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17. *Analytical Expressions of Amplitudes for Principal Tidal Components.*

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Abstract

The tide-generating potential of the second order by the moon and the sun is expanded analytically and the amplitude coefficients are expressed in astronomical parameters such as orbital elements.

First, the expansion is carried out on the assumption that the eccentricity and the inclination of the orbit are constant while the longitude of the ascending node and the argument of perigee of the moon vary in proportion to the time. Numerical calculations show the amplitudes agree with the values of the preceding precise computation with an accuracy of 2.3 per cent.

After that, the effect of periodic changes of the elements is taken into consideration. Here, the method of variation of constants is applied. This method, on the one hand, gives the means of calculation of amplitude for new components generated by the perturbation. On the other hand, it improves the amplitude expressions obtained by the above assumption with an accuracy of 0.1 per cent.

1. Introduction

The tidal potential generated by the moon and the sun is used for calculating the theoretical standard in the analysis and in the prediction of earth tide as well as oceanic tide. The potential is conveniently expressed as the series of periodic terms of which arguments are linear functions of the time. To obtain such a representation, the most appropriate treatment is the method of harmonic expansion.

The harmonic expansion of the tide-generating potential was already accomplished and the amplitude coefficients were calculated by DOODSON (1922). It was recomputed by CARTWRIGHT and TAYLER (1971) by modern technics and their results were corrected by CARTWRIGHT and EDDEN (1973). These calculations were, however, carried out numerically. As a result, the relation between the amplitude and the astronomical parameters related to the motion of the moon and the sun remains unclear. From the theoretical point of view, it is more desirable that the tidal potential is expanded in the analytical form.

In this paper, the analytical expansion is attempted for the tide-generating potential of the second order. The amplitude coefficients of several principal components are tried to be written in analytical forms by making use of astronomical parameters.

The motion of the moon around the earth is roughly approximated by an elliptic orbit and the orbit is generally represented by orbital elements such as the semi-major axis, the eccentricity, the inclination, etc. Even if the orbit is fixed to the space as an ellipse, numerous components are produced in the tide-generating potential. Actually, the orbit of the moon is considerably affected by the attraction of the sun. Consequently, the values of the elements continuously vary and oscillate in various periods. More or less, the situation is the same for the solar orbit. Accordingly, still more components are added to the tidal potential.

To understand this better, we consider first the case where the eccentricity and the inclination of the orbit are constant while the longitude of the ascending node and the argument of perigee vary in proportion to the time. These assumptions are close to the real ones and give good approximations for the amplitudes of tidal components. After that, the effect of perturbation to the orbital elements is taken into consideration.

In this paper, we adopt the local sidereal time in the argument of diurnal and semi-diurnal components instead of mean lunar time taken by DOODSON (1922). This is quite convenient for comparing the phases of observed results with that of theoretical ones because the local sidereal time is easy to obtain for a place of observation.

2. The Expansion for the Semi-Diurnal Wave

The semi-diurnal wave of the lunar tide-generating potential of the second order is expressed as

$$G(a/r)^3 \cos^2 \phi \cos^2 \delta \cos 2H\,,$$

where G : Doodson's constant,

a : the semi-major axis of the lunar orbit,

r : the distance between the earth's center and the moon,

ϕ : the latitude of the place of observation,

δ : the declination of the moon,

and H: the hour angle of the moon.

Since the factor $G\cos^2\phi$ is constant for a station, it is sufficient to treat $(a/r)^3\cos^2\delta\cos 2H$ as time depending. Then, we try to expand that into a series of periodic terms of which arguments are linear functions of time, t.

When the right ascension of the moon is denoted by α and the sidereal time by θ, the hour angle of the moon, H, is expressed as

$$H=\theta-\alpha\,, \tag{1}$$

then, it becomes

$$\begin{aligned}&(a/r)^3\cos^2\delta\cos 2H\\&=(a/r)^3\cos^2\delta\cos 2\alpha\cos 2\theta+(a/r)^3\cos^2\delta\sin 2\alpha\sin 2\theta\,.\end{aligned} \tag{2}$$

By applying the formulae of spherical triangle, the following relations hold.

$$\left.\begin{aligned}&\cos^2\delta\cos 2\alpha=a_{11}+a_{12}\cos 2f+a_{13}\sin 2f\,,\\ \text{and}\quad&\cos^2\delta\sin 2\alpha=a_{21}+a_{22}\cos 2f+a_{23}\sin 2f\,.\end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned}a_{11}&=\frac{1}{2}\sin^2 I_q\cos 2\Omega_q\,,\\ a_{12}&=\frac{1}{2}\{(1+\cos^2 I_q)\cos 2\Omega_q\cos 2\omega_q-2\cos I_q\sin 2\Omega_q\sin 2\omega_q\}\,,\\ a_{13}&=-\frac{1}{2}\{(1+\cos^2 I_q)\cos 2\Omega_q\sin 2\omega_q+2\cos I_q\sin 2\Omega_q\cos 2\omega_q\}\,,\\ a_{21}&=\frac{1}{2}\sin^2 I_q\sin 2\Omega_q\,,\\ a_{22}&=\frac{1}{2}\{2\cos I_q\cos 2\Omega_q\sin 2\omega_q+(1+\cos^2 I_q)\sin 2\Omega_q\cos 2\omega_q\}\,,\\ a_{23}&=\frac{1}{2}\{2\cos I_q\cos 2\Omega_q\cos 2\omega_q-(1+\cos^2 I_q)\sin 2\Omega_q\sin 2\omega_q\}\,.\end{aligned}\right\} \tag{4}$$

It the above expressions,

Ω_q: the right ascension of the node, N_q,

I_q: the inclination of the orbit to the equator,

ω_q: the argument of perigee, A,

and f: the true anomaly of the moon.

The lunar orbital elements Ω_q, I_q and ω_q are defined with respect to the equator. The meaning of these elements are schematically shown in Fig. 1.

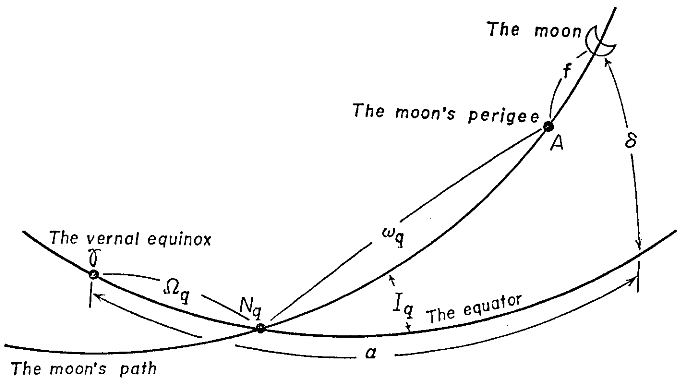

Fig. 1. Lunar orbital elements defined with respect to the equator.

The relation (2) is transformed by the relations (3) into the form as

$$\begin{aligned}(a/r)^3 \cos^2 \delta \cos 2H = &(a_{11} \cos 2\theta + a_{21} \sin 2\theta)(a/r)^3 \\ &+ (a_{12} \cos 2\theta + a_{22} \sin 2\theta)(a/r)^3 \cos 2f \\ &+ (a_{13} \cos 2\theta + a_{23} \sin 2\theta)(a/r)^3 \sin 2f . \qquad (5)\end{aligned}$$

Orbital elements are generally defined with respect to the ecliptic. For the elements to the ecliptic, the following notations are used;

Ω: the longitude of the node, N,

I: the inclination of the orbit to the ecliptic,

and ω: the argument of perigee, A.

The relation of these two kinds of orbital elements is shown in Fig. 2.

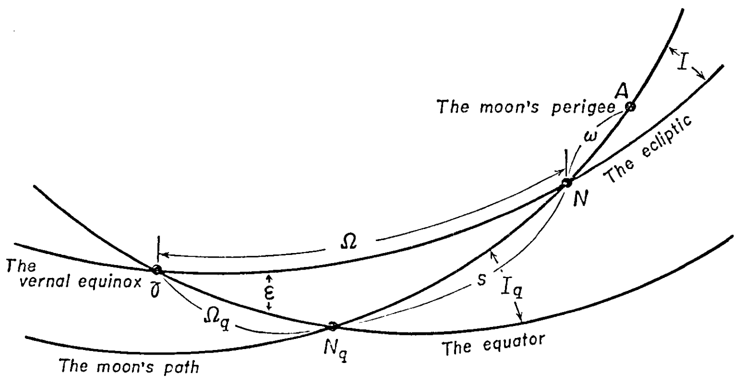

Fig. 2. Relations between two kinds of lunar elements.

By formulae of spherical triangle, the following relations hold among them.

$$
\left.
\begin{aligned}
&\cos I_q = -\sin I \sin \varepsilon \cos \Omega + \cos I \cos \varepsilon\,,\\
&\sin I_q \sin \Omega_q = \sin I \sin \Omega\,,\\
&\sin I_q \sin s = \sin \varepsilon \sin \Omega\,,\\
&\sin I_q \cos \Omega_q = \sin I \cos \varepsilon \cos \Omega + \cos I \sin \varepsilon\,,\\
&\sin I_q \cos s = \cos I \sin \varepsilon \cos \Omega + \sin I \cos \varepsilon\,,\\
&\cos I_q \cos s \sin \Omega_q + \sin s \cos \Omega_q = \cos I \sin \Omega\,,\\
&\cos I_q \sin s \cos \Omega_q + \cos s \sin \Omega_q = \cos I \sin \Omega\,,\\
&\cos I_q \cos s \cos \Omega_q - \sin s \sin \Omega_q = \cos I \cos \varepsilon \cos \Omega - \sin I \sin \varepsilon\,,\\
&\cos I_q \sin s \sin \Omega_q - \cos s \cos \Omega_q = -\cos \Omega\,,
\end{aligned}
\right\}
\qquad (6)
$$

where ε indicates the obliquity of the ecliptic and s the angular distance between N and N_q. The sum of s and ω becomes ω_q. By applying the above relations, expressions (4) are transformed into the relations as follows:

$$
\begin{aligned}
a_{11} = {} & \frac{1}{4} \sin^2 \varepsilon (2 - 3 \sin^2 I)\\
& + \frac{1}{4} \sin 2\varepsilon \sin 2I \cos \Omega\\
& + \frac{1}{4} (1 + \cos^2 \varepsilon) \sin^2 I \cos 2\Omega\,,\\
a_{12} = {} & \frac{3}{4} \sin^2 \varepsilon \sin^2 I \cos 2\omega\\
& - \frac{1}{4} \sin 2\varepsilon \sin I (1 + \cos I) \cos (\Omega + 2\omega)\\
& - \frac{1}{4} \sin 2\varepsilon \sin I (1 - \cos I) \cos (\Omega - 2\omega)\\
& + \frac{1}{8} (1 + \cos^2 \varepsilon)(1 + \cos I)^2 \cos (2\Omega + 2\omega)\\
& + \frac{1}{8} (1 + \cos^2 \varepsilon)(1 - \cos I)^2 \cos (2\Omega - 2\omega)\,,\\
a_{13} = {} & -\frac{3}{4} \sin^2 \varepsilon \sin^2 I \sin 2\omega\\
& + \frac{1}{4} \sin 2\varepsilon \sin I (1 + \cos I) \sin (\Omega + 2\omega)
\end{aligned}
$$

$$
\left.\begin{aligned}
&+\frac{1}{4}\sin 2\varepsilon \sin I(1-\cos I)\sin(\Omega-2\omega)\\
&-\frac{1}{8}(1+\cos^2\varepsilon)(1+\cos I)^2\sin(2\Omega+2\omega)\\
&+\frac{1}{8}(1+\cos^2\varepsilon)(1-\cos I)^2\sin(2\Omega-2\omega)\,,\\
a_{21}=&\ \ \frac{1}{2}\sin\varepsilon\sin 2I\sin\Omega\\
&+\frac{1}{2}\cos\varepsilon\sin^2 I\sin 2\Omega\,,\\
a_{22}=&-\frac{1}{2}\sin\varepsilon\sin I(1+\cos I)\sin(\Omega+2\omega)\\
&-\frac{1}{2}\sin\varepsilon\sin I(1-\cos I)\sin(\Omega-2\omega)\\
&+\frac{1}{4}\cos\varepsilon(1+\cos I)^2\sin(2\Omega+2\omega)\\
&+\frac{1}{4}\cos\varepsilon(1-\cos I)^2\sin(2\Omega-2\omega)\,,\\
a_{23}=&-\frac{1}{2}\sin\varepsilon\sin I(1+\cos I)\cos(\Omega+2\omega)\\
&-\frac{1}{2}\sin\varepsilon\sin I(1-\cos I)\cos(\Omega-2\omega)\\
&+\frac{1}{4}\cos\varepsilon(1+\cos I)^2\cos(2\Omega+2\omega)\\
&-\frac{1}{4}\cos\varepsilon(1-\cos I)^2\cos(2\Omega-2\omega)\,.
\end{aligned}\right\} \tag{7}
$$

In the above relations, the longitude of ascending node, Ω, and the argument of perigee, ω, are separated from the other factors in each term as the argument of the cosine or sine function.

On the other hand, $(a/r)^3$, $(a/r)^3\cos 2f$ and $(a/r)^3\sin 2f$ can be expanded in terms of mean anomaly, l, by means of ordinary elliptic expansion. That is,

$$
\left.\begin{aligned}
(a/r)^3 &= t_0+t_1\cos l+t_2\cos 2l+t_3\cos 3l+\cdots\,,\\
(a/r)^3\cos 2f &= \quad c_1\cos l+c_2\cos 2l+c_3\cos 3l+\cdots\,,\\
(a/r)^3\sin 2f &= \quad s_1\sin l+s_2\sin 2l+s_3\sin 3l+\cdots\,.
\end{aligned}\right\} \tag{8}
$$

Here, l is also the element which varies in proportion to the time, t, and $t_0, t_1, t_2, \cdots, c_1, c_2, c_3, \cdots$, and $s_1, s_2, s_3, \cdots$, are power series of the eccentricity, e, of the lunar orbit. Neglecting the powers of e above the seventh, they can be written as

$$\left.\begin{aligned}
t_0&=1+\frac{3}{2}e^2+\frac{15}{8}e^4+\frac{35}{16}e^6+\cdots\cdots,\\
t_1&=3e+\frac{27}{8}e^3+\frac{261}{64}e^5+\frac{14309}{3072}e^7+\cdots\cdots,\\
t_2&=\frac{9}{2}e^2+\frac{7}{2}e^4+\frac{141}{32}e^6+\cdots\cdots,\\
t_3&=\frac{53}{8}e^3+\frac{393}{128}e^5+\frac{24753}{5120}e^7+\cdots\cdots,\\
t_4&=\frac{77}{8}e^4+\frac{129}{80}e^6+\cdots\cdots,\\
t_5&=\frac{1773}{128}e^5-\frac{4987}{3072}e^7+\cdots\cdots.
\end{aligned}\right\}\qquad(9)$$

$$\left.\begin{aligned}
c_1&=-\frac{1}{2}e+\frac{1}{12}e^3+\frac{1}{768}e^5+\frac{1}{2880}e^7+\cdots\cdots,\\
c_2&=1-\frac{5}{2}e^2+\frac{41}{48}e^4-\frac{133}{1440}e^6+\cdots\cdots,\\
c_3&=\frac{7}{2}e-\frac{123}{16}e^3+\frac{4971}{1280}e^5-\frac{841}{1024}e^7+\cdots\cdots,\\
c_4&=\frac{17}{2}e^2-\frac{115}{6}e^4+\frac{9079}{720}e^6+\cdots\cdots,\\
c_5&=\frac{845}{48}e^3-\frac{32525}{768}e^5+\frac{2194175}{64512}e^7+\cdots\cdots.
\end{aligned}\right\}\qquad(10)$$

$$\left.\begin{aligned}
s_1&=-\frac{1}{2}e+\frac{1}{24}e^3-\frac{7}{256}e^5-\frac{827}{46080}e^7+\cdots\cdots,\\
s_2&=1-\frac{5}{2}e^2+\frac{37}{48}e^4-\frac{217}{1440}e^6+\cdots\cdots,\\
s_3&=\frac{7}{2}e-\frac{123}{16}e^3+\frac{4809}{1280}e^5-\frac{461}{512}e^7+\cdots\cdots,\\
s_4&=\frac{17}{2}e^2-\frac{115}{6}e^4+\frac{8951}{720}e^6+\cdots\cdots,\\
s_5&=\frac{845}{48}e^3-\frac{32525}{768}e^5+\frac{1089275}{32256}e^7+\cdots\cdots.
\end{aligned}\right\}\qquad(11)$$

By substituting the relations (7) and (8) into (5), $(a/r)^3\cos^2\delta\cos 2H$ can finally be rewritten in the following form.

$$
\begin{aligned}
&(a/r)^3 \cos^2\delta \cos 2H \\
&= \sum_{j=0}^{\infty} \Big\{ \frac{1}{8} \sin^2\varepsilon(2-3\sin^2 I)t_j \cos(2\theta \pm jl) \\
&\quad -\frac{1}{8} \sin\varepsilon(1-\cos\varepsilon) \sin 2I \cdot t_j \cos(2\theta \pm jl + \Omega) \\
&\quad +\frac{1}{8} \sin\varepsilon(1+\cos\varepsilon) \sin 2I \cdot t_j \cos(2\theta \pm jl - \Omega) \\
&\quad +\frac{1}{16}(1-\cos\varepsilon)^2 \sin^2 I \cdot t_j \cos(2\theta \pm jl + 2\Omega) \\
&\quad +\frac{1}{16}(1+\cos\varepsilon)^2 \sin^2 I \cdot t_j \cos(2\theta \pm jl - 2\Omega) \Big\} \\
&+ \sum_{j=1}^{\infty} \Big\{ \frac{3}{16} \sin^2\varepsilon \sin^2 I (c_j \pm s_j) \cos(2\theta \pm jl + 2\omega) \\
&\quad +\frac{3}{16} \sin^2\varepsilon \sin^2 I (c_j \mp s_j) \cos(2\theta \pm jl - 2\omega) \\
&\quad +\frac{1}{8} \sin\varepsilon(1-\cos\varepsilon) \sin I(1+\cos I)(c_j \pm s_j) \cos(2\theta \pm jl + \Omega + 2\omega) \\
&\quad -\frac{1}{8} \sin\varepsilon(1+\cos\varepsilon) \sin I(1+\cos I)(c_j \mp s_j) \cos(2\theta \pm jl - \Omega - 2\omega) \\
&\quad +\frac{1}{8} \sin\varepsilon(1-\cos\varepsilon) \sin I(1-\cos I)(c_j \pm s_j) \cos(2\theta \pm jl + \Omega - 2\omega) \\
&\quad -\frac{1}{8} \sin\varepsilon(1+\cos\varepsilon) \sin I(1-\cos I)(c_j \mp s_j) \cos(2\theta \pm jl - \Omega + 2\omega) \\
&\quad +\frac{1}{32}(1-\cos\varepsilon)^2(1+\cos I)^2(c_j \pm s_j) \cos(2\theta \pm jl + 2\Omega + 2\omega) \\
&\quad +\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_j \mp s_j) \cos(2\theta \pm jl - 2\Omega - 2\omega) \\
&\quad +\frac{1}{32}(1-\cos\varepsilon)^2(1-\cos I)^2(c_j \mp s_j) \cos(2\theta \pm jl + 2\Omega - 2\omega) \\
&\quad +\frac{1}{32}(1+\cos\varepsilon)^2(1-\cos I)^2(c_j \pm s_j) \cos(2\theta \pm jl - 2\Omega + 2\omega) \Big\}. \qquad (12)
\end{aligned}
$$

In the above expression, each term should be employed twice. First, by taking the upper signs from signs $\pm$ and $\mp$, and second, by taking the lower. The relation (12) shows the expanded form of the semi-diurnal lunar tide-generating potential of the second order.

As for the solar tide-generating potential, the expanded form of semi-diurnal wave can be directly written from relation (12) by replacing the elements of the moon for that of the sun. As the inclination of orbit, I', and the longitude of the ascending node, Ω', can be considered to be zero in the solar orbit, the potential is written in the following form.

$$
\begin{aligned}
(a'/r')^3 \cos^2\delta' \cos 2H' = & \sum_{j=0}^{\infty} \frac{1}{4} \sin^2\varepsilon \cdot t'_j \cos(2\theta \pm jl') \\
& + \sum_{j=1}^{\infty} \left\{ \frac{1}{8}(1-\cos\varepsilon)^2 (c'_j \pm s'_j) \cos(2\theta \pm jl' + 2\omega') \right. \\
& \left. + \frac{1}{8}(1+\cos\varepsilon)^2 (c'_j \mp s'_j) \cos(2\theta \pm jl' - 2\omega') \right\}, \quad (13)
\end{aligned}
$$

where letters with a prime designate the corresponding elements for the sun. Each term should be employed twice as in the case of relation (12).

The terms in these expansions may be compared with the tidal components already given by DOODSON (1922), CARTWRIGHT and TAYLER (1971) and CARTWRIGHT and EDDEN (1973). For the comparison, the argument employed by each author and their relations are summarized in Table 1. By the table, respective arguments can easily be rewritten

Table 1. Interrelations of the elements appeared in the arguments of tidal components.

	Doodson	Cartwright & Tayler	Present paper
The local sidereal time	$(\tau+s)$*	—	θ
The mean lunar time	τ	—	—
The longitude of the ascending node of the moon	$-N'$	Ω	Ω
The argument of lunar perigee	$p+N'$	$\tilde{\omega}-\Omega$	ω
The mean anomaly of the moon	$s-p$	$L-\tilde{\omega}$	l
The longitude of lunar perigee	p	$\tilde{\omega}$	$\tilde{\omega}$
The mean longitude of the moon	s	L	λ
Tne longitude of solar perigee	p_s	$\tilde{\omega}'$	$\tilde{\omega}'$
The mean anomaly of the sun	$h-p_s$	$L'-\tilde{\omega}'$	l'
The mean longitude of the sun	h	L'	λ'

* The expression in parentheses is approximate one.

into other ones and corresponding components can be found. For example, an argument $(2\tau-s+p)$ in DOODSON's representation (N_2 component, DOODSON's argument number 245.655) corresponds to $(2\theta-3l-2\Omega-2\omega)$ or $(2\theta-3\lambda+\tilde{\omega})$ in this paper.

3. The Expansions for the Diurnal and the Long Period Wave

For the diurnal and the long period wave, the expansions are carried out in the same manner as that of the semi-diurnal one. As the expressions of the potentials are

$$G(a/r)^3 \sin 2\phi \sin 2\delta \cos H \qquad \text{for the diurnal wave}$$

and

$$G(a/r)^3 3\left(\sin^2\phi-\frac{1}{3}\right)\left(\sin^2\delta-\frac{1}{3}\right) \qquad \text{for the long period wave,}$$

the factors $(a/r)^3 \sin 2\delta \cos H$ and $\frac{2}{3}(a/r)^3(1-3\sin^2\delta)$ are treated here as time depending potions for respective wave. To avoid the complexity, only the results are presented in what follows.

For the lunar diurnal wave;

$$\begin{aligned}
&(a/r)^3 \sin 2\delta \cos H \\
&= \sum_{j=0}^{\infty}\left\{\frac{1}{8}\sin 2\varepsilon(2-3\sin^2 I)t_j \sin(\theta \pm jl)\right. \\
&\quad -\frac{1}{8}(1-\cos\varepsilon)(2\cos\varepsilon+1)\sin 2I\cdot t_j \sin(\theta\pm jl+\Omega) \\
&\quad +\frac{1}{8}(1+\cos\varepsilon)(2\cos\varepsilon-1)\sin 2I\cdot t_j \sin(\theta\pm jl-\Omega) \\
&\quad +\frac{1}{8}\sin\varepsilon(1-\cos\varepsilon)\sin^2 I\cdot t_j \sin(\theta\pm jl+2\Omega) \\
&\quad \left.-\frac{1}{8}\sin\varepsilon(1+\cos\varepsilon)\sin^2 I\cdot t_j \sin(\theta\pm jl-2\Omega)\right\} \\
&+\sum_{j=1}^{\infty}\left\{\frac{3}{16}\sin 2\varepsilon \sin^2 I(c_j\pm s_j)\sin(\theta\pm jl+2\omega)\right. \\
&\quad +\frac{3}{16}\sin 2\varepsilon\sin^2 I(c_j\mp s_j)\sin(\theta\pm jl-2\omega) \\
&\quad +\frac{1}{8}(1-\cos\varepsilon)(2\cos\varepsilon+1)\sin I(1+\cos I)(c_j\pm s_j)\sin(\theta\pm jl+\Omega+2\omega) \\
&\quad -\frac{1}{8}(1+\cos\varepsilon)(2\cos\varepsilon-1)\sin I(1+\cos I)(c_j\mp s_j)\sin(\theta\pm jl-\Omega-2\omega) \\
&\quad -\frac{1}{8}(1-\cos\varepsilon)(2\cos\varepsilon+1)\sin I(1-\cos I)(c_j\mp s_j)\sin(\theta\pm jl+\Omega-2\omega) \\
&\quad +\frac{1}{8}(1+\cos\varepsilon)(2\cos\varepsilon-1)\sin I(1-\cos I)(c_j\pm s_j)\sin(\theta\pm jl-\Omega+2\omega) \\
&\quad +\frac{1}{16}\sin\varepsilon(1-\cos\varepsilon)(1+\cos I)^2(c_j\pm s_j)\sin(\theta\pm jl+2\Omega+2\omega)
\end{aligned}$$

$$-\frac{1}{16}\sin\varepsilon(1+\cos\varepsilon)(1+\cos I)^2(c_j\mp s_j)\sin(\theta\pm jl-2\Omega-2\omega)$$

$$+\frac{1}{16}\sin\varepsilon(1-\cos\varepsilon)(1-\cos I)^2(c_j\mp s_j)\sin(\theta\pm jl+2\Omega-2\omega)$$

$$\left.-\frac{1}{16}\sin\varepsilon(1+\cos\varepsilon)(1-\cos I)^2(c_j\pm s_j)\sin(\theta\pm jl-2\Omega+2\omega)\right\}. \tag{14}$$

For the solar diurnal wave;

$$(a'/r')^3\sin 2\delta'\cos H'$$

$$=\sum_{j=0}^{\infty}\frac{1}{4}\sin 2\varepsilon\cdot t'_j\sin(\theta\pm jl')$$

$$+\sum_{j=1}^{\infty}\left\{\frac{1}{4}\sin\varepsilon(1-\cos\varepsilon)(c'_j\pm s'_j)\sin(\theta\pm jl'+2\omega')\right.$$

$$\left.-\frac{1}{4}\sin\varepsilon(1+\cos\varepsilon)(c'_j\mp s'_j)\sin(\theta\pm jl'-2\omega')\right\}. \tag{15}$$

For the lunar long period wave;

$$\frac{2}{3}(a/r)^3(1-3\sin^2\delta)$$

$$=\sum_{j=0}^{\infty}\left\{\frac{1}{2}\left(\frac{4}{3}-\sin^2\varepsilon-2\sin^2 I\cos^2\varepsilon-\cos^2 I\sin^2\varepsilon\right)t_j\cos(jl)\right.$$

$$-\frac{1}{4}\sin 2\varepsilon\sin 2I\cdot t_j\cos(\pm jl+\Omega)$$

$$\left.+\frac{1}{4}\sin^2\varepsilon\sin^2 I\cdot t_j\cos(\pm jl+2\Omega)\right\}$$

$$+\sum_{j=1}^{\infty}\left\{\frac{1}{4}\sin 2\varepsilon\sin I(1+\cos I)(c_j\pm s_j)\cos(\pm jl+\Omega+2\omega)\right.$$

$$-\frac{1}{4}\sin 2\varepsilon\sin I(1-\cos I)(c_j\mp s_j)\cos(\pm jl+\Omega-2\omega)$$

$$+\frac{1}{8}\sin^2\varepsilon(1+\cos I)^2(c_j\pm s_j)\cos(\pm jl+2\Omega+2\omega)$$

$$\left.+\frac{1}{8}\sin^2\varepsilon(1-\cos I)^2(c_j\mp s_j)\cos(\pm jl+2\Omega-2\omega)\right\}. \tag{16}$$

For the solar long period wave;

$$\frac{2}{3}(a'/r')^3(1-3\sin^2\delta')$$

$$=\sum_{j=0}^{\infty}\left(\frac{2}{3}-\sin^2\varepsilon\right)t'_j\cos(jl')$$

$$+\sum_{j=1}^{\infty}\left\{\frac{1}{2}\sin^2\varepsilon(c'_j\pm s'_j)\cos(\pm jl'+2\omega')\right.$$

$$\left.+\frac{1}{2}\sin^2\varepsilon(c'_j\mp s'_j)\cos(\pm jl'-2\omega')\right\}. \tag{17}$$

In the above expressions, the terms which contain the sign $\pm$ or $\mp$ should be employed twice in the same manner as described for the relation (12).

4. Analytical Forms of Amplitudes for Principal Tidal Components

According to the expanded forms given in the previous sections, tide-generating potential has a number of terms. In these terms, however, there are not many terms which have actual meaning as tidal components because the values of $\sin I$, $1-\cos I$ and c_j-s_j are considerably small. We can pick up about 130 components which are contained in the tables by CARTWRIGHT and TAYLER (1971). Among them, the analytical expressions of the amplitudes for only principal components are given in the following Tables 2, 3 and 4. We also give three kinds of argument

Table 2. Analytical expressions of amplitude for semi-diurnal components.

Wave	Amplitude	Argument		
M_2	$\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_2+s_2)$	$2\theta-2l-2\Omega-2\omega$	$2\theta-2\lambda$	2τ
S_2	$\frac{C}{8}(1+\cos\varepsilon)^2(c'_2+s'_2)$	$2\theta-2l'\quad-2\Omega'$	$2\theta-2\lambda'$	$2\tau+2s-2h$
N_2	$\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_3+s_3)$	$2\theta-3l-2\Omega-2\omega$	$2\theta-3\lambda+\tilde{\omega}$	$2\tau-s+p$
${}^mK_2$	$\frac{1}{4}\sin^2\varepsilon(2-3\sin^2 I)t_0$	2θ	2θ	$2\tau+2s$
${}^sK_2$	$\frac{C}{2}\sin^2\varepsilon\cdot t'_0$	2θ	2θ	$2\tau+2s$
275.565	$\frac{1}{4}\sin\varepsilon(1+\cos\varepsilon)\sin 2I\cdot t_0$	$2\theta\quad-\Omega$	$2\theta-\Omega$	$2\tau+2s+N'$
255.545	$-\frac{1}{8}\sin\varepsilon(1+\cos\varepsilon)\sin I(1+\cos I)(c_2+s_2)$	$2\theta-2l-\Omega\quad-2\omega$	$2\theta-2\lambda+\Omega$	$2\tau-N'$
L_2	$\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_1+s_1)$	$2\theta\quad-l-2\Omega-2\omega$	$2\theta-\lambda-\tilde{\omega}$	$2\tau+s-p$
T_2	$\frac{C}{8}(1+\cos\varepsilon)^2(c'_3+s'_3)$	$2\theta-3l'\quad-2\omega'$	$2\theta-3\lambda'+\tilde{\omega}'$	$2\tau+2s-3h+p_s$
$2N_2$	$\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_4+s_4)$	$2\theta-4l-2\Omega-2\omega$	$2\theta-4\lambda+2\tilde{\omega}$	$2\tau-2s+2p$

Table 3. Analytical expressions of amplitudes for diurnal components.

Wave	Amplitude	Argument		
O_1	$-\frac{1}{16}\sin\varepsilon(1+\cos\varepsilon)(1+\cos I)^2(c_2+s_2)$	$\theta-2l-2\Omega-2\omega$	$\theta-2\lambda$	$\tau-s$
${}^mK_1$	$\frac{1}{4}\sin 2\varepsilon(2-3\sin^2 I)t_0$	θ	θ	$\tau+s$
${}^sK_1$	$\frac{C}{2}\sin 2\varepsilon\cdot t_0'$	θ	θ	$\tau+s$
P_1	$-\frac{C}{4}\sin\varepsilon(1+\cos\varepsilon)(c_2'+s_2')$	$\theta-2l' \quad -2\omega'$	$\theta-2\lambda'$	$\tau+s-2h$
Q_1	$-\frac{1}{16}\sin\varepsilon(1+\cos\varepsilon)(1+\cos I)^2(c_3+s_3)$	$\theta-3l-2\Omega-2\omega$	$\theta-3\lambda+\varpi$	$\tau-2s+p$
165.565	$\frac{1}{4}(1+\cos\varepsilon)(2\cos\varepsilon-1)\sin 2I\cdot t_0$	$\theta \quad -\Omega$	$\theta-\Omega$	$\tau+s+N'$
145.545	$-\frac{1}{8}(1+\cos\varepsilon)(2\cos\varepsilon-1)\sin I \times(1+\cos I)(c_2+s_2)$	$\theta-2l \quad -\Omega-2\omega$	$\theta-2\lambda+\Omega$	$\tau-s-N'$
M_1	$\frac{1}{8}\sin 2\varepsilon(2-3\sin^2 I)t_1$	$\theta \quad -l$	$\theta-\lambda+\varpi$	$\tau+p$
J_1		$\theta \quad +l$	$\theta+\lambda-\varpi$	$\tau+2s-p$
OO_1	$\frac{1}{16}\sin\varepsilon(1-\cos\varepsilon)(1+\cos I)^2(c_2+s_2)$	$\theta+2l+2\Omega+2\omega$	$\theta+2\lambda$	$\tau+3s$
135.645	$-\frac{1}{8}(1+\cos\varepsilon)(2\cos\varepsilon-1)\sin I \times(1+\cos I)(c_1+s_1)$	$\theta-3l \quad -\Omega-2\omega$	$\theta-3\lambda+\varpi+\Omega$	$\tau-2s+p-N'$
155.455	$-\frac{1}{16}\sin\varepsilon(1+\cos\varepsilon)(1+\cos I)^2(c_1+s_1)$	$\theta \quad -l-2\Omega-2\omega$	$\theta-\lambda-\varpi$	$\tau-p$
165.545	$-\frac{1}{4}(1-\cos\varepsilon)(2\cos\varepsilon+1)\sin 2I\cdot t_0$	$\theta \quad +\Omega$	$\theta+\Omega$	$\tau+s-N'$
185.565	$\frac{1}{8}(1-\cos\varepsilon)(2\cos\varepsilon+1)\sin I \times(1+\cos I)(c_2+s_2)$	$\theta+2l \quad +\Omega+2\omega$	$\theta+2\lambda-\Omega$	$\tau+3s+N'$
π_1	$-\frac{C}{4}\sin\varepsilon(1+\cos\varepsilon)(c_3'+s_3')$	$\theta-3l' \quad -2\omega'$	$\theta-3\lambda'+\varpi'$	$\tau+s-3h+p_s$

representations for each component in the tables. Here, the letter C denotes the ratio of tidal force of the sun to that of the moon for the tidal potential of the second order. The numerical value is

$$C=0.459235\,.$$

In order to calculate numerical values of the amplitudes, it seems appropriate to take mean values of the eccentricity, e, and the inclination, I, for expressions in Tables 2, 3 and 4. Calculated results of amplitudes are given in Table 5 with that of CARTWRIGHT and EDDEN (1973). They agree with an accuracy of 2.3 per cent. Numerical values frequently used in the present calculation are summarized in Table 6.

Table 4. Analytical expressions of amplitudes for long period components.

Wave	Amplitude	Argument		
M_0	$\frac{1}{2}\left(\frac{4}{3}-\sin^2\varepsilon-2\sin^2 I\cos^2\varepsilon-\cos^2 I\sin^2\varepsilon\right)t_0$	—	—	—
S_0	$C\left(\frac{2}{3}-\sin^2\varepsilon\right)t_0'$	—	—	—
M_f	$\frac{1}{8}\sin^2\varepsilon(1+\cos I)^2(c_2+s_2)$	$2l+2\Omega+2\omega$	2λ	$2s$
M_m	$\frac{1}{2}\left(\frac{4}{3}-\sin^2\varepsilon-2\sin^2 I\cos^2\varepsilon-\cos^2 I\sin^2\varepsilon\right)t_1$	l	$\lambda-\tilde{\omega}$	$s-p$
S_{sa}	$\frac{C}{2}\sin^2\varepsilon(c_2'+s_2')$	$2l' \quad +2\omega'$	$2\lambda'$	$2h$
055.565	$-\frac{1}{2}\sin 2\varepsilon\sin 2I\cdot t_0$	Ω	Ω	N'
075.565	$\frac{1}{4}\sin 2\varepsilon\sin I(1+\cos I)(c_2+s_2)$	$2l\ +\Omega+2\omega$	$2\lambda-\Omega$	$2s+N'$
085.455	$\frac{1}{8}\sin^2\varepsilon(1+\cos I)^2(c_3+s_3)$	$3l+2\Omega+2\omega$	$3\lambda-\tilde{\omega}$	$3s-p$
085.465	$\frac{1}{4}\sin 2\varepsilon\sin I(1+\cos I)(c_3+s_3)$	$3l\ +\Omega+2\omega$	$3\lambda-\tilde{\omega}-\Omega$	$3s-p+N'$
S_a	$2C\left(\frac{2}{3}-\sin^2\varepsilon\right)t_1'$	l'	$\lambda'-\tilde{\omega}'$	$h-p_s$

5. The Components Generated by the Perturbation

Although the expansions already given explain the origin of almost all principal components, there exist a number of other components which are not found in these expansions. For example, the expansions do not contain the well known components, ν_2, μ_2 and λ_2 waves. This fact originates from the omission of the effect of the perturbed motion of the moon. By the perturbation, the values of the elements of the moon oscillate in various periods. By considering the effect of the perturbation, numbers of new components are produced and the amplitude of previously obtained components are affected. These situations will be explained in this and the following sections with examples.

Considering the process of the derivation of expanded forms of the potential, it is clear that relations (12), (13), (14), (15), (16) and (17) hold just as they are, even in the case where all the elements are considered to be osculating ones or instantaneous elements. Then, the effect of the perturbation is estimated by replacing the elements by osculating ones in the expressions of components in the previous sections. In later discussions, elements λ and $\tilde{\omega}$ are mainly used instead of l and ω,

Table 5. Numerical values of amplitudes compared with that of Cartwright and Edden.

Wave	Present results	Results by Cartwright & Edden	Wave	Present results	Results by Cartwright & Edden
M_2	0.90858	0.90809	OO_1	0.01623	0.01624
S_2	0.42182	0.42248	135.645	−0.01368	−0.01360
N_2	0.17475	0.17386	155.455	0.01042	0.01066
${}^mK_2$	0.07852	0.11498 (for ${}^mK_2$ and ${}^sK_2$)	165.545	−0.01050	−0.01051
${}^sK_2$	0.03635		185.565	0.01039	0.01039
275.565	0.03422	0.03426	π_1	−0.01024	−0.01028
255.545	−0.03388	−0.03390			
L_2	−0.02512	−0.02567	M_0	0.50455	0.73806 (for M_0 and S_0)
T_2	0.02468	0.02476	S_0	0.23358	
$2N_2$	0.02329	0.02301	M_f	0.15643	0.15642
			M_m	0.08301	0.08254
O_1	−0.37701	−0.37694*	S_{sa}	0.07263	0.07281
${}^mK_1$	0.36221	0.53011 (for ${}^mK_1$ and ${}^sK_1$)	055.565	−0.06550	−0.06556
${}^sK_1$	0.16768		075.565	0.06484	0.06483
P_1	−0.17503	−0.17543	085.455	0.03009	0.02996
Q_1	−0.07251	−0.07106	085.465	0.01247	0.01241
165.565	0.07182	0.07186	S_a	0.01171	0.01156
145.545	−0.07111	−0.07106			
M_1	0.02980	0.02964			
J_1	0.02980	0.02964			

* For the comparison, the signs of values by Cartwright and Edden are inverted in the cases of diurnal waves.

Table 6. Numerical values for astronomical parameters.

$\cos\varepsilon$:	0.917464	$\sin\varepsilon$:	0.397819
$\cos I$:	0.995970	$\sin I$:	0.089683
e :	0.054900	e' :	0.016718
t_0 :	1.004538	t_0' :	1.000419
t_1 :	0.165262	t_1' :	0.050168
t_2 :	0.013595	t_2' :	0.001258
c_1+s_1 :	−0.054880	$c_1'+s_1'$:	−0.016717
c_2+s_2 :	1.984944	$c_2'+s_2'$:	1.998603
c_3+s_3 :	0.381763	$c_3'+s_3'$:	0.116951
c_4+s_4 :	0.050892	$c_4'+s_4'$:	0.004748

$$\left.\begin{aligned} \text{where} \qquad \lambda &= l+\Omega+\omega\,, \\ \tilde{\omega} &= \Omega+\omega\,. \end{aligned}\right\} \tag{18}$$

The periodic changes of orbital elements are treated here by the method of variation of constants. In the method, time derivatives of orbital elements are given in the following forms (BROUWER and CLEMENCE, 1961a).

$$\left.\begin{aligned} \frac{de}{dt} &= -\frac{p(1-p)}{na^2e}\frac{\partial R}{\partial\lambda}-\frac{p}{na^2e}\frac{\partial R}{\partial\tilde{\omega}}\,, \\ \frac{dI}{dt} &= -\frac{\tan(I/2)}{na^2p}\left(\frac{\partial R}{\partial\lambda}+\frac{\partial R}{\partial\tilde{\omega}}\right)-\frac{1}{na^2p\sin I}\frac{\partial R}{\partial\Omega}\,, \\ \frac{d\sigma}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a}+\frac{p(1-p)}{na^2e}\frac{\partial R}{\partial e}+\frac{\tan(I/2)}{na^2p}\frac{\partial R}{\partial I}\,, \\ \frac{d\tilde{\omega}}{dt} &= \frac{p}{na^2e}\frac{\partial R}{\partial e}+\frac{\tan(I/2)}{na^2p}\frac{\partial R}{\partial I}\,, \\ \frac{d\Omega}{dt} &= \frac{1}{na^2p\sin I}\frac{\partial R}{\partial I}\,, \\ \frac{d^2\rho}{dt^2} &= -\frac{3}{a^2}\frac{\partial R}{\partial\lambda}\,, \end{aligned}\right\} \tag{19}$$

where σ : the mean longitude at epoch,
R : the disturbing function,
and $p=(1-e^2)^{1/2}$.

When the mean motion is denoted by n, ρ is expressed by

$$\rho=\int n\,dt\,, \tag{20}$$

and then, λ is also given by the relation

$$\lambda=\rho+\sigma\,. \tag{21}$$

On the other hand, by expressing the most significant terms only, the disturbing function of the moon's motion is given in the following form (BROWER and CLEMENCE, 1961b).

$$\begin{aligned} R=n'^2a^2\Big[&\left(\frac{1}{4}+\frac{3}{8}e^2+\frac{3}{8}e'^2-\frac{3}{8}\gamma^2\right) \\ &+\left(\frac{3}{4}-\frac{15}{8}e^2-\frac{15}{8}e'^2-\frac{3}{8}\gamma^2\right)\cos(2\lambda-2\lambda') \\ &-\frac{1}{2}e\cos(\lambda-\tilde{\omega})-\frac{9}{4}e\cos(\lambda-2\lambda'+\tilde{\omega})+\frac{3}{4}e\cos(3\lambda-2\lambda'-\tilde{\omega}) \\ &+\frac{3}{4}e'\cos(\lambda'-\tilde{\omega}')-\frac{3}{8}e'\cos(2\lambda-\lambda'-\tilde{\omega})+\frac{21}{8}e'\cos(2\lambda-3\lambda'+\tilde{\omega}') \end{aligned}$$

$$-\frac{1}{8}e^2\cos(2\lambda-2\tilde{\omega})+\frac{15}{8}e^2\cos(2\lambda'-2\tilde{\omega})+\frac{3}{4}e^2\cos(4\lambda-2\lambda'-2\tilde{\omega})$$

$$+\frac{3}{8}\gamma^2\cos(2\lambda-2\Omega)+\frac{3}{8}\gamma^2\cos(2\lambda'-2\Omega)$$

$$+\frac{3}{8}\frac{a}{a'}\cos(\lambda-\lambda')+\frac{5}{8}\frac{a}{a'}\cos(3\lambda-3\lambda')-\frac{15}{16}\frac{a}{a'}e\cos(\lambda'-\tilde{\omega})$$

$$+\cdots\Big]. \qquad (22)$$

where letters without a prime designate the elements of the moon and letters with a prime that of the sun. In the disturbing function, $\sin I$ is represented as γ.

By integrating equations (19) after applying the function, R, periodic change of each element can be obtained. In actual calculation of an amplitude of certain component, it is sufficient to take only combinations of component and disturbing terms which constitute the argument of considering component.

Let us now consider the effect of a term of which argument has the period T in the disturbing function. This term will bring about changes of some elements with the period T. When the elements are contained in the expression of the amplitude or in the argument of a tidal component and when the component has the period S, the component will yield two different components of periods $T+S$ and $T-S$. This fact is easily understood by formulae of the elementary trigonometry or simply known as beat phenomena. Here, the amplitude of ν_2 wave is taken into consideration as an example.

Since the argument of ν_2 wave has the form $(2\theta-3\lambda-\tilde{\omega}+2\lambda')$, it is probable that the argument will be produced from the following three cases of combination between tidal components and terms of the disturbing function because the sum or difference of both arguments become $(2\theta-3\lambda-\tilde{\omega}+2\lambda')$.

Case 1.

disturbing term $R_1=\frac{15}{8}n'^2a^2e^2\cos(2\lambda'-2\tilde{\omega})$,

component $N_2=\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_3+s_3)\cos(2\theta-3\lambda+\tilde{\omega})$.

Case 2.

disturbing term $R_2=-\frac{9}{4}n'^2a^2e\cos(\lambda-2\lambda'+\tilde{\omega})$,

component $M_2=\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_2+s_2)\cos(2\theta-2\lambda)$.

Case 3.

$$\begin{cases} \text{disturbing term} & R_3 = n'^2 a^2\left(\frac{3}{4} - \frac{15}{8}e^2 - \frac{15}{8}e'^2 - \frac{3}{8}\gamma^2\right)\cos(2\lambda - 2\lambda'), \\ \text{component} & L_2 = \frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I)^2(c_1+s_1)\cos(2\theta - \lambda - \tilde{\omega}), \end{cases}$$

here, the names of the waves N_2, M_2, etc. are used as analytical expressions of components.

In the Case 1, by applying R_1 instead of R, equations (19) yield

$$\left.\begin{aligned} \frac{de}{dt} &= -\frac{n'^2}{n}\frac{15}{4}pe\sin(2\lambda' - 2\tilde{\omega}), \\ \frac{dI}{dt} &= -\frac{n'^2}{n}\frac{15}{4}\frac{\tan(I/2)}{p}e^2\sin(2\lambda' - 2\tilde{\omega}), \\ \frac{d\sigma}{dt} &= \frac{n'^2}{n}\frac{15}{4}\{p(1-p) - 2e^2\}\cos(2\lambda' - 2\tilde{\omega}), \\ \frac{d\tilde{\omega}}{dt} &= \frac{n'^2}{n}\frac{15}{4}p\cos(2\lambda' - 2\tilde{\omega}), \\ \frac{d^2\rho}{dt} &= 0. \end{aligned}\right\} \tag{23}$$

By integration, periodic changes of the elements δe_1, δI_1, $\delta\sigma_1$ $\delta\tilde{\omega}_1$ and $\delta\rho_1$ produced from R_1 may be written as

$$\left.\begin{aligned} \delta e_1 &= \frac{m^2}{m-(1-c)}\frac{15}{8}pe\cos(2\lambda' - 2\tilde{\omega}), \\ \delta I_1 &= \frac{m^2}{m-(1-c)}\frac{15}{8}\frac{\tan(I/2)}{p}e^2\cos(2\lambda' - 2\tilde{\omega}), \\ \delta\sigma_1 &= \frac{m^2}{m-(1-c)}\frac{15}{8}\{p(1-p) - 2e^2\}\sin(2\lambda' - 2\tilde{\omega}\}, \\ \delta\tilde{\omega}_1 &= \frac{m^2}{m-(1-c)}\frac{15}{8}p\cos(2\lambda' - 2\tilde{\omega}), \\ \delta\rho_1 &= 0, \end{aligned}\right\} \tag{24}$$

where $m = n'/n = 0.07480\ 13243$
and $n(1-c)$ represents the mean angular velocity of the forward motion of the moon's perigee. It is known that $(1-c)$ is expressed as a power series of m (BROWER and CLEMENCE, 1961b). That is,

$$1 - c = \frac{3}{4}m^2 + \frac{225}{32}m^3 + \frac{4071}{128}m^4 + \frac{265493}{2048}m^5 + \cdots\cdots. \tag{25}$$

Although the convergence of the series is rather slow, the value finally becomes

$$1-c=0.00857\ 25730\,.$$

As $\delta\rho_1$ is equal to zero, the relation

$$\delta\lambda_1=\delta\sigma_1 \tag{26}$$

hold in this case.

The total effect of periodic changes of the elements for N_2 wave produced by R_1 may be written in the first order of approximation as

$$\begin{aligned}
&\frac{\partial N_2}{\partial e}\delta e_1+\frac{\partial N_2}{\partial I}\delta I_1+\frac{\partial N_2}{\partial \lambda}\delta\lambda_1+\frac{\partial N_2}{\partial \tilde{\omega}}\delta\tilde{\omega}_1\\
&\quad=\frac{1}{32}(1+\cos I)^2(1+\cos\varepsilon)^2(c_3+s_3)\times\frac{15}{8}\frac{m^2}{m-(1-c)}\\
&\quad\times\Big[\frac{pe}{c_3+s_3}\frac{\partial}{\partial e}(c_3+s_3)\cos(2\theta-3\lambda+\tilde{\omega})\cos(2\lambda'-2\tilde{\omega})\\
&\quad-\frac{2e^2}{p}\tan^2(I/2)\cos(2\theta-3\lambda+\tilde{\omega})\cos(2\lambda'-2\tilde{\omega})\\
&\quad+\{3p(1-p)-6e^2\}\sin(2\theta-3\lambda+\tilde{\omega})\sin(2\lambda'-2\tilde{\omega})\\
&\quad-p\sin(2\theta-3\lambda+\tilde{\omega})\sin(2\lambda'-2\tilde{\omega})\Big]\,.
\end{aligned}\tag{27}$$

Then, the potion of ν_2 component separated from N_2 wave is obtained by taking out the terms of which argument is $(2\theta-3\lambda-\tilde{\omega}+2\lambda')$ from the relation (27). This potion is denoted by $\nu_2(N_2)$ hereafter. As the values of e and γ are in the same order of magnitude, it is convenient for the coefficients in the bracket to be expressed as a power series of them. After some calculations, it becomes

$$\begin{aligned}
\nu_2(N_2)&=\frac{1}{32}(1+\cos I_0)^2(1+\cos\varepsilon)^2(c_3+s_3)_0\times\frac{15}{8}\frac{m^2}{m-(1-c)}\\
&\quad\times\left(1-\frac{55}{56}e_0^2-\frac{4643}{3136}e_0^4-\frac{1}{4}e_0^2\gamma_0^2+\cdots\right)\cos(2\theta-3\lambda_0-\tilde{\omega}_0+2\lambda'_0)\,,
\end{aligned}\tag{28}$$

where suffix "0" designates the average values for the elements e, I and γ, and the potions of linear function of the time for λ, $\tilde{\omega}$ and λ'. The notation $(c_j+s_j)_0$ represents (c_j+s_j) replaced e with e_0.

In the same manner as above, the potions of ν_2 component from Case 2 and Case 3 can be calculated. The results are as follows;

$$\begin{aligned}
\nu_2(M_2)&=\frac{1}{32}(1+\cos I_0)^2(1+\cos\varepsilon)^2(c_2+s_2)_0\times\frac{9m^2}{1-2m+(1-c)}\\
&\quad\times e_0\left(1+\frac{3}{2}m+\frac{39}{16}m^2-\frac{1}{16}\gamma_0^2-\frac{9}{8}e_0^2+\cdots\right)\cos(2\theta-3\lambda_0-\tilde{\omega}_0+2\lambda'_0)\,,
\end{aligned}$$

$$\left.\begin{aligned}\nu_2(L_2)=&\frac{1}{32}(1+\cos I_0)^2(1+\cos\varepsilon)^2(c_1+s_1)_0\times\left(-\frac{39}{16}\right)\frac{m^2}{1-m}\\&\times\left(1+\frac{3}{13}m+\frac{3}{13}m^2-\frac{4}{13}\gamma_0^2-\frac{41}{26}e_0^2-\frac{20}{13}e_0'^2+\cdots\right)\\&\qquad\qquad\times\cos(2\theta-3\lambda_0-\tilde{\omega}_0+2\lambda_0')\,.\end{aligned}\right\}\qquad(29)$$

The meaning of the new notations are self-explanatory.

The total amplitude of ν_2 wave can be obtained as the sum of these three results. It is given as

(The amplitude of ν_2 wave)

$$=\frac{105}{256}(1+\cos I_0)^2(1+\cos\varepsilon)^2\left(1+\frac{323}{140}m+\frac{2837}{224}m^2-\frac{37}{14}e_0^2+\cdots\right).\qquad(30)$$

The convergence of the series in the last parentheses is slow and inadequate for the numerical calculation of the amplitude. However, the precise computation for the above three cases explains 95 per cent of amplitude of ν_2 wave given by CARTWRIGHT and EDDEN (1973). The remaining 5 per cent is supposed to be caused by the effect of higher terms of the perturbing function or of perturbations above the second order.

6. The Corrections for the Amplitude Coefficient

The numerical value of the amplitude of each component may be obtained by substituting the mean values of the eccentricity, e_0, and the inclination, I_0, into the relations given in Section 4. The agreement with the results given by CARTWRIGHT and EDDEN (1973) is generally excellent. There are, however, some components which show slight differences. For example, the amplitude coefficient of N_2 wave becomes 0.17475 in our calculation while the result of CARTWRIGHT and EDDEN is 0.17386. The difference reaches about 0.5 per cent of the amplitude. However, it is explained by considering the effect of the perturbation for the orbital elements of the moon.

Since N_2 wave has the argument $(2\theta-3\lambda+\tilde{\omega})$, it is evident that the components of the same argument are also produced from the combination of other wave components and terms of the disturbing function. Considering the disturbing function given in (22), the following five combinations probably produce the components of which argument is $(2\theta-3\lambda+\tilde{\omega})$. That is;

wave	argument	term of the disturbing function
M_2	$(2\theta-2\lambda)$	$-\frac{1}{2}n'^2a^2e\cos(\lambda-\tilde{\omega})$,
$2N_2$	$(2\theta-4\lambda+2\tilde{\omega})$	〃
L_2	$(2\theta-\lambda-\tilde{\omega})$	$-\frac{1}{8}n'^2a^2e^2\cos(2\lambda-2\tilde{\omega})$,
225.855	$(2\theta-5\lambda+3\tilde{\omega})$	〃
265.675	$(2\theta-\lambda+\tilde{\omega}+2\Omega)$	$\frac{3}{8}n'^2a^2\gamma^2\cos(2\lambda-2\Omega)$.

The effect of these perturbation can be calculated as in the same manner as given in the previous section. By actual calculation, the correction factor for the amplitude of N_2 wave is obtained. The correction comes mainly from the first three combinations. They are written as

$$\left.\begin{aligned} N_2(2N_2)&=(\text{The amplitude of } 2N_2 \text{ wave})\\ &\quad\times\left(-\frac{m^2}{e}\right)\left(1+\frac{3}{4}m^2+\frac{621}{68}e^2+\frac{225}{32}m^3+\cdots\right),\\ N_2(M_2)&=(\text{The amplitude of } M_2 \text{ wave})\\ &\quad\times\frac{9}{2}m^2e\left(1+m^2+\frac{1}{4}e^2+\frac{75}{8}m^3+\cdots\right),\\ N_2(L_2)&=(\text{The amplitude of } L_2 \text{ wave})\\ &\quad\times\left(-\frac{1}{8}m^2\right)\left(1+\frac{3}{4}m^2-\frac{19}{8}e^2+\frac{225}{32}m^3+\cdots\right). \end{aligned}\right\} \tag{31}$$

By adding the corrections, the analytical form of N_2 wave becomes

$$N_2=\frac{1}{32}(1+\cos\varepsilon)^2(1+\cos I_0)^2(c_3+s_3)_0\left(1-\frac{63}{56}m^2+\frac{123}{224}m^4+\cdots\right)$$
$$\times\cos(2\theta-3\lambda+\tilde{\omega}), \tag{32}$$

and the correction factor for the amplitude of N_2 wave in Section 4 takes the form

$$1-\frac{63}{56}m^2+\frac{123}{224}m^4+\cdots.$$

The numerical calculation of the amplitude thus obtained agrees with the result of CARTWRIGHT and EDDEN with an accuracy of 0.1 per cent. The disagreements of numerical values found in the other components will be explained in the same way.

7. Conclusion

The analytical forms of principal components in the tide-generating potential are presented in this paper. They are given in Tables 2, 3 and

4. In fact, the results given in Section 4 are only approximate ones and slight corrections are necessary. For obtaining results of higher accuracy, we must taken into consideration the effect of the perturbation for the orbital elements. However, the calculation of correction factors given here is quite troublesome and mistakes are apt to be made. Moreover, the expansion by a power series of m, e and γ is not always the best because the convergence is rather slow, The situation is the same for the calculation of components generated as the results of beat such as ν_2 and μ_2 waves. As a result, only a few expressions are given which show the effect of the perturbation in this paper. It seems better to find some other method of calculation in considering the effect of the perturbation.

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17. 起潮力ポテンシャルにおける分潮振幅の解析的表現

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起潮力ポテンシャルの調和展開は，数値的にはすでに完了していて，各分潮の振幅係数が決定されている。しかし，その振幅が解析的にどのような形をとっているのかということについては，不明の点が多かった。ここでは，起潮力ポテンシャルを解析的に展開することを試み，月，太陽の軌道要素などをパラメーターとして，振幅係数の解析的な形を求めることを行った。

計算を容易にするため，はじめは，月の軌道要素のうち，離心率 e，軌道傾斜角 I は一定とし，昇交点黄経 Ω，近地点引数 ω は時間に関して一様に変化するという仮定をおいた。この場合の計算は比較的に簡単で，主要分潮のほとんどについて，その振幅を解析的な形に表現することができた。太陽に起因する分潮も，ほぼ同様な方法で計算ができた。この近似は現実にかなり近いので，数値的に計算すると，得られた振幅は，過去に求められた振幅の数値と，2.3% 以内の誤差で一致する。

しかし，現実には存在するが，上記の近似だけではあらわれてこない種類の分潮もたくさんある。これらの分潮についての計算をするためには，太陽の引力によって月の軌道が摂動を受け，軌道要素が周期的に変化するという条件を考慮しなければならない。ここでは，定数変化法によってその計算を行った。この方法はまた，最初の近似で得られた振幅を，より精度の高い形に修正するのにも応用できる。

摂動をとりいれるこの計算はやや手間がかかるが，一次の摂動だけを考慮した 2, 3 の計算例では，摂動によって生ずる項の振幅の 95% を説明することができ，また，はじめの近似で得た分潮の振幅を，0.1% ていどの誤差にまで高めることができた。