[&(i/2)\{R^0_{pp}e^{i\bar{Q}_a}+e^{-i\bar{Q}_a}\}\{R^1_{pp}e^{i\bar{P}_c}-e^{-i\bar{P}_c}\}e^{-\delta\hat{P}+\hat{Q}_b}\\&+\{R^0_{pp}e^{i\bar{Q}_a}-e^{-i\bar{Q}_a}\}\{R^2_{pp}e^{i\bar{P}_c}-R^2_{ss}e^{-i\bar{P}_c}\}e^{-\delta\hat{P}-\hat{Q}_b}\\&-(i/2)\{e^{i\bar{Q}_a}+R^0_{pp}e^{-i\bar{Q}_a}\}\{R^2_{ss}e^{i\bar{P}_c}-R^2_{pp}e^{-i\bar{P}_c}\}e^{\delta\hat{P}+\hat{Q}_b}\\&-\{e^{i\bar{Q}_a}-R^0_{pp}e^{-i\bar{Q}_a}\}\{e^{i\bar{P}_c}-R^1_{pp}e^{-i\bar{P}_c}\}e^{\delta\hat{P}-\hat{Q}_b}]\\&-R_0\{e^{i\bar{P}_c}-e^{-i\bar{P}_c}\}=0\,. \end{aligned}\tag{17.1}$$

Equation (5.8) is helpful in finding the most predominant term in the equation. Then, the fourth term in the bracket provides two independent equations,

$$1-R^0_{ss}e^{-2i\bar{Q}_a}=0\,,$$
$$1-R^1_{pp}e^{-2i\bar{P}_c}=0\,, \tag{17.2}$$

where R^0_{pp} is replaced by R^0_{ss} and then the first equation is identical with Eq. (15.2).

18. Asymptotic frequency equation when $h_1b>\nu>k_2a$

Now the angle i_1 is defined as a real number but all the other angles, i_0, i_2, f_0 and f_2, as complex numbers, which means that there exists only P rays in the core for a given ray parameter (see Fig. 3k).

Then we get

$$\begin{aligned}&-i(8K_0/\mu_2^3)\mathfrak{F}_a=\\&\Delta_0\Delta_2[-\{R^1_{pp}e^{i\bar{P}_c}-e^{-i\bar{P}_c}\}e^{-\delta\hat{P}-\delta\hat{Q}}+R^0_{pp}\{R^2_{pp}e^{i\bar{P}_c}-R^2_{ss}e^{-i\bar{P}_c}\}e^{-\delta\hat{P}+\delta\hat{Q}}\\&\quad+R^0_{pp}\{R^2_{ss}e^{i\bar{P}_c}-R^2_{pp}e^{-i\bar{P}_c}\}e^{\delta\hat{P}-\delta\hat{Q}}-\{e^{i\bar{P}_c}-R^1_{pp}e^{-i\bar{P}_c}\}e^{\delta\hat{P}+\delta\hat{Q}}]\\&-R_0\{e^{i\bar{P}_c}-e^{-i\bar{P}_c}\}=0\end{aligned}\tag{18.1}$$

Equation (5.8) indicates that the fourth term in the bracket is superior to other terms in their contributions to the equation. Hence, we have

$$1-R_{pp}^{1}e^{-2i\bar{P}_c}=0\,. \tag{18.2}$$

This equation is identical with the second one of both Eq. (16.8) and Eq. (17.2), and it represents the frequency equation of the spheroidal oscillations of the Earth attributable to multiple total-reflections of P rays in the liquid core. Hence, disturbances associated with these modes are restricted within the core and in the vicinity of its boundary in the mantle.

19. Summary

Two kinds of the asymptotic formulas of the spherical Bessel functions and spherical Neumann functions are successfully employed for deriving asymptotic frequency equations of the spheroidal oscillations of the Earth with a uniform mantle and core. Then, we have assumed both frequency and angular order to be very large while keeping its ratio (or phase velocity) finite.

The equations are expressed in different forms corresponding to different ray-geometries in the Earth and are denoted in terms of the reflection and transmission coefficients and the intercept times of relevant P and S rays incident on the free surface and mantle-core boundary. This fact surely proves the strong connection between high radial modes of free oscillations and body waves.

Here it should be noted that it will be very difficult to expand the asymptotic frequency equations of the spheroidal oscillations into ray series transmitted in the Earth unless we have certain information on the ratio between the displacements associated with P and S waves. This situation will be made clear in a future paper [ODAKA (1980b)], in which the frequency equations are derived in terms of ray theory under a certain interference condition of body waves traveling in the Earth. The paper will also give further reduction of the equations, discussion on the distribution of their solutions (eigenfrequencies) and numerical computations.

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15. 均質なマントルと核より成る弾性球の伸び縮み振動に対する特性方程式の漸近形
—有限な位相速度のモード—

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弾性球の伸び縮み振動の特性方程式は，高振動数，高位相速度の仮定のもとでは大変簡単になる。これは，波線理論的には球内のP波とS波が分離するからである。しかし位相速度を有限に保つ場合には，P波とS波が coupling をしている状態にあるので，その漸近式（高振動数近似）も複雑になる。

ここでは単純なモデル（均質な固体マントルと均質な流体核より成る）で満足することにして，後者の場合に対する式の導出を試みた。方法は，正規モード理論による特性方程式（球 Bessel 関数，球 Neumann 関数とそれらの一次微分関数で表現される）を，そこに現れる関数の漸近式を用いて展開するというものである。

方程式は，最終的には，実体波の反射係数，透過係数，intercept time というような波線理論的な量で表現されており，それは球内の異なった波線伝播様式に対してそれぞれ異なっている。